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On equitorsion geodesic mappings of general affine connection spaces onto generalized Riemannian spaces

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ABSTRACT

In the papers Minčić (1973) [15], Minčić (1977) [16], several Ricci type identities are obtained by using non-symmetric affine connection. Four kinds of covariant derivatives appear in these identities.

In the present work, we consider equitorsion geodesic mappings f of two spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$, where $\mathbb{G}\overline{\mathbb{R}}_N$ has a non-symmetric metric tensor, i.e. we study the case when $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ have the same torsion tensors at corresponding points. Such a mapping is called an equitorsion mapping Minčić (1997) [12], Stanković et al. (2010) [14], Stanković (in press) [13].

The existence of a mapping of such type implies the existence of a solution of the fundamental equations. We find several forms of these fundamental equations. Among these forms a particularly important form is system of partial differential equations of Cauchy type.

1. Introduction

A generalized Riemannian space $\mathbb{GR}_N$ is a differentiable N-dimensional manifold, equipped with non-symmetric metric tensor g_{ij}. Generalized Cristoffel's symbols of the first kind of the space $\mathbb{GR}_N$ are given by the formula

$$\Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}),$$

where, for example, $g_{ij,k} = \partial g_{ij}/\partial x^k$. Connection coefficients of the space $\mathbb{GR}_N$ are the generalized Cristoffel's symbols of the second kind $\Gamma^i_{jk} = g^{\underline{ip}}\Gamma_{p.jk}$, where $(g^{\underline{ij}}) = (g_{\underline{ij}})^{-1}$ and $\underline{ij}$ denotes a symmetrization with division with respect to the indices i and j. Generally we have $\Gamma^i_{jk} \neq \Gamma^i_{kj}$. We suppose that $\underline{\overline{g}} = \det(\overline{g}_{\underline{ij}}) \neq 0$, $\underline{g} = \det(g_{\underline{ij}}) \neq 0$. A general affine connection space $\mathbb{GA}_N$ is a differentiable N-dimensional manifold, with non-symmetric connection coefficients L^i_{jk}.

Geodesic mappings and their generalizations were investigated by many authors, for example: Sinyukov [1], Mikeš [2–6], Kiosak [5], Vanžurová [5,6], Berezovski [4], Hinterleitner [6], Hall and Lonie [7–9], Prvanović [10], Minčić [11–13], Stanković [11–14] and many others.

Many authors asked if it makes sense to consider geodesic mappings between two spaces with non-symmetric connections whereas the definition of geodesics includes only symmetric connections. In [11], Minčić and Stanković showed that it is possible. This fact enables further consideration of geodesic mappings when the connection is non-symmetric (see [11–13]).

Let us consider two N-dimensional manifolds $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ and differentiable mapping

$$f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{R}}_N.$$

We can consider these manifolds in the *common system of local coordinates* with respect to this mapping (see Fig. 1.1). Namely, if $f : M \in \mathbb{GA}_N \to \overline{M} \in \mathbb{G}\overline{\mathbb{R}}_N$ and if $(\mathcal{U}, \varphi)$ is local chart around the point M, it will be $\varphi(M) = x = (x^1, \ldots, x^N)$

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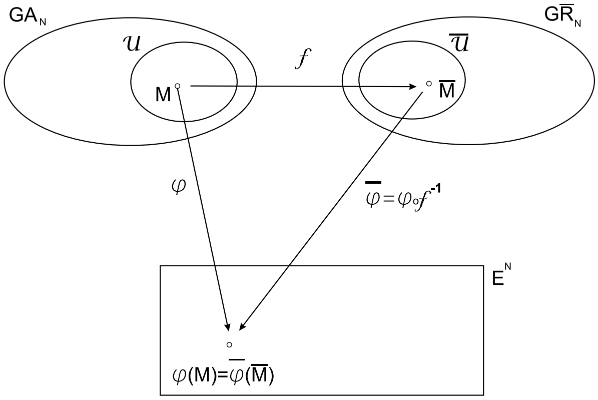

Fig. 1.1. Manifolds in the common system of local coordinates.

$\in \mathbb{E}^N$ (Euclidean N-space). In this case, we define for the coordinate mapping in the $\mathbb{G}\overline{\mathbb{R}}_N$ the mapping $\overline{\varphi} = \varphi \circ f^{-1}$, and then

$$\overline{\varphi}(\overline{M}) = (\varphi \circ f^{-1})\,(f(M)) = \varphi(M) = x = (x^1, \dots, x^N) \in \mathbb{E}^N, \tag{1.1}$$

wherefore the points M and $\overline{M} = f(M)$ have the same local coordinates.

A *geodesic mapping* [3–5,11,12] of $\mathbb{GA}_N$ onto $\mathbb{G}\overline{\mathbb{R}}_N$ is a diffeomorphism $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ under which the geodesics of the space $\mathbb{GA}_N$ correspond to the geodesics of the space $\mathbb{G}\overline{\mathbb{R}}_N$. At the corresponding points M and $\overline{M}$, we can put

$$\overline{\Gamma}^i_{jk} = L^i_{jk} + P^i_{jk}, \quad (i, j, k = 1, \dots, N), \tag{1.2}$$

where P^i_{jk} is *the deformation tensor* of the connection L^i_{jk} of $\mathbb{GA}_N$ according to the mapping $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{R}}_N$. The tensor P^i_{jk} is non-symmetric with respect to the indices j and k, because L^i_{jk} and $\overline{\Gamma}^i_{jk}$ are non-symmetric.

A necessary and sufficient condition for the mapping f to be geodesic [11] is that the deformation tensor P^i_{jk} from (1.2) has the form

$$P^i_{jk} = \delta^i_j \psi_k + \delta^i_k \psi_j + \xi^i_{jk}, \tag{1.3}$$

where

$$\psi_i = \frac{1}{N+1}(\overline{\Gamma}^\alpha_{i\alpha} - L^\alpha_{i\alpha}), \qquad \xi^i_{jk} = P^i_{\underset{\vee}{jk}} = \frac{1}{2}(P^i_{jk} - P^i_{kj}). \tag{1.4}$$

We remark that in $\mathbb{GR}_N$ the condition below holds true (see [11]):

$$\Gamma^\alpha_{\underset{\vee}{i\alpha}} = \frac{1}{2}\Gamma^\alpha_{[i\alpha]} = 0, \tag{1.5}$$

where $\underline{ij}$ denotes the symmetrization, $\underset{\vee}{ij}$-antisymmetrization, $[i \dots j]$ denotes the antisymmetrization without division with respect to the indices i, j, and also $(i \dots j)$ denotes the symmetrization without division with respect to the indices i, j.

In $\mathbb{GA}_N$ ($\mathbb{GR}_N$), one can define four kinds of covariant derivatives [15,16]. For example, for a tensor a^i_j, we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + L^i_{pm}a^p_j - L^p_{jm}a^i_p, \qquad a^i_{j\underset{2}{|}m} = a^i_{j,m} + L^i_{mp}a^p_j - L^p_{mj}a^i_p,$$

$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + L^i_{pm}a^p_j - L^p_{mj}a^i_p, \qquad a^i_{j\underset{4}{|}m} = a^i_{j,m} + L^i_{mp}a^p_j - L^p_{jm}a^i_p.$$

Remark 1.1. Let $\mathbb{GA}_N$ be an N-dimensional differentiable manifold, on which a non-symmetric affine connection L^i_{jk} is introduced. Because of the non-symmetry of the connection L^i_{jk}, another connection can be defined by $\widetilde{L}^i_{jk} = L^i_{kj}$.

Denote by $\underset{\theta}{|}, \underset{\theta}{\overline{|}}$ the covariant derivative of the kind θ, $(\theta = 1, 2, 3, 4)$ in $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$, respectively.

Whereas in a Riemannian space (the space of General Relativity Theory), the connection coefficients are expressed in terms of the symmetric metric tensor g_{ij}, in Einstein's work in Unified Field Theories (1950–1955), the relation between these magnitudes is determined by the following equation:

$$g_{\underset{+-}{ij};m} \equiv g_{ij,m} - \Gamma^p_{im}g_{pj} - \Gamma^p_{mj}g_{ip} = 0, \quad \left(g_{ij,m} = \frac{\partial g_{ij}}{\partial x^m}\right). \tag{1.6}$$

In the Eq. (1.6), the index i behaves in the sense of the first kind of derivative $(\underset{1}{|})$, and the index j in the sense of the second one $(\underset{2}{|})$.

Einstein in [17], 1950, for the covariant curvature tensor in his theory obtains a Bianchi-type identity:

$$R_{\underset{-+-+}{iklm};n} + R_{\underset{-+++}{ikmn};l} + R_{\underset{-+--}{iknl};m} = 0, \tag{1.7}$$

where $R_{iklm} = g_{pi}R^{p}_{klm}$, and the indices behave in the sense as explained in the comment just below relation (1.6).

In the case of the space $\mathbb{GA}_N$ ($\mathbb{GR}_N$), we have five independent curvature tensors [18,19] (in [18] $\underset{5}{R}$ is denoted by $\underset{2}{\tilde{R}}$):

$$\begin{aligned}
\underset{1}{R}{}^{i}_{jmn} &= L^{i}_{j[m,n]} + L^{p}_{j[m}L^{i}_{pn]},\\
\underset{2}{R}{}^{i}_{jmn} &= L^{i}_{[mj,n]} + L^{p}_{[mj}L^{i}_{n]p},\\
\underset{3}{R}{}^{i}_{jmn} &= L^{i}_{jm,n} - L^{i}_{nj,m} + L^{p}_{jm}L^{i}_{np} - L^{p}_{nj}L^{i}_{pm} + L^{p}_{nm}L^{i}_{[pj]},\\
\underset{4}{R}{}^{i}_{jmn} &= L^{i}_{jm,n} - L^{i}_{nj,m} + L^{p}_{jm}L^{i}_{np} - L^{p}_{nj}L^{i}_{pm} + L^{p}_{mn}L^{i}_{[pj]},\\
\underset{5}{R}{}^{i}_{jmn} &= \frac{1}{2}(L^{i}_{j[m,n]} + L^{i}_{[mj,n]} + L^{p}_{jm}L^{i}_{pn} + L^{p}_{mj}L^{i}_{np} - L^{p}_{jn}L^{i}_{mp} - L^{p}_{nj}L^{i}_{pm}).
\end{aligned} \tag{1.8}$$

In a Riemannian space, the Eq. (1.3) is equivalent to Levi-Civita's equation (see [1]):

$$\overline{g}_{ij;k} = 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik}, \tag{1.9}$$

where (;) is the covariant derivative in the space $\mathbb{R}_N$, i.e. $\overline{g}_{ij;k} = \partial\overline{g}_{ij}/\partial x^k - \Gamma^{p}_{ik}\overline{g}_{pj} - \Gamma^{p}_{jk}\overline{g}_{ip}$, and Γ is the Levi-Civita's connection.

Theorem 1.1 (*[11]*)**.** *Generalized Riemannian space $\mathbb{GR}_N$ admits nontrivial geodesic mappings onto generalized Riemannian space $\mathbb{G}\overline{\mathbb{R}}_N$ if and only if for the metric tensor of the space $\mathbb{G}\overline{\mathbb{R}}_N$ is valid:*

$$\overline{g}_{ij\underset{1}{|}k} = \overline{g}_{\underset{\vee}{ij}\,\underset{1}{\bar{|}}k} + 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{kj} + \psi_j\overline{g}_{ik} + \xi^{p}_{ik}\overline{g}_{pj} + \xi^{p}_{jk}\overline{g}_{ip}, \tag{1.10}$$

where $(\underset{1}{|})$ and $(\underset{1}{\bar{|}})$ are covariant derivatives in the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$, respectively.

The condition (1.3) is equivalent to (1.10). It can easily be seen that for the second, third and fourth kind of covariant derivatives equations similar to (1.10) can be derived.

2. Equitorsion geodesic mappings

A geodesic mapping $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ is an *equitorsion geodesic mapping* if the torsion tensors of the spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ are equal in the common local coordinates. Then from (1.2)–(1.4), we get

$$\overline{\Gamma}^{h}_{\underset{\vee}{ij}} - L^{h}_{\underset{\vee}{ij}} = \xi^{h}_{ij} = 0, \tag{2.1}$$

where $\underset{\vee}{ij}$ denotes the antisymmetrization with respect to the indices i, j (see [12–14]).

Mikeš and Berezovski proved in [3–5] the following theorem:

Theorem 2.1. *The manifold with affine connection $\mathbb{A}_N$ admits geodesic mapping onto Riemannian manifold $\overline{R}_N$ with the metric tensor $\overline{g}_{ij}$ if and only if the following set of differential equations of Cauchy type with covariant derivatives has a solution with respect to the symmetric tensor: $\overline{g}_{ij}$, $(\det(\overline{g}_{ij}) \neq 0)$, the covector ψ_i and the function μ.*

$$\begin{aligned}
&\text{(a) } \overline{g}_{ij;k} = 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik};\\
&\text{(b) } N\psi_{i;j} = N\psi_i\psi_j + \mu\overline{g}_{ij} - \overline{g}^{\beta\gamma}\overline{g}_{i\alpha}R^{\alpha}_{\beta\gamma j} - R_{ij} - \frac{2}{N+1}R^{\alpha}_{\alpha ij};\\
&\text{(c) } (N-1)\mu_{;i} = 2(N-1)\psi_\alpha\overline{g}^{\beta\gamma}R^{\alpha}_{\beta\gamma i} + \psi_\alpha\overline{g}^{\alpha\beta}\left(5R_{\beta i} + \frac{6}{N+1}R^{\gamma}_{\gamma\beta i} - R_{i\beta}\right)\\
&\qquad\qquad + \overline{g}^{\alpha\beta}\left(R^{\gamma}_{\alpha\beta i;\gamma} - R_{\alpha i;\beta} - \frac{2}{N+1}R^{\gamma}_{\gamma\alpha i}\right),
\end{aligned} \tag{2.2}$$

where (;) denotes covariant derivative with respect to the connection of $\mathbb{A}_N$, $(\overline{g}^{ij})$ is the matrix inverse to $(\overline{g}_{ij})$, R^{h}_{ijk}, R_{ij} are respectively Riemannian and Ricci tensors of the manifold $\mathbb{A}_N$, and $R^{\alpha}_{j} = g^{\beta\gamma}R^{\alpha}_{\beta\gamma j}$, $R^{\alpha\ \beta}_{ij.} = g^{\beta\gamma}R^{\alpha}_{ij\gamma}$, $R^{\alpha\ \beta}_{i;.} = g^{\beta\gamma}R^{\alpha}_{i;\gamma}$ and $R^{\alpha\ \beta}_{.;i} = g^{\beta\gamma}R^{\alpha}_{\gamma;i}$.

We give some generalizations of this theorem in the case of manifolds with a non-symmetric metric tensor. From (1.10) and (2.1), we have

$$\overline{g}_{\underline{ij}\underset{1}{|}k} = 2\psi_k\overline{g}_{\underline{ij}} + \psi_i\overline{g}_{\underline{kj}} + \psi_j\overline{g}_{\underline{ik}} = \overline{g}_{\underline{ij}\underset{2}{|}k}. \tag{2.3}$$

Further, we obtain

$$\overline{g}_{\underline{ij}\underset{1}{|}ks} - \overline{g}_{\underline{ij}\underset{1}{|}sk} = 2\overline{g}_{\underline{ij}}\underset{1}{\psi}_{[ks]} + \overline{g}_{\underline{k(i}}\underset{1}{\psi}_{j)s} - \overline{g}_{\underline{s(i}}\underset{1}{\psi}_{j)k}, \tag{2.4}$$

where $\underset{1}{\psi}_{jk} = \psi_{j\underset{1}{|}k} - \psi_j\psi_k$. Using the appropriate Ricci identity [15], from (2.4), one gets

$$\overline{g}_{\underline{ij}\underset{1}{|}ks} - \overline{g}_{\underline{ij}\underset{1}{|}sk} = -\overline{g}_{\underline{i\alpha}}\underset{1}{R}^{\alpha}_{jks} - \overline{g}_{\underline{j\alpha}}\underset{1}{R}^{\alpha}_{iks} - L^p_{[ks]}\overline{g}_{\underline{ij}\underset{1}{|}p},$$

i.e.

$$-\overline{g}_{\underline{i\alpha}}\underset{1}{R}^{\alpha}_{jks} - \overline{g}_{\underline{j\alpha}}\underset{1}{R}^{\alpha}_{iks} - L^p_{[ks]}(2\psi_p\overline{g}_{\underline{ij}} + \psi_i\overline{g}_{\underline{pj}} + \psi_j\overline{g}_{\underline{ip}}) = 2\overline{g}_{\underline{ij}}\underset{1}{\psi}_{[ks]} + \overline{g}_{\underline{k(i}}\underset{1}{\psi}_{j)s} - \overline{g}_{\underline{s(i}}\underset{1}{\psi}_{j)k}. \tag{2.5}$$

Transvecting the last equation by $\overline{g}^{\underline{ij}}$, we get

$$\underset{1}{\psi}_{\{ks\}} = -\frac{1}{N+1}\underset{1}{R}^{\alpha}_{\alpha ks}, \tag{2.6}$$

where $\underset{1}{\psi}_{\{ks\}} = \underset{1}{\psi}_{[ks]} + \psi_p L^p_{[ks]}$. Replacing (2.6) in (2.5), we obtain

$$-\overline{g}_{(\underline{i\alpha}}\underset{1}{R}^{\alpha}_{j)ks} + \frac{2}{N+1}\overline{g}_{\underline{ij}}\underset{1}{R}^{\alpha}_{\alpha ks} - L^p_{[ks]}\psi_{(i}\overline{g}_{\underline{pj})} = \overline{g}_{\underline{k(i}}\underset{1}{\psi}_{j)s} - \overline{g}_{\underline{s(i}}\underset{1}{\psi}_{j)k}. \tag{2.7}$$

Transvecting this equation by $\overline{g}^{\underline{jk}}$, we get

$$-\overline{g}^{\underline{jk}}\overline{g}_{\underline{i\alpha}}\underset{1}{R}^{\alpha}_{jks} + \underset{1}{R}_{is} + \frac{2}{N+1}\underset{1}{R}^{\alpha}_{\alpha is} - L^p_{[ps]}\psi_i - \overline{g}^{\underline{jk}}\overline{g}_{\underline{ip}}L^p_{[ks]}\psi_j = N\underset{1}{\psi}_{is} - \overline{g}^{\underline{jk}}\overline{g}_{\underline{si}}\underset{1}{\psi}_{jk}. \tag{2.8}$$

Using (1.5) and (2.1), we get

$$N\psi_{i\underset{1}{|}j} = N\psi_i\psi_j + \underset{1}{\mu}\,\overline{g}_{\underline{ij}} - \overline{g}^{\underline{\beta\gamma}}\overline{g}_{\underline{i\alpha}}\underset{1}{R}^{\alpha}_{\beta\gamma j} + \underset{1}{R}_{ij} + \frac{2}{N+1}\underset{1}{R}^{\alpha}_{\alpha ij} - \psi^{\beta}L_{i.[\beta j]}, \tag{2.9}$$

where $\underset{1}{\mu} = \overline{g}^{\underline{jk}}\underset{1}{\psi}_{jk}$ and $\psi^j = \overline{g}^{\underline{ij}}\psi_i$. Because of $\overline{g}^{\underline{ik}}\overline{g}_{\underline{ij}} = \delta^k_j$, one obtains

$$\overline{g}^{\underline{ij}}_{\underset{1}{|}k} = -2\psi_k\overline{g}^{\underline{ij}} - \delta^i_k\psi^j - \delta^j_k\psi^i = \overline{g}^{\underline{ij}}_{\underset{2}{|}k}. \tag{2.10}$$

From (2.9), we obtain

$$\begin{aligned} N\psi_{i\underset{1}{|}jk} - N\psi_{i\underset{1}{|}kj} = {} & N\psi_{i\underset{1}{|}k}\psi_j + N\psi_i\psi_{j\underset{1}{|}k} + \underset{1}{\mu}_{\underset{1}{|}k}\overline{g}_{\underline{ij}} + \underset{1}{\mu}\,\overline{g}_{\underline{ij}\underset{1}{|}k} - \overline{g}^{\underline{\beta\gamma}}_{\underset{1}{|}k}\overline{g}_{\underline{i\alpha}}\underset{1}{R}^{\alpha}_{\beta\gamma j} - \overline{g}^{\underline{\beta\gamma}}\overline{g}_{\underline{i\alpha}\underset{1}{|}k}\underset{1}{R}^{\alpha}_{\beta\gamma j} \\ & - \overline{g}^{\underline{\beta\gamma}}\overline{g}_{\underline{i\alpha}}\underset{1}{R}^{\alpha}_{\beta\gamma j\underset{1}{|}k} + \underset{1}{R}_{ij\underset{1}{|}k} + \frac{2}{N+1}\underset{1}{R}^{\alpha}_{\alpha ij\underset{1}{|}k} - \psi^{\beta}_{\underset{1}{|}k}L_{i.[\beta j]} - \psi^{\beta}L_{i.[\beta j]\underset{1}{|}k} \\ & - N\psi_{i\underset{1}{|}j}\psi_k - N\psi_i\psi_{k\underset{1}{|}j} - \underset{1}{\mu}_{\underset{1}{|}j}\overline{g}_{\underline{ik}} - \underset{1}{\mu}\,\overline{g}_{\underline{ik}\underset{1}{|}j} + \overline{g}^{\underline{\beta\gamma}}_{\underset{1}{|}j}\overline{g}_{\underline{i\alpha}}\underset{1}{R}^{\alpha}_{\beta\gamma k} + \overline{g}^{\underline{\beta\gamma}}\overline{g}_{\underline{i\alpha}\underset{1}{|}j}\underset{1}{R}^{\alpha}_{\beta\gamma k} \\ & + \overline{g}^{\underline{\beta\gamma}}\overline{g}_{\underline{i\alpha}}\underset{1}{R}^{\alpha}_{\beta\gamma k\underset{1}{|}j} - \underset{1}{R}_{ik\underset{1}{|}j} - \frac{2}{N+1}\underset{1}{R}^{\alpha}_{\alpha ik\underset{1}{|}j} + \psi^{\beta}_{\underset{1}{|}j}L_{i.[\beta k]} + \psi^{\beta}L_{i.[\beta k]\underset{1}{|}j}. \end{aligned} \tag{2.11}$$

Taking into account (2.3), (2.9), (2.10), contracting with $\overline{g}^{\underline{ij}}$ in (2.11) and using the corresponding Ricci identity [15], we get that the left side of the Eq. (2.11) is

$$\mathcal{L} = -N\overline{g}^{\underline{ij}}\psi_p\underset{1}{R}^{p}_{ijk} - N\overline{g}^{\underline{ij}}L^p_{[jk]}\psi_{i\underset{1}{|}p}$$

and the right side is

$$\begin{aligned} \mathcal{D} = {} & (N-2)\overline{g}^{\underline{\beta\gamma}}\underset{1}{R}^{\alpha}_{\beta\gamma k}\psi_\alpha + 4\overline{g}^{\underline{ij}}\underset{1}{R}_{ik}\psi_j + \frac{6}{N+1}\overline{g}^{\underline{ij}}\underset{1}{R}^{\alpha}_{\alpha ik}\psi_j + (N-3)\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\psi_\alpha\psi_\gamma \\ & + (N-1)\underset{1}{\mu}_{\underset{1}{|}k} - \psi^{\gamma}\underset{1}{R}^{\alpha}_{\alpha\gamma k} + \overline{g}^{\underline{\beta\gamma}}\underset{1}{R}^{\alpha}_{\beta\gamma k\underset{1}{|}\alpha} - \overline{g}^{\underline{ij}}\underset{1}{R}_{ik\underset{1}{|}j} - \frac{2}{N+1}\overline{g}^{\underline{ij}}\underset{1}{R}^{\alpha}_{\alpha ik\underset{1}{|}j} + \overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]\underset{1}{|}\gamma}\psi_\alpha + \overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\psi_{\alpha\underset{1}{|}\gamma}. \end{aligned} \tag{2.12}$$

From $\mathcal{L} = \mathcal{D}$, we get

$$\begin{aligned}(N-1)\underset{1}{\mu}_{\underset{1}{|}k} = {} & -2(N-1)\overline{g}^{\underline{\beta\gamma}}\underset{1}{R}{}^{\alpha}_{\beta\gamma k}\psi_\alpha - 4\overline{g}^{\underline{ij}}\underset{1}{R}_{ik}\psi_j - \frac{6}{N+1}\overline{g}^{\underline{ij}}\underset{1}{R}{}^{\alpha}_{\alpha ik}\psi_j - (N-3)\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\psi_\alpha\psi_\gamma \\ & + \psi^\gamma\underset{1}{R}{}^{\alpha}_{\alpha\gamma k} - \overline{g}^{\underline{\beta\gamma}}\underset{1}{R}{}^{\alpha}_{\beta\gamma k\underset{1}{|}\alpha} + \overline{g}^{\underline{ij}}\underset{1}{R}_{ik\underset{1}{|}j} + \frac{2}{N+1}\overline{g}^{\underline{ij}}\underset{1}{R}{}^{\alpha}_{\alpha ik\underset{1}{|}j} - \overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]\underset{1}{|}\gamma}\psi_\alpha \\ & - \frac{(N+1)}{N}\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\left(N\psi_\alpha\psi_\gamma - \overline{g}^{\underline{sq}}\overline{g}_{\underline{\alpha p}}\underset{1}{R}{}^{p}_{sq\gamma} + \underset{1}{R}_{\alpha\gamma} + \frac{2}{N+1}\underset{1}{R}{}^{p}_{p\alpha\gamma}\right).\end{aligned} \tag{2.13}$$

In $\mathbb{GA}_N$, (see [20]), the following is valid:

$$\underset{jmn}{\mathfrak{S}}\underset{1}{R}{}^{i}_{jmn} = \underset{1}{R}{}^{i}_{jmn} + \underset{1}{R}{}^{i}_{mnj} + \underset{1}{R}{}^{i}_{njm} = \underset{jmn}{\mathfrak{S}}(L^{i}_{[jm],n} + L^{p}_{[jm]}L^{i}_{pn}), \tag{2.14}$$

and finally, replacing in (2.13), we get

$$\begin{aligned}(N-1)\underset{1}{\mu}_{\underset{1}{|}k} = {} & -2(N-1)\overline{g}^{\underline{\alpha\beta}}\psi_p\underset{1}{R}{}^{p}_{\alpha\beta k} - \psi_\alpha\overline{g}^{\alpha\beta}\left(5\underset{1}{R}_{\beta k} + \frac{6}{N+1}\underset{1}{R}{}^{\gamma}_{\gamma\beta k} - \underset{1}{R}_{k\beta}\right) \\ & - \overline{g}^{\alpha\beta}\left(\underset{1}{R}{}^{\gamma}_{\alpha\beta k\underset{1}{|}\gamma} - \underset{1}{R}_{\alpha k\underset{1}{|}\beta} - \frac{2}{N+1}\underset{1}{R}{}^{\gamma}_{\gamma\alpha k\underset{1}{|}\beta}\right) - 2(N-1)\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\psi_\alpha\psi_\gamma - \overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]\underset{1}{|}\gamma}\psi_\alpha \\ & - \frac{(N+1)}{N}\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\left(-\overline{g}^{\underline{sq}}\overline{g}_{\underline{\alpha p}}\underset{1}{R}{}^{p}_{sq\gamma} + \underset{1}{R}_{\alpha\gamma} + \frac{2}{N+1}\underset{1}{R}{}^{p}_{p\alpha\gamma}\right) + \psi_\alpha\overline{g}^{\alpha\beta}\underset{q\beta k}{\mathfrak{S}}L^{p}_{[q\beta]}L^{q}_{pk}.\end{aligned} \tag{2.15}$$

So, the next theorem is proved.

Theorem 2.2. *If the manifold with general affine connection $\mathbb{GA}_N$ admits equitorsion geodesic mapping onto generalized Riemannian manifold $\mathbb{G}\overline{\mathbb{R}}_N$ with the metric tensor $\overline{g}_{ij}$, then the following set of differential equations with covariant derivatives of the first kind of Cauchy type has a solution with respect to the symmetric tensor $\overline{g}_{\underline{ij}}$, the covector ψ_i and the function $\underset{1}{\mu}$:*

$$\begin{aligned} & \text{(a) } \overline{g}_{\underline{ij}\underset{1}{|}k} = 2\psi_k\overline{g}_{\underline{ij}} + \psi_i\overline{g}_{\underline{kj}} + \psi_j\overline{g}_{\underline{ik}}; \\ & \text{(b) } N\psi_{i\underset{1}{|}j} = N\psi_i\psi_j + \underset{1}{\mu}\,\overline{g}_{\underline{ij}} - \overline{g}^{\underline{\beta\gamma}}\overline{g}_{\underline{i\alpha}}\underset{1}{R}{}^{\alpha}_{\beta\gamma j} + \underset{1}{R}_{ij} + \frac{2}{N+1}\underset{1}{R}{}^{\alpha}_{\alpha ij} - \overline{g}^{\underline{\alpha\beta}}\overline{g}_{\underline{\gamma i}}L^{\gamma}_{[\beta j]}\psi_\alpha; \\ & \text{(c) } (N-1)\underset{1}{\mu}_{\underset{1}{|}k} = -2(N-1)\overline{g}^{\underline{\alpha\beta}}\psi_p\underset{1}{R}{}^{p}_{\alpha\beta k} - \psi_\alpha\overline{g}^{\alpha\beta}\left(5\underset{1}{R}_{\beta k} + \frac{6}{N+1}\underset{1}{R}{}^{\gamma}_{\gamma\beta k} - \underset{1}{R}_{k\beta}\right) \\ & \qquad - \overline{g}^{\alpha\beta}\left(\underset{1}{R}{}^{\gamma}_{\alpha\beta k\underset{1}{|}\gamma} - \underset{1}{R}_{\alpha k\underset{1}{|}\beta} - \frac{2}{N+1}\underset{1}{R}{}^{\gamma}_{\gamma\alpha k\underset{1}{|}\beta}\right) - 2(N-1)\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\psi_\alpha\psi_\gamma - \overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]\underset{1}{|}\gamma}\psi_\alpha \\ & \qquad - \frac{(N+1)}{N}\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\left(-\overline{g}^{\underline{sq}}\overline{g}_{\underline{\alpha p}}\underset{1}{R}{}^{p}_{sq\gamma} + \underset{1}{R}_{\alpha\gamma} + \frac{2}{N+1}\underset{1}{R}{}^{p}_{p\alpha\gamma}\right) + \psi_\alpha\overline{g}^{\alpha\beta}\underset{q\beta k}{\mathfrak{S}}L^{p}_{[q\beta]}L^{q}_{pk}.\end{aligned} \tag{2.16}$$

Following this procedure, the next theorems can be proved.

Theorem 2.3. *If the manifold with general affine connection $\mathbb{GA}_N$ admits equitorsion geodesic mapping onto generalized Riemannian manifold $\mathbb{G}\overline{\mathbb{R}}_N$ with the metric tensor $\overline{g}_{ij}$, then the following set of differential equations with covariant derivatives of the second kind of Cauchy type has a solution with respect to the symmetric tensor $\overline{g}_{\underline{ij}}$, the covector ψ_i and the function:* $\underset{2}{\mu} = \overline{g}^{\underline{jk}}\underset{2}{\psi}_{jk}$

$$\begin{aligned} & \text{(a) } \overline{g}_{\underline{ij}\underset{2}{|}k} = 2\psi_k\overline{g}_{\underline{ij}} + \psi_i\overline{g}_{\underline{kj}} + \psi_j\overline{g}_{\underline{ik}}; \\ & \text{(b) } N\psi_{i\underset{2}{|}j} = N\psi_i\psi_j + \underset{2}{\mu}\,\overline{g}_{\underline{ij}} - \overline{g}^{\underline{\beta\gamma}}\overline{g}_{\underline{i\alpha}}\underset{2}{R}{}^{\alpha}_{\beta\gamma j} + \underset{2}{R}_{ij} + \frac{2}{N+1}\underset{2}{R}{}^{\alpha}_{\alpha ij} + \overline{g}^{\underline{\alpha\beta}}\overline{g}_{\underline{\gamma i}}L^{\gamma}_{[\beta j]}\psi_\alpha; \\ & \text{(c) } (N-1)\underset{2}{\mu}_{\underset{2}{|}k} = -2(N-1)\overline{g}^{\underline{\alpha\beta}}\psi_p\underset{2}{R}{}^{p}_{\alpha\beta k} - \psi_\alpha\overline{g}^{\alpha\beta}\left(5\underset{2}{R}_{\beta k} + \frac{6}{N+1}\underset{2}{R}{}^{\gamma}_{\gamma\beta k} - \underset{2}{R}_{k\beta}\right) \\ & \qquad - \overline{g}^{\alpha\beta}\left(\underset{2}{R}{}^{\gamma}_{\alpha\beta k\underset{2}{|}\gamma} - \underset{2}{R}_{\alpha k\underset{2}{|}\beta} - \frac{2}{N+1}\underset{2}{R}{}^{\gamma}_{\gamma\alpha k\underset{2}{|}\beta}\right) + 2(N-1)\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\psi_\alpha\psi_\gamma + \overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]\underset{2}{|}\gamma}\psi_\alpha \\ & \qquad + \frac{(N+1)}{N}\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\left(-\overline{g}^{\underline{sq}}\overline{g}_{\underline{\alpha p}}\underset{2}{R}{}^{p}_{sq\gamma} + \underset{2}{R}_{\alpha\gamma} + \frac{2}{N+1}\underset{2}{R}{}^{p}_{p\alpha\gamma}\right) + \psi_\alpha\overline{g}^{\alpha\beta}\underset{q\beta k}{\mathfrak{S}}L^{p}_{[\beta q]}L^{q}_{kp}.\end{aligned} \tag{2.17}$$

Theorem 2.4. *If the manifold with general affine connection $\mathbb{GA}_N$ admits equitorsion geodesic mapping onto generalized Riemannian manifold $\mathbb{G}\overline{\mathbb{R}}_N$ with the metric tensor $\overline{g}_{ij}$, then the following set of differential equations with covariant derivatives of the third kind of Cauchy type has a solution with respect to the symmetric tensor $\overline{g}_{\underline{ij}}$, the covector ψ_i and the function $\underset{3}{\mu} = \overline{g}^{\underline{jk}} \underset{3}{\psi}_{jk}$*

$$
\begin{aligned}
&\text{(a) } \overline{g}_{\underline{ij}\underset{3}{|}k} = 2\psi_k \overline{g}_{\underline{ij}} + \psi_i \overline{g}_{\underline{kj}} + \psi_j \overline{g}_{\underline{ik}};\\
&\text{(b) } N\psi_{i\underset{3}{|}j} = N\psi_i\psi_j + \underset{3}{\mu}\,\overline{g}_{\underline{ij}} - \overline{g}^{\underline{\beta\gamma}}\overline{g}_{\underline{i\alpha}}\underset{2}{R}{}^{\alpha}_{\beta\gamma j} + \underset{2}{R}{}_{ij} + \frac{2}{N+1}\underset{2}{R}{}^{\alpha}_{\alpha ij} + \overline{g}^{\underline{\alpha\beta}}\overline{g}_{\underline{\gamma i}}L^{\gamma}_{[\beta j]}\psi_\alpha;\\
&\text{(c) } (N-1)\underset{3}{\mu}{}_{\underset{3}{|}k} = -2(N-1)\overline{g}^{\underline{\alpha\beta}}\psi_p\underset{2}{R}{}^{p}_{\alpha\beta k} - \psi_\alpha\overline{g}^{\alpha\beta}\left(5\underset{2}{R}{}_{\beta k} + \frac{6}{N+1}\underset{2}{R}{}^{\gamma}_{\gamma\beta k} - \underset{2}{R}{}_{k\beta}\right)\\
&\qquad - \overline{g}^{\alpha\beta}\left(\underset{2}{R}{}^{\gamma}_{\alpha\beta k\underset{3}{|}\gamma} - \underset{2}{R}{}_{\alpha k\underset{3}{|}\beta} - \frac{2}{N+1}\underset{2}{R}{}^{\gamma}_{\gamma\alpha k\underset{3}{|}\beta}\right) + 2(N-1)\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\psi_\alpha\psi_\gamma + \overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]\underset{3}{|}\gamma}\psi_\alpha\\
&\qquad + \frac{(N+1)}{N}\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\left(-\overline{g}^{\underline{sq}}\overline{g}_{\underline{\alpha p}}\underset{2}{R}{}^{p}_{sq\gamma} + \underset{2}{R}{}_{\alpha\gamma} + \frac{2}{N+1}\underset{2}{R}{}^{p}_{p\alpha\gamma}\right) + \psi_\alpha\overline{g}^{\alpha\beta}\underset{q\beta k}{\mathfrak{S}}L^{p}_{[\beta q]}L^{q}_{kp}.
\end{aligned}
\tag{2.18}
$$

Theorem 2.5. *If the manifold with general affine connection $\mathbb{GA}_N$ admits equitorsion geodesic mapping onto generalized Riemannian manifold $\mathbb{G}\overline{\mathbb{R}}_N$ with the metric tensor $\overline{g}_{ij}$, then the following set of differential equations with covariant derivatives of the fourth kind of Cauchy type has a solution with respect to the symmetric tensor $\overline{g}_{\underline{ij}}$, the covector ψ_i and the function $\underset{4}{\mu} = \overline{g}^{\underline{jk}} \underset{4}{\psi}_{jk}$*

$$
\begin{aligned}
&\text{(a) } \overline{g}_{\underline{ij}\underset{4}{|}k} = 2\psi_k \overline{g}_{\underline{ij}} + \psi_i \overline{g}_{\underline{kj}} + \psi_j \overline{g}_{\underline{ik}};\\
&\text{(b) } N\psi_{i\underset{4}{|}j} = N\psi_i\psi_j + \underset{4}{\mu}\,\overline{g}_{\underline{ij}} - \overline{g}^{\underline{\beta\gamma}}\overline{g}_{\underline{i\alpha}}\underset{1}{R}{}^{\alpha}_{\beta\gamma j} + \underset{1}{R}{}_{ij} + \frac{2}{N+1}\underset{1}{R}{}^{\alpha}_{\alpha ij} - \overline{g}^{\underline{\alpha\beta}}\overline{g}_{\underline{\gamma i}}L^{\gamma}_{[\beta j]}\psi_\alpha;\\
&\text{(c) } (N-1)\underset{4}{\mu}{}_{\underset{4}{|}k} = -2(N-1)\overline{g}^{\underline{\alpha\beta}}\psi_p\underset{1}{R}{}^{p}_{\alpha\beta k} - \psi_\alpha\overline{g}^{\alpha\beta}\left(5\underset{1}{R}{}_{\beta k} + \frac{6}{N+1}\underset{1}{R}{}^{\gamma}_{\gamma\beta k} - \underset{1}{R}{}_{k\beta}\right)\\
&\qquad - \overline{g}^{\alpha\beta}\left(\underset{1}{R}{}^{\gamma}_{\alpha\beta k\underset{4}{|}\gamma} - \underset{1}{R}{}_{\alpha k\underset{4}{|}\beta} - \frac{2}{N+1}\underset{1}{R}{}^{\gamma}_{\gamma\alpha k\underset{4}{|}\beta}\right) - 2(N-1)\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\psi_\alpha\psi_\gamma - \overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]\underset{4}{|}\gamma}\psi_\alpha\\
&\qquad - \frac{(N+1)}{N}\overline{g}^{\underline{\alpha\beta}}L^{\gamma}_{[\beta k]}\left(-\overline{g}^{\underline{sq}}\overline{g}_{\underline{\alpha p}}\underset{1}{R}{}^{p}_{sq\gamma} + \underset{1}{R}{}_{\alpha\gamma} + \frac{2}{N+1}\underset{1}{R}{}^{p}_{p\alpha\gamma}\right) + \psi_\alpha\overline{g}^{\alpha\beta}\underset{q\beta k}{\mathfrak{S}}L^{p}_{[q\beta]}L^{q}_{pk}.
\end{aligned}
\tag{2.19}
$$

Systems (2.16)–(2.19) have no more than one solution for the following initial condition at the point x_0:

$$\overline{g}_{\underline{ij}}(x_0) = \overset{0}{\overline{g}}_{\underline{ij}}, \qquad \psi_i(x_0) = \overset{0}{\psi}_i, \qquad \underset{\theta}{\mu}(x_0) = \underset{\theta}{\overset{0}{\mu}}, \quad \theta = 1, 2, 3, 4.$$

General solutions of Eqs. (2.16)–(2.19) depend on a finite number of substantial parameters

$$r \le r_0 \equiv \frac{(N+1)\,(N+2)}{2}.$$

Finding all solutions of (2.16)–(2.19) requires considering their integrability conditions and differential extensions, which form a set of algebraic equations with respect to the unknown functions $\overline{g}_{\underline{ij}}$, ψ_i and $\underset{\theta}{\mu}$, $\theta = 1, 2, 3, 4$, with coefficient from $\mathbb{GA}_N$. But this would certainly be a fairly difficult work to be done.

3. Conclusion

We consider equitorsion geodesic mappings [12–14] and give new generalizations of the mapping $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{R}}_N$. In this way, we extend some recently obtained results from [3–6] where geodesic mappings were investigated of an affine connected space onto a Riemannian space (in the symmetric case).

As corollaries, we get extensions of the corresponding results concerning geodesic mappings of an affine connected space onto a Riemannian space from [3–6] using a non-symmetric metric tensor and the four kinds of covariant derivatives. We also use the techniques developed in cited papers.

We emphasize the following results of the paper:

It is possible to extend the concept of a geodesic mapping of an affine connected space onto a Riemannian space, by considering equitorsion geodesic mappings. In this way, equitorsion geodesic mappings are available for a wider class of metrics. It is reasonable to expect that these facts will be a motivation in some further investigations of geodesic mappings, and generally for all extensions from the $(\mathbb{A}_N)$ $\mathbb{R}_N$ into the $(\mathbb{GA}_N)$ $\mathbb{GR}_N$ spaces.

In this paper, we got four systems of PDEs of Cauchy type in $\mathbb{GA}_N$. Perhaps in future work we can consider solutions of these systems.

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