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Note

On the crossing numbers of certain generalized Petersen graphs*

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Abstract

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In his paper on the crossing numbers of generalized Petersen graphs, Fiorini proves that $P(8, 3)$ has crossing number 4 and claims at the end that $P(10, 3)$ also has crossing number 4. In this article, we give a short proof of the first claim and show that the second claim is false. The techniques are interesting in that they focus on disjoint cycles, which must cross each other an even number of times.

1. Introduction

In his very interesting paper [3], Fiorini needs, as an inductive base, that the generalized Petersen graph $P(8, 3)$ has crossing number 4. His proof [3, pp. 234–236] is a tedious case-by-case analysis, and ends with the remark, 'The other cases are similarly dealt with'. One of the goals of this article is to provide a short, complete proof of this result.

In the concluding section of the article, Fiorini states, 'It follows from our conclusions that $\mathrm{cr}(3h+1, 3) = \cdots = h+1$, where $\mathrm{cr}(G)$ is the crossing number of the graph G [3, p. 240]. This implies that the graph $P(10, 3)$ has crossing number 4. However, in the article, it is only proved that $4 \leqslant \mathrm{cr}(P(10, 3)) \leqslant 6$. The other goal of the present work is to show that the concluding statement is false, by providing a short proof that $\mathrm{cr}(P(10, 3)) \geqslant 5$.

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The techniques we use are based on the rather obvious idea that if C and C' are vertex-disjoint cycles, then, in any drawing Φ in the plane, $|\Phi(C) \cap \Phi(C')|$ must be even. To prove the above results, then, it suffices to find appropriate cycles in the graphs.

The particular graphs are symmetric in the sense that for any two vertices u and v and any two edges e and f, there are automorphisms θ and φ such that $\theta(u) = v$ and $\varphi(e) = f$. This symmetry is a big help in the analysis. (The graph $P(8, 3)$ is number 16 in [1], while $P(10, 3)$ is 20B.)

One further point should be made: the graph $P(8, 3)$ is a counterexample to [5, Conjecture p. 374]. That is, $P(8, 3)$ has crossing number $k = 4$, but no proper subgraph has crossing number 3. In [5], it is proved that every cubic graph with crossing number at least 3 has a subgraph with crossing number exactly 2. Moreover, the cartesian product $C_3 \times C_3$ has crossing number 3, but every proper subgraph has crossing number at most 1. Thus, the degree restriction in [5] is essential.

In Sections 2 and 3, we prove that $P(8, 3)$ has crossing number 4, while in Section 4, we prove that $\mathrm{cr}(P(10, 3)) \geqslant 5$.

2. Technical results for $P(8, 3)$

We shall make repeated use of the following facts about $P = P(8, 3)$.

Fact 1. *If $p = (u, v, w)$ and $q = (x, y, z)$ are any two paths of length 2 in P, then there is an automorphism θ of P such that $\theta(u)$, $\theta(v)$ and $\theta(w)$ are, respectively, x, y and z.*

Reason. This is an easy consequence of [6, Theorem 7.54] □

Fact 2. *P is bipartite.*

For the labelling of P, refer to Fig. 1.

Fact 3. *The only paths of length at most 4 joining the vertices v_3 and v_8 are (v_8, v_1, v_2, v_3) and (v_8, u_8, u_1, v_3).*

Reason. Because P is bipartite, all paths have the same parity, so we need only concern ourselves with paths of length 3. Enumerate them. □

Fact 4. *The only paths of length at most 4 joining v_2 and v_8 are (v_2, v_1, v_8), $(v_2, u_6, u_5, v_7, v_8)$, $(v_2, u_6, u_7, u_8, v_8)$ and $(v_2, v_3, u_1, u_8, v_8)$.*

Fact 5. *The girth of P is 6.*

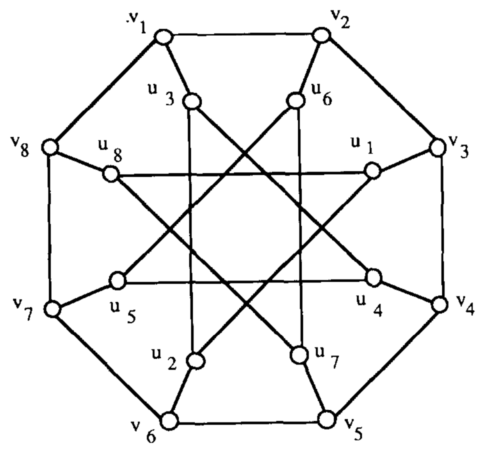

Fig. 1.

3. Crossing number of $P(8, 3)$ is four

In this section, we shall prove the following result, which is the first goal of this article.

Theorem 6. $\mathrm{cr}(P) = 4$ *and if* G *is any proper subgraph of* P, *then* $\mathrm{cr}(G) \leq 2$.

Our initial attack is via the *removal number* of a graph. For a graph G, the *removal number* $r(G)$ of G is the smallest nonnegative integer r such that the removal of some r edges from G results in a planar subgraph of G. For a drawing Φ of G, let $\mathrm{cr}(\Phi)$ denote the number of crossings in Φ. By removing an edge from each crossing of a drawing of G in the plane we get a set of edges whose removal leaves a planar subgraph. Thus we have the following.

Lemma 7. *For any drawing* Φ *of* G, $\mathrm{cr}(\Phi) \geq r(G)$.

Lemma 8. $r(P) \geq 3$.

Proof. Let $r = r(P)$ and let P' be a planar subgraph of P having $24 - r$ edges. It is easy to see that P' is a connected spanning subgraph of P. By Euler's formula, in any planar drawing of P', there are $10 - r$ faces. Since P' has girth at least $6, 6(10 - r) \leq 2(24 - r)$, so $r \geq 3$. □

We will encounter this type of argument several times. To save space, we shall simply say: 'Use Euler's formula'.

Corollary 8.1. *If* $\mathrm{cr}(P) \leq 3$, *then, in an optimal drawing of* P, *no edge is crossed twice.*

Lemma 9. *If v is any vertex of P, then $r(P-v) \geqslant 2$.*

This can be proved using Euler's formula.

Corollary 9.1. *If Φ is a drawing of P with* $\mathrm{cr}(\Phi)=3$, *then the* 6 *edges involved form a matching in P.*

Proof of Theorem 6. Let Φ be a drawing of P such that $\mathrm{cr}(\Phi)=\mathrm{cr}(P)$. We assume $\mathrm{cr}(\Phi) \leqslant 3$ and derive a contradiction, which will prove that $\mathrm{cr}(P) \geqslant 4$. By Lemma 7, there must be at least 3 crossings in the drawing Φ. Hence, we can assume $\mathrm{cr}(\Phi)=3$.

Because P is edge-transitive, we can assume that $e=v_1v_2$ crosses some edge f in the drawing Φ. Delete e and an edge from each of the other two crossings. This produces a planar subgraph P', with an embedding in the plane inherited from Φ. By Euler's formula, every face of this drawing of P' is bounded by a 6-cycle. Let C and C' be the face boundaries containing the edge f. As no edge is crossed twice in Φ, v_1 and v_2 each occur in one of C and C', one in each. Without loss of generality, we can assume that the drawing of $(C \cup C')+e$ is that shown in Fig. 2. (Any other possible addition of e either is equivalent or introduces a cycle of length less than 6 to P.)

By considering all the possibilities (making good use of Facts 3 and 4), one finds that, up to isomorphism, there is only one possibility for the labelling of $(C \cup C')+e$ (this is shown in Fig. 3). Therefore, we can assume that one of the three crossings of Φ is v_1v_2 with u_1u_8.

Let C_1, C_2 and C_3 be the 6-cycles $(v_1, v_2, v_3, v_4, u_4, u_3)$, $(v_1, v_2, u_6, u_5, v_7, v_8)$ and $(u_1, u_8, u_7, v_5, v_6, u_2)$, respectively. We note that each of C_1 and C_2 is vertex-disjoint from C_3. Both of them cross C_3 at the known crossing. Since disjoint cycles cannot cross only once, each C_i must cross C_3 somewhere else,

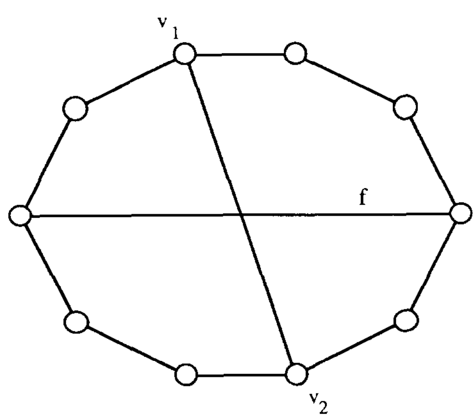

Fig. 2.

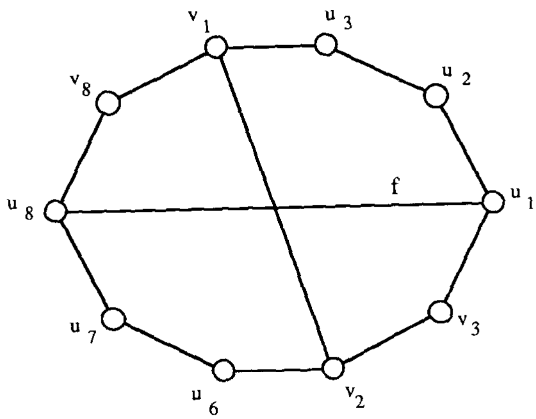

Fig. 3.

$i=1, 2$. Let this be at the edge e_i of C_3. By Corollary 8.1, and the fact that C_1 and C_2 have only the edge v_1v_2 in common, the edges u_1u_8, e_1 and e_2 are all distinct. Moreover, every crossing of Φ involves one of them.

By Corollary 9.1, no two of the edges u_1u_8, e_1 and e_2 is incident with a common vertex. Therefore, $\{e_1, e_2\} = \{u_2v_6, u_7v_5\}$. A subdivision of $K_{3,3}$ contained in $P - \{u_1u_8, u_7v_5, u_2v_6\}$ is exhibited in Fig. 4. This contradiction completes the proof that $\text{cr}(\Phi) \geqslant 4$.

We now provide the finishing touches to our analysis of P. We see in Fig. 5 that $\text{cr}(P) \leqslant 4$ and, moreover that the edge $e = v_8u_8$ is crossed twice in a drawing having only 4 crossings. It follows that $\text{cr}(P) = 4$ and that $\text{cr}(P - e) = 2$. Since P is edge-transitive, this last equation holds for any edge of P. Therefore, no proper subgraph of P has crossing number 3. □

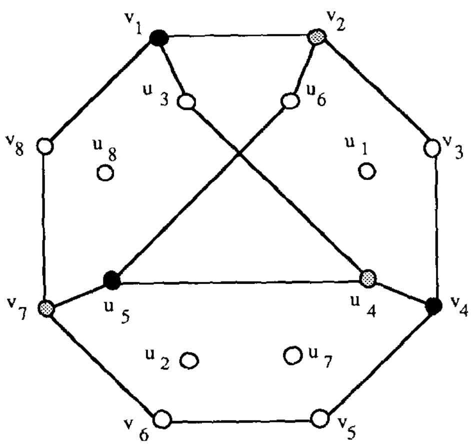

Fig. 4.

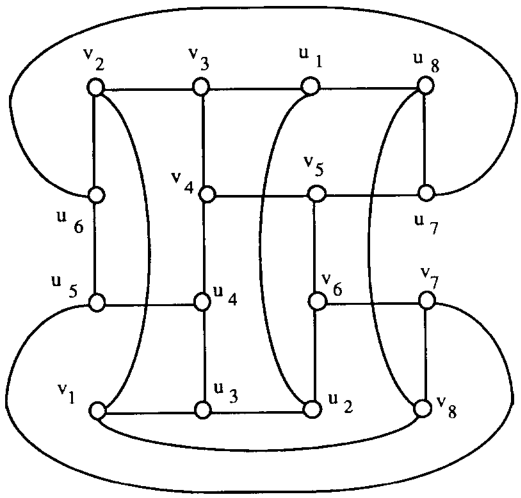

Fig. 5.

4. Crossing number of $P(10, 3)$ is at least five

In this section, we shall prove the following.

Theorem 10. $\mathrm{cr}(P(10, 3)) \geqslant 5$.

The proof is based on the following facts. For brevity, let $Q = P(10, 3)$.

Fact 11. *If $p = (x_1, x_2, x_3, x_4)$ and $q = (y_1, y_2, y_3, y_4)$ are any two paths of length 3 in Q, then there is an automorphism θ of Q such that $\theta(x_i) = y_i$, for $i = 1, 2, 3, 4$.*

Fact 12. *For any two edges e and f of Q, not both incident with the same vertex, there are 6-cycles C_1, C_2 and C_3 such that $C_1 \cup C_2$ is vertex-disjoint from C_3, $C_1 \cap C_2$ consists of a path of length 2 containing e and f is in C_3.*

Reason. We refer to the labelling in Fig. 6. Without loss of generality, we can assume $e = u_1v_1$. Let θ be the automorphism of Q that reflects about the line through e and u_6v_6.

For any edge f not incident with either u_1 or v_1, either f or $\theta(f)$ is in one of the 6-cycles $C_1 = (v_8, v_9, u_7, u_8, u_9, u_{10})$, $C_2 = (v_6, v_7, v_8, v_9, u_7, u_6)$ and $C_3 = (v_7, v_8, v_9, v_{10}, u_4, u_3)$.

The 6-cycles containing e of interest are $C_4 = (v_1, v_2, v_3, v_4, u_2, u_1)$, $C_5 = (v_1, u_1, u_2, u_3, u_4, v_{10})$ and $C_6 = (v_1, v_2, u_8, u_9, u_{10}, u_1)$. Observe that $C_4 \cup C_5$ is disjoint from C_1 and C_2, while $C_4 \cap C_6$ is disjoint from C_3. $\square$

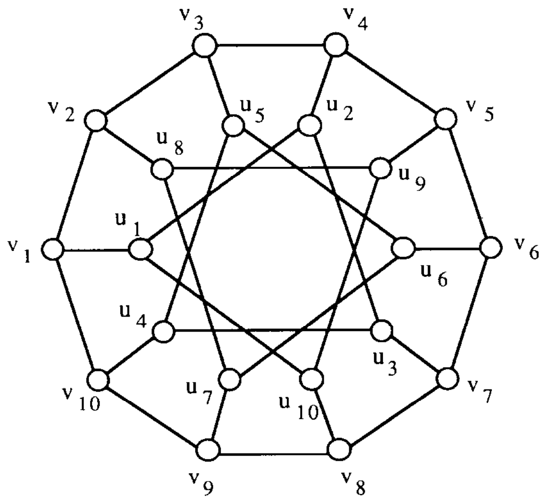

Fig. 6.

Fact 13. *Let e and f be edges of Q such that some 6-cycle has edge-sequence $(e, e_2, f, e_4, e_5, e_6)$. Then there is an edge g of Q such that $Q - \{e, f, g\}$ is a subdivision of $P(7, 3)$.*

Reason. The deletion of the edges v_1u_1, v_2u_8, v_3u_5 yields a subdivision of $P(7, 3)$. By Fact 11, there is an automorphism of Q that maps the path with edge-sequence (u_1v_1, v_1v_2, v_2u_8) onto the path with edge-sequence (e, e_2, f), so u_1v_1 maps to e and v_2u_8 maps to f. Then g is the image of v_3u_5. □

The final fact we need is less immediate.

Lemma 14. *If u is any vertex of Q, then* $\mathrm{cr}(Q - u) = 3$.

Proof. Without loss of generality, we can assume $u = u_4$. See Fig. 7 for a drawing of $Q - u$ with only three crossings. Partition $E(Q - u)$ into the sets (suppressing the three degree-two vertices):

$$A_1 = \{v_1u_1, u_2v_4, v_3v_2, v_9v_8, v_7v_6, u_6u_7\};$$
$$A_2 = \{u_1u_2, v_4v_3, v_2v_1, v_8v_7, v_6u_6, u_7v_9\};$$
$$A_3 = \{v_1v_9, u_2v_7, v_3u_6\};$$
$$A_4 = \{u_1u_{10}, u_{10}v_8, v_4v_5, v_5v_6, v_2u_8, u_8u_7\};$$
$$A_5 = \{u_9u_{10}, u_9v_5, u_9u_8\}.$$

It is readily observed from Fig. 7 that if $e, f \in A_i$, then there is an automorphism θ of $Q - u$ such that $\theta(e) = f$. Every crossing in an optimal drawing of $Q - u$ must involve an edge not in A_5. Thus, it suffices to show that if $e \in \bigcup_{i=1}^{4} A_i$, then $\mathrm{cr}(Q - u - e) \geqslant 2$.

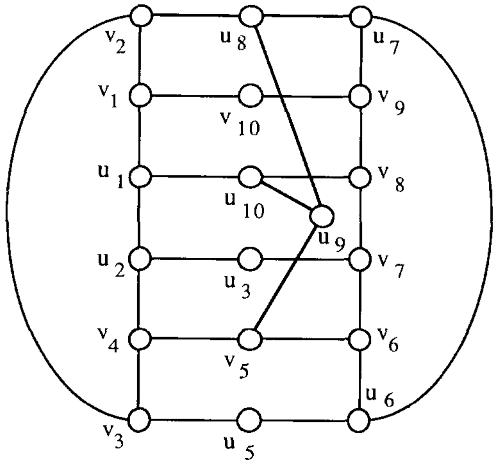

Fig. 7.

Let $G_1 = Q - u - \{v_2v_3, v_7v_8, v_1v_9\}$. Then G_1 is a subdivision of the graph G_5 of [4, Fig. 1], which is an irreducible graph for the real projective plane. See Fig. 8(a). As explained in [5], this graph has crossing number 2. This accounts for e in one of A_1, A_2, and A_3. Similarly, the graph $G_2 = Q - u - \{u_1u_{10}, v_4v_3, v_9v_8\}$ is a subdivision of the same G_5. See Fig. 8(b). This accounts for e in A_4.

Since the deletion of any edge of $Q - u$ in $\bigcup_{i=1}^{4} A_i$ produces a graph with crossing number at least two, we see that $Q - u$ must have crossing number at least 3, as claimed. □

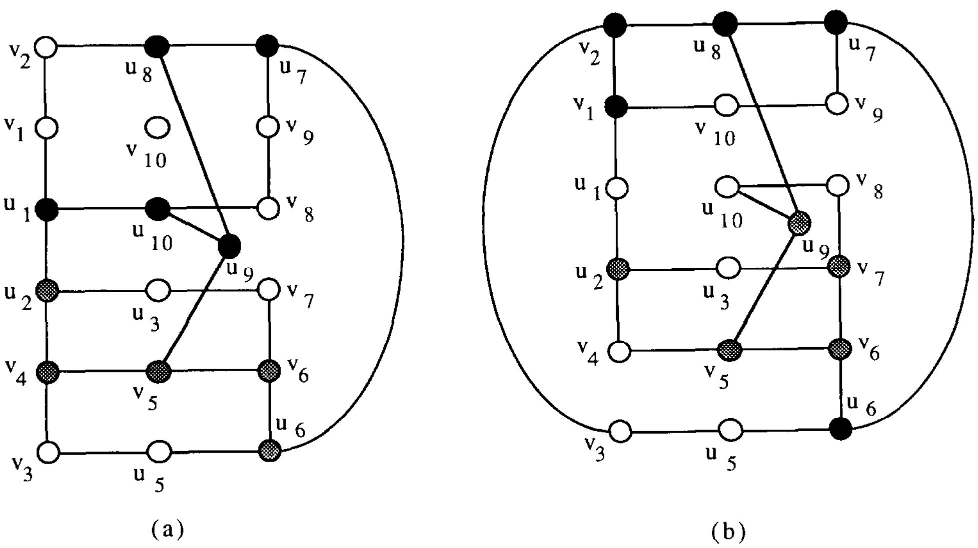

Fig. 8.

We are now prepared to prove Theorem 10.

Proof of Theorem 10. Suppose there is a drawing Φ of Q such that $\mathrm{cr}(\Phi) \leq 4$. If u is any vertex of Q, then Φ induces a drawing of $Q-u$, which, by Lemma 14, must have at least three crossings. Therefore, among the edges of Q incident with u, there can be at most one crossing in total.

Now suppose e and f cross in Φ. Then e and f are not incident with a common vertex, so, by Fact 12, there are 6-cycles C_1, C_2 and C_3 with C_3 disjoint from $C_1 \cup C_2$, f in C_3 and $C_1 \cap C_2$ consisting of a path of length 2 containing e.

Since C_1 and C_3 cross (e crosses f), and C_1 is disjoint from C_3, there is another crossing involving other edges of C_1 and C_3. Similarly, there is another crossing involving other edges of C_2 and C_3. These two other crossings must, in fact, be distinct, since not both the edges in their common path can be in crossings, by a remark in the first paragraph of the proof. Let C_3 have the edge-sequence $(f, f_2, f_3, f_4, f_5, f_6)$; there are at least three edges crossed in this cycle. Since no two of them are incident with a common vertex and one of them is f, they must be f, f_3 and f_5.

Observe that f and f_3 satisfy the hypotheses of Fact 13, so that $Q' = Q - \{f, f_3\}$ contains a subdivision of $P(7, 3)$. There is a drawing of Q' induced by Φ that has $\mathrm{cr}(\Phi) - 2$ crossings—this number is at most 2. Therefore, $\mathrm{cr}(P(7, 3)) \leq 2$. But this contradicts the known fact that $\mathrm{cr}(P(7, 3)) = 3$ [2]. This shows no such drawing Φ exists and we conclude that $\mathrm{cr}(P(10, 3)) \geq 5$. □

5. Concluding remarks

The question of finding a formula for $\mathrm{cr}(P(n, k))$ was raised in [3]. In [2], $\mathrm{cr}(P(n, 2))$ is evaluated and in [3], $\mathrm{cr}(P(n, 3))$ is computed for $n \not\equiv 1 \pmod 3$. In contrast to the claim at the end of [3], we make the following conjecture.

Conjecture. *For* $n > 2$, $\mathrm{cr}(P(3n+1, 3)) = n + 3$.

In particular, we believe that $\mathrm{cr}(P(10, 3)) = 6$. In support of this, we have been able to prove, using an argument very similar to that used to prove Lemma 14, that if e is any edge of $P(10, 3)$, then $\mathrm{cr}(P(10, 3) - e) = 4$. Of course, this provides a different proof of Theorem 10, but, as we use Lemma 14 to prove this sharper result, this proof of Theorem 10 is a longer than the one given here.

In general, Fiorini has shown that, for $n > 2$, $\mathrm{cr}(P(3n+1, 3))$ is either $n+1$, $n+2$ or $n+3$.

Acknowledgement

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