Stability of Pexiderized homogeneity almost everywhere

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Received 13 February 2007
Available online 20 June 2007
Submitted by Steven G. Krantz

Abstract
In the paper we examine stability of Pexiderized \( \phi \)-homogeneity equation

\[ f(\alpha x) = \phi(\alpha)g(x) \]

almost everywhere. In particular we prove, that if \((G, \cdot, 0)\) is a group with zero, \((G, X)\) is a \(G\)-space, \(Y\) is a locally convex vector space over \(K \in \{\mathbb{R}, \mathbb{C}\}\) and for functions \(\phi : G \to K\), \(f, g : X \to Y\) the difference

\[ f(\alpha x) - \phi(\alpha)g(x) \]

is suitably bounded almost everywhere in \(G \times X\), then, under certain assumptions on \(f, \phi, g\), the function \(\phi\) is almost everywhere in \(G\) equal to \(c\tilde{\phi}\), where \(c \in K \setminus \{0\}\) is a constant and \(\tilde{\phi} : G \to K\) a multiplicative function, the function \(g\) is almost everywhere in \(X\) equal to a \(\tilde{\phi}\)-homogeneous function \(F : X \to Y\), and the difference \(f - cF\) in some sense bounded almost everywhere in \(X\). From this result we derive the stability of Pexiderized multiplicativity almost everywhere.

1. Introduction
Since the time, when S. Ulam [15] posed his celebrated problem concerning the stability of the equation of homomorphism and D.H. Hyers [6] gave the first its solution, many papers have been devoted to this subject (for a wide bibliography we refer the Reader to [4,7]). The classical question about the stability of a functional equation looks as follows. Let a function \(f\) satisfy a given equation with some accuracy measured in a different way, mostly by the norm of the difference between left- and right-hand side of the equation (such functions \(f\) are often called approximate solutions of the equation). The question is: whether, and under what assumptions, for \(f\) we can find a solution of the equation, which is close to \(f\). If it is the case, then the equation is called stable in the Hyers–Ulam sense. Sometimes it happens that if an approximate solution is unbounded, then it must be a solution of this equation. In this case we say that the equation is superstable.

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doi:10.1016/j.jmaa.2007.06.018
Following another famous problem (posed by P. Erdős [3]) concerning, in general, functional equations assumed to hold “almost everywhere,” R. Ger [5], and next J. Tabor [14], considered “almost approximate” additive mappings, i.e. functions satisfying the Cauchy equation with some accuracy and almost everywhere in a product space.

In the paper we describe functions \( f, \phi, g \) such that the difference

\[
f(\alpha x) - \phi(\alpha)g(x)
\]

is suitably bounded almost everywhere in a product \( G \times X \) of a group \( G \) and a \( G \)-space \( X \). To cover classical cases of homogeneity, we consider an action of group with zero on a set \( X \). Since on \( X \) we have a group action only, we will need a new construction (in comparison with additivity almost everywhere) of ideals which are conjugate.

We begin with definitions of a group \( G \) with zero, linearly independent ideals in such the group, a \( G \)-space \( X \) and linearly independent ideals in that space, and at last, a notion of conjugate ideals in the product \( G \times X \). Finally, we consider the stability of Pexiderized \( \phi \)-homogeneity and Pexiderized multiplicativity almost everywhere.

2. Basic notions and auxiliary results

*Groups with zero, multiplicative functions.* By a group with zero we mean a structure \((G, \cdot, 0)\) where \( G^* := G \setminus \{0\} \neq \emptyset \), \((G^*, |G^* \times G^*\})\) is a group in the classical meaning and \( \alpha \cdot 0 = 0 \cdot \alpha = 0 \) for every \( \alpha \in G \).

The following lemma describes properties of homomorphisms between groups with zero (such homomorphisms we will call multiplicative functions), i.e. functions \( \phi : G \to H \) mapping a group with zero \((G, \cdot, 0)\) into a group with zero \((H, \cdot, 0)\) such that

\[
\phi(\alpha \beta) = \phi(\alpha)\phi(\beta) \quad \text{for } \alpha, \beta \in G.
\]

**Lemma 1.** (Cf. [8, Lemma 1].) Let \((G, \cdot, 0)\) and \((H, \cdot, 0)\) be groups with zero and assume that \( \phi : G \to H \) is a multiplicative function. Then \( \phi(0) \in \{0, 1\} \). Next, if \( \phi(0) = 0 \) for some \( \alpha_0 \in G^* \), then \( \phi = 0 \). Further, if \( \phi(0) = 1 \), then \( \phi|_{G^*} = 1 \), and \( \phi|_{G^*} \neq 1 \) implies \( \phi(0) = 0 \). Finally, if \( \phi \neq 0 \), then \( \phi(1) = 1 \), \( \phi(G^*) \subset H^* \) and \( \phi(\alpha^{-1}) = \phi(\alpha)^{-1} \) for every \( \alpha \in G^* \).

*G-spaces.* Assume that \((G, \cdot, 0)\) is a group with zero and let \( X \) be a nonempty set with a fixed element \( \theta \). Assume that on the set \( X \) we are given an action of the group \( G \), i.e. let \( \cdot : G \times X \to X \) satisfy

\[
(g_1 g_2)x = g_1(g_2x) \quad \text{for } g_1, g_2 \in G, \ x \in X,
\]

\[
1x = x \quad \text{for } x \in X,
\]

\[
g\theta = \theta \quad \text{for } g \in G,
\]

\[
0x = \theta \quad \text{for } x \in X.
\]

The structure \((X, G)\) satisfying these conditions will be called a \( G \)-space. A \( G \)-space \( X \) is called trivial provided \( X = \{\theta\} \). As it is easy to see, the group \( G \) is a \( G \)-space itself. Moreover a pair \((\mathbb{K}^n, \mathbb{K})\), where \( \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} \), with a multiplication of vectors by scalars is a classical example of a \( \mathbb{K} \)-space.

*Ideals, linear invariance.* Let \( X \) be a nonempty set. A nonempty family \( \mathcal{J}(X) \subset 2^X \) is called an ideal in \( X \) provided

\[
A \in \mathcal{J}(X), \ B \subset A \implies B \in \mathcal{J}(X),
\]

\[
A, B \in \mathcal{J}(X) \implies A \cup B \in \mathcal{J}(X).
\]

An ideal \( \mathcal{J}(X) \) is said to be proper, if \( X \notin \mathcal{J}(X) \). An ideal \( \mathcal{J}(X) = \{\emptyset\} \) is called trivial. Otherwise we say that \( \mathcal{J}(X) \) is nontrivial.

Let \( \mathcal{J}(X) \) be an ideal in a nonempty set \( X \). We say that a condition \( W \), defined on a set \( A \subset X \), holds \( \mathcal{J}(X) \)-almost everywhere in \( A \) (we will write \( \mathcal{J}(X) \)-a.e. in \( A \)), if there exists a set \( U \in \mathcal{J}(X) \) such that for every \( x \in A \setminus U \) we have \( W(x) \).

Now, let \((G, \cdot, 0)\) be a group with zero. An ideal \( \mathcal{J}(G) \subset 2^G \) is called linearly invariant if \( \alpha(A^*)^{-1} = \{\alpha \beta^{-1} : \beta \in A^*\} \in \mathcal{J}(G) \) for every \( \alpha \in G, A \in \mathcal{J}(G) \).
Remark 2. The notion of linearly invariant ideals has been introduced for groups in the classical meaning. If \( J(G) \) is an linearly invariant ideal in a group with zero then either \( J(G) = \{\emptyset\} \) or \( \{0\} \in J(G) \). Then, in the case when \( J(G) \) is proper, we have \( G^* \notin J(G) \). Moreover (cf. [11, p. 438])

\[
(A^*)^{-1} \alpha, \alpha A, A \alpha \in J(G) \quad \text{for} \quad \alpha \in G, \; A \in J(G).
\]

Let \( X \) be a \( G \)-space. An ideal \( J(X) \) in \( X \) is said to be linearly invariant provided \( \alpha U \in J(X) \) for every \( \alpha \in G, \; U \in J(X) \). The same name for two different notions will not cause mistakes because in each case we will mark whether considered set is a member of an ideal in a group \( G \) or in a \( G \)-space \( X \).

Conjugate ideals. Let \((G, \cdot, 0)\) be a group with zero and let \( X \) be a \( G \)-space. Assume that we are given proper linearly invariant ideals \( J(G) \) and \( J(X) \) in the group \( G \) and in the \( G \)-space \( X \), respectively. In the product \( G \times X \) we need a \((J(G), J(X))\)-ideal, which is, in some sense, conjugate with given ideals in \( G \) and \( X \).

Definition 3. By a \((J(G), J(X))\)-ideal we will mean an ideal \( J(G \times X) \subset 2^{G \times X} \) satisfying the following conditions:

1. \( U_1 \times X, G \times U_2 \in J(G \times X) \) for \( U_1 \in J(G) \) and \( U_2 \in J(X) \);
2. if \( M \in J(G \times X) \) then there exist sets \( U_1 \in J(G) \) and \( U_2 \in J(X) \) such that
   \[
   M^α := \{ x \in X : (α, x) \in M \} \in J(X) \quad \text{for} \quad α \in G \setminus U_1, \\
   M_x := \{ α \in G : (α, x) \in M \} \in J(G) \quad \text{for} \quad x \in X \setminus U_2;
   \]
3. \( \{ (α, x) \in G \times X : αx \in U \} \in J(G \times X) \) for every \( U \in J(X) \).

As one can easily check, from the condition (2) and from the fact that both ideals \( J(G) \) and \( J(X) \) are proper, we obtain that also the \((J(G), J(X))\)-ideal \( J(G \times X) \) is proper. Some examples of such ideals in some \( G \)-spaces \( X \) are given in [8].

Fix \( α \in G^* \), and let \( Φ_α : G \times X \to G \times X, \; Φ_α(β, x) = (αβ, x) \) for \( (β, x) \in G \times X \). We will say that the family \( \{ Φ_α \}_{α \in G^*} \) preserves an ideal \( J(G \times X) \) in \( G \times X \) whenever \( Φ_α^{-1}(M) \in J(G \times X) \) for every \( M \in J(G \times X) \) and \( α \in G^* \). As one can check, in each example in [8], the family \( \{ Φ_α \}_{α \in G^*} \) preserves given there ideal \( J(G \times X) \).

Auxiliary results. We prove some results which will be useful in the sequel.

Lemma 4. Let \((G, \cdot, 0)\) be a group with zero, and let \( J(G) \) be a proper linearly invariant ideal in \( G \). If \( U \in J(G) \), then for every \( g \in G^* \) we find \( x, y \in G^* \setminus U \) such that \( g = xy \).

Proof. Let \( U \in J(G) \) and put \( V = G^* \setminus U \). First we prove that \( G^* = \{ g \in G^* : (gV^{-1}) \cap V \neq \emptyset \} \). Indeed, for fixed \( g \in G^* \) we have

\[
G^* \setminus ((gV^{-1}) \cap V) = \left( \left( G^* \setminus gV^{-1} \right) \cup gV^{-1} \right) \setminus ((gV^{-1}) \cap V) \subset \left( G^* \setminus gV^{-1} \right) \cup (gV^{-1} \setminus V)
\]

and, since \( G^* \notin J(G) \) (cf. Remark 2), so \((gV^{-1}) \cap V \neq \emptyset \).

Now, fix \( g \in G^* \). Then \((gV^{-1}) \cap V \neq \emptyset \). Hence we find \( x \in V = G^* \setminus U \) such that \( x \in gV^{-1} \). Thus \( x^{-1}g \in V = G^* \setminus U \), which means that there exists \( y \in G^* \setminus U \) such that \( x^{-1}g = y \). \( \square \)

Lemma 5. Let \((G, \cdot, 0)\) and \((H, \cdot, 0)\) be groups with zero and assume that \( H \) is abelian. Let moreover \( J(G) \) be a proper linearly invariant ideal in \( G \). If nonzero multiplicative functions \( φ_1, φ_2 : G \to H \) \((φ_i(G^*) \subset H^* \) for \( i = 1, 2 \)\) satisfy

\[
φ_1(β) = cφ_2(β) \quad J(G) \text{-a.e. in } G,
\]

with some \( c \in H^* \), then \( φ_1(β) = φ_2(β) \) for \( β \in G^* \).
Theorem 7. If for each $\beta \in G^* \setminus U$ we get
\[ \phi_1(\beta) = c\phi_2(\beta) \quad \text{for all } \beta \in G \setminus U. \] (1)

Fix $\beta \in G^*$. Then, by Lemma 4 we find $x, y \in G^* \setminus U$ such that $g = xy$. Since $\phi_1, \phi_2$ are multiplicative functions, so by (1) we get
\[ \phi_1(\beta) = \phi_1(xy) = \phi_1(x)\phi_1(y) = c\phi_2(x)\phi_2(y) = c^2\phi_2(xy) = c^2\phi_2(\beta). \] (2)

Now, let for the moment $\beta \in G^* \setminus U$. Then, by (1) and (2) we get
\[ c^2\phi_2(\beta) = \phi_1(\beta) = c\phi_2(\beta), \]
and since $\phi_2(\beta) \in H^*$, so $c = 1$. Thus, by (2), $\phi_1(\beta) = \phi_2(\beta)$ for $\beta \in G^*$. \(\square\)

Lemma 6. (See [9, Lemma 2].) Let $(G, \cdot, 0)$ be a group with zero and let $X$ be a $G$-space. Assume that $\mathcal{J}(G)$ and $\mathcal{J}(X)$ are proper linearly invariant ideals in $G$ and $X$, respectively. Let moreover $\mathcal{J}(G \times X)$ be a $(\mathcal{J}(G), \mathcal{J}(X))$-ideal in $G \times X$ and assume that the family $\{\Phi_\alpha\}_{\alpha \in G^*}$ preserves $\mathcal{J}(G \times X)$. If a property $W$, defined on $G \times X$, satisfies a condition
\[ W(\alpha, x) \text{ for } (\alpha, x) \in (G \times X) \setminus M, \]
with some $M \in \mathcal{J}(G \times X)$, then there exists a set $A \in \mathcal{J}(G)$ such that for every $\alpha \in G \setminus (A \cup \{0\})$ there is a set $N_\alpha \in \mathcal{J}(G \times X)$ with the property
\[ W(\alpha\beta, x) \wedge W(\alpha, \beta x) \text{ for each } (\beta, x) \in (G \times X) \setminus N_\alpha. \]

Superstability of $\phi$-homogeneity. Now, for the convenience of the Reader, we quote results concerning superstability of the $\phi$-homogeneity equation.

From now on, if it will not be assumed otherwise, $(G, \cdot, 0)$ is a group with zero, $X$ is a $G$-space, and $\mathcal{J}(G)$ and $\mathcal{J}(X)$ are proper linearly invariant ideals in $G$ and $X$, respectively. Assume that $\mathcal{J}(G \times X)$ and $\mathcal{J}(G \times X)$ are $(\mathcal{J}(G), \mathcal{J}(X))$- and $(\mathcal{J}(G), \mathcal{J}(X))$-ideals in $G \times G$ and $G \times X$, respectively. Let $Y$ be a locally convex linear topological space over $\mathbb{K}$. By $B(Y)$ we denote the family of all bounded subsets of the space $Y$. Let $\delta : G \to \mathbb{K}$ and $V \in B(Y)$. Assume moreover that $\psi : G \to [0, \infty)$ is multiplicative function and let a function $K : X \to \mathbb{K}$ satisfy the inequality
\[ |K(\alpha x)| \leq \psi(\alpha)|K(x)| \quad \text{for } \alpha \in G, \ x \in X. \]

Without loss of generality we may assume that $\psi \neq 0$ (which jointly with Lemma 1 implies $\psi(G^*) \subset (0, \infty)$), since otherwise $K = 0$, and then considered here stability conditions (3) and (8) are, in fact, equalities almost everywhere, which have been considered in [8,9]. Finally, for any $V \subset Y$, by $\text{aconv} V$ we denote absolutely convex hull of the set $V$, i.e. the smallest convex and balanced set containing $V$. It is known that if $r \in \mathbb{K}$, $|r| \leq 1$, $V \subset Y$, then $rV \subset \text{aconv} V$. Moreover, if $V \in B(Y)$, then also $\text{aconv} V \in B(Y)$.

Theorem 7. (See [10, Theorem 1].) Assume that $\phi : G \to \mathbb{K}$ is a nonzero multiplicative function, (i.e. $\phi(G^*) \subset \mathbb{K}^* := \mathbb{K} \setminus \{0\}$), and let a function $f : X \to Y$ satisfy the condition
\[ f(\alpha x) - \phi(\alpha)f(x) \in \delta(\alpha)K(x)V \quad \mathcal{J}(G \times X)$-a.e. in $G \times X. \] (3)

If for each $\alpha \in G^*$
\[ \mathcal{J}(G) = \text{ess inf}_{\beta \in G^*} \frac{\delta(\beta)|\psi(\alpha) + |\delta(\alpha\beta)|}{|\phi(\beta)|} = 0, \] (4)
then there exists a function $F : X \to Y$ such that
\[ F(\alpha x) = \phi(\alpha)F(x) \quad \text{for } (\alpha, x) \in G^* \times X, \]
\[ f(x) = F(x) \quad \mathcal{J}(X)$-a.e. in $X. \]
It appears that the assumption that \( \phi \) is a multiplicative function may be replaced in Theorem 7 with \( \mathcal{J}(X) \)-essentially \( K \)-unboundedness of \( f \).

**Definition 8.** A function \( f : X \to Y \) is called \( \mathcal{J}(X) \)-essentially \( K \)-bounded provided there exist sets \( U \in \mathcal{J}(X) \) and \( W \in \mathcal{B}(Y) \) such that

\[
f(x) \in K(x)W \quad \text{for } x \in X \setminus U.
\]

Otherwise \( f \) is said to be \( \mathcal{J}(X) \)-essentially \( K \)-unbounded. In the case \( K = 1 \) we will say that \( f \) is (or is not) \( \mathcal{J}(X) \)-essentially bounded.

For a function \( \phi : G \to \mathbb{K} \) let us denote \( \text{supp} \phi := \{ \alpha \in G : \phi(\alpha) \neq 0 \} \).

**Theorem 9.** (See [10, Theorem 2].) Assume that functions \( \phi : G \to \mathbb{K} \) and \( f : X \to Y \) satisfy the condition (3). If the function \( f \) is \( \mathcal{J}(X) \)-essentially \( K \)-unbounded, then there exists a multiplicative function \( \tilde{\phi} : G \to \mathbb{K} \) such that

\[
\phi(\beta) = \tilde{\phi}(\beta) \quad \mathcal{J}(G) \text{-a.e. in } G.
\]

Furthermore, if \( \text{supp} \phi \notin \mathcal{J}(G) \) and for each \( \alpha \in G^* \)

\[
\mathcal{J}(G) - \text{ess inf} \frac{|\delta(\beta)| \psi(\alpha) + |\delta(\alpha \beta)|}{|\phi(\beta)|} = 0,
\]

then there exists a function \( F : X \to Y \) such that

\[
F(\alpha x) = \tilde{\phi}(\alpha) F(x) \quad \text{for } (\alpha, x) \in G^* \times X,
\]

\[
f(x) = F(x) \quad \mathcal{J}(X) \text{-a.e. in } X.
\]

3. Stability of Pexiderized \( \phi \)-homogeneity

We assume here additionally that the family \( \{\Phi_\alpha\}_{\alpha \in G^*} \) preserves \( \mathcal{J}(G \times X) \).

We begin with the proposition which shows that the stability of the Pexiderized \( \phi \)-homogeneity almost everywhere may be reduced to the stability of \( \phi \)-homogeneity almost everywhere.

**Proposition 10.** Assume that functions \( \phi : G \to \mathbb{K} \), \( \text{supp} \phi \notin \mathcal{J}(G) \), and \( f, g : X \to Y \) satisfy the condition

\[
f(\alpha x) - \phi(\alpha) g(x) \in \delta(\alpha) K(x) V \quad \text{for every } (\alpha, x) \in (G \times X) \setminus M,
\]

where \( M \in \mathcal{J}(G \times X) \). Then there exists a set \( U_1 \in \mathcal{J}(G) \) such that for every \( \alpha \in (G^* \cap \text{supp} \phi) \setminus U_1 \) there exists \( N_\alpha \in \mathcal{J}(G \times X) \) with

\[
g(\beta x) - \phi(\alpha)^{-1} \phi(\alpha \beta) g(x) \in \delta_\alpha(\beta) K(\beta x) V \quad \text{for } (\beta, x) \in (G \times X) \setminus N_\alpha,
\]

where \( \delta_\alpha(\beta) := \frac{|\delta(\alpha \beta)| + |\delta(\alpha)| |\psi(\beta)|}{|\phi(\alpha)|} \). Moreover, there exists a set \( U_2 \in \mathcal{J}(G) \) such that for every \( \alpha \in G^* \setminus U_2 \) we have \( \alpha M^\alpha \in \mathcal{J}(X) \) and

\[
f(x) - \phi(\alpha) g(\alpha^{-1} x) \in |\delta(\alpha)| |\phi(\alpha)|^{-1} |K(\alpha)| V \quad \text{for } x \in X \setminus \alpha M^\alpha.
\]

**Proof.** From Lemma 6, there exists a set \( U_1 \in \mathcal{J}(G) \) such that for every \( \alpha \in G^* \setminus U_1 \) there exists a set \( N_\alpha \in \mathcal{J}(G \times X) \) with

\[
f(\alpha \beta x) - \phi(\alpha \beta) g(x) \in \delta(\alpha \beta) K(\alpha \beta x) V \subset |\delta(\alpha \beta)| |K(\alpha \beta)| V,\]

\[
f(\alpha \beta x) - \phi(\alpha) g(\beta x) \in \delta(\alpha) K(\beta x) V \subset |\delta(\alpha)| |\psi(\beta)| |K(\beta)| V
\]

for every \( (\beta, x) \in (G \times X) \setminus N_\alpha \). Hence, for \( \alpha \in (G^* \cap \text{supp} \phi) \setminus U_1 \) (clearly \( (G^* \cap \text{supp} \phi) \setminus U_1 \neq \emptyset \), since \( \text{supp} \phi \notin \mathcal{J}(G) \)), we have

\[
g(\beta x) - \phi(\alpha)^{-1} \phi(\alpha \beta) g(x) \in \frac{|\delta(\alpha \beta)| + |\delta(\alpha)| |\psi(\beta)|}{|\phi(\alpha)|} |K(\alpha)| V
\]

for \( (\beta, x) \in (G \times X) \setminus N_\alpha \), which proves (6).
Finally, since $M \in \mathcal{J}(G \times X)$, so there exists $U_2 \in \mathcal{J}(G)$ such that $M^\alpha \in \mathcal{J}(X)$ for every $\alpha \in G \setminus U_2$. Fix $\alpha \in G^* \setminus U_2$ arbitrarily, and let $x \in X \setminus \alpha M^\alpha$ (clearly $\alpha M^\alpha \in \mathcal{J}(X)$). Then $(\alpha, \alpha^{-1} x) \notin \mathcal{J}$, and by (5) we get

$$f(x) - \phi(\alpha)g(\alpha^{-1} x) \in \delta(\alpha)K(\alpha^{-1} x)V \subset |\delta(\alpha)|\psi(\alpha)^{-1}|K(x)| \text{aconv } V,$$

which completes the proof. \qed

Now we are in a position to prove the first of our main results.

**Theorem 11.** Assume that a nonzero multiplicative function $\phi : G \to \mathbb{K}$ and functions $f, g : X \to Y$ satisfy the condition

$$f(\alpha x) - \phi(\alpha)x \in \delta(\alpha)K(x)V \quad \mathcal{J}(G \times X)$-$a.e.$ in $G \times X.$

(8)

There exists a set $U_1 \in \mathcal{J}(G)$ such that if for some $\alpha \in G^* \setminus U_1$ and every $\gamma \in G^*$

$$\mathcal{J}(G) - \text{ess inf}_{\beta \in G^*} \frac{\delta(\alpha \beta)|\psi(\gamma)| + |\delta(\alpha \gamma \beta)| + 2|\delta(\alpha)|\psi(\beta \gamma)}{|\phi(\alpha)\phi(\beta)|} = 0,$$

(9)

then there exists a function $F : X \to Y$ such that

$$F(\beta x) = \phi(\beta)F(x) \quad \text{for } (\beta, x) \in G^* \times X,$n
$$g(x) = F(x) \quad \mathcal{J}(X)$-$a.e.$ in $X,$
$$f(x) - F(x) \in C|K(x)| \text{aconv } V \quad \mathcal{J}(X)$-$a.e.$ in $X,$

(10)

for every $C > \mathcal{J}(G) - \text{ess inf}_{\beta \in G^*} |\delta(\beta)|\psi(\beta)^{-1}$.

**Proof.** From (8) it follows that there exists a set $M \in \mathcal{J}(G \times X)$ such that (5) is satisfied. Since $\phi$ is a nonzero multiplicative function, by Lemma 1, supp $\phi = G^*$. Thus (9) is properly defined, and (9) implies (4) for functions $\phi$ and $\delta$. Let $U_1 \in \mathcal{J}(G)$ be as in Proposition 10 and fix $\alpha \in G^* \setminus U_1$ such that (9) holds. Then, by Proposition 10, there exists $N_\alpha \in \mathcal{J}(X)$ such that

$$g(\beta x) - \phi(\beta)g(x) \in \delta(\beta)|K(x)| \text{aconv } V \quad \text{for } (\beta, x) \in (G \times X) \setminus N_\alpha.$$

On account of Theorem 7 we obtain that there exists a function $F : X \to Y$ satisfying (10) and such that $g(x) = F(x)$ for $x \in X \setminus S$ with some $S \in \mathcal{J}(X)$.

Now, let also $U_2 \in \mathcal{J}(G)$ be as in Proposition 10. Fix $\beta \in G^* \setminus U_2$ and $x \in X \setminus \beta(M^\beta \cup S)$. Then $\beta^{-1} x \notin S$, $(\beta, \beta^{-1} x) \notin \mathcal{J}$, and on account of (7) we get

$$f(x) - F(x) = f(x) - \phi(\beta)F(\beta^{-1} x) = f(x) - \phi(\beta)g(\beta^{-1} x) \in |\delta(\beta)|\psi(\beta)^{-1}|K(x)| \text{aconv } V.$$

Fix $C > \mathcal{J}(G) - \text{ess inf}_{\beta \in G^*} |\delta(\beta)|\psi(\beta)^{-1}$ and let $\beta \in G^* \setminus U_2$ be such that $|\delta(\beta)|\psi(\beta)^{-1} < C$. Then

$$f(x) - F(x) \in C|K(x)| \text{aconv } V \quad \text{for every } x \in X \setminus \beta(M^\beta \cup S),$$

which finishes the proof. \qed

Now we will prove, similarly as in the case of $\phi$-homogeneity, that in Theorem 11 the assumption that $\phi$ is a multiplicative function may be replaced with the one on $\mathcal{J}(X)$-essentially $K$-unboundedness of the function $g$.

**Theorem 12.** Assume that functions $\phi : G \to \mathbb{K}$, supp $\phi \notin \mathcal{J}(G)$, and $f, g : X \to Y$ satisfy the condition (8). There exists $U_1 \in \mathcal{J}(G)$ such that if for some $\alpha \in (G^* \cap \text{supp } \phi) \setminus U_1$ and every $\gamma \in G^*$

$$\mathcal{J}(G) - \text{ess inf}_{\beta \in \text{supp } \phi} \frac{|\delta(\alpha \beta)|\psi(\gamma) + |\delta(\alpha \gamma \beta)| + 2|\delta(\alpha)|\psi(\beta \gamma)}{|\phi(\alpha)\phi(\beta)|} = 0,$$

(11)

and the function $g$ is $\mathcal{J}(X)$-essentially $K$-unbounded, then there exists a multiplicative function $\tilde{\phi} : G \to \mathbb{K}$, a constant $c \in \mathbb{K}^*$ and a function $F : X \to Y$ such that
\[
F(\beta x) = \tilde{\phi}(\beta) F(x) \quad \text{for } (\beta, x) \in G^* \times X,
\]
\[
\phi(\beta) = c\tilde{\phi}(\beta) \quad \mathcal{J}(G)\text{-a.e. in } G,
\]
\[
g(x) = F(x) \quad \mathcal{J}(X)\text{-a.e. in } X,
\]
\[
f(x) - cF(x) \in C \setminus K(x) \setminus \text{aconv } V \quad \mathcal{J}(X)\text{-a.e. in } X,
\]
for every \( C > \mathcal{J}(G) - \text{ess inf}_{\beta \in G} |\delta(\beta)|\psi(\beta)^{-1}. \)

**Proof.** From (8) it follows the existence of \( M \in \mathcal{J}(G \times X) \) such that (5) holds. Let \( U_1 \in \mathcal{J}(G) \) be as in Proposition 10. Fix \( \alpha \in (G^* \cap \text{supp } \phi) \setminus U_1 \) such that (11) holds, and let us denote \( \phi_\alpha : G \to \mathbb{K}, \phi_\alpha(\beta) := \phi(\alpha)^{-1}\phi(\alpha\beta). \) On account of Proposition 10, there exists \( N_\alpha \in \mathcal{J}(G \times X) \) such that
\[
g(\beta x) - \phi_\alpha(\beta)g(x) \in \delta_\alpha(\beta) K(x) \setminus \text{aconv } V \quad \text{for } (\beta, x) \in (G \times X) \setminus N_\alpha.
\]
Then, by Theorem 9, from the \( \mathcal{J}(X)\)-essentially \( K \)-unboundedness of \( g \) we obtain that there exists a multiplicative function \( \tilde{\phi}_\alpha : G \to \mathbb{K} \) such that
\[
\phi(\alpha)^{-1}\phi(\alpha\beta) = \phi_\alpha(\beta) = \tilde{\phi}_\alpha(\beta) \quad \text{for } \beta \in G \setminus S_\alpha,
\]
with some \( S_\alpha \in \mathcal{J}(G). \) Since \( \text{supp } \phi \notin \mathcal{J}(G) \), by (13) also \( \text{supp } \tilde{\phi}_\alpha \notin \mathcal{J}(G) \), but \( \tilde{\phi}_\alpha \) is a multiplicative function, so \( \tilde{\phi}_\alpha \neq 0 \), which means that \( \tilde{\phi}_\alpha(G^*) \subset \mathbb{K}^*. \)

Let \( \tilde{\phi} := \tilde{\phi}_\alpha. \) By (13) there exists a constant \( c \in \mathbb{K}^* \) such that
\[
\phi(\beta) = c\tilde{\phi}(\beta) \quad \text{for every } \beta \in G \setminus \alpha S_\alpha.
\]
Then, with \( \tilde{N}_\alpha := N_\alpha \cup (S_\alpha \times X) \in \mathcal{J}(G \times X), \) from (5) we get
\[
f(\alpha x) - c\tilde{\phi}(\alpha)g(x) \in \delta(\alpha) K(x) V \quad \text{for } (\alpha, x) \in (G \times X) \setminus \tilde{N}_\alpha,
\]
which gives
\[
c^{-1} f(\alpha x) - \tilde{\phi}(\alpha)g(x) \in c^{-1}\delta(\alpha) K(x) V \quad \text{for } (\alpha, x) \in (G \times X) \setminus \tilde{N}_\alpha.
\]
Since \( \text{supp } \tilde{\phi} = \text{supp } \tilde{\phi}_\alpha = G^* \), by (14), \( G \setminus \text{supp } \phi \in \mathcal{J}(G). \) From (11)
\[
\mathcal{J}(G) - \text{ess inf}_{\beta \in \text{supp } \phi} \frac{c^{-1}(|\delta(\alpha\beta)|\psi(\gamma) + |\delta(\alpha\gamma\beta)| + 2|\delta(\alpha)|\psi(\beta\gamma))}{|\phi(\alpha)\phi(\beta)|} = 0,
\]
which with \( G \setminus \text{supp } \phi \in \mathcal{J}(G) \) implies (9), and Theorem 11 finishes the proof. \( \square \)

**Remark 13.** Note that if \( \tilde{\phi} : G \to \mathbb{K} \) is a nonzero multiplicative function, \( c \in \mathbb{K}^* \) and a function \( F : X \to Y \) satisfies (12), then functions \( \tilde{\phi} : G \to \mathbb{K}, \tilde{\phi}(\alpha) = c\tilde{\phi}(\alpha), F : X \to Y \) and \( H : X \to Y, H(x) = cF(x) \), satisfy Pexiderized \( \tilde{\phi} \)-homogeneity equation
\[
H(\alpha x) = \tilde{\phi}(\alpha) F(x) \quad \text{for } (\alpha, x) \in G^* \times X.
\]
Then the statement of the above Theorem 12 may be formulated in way, which is closer to stability results, i.e. functions which are approximate solutions of Pexiderized \( \phi \)-homogeneity must be close to solutions of Pexiderized \( \tilde{\phi} \)-homogeneity. This statement looks as follows.

Assume that functions \( \phi : G \to \mathbb{K}, \text{supp } \phi \notin \mathcal{J}(G), \) and \( f, g : X \to Y \) satisfy the condition (8). There exists \( U_1 \in \mathcal{J}(G) \) such that if for some \( \alpha \in (G^* \setminus \text{supp } \phi) \setminus U_1 \) and every \( \gamma \in G^* \) the condition (11) is satisfied and the function \( g \) is \( \mathcal{J}(X)\)-essentially \( K \)-unbounded, then there exists a solution \( \tilde{\phi} : G \to \mathbb{K}, F : X \to Y \) and \( H : X \to Y \) of the Pexiderized \( \tilde{\phi} \)-homogeneity equation (15) satisfying
\[
\phi(\beta) = \tilde{\phi}(\beta) \quad \mathcal{J}(G)\text{-a.e. in } G,
\]
\[
g(x) = F(x) \quad \mathcal{J}(X)\text{-a.e. in } X,
\]
\[
f(x) - H(x) \in C \setminus K(x) \setminus \text{aconv } V \quad \mathcal{J}(X)\text{-a.e. in } X,
\]
for every \( C > \mathcal{J}(G) - \text{ess inf}_{\beta \in G} |\delta(\beta)|\psi(\beta)^{-1}. \)
From Theorem 12 we obtain the following corollary ($\delta = v, \psi = 1, K = 1$).

**Corollary 14.** Let $(Y, \| \cdot \|)$ be a normed space over $\mathbb{K}$ and let $v : G \to [0, \infty)$ be given. Assume that functions $\phi : G \to \mathbb{K},$ supp $\phi \notin \mathcal{J}(G),$ and $f, g : X \to Y$ satisfy the inequality
\[
\| f(\alpha x) - \phi(\alpha) g(x) \| \leq v(\alpha) J(G \times X) \text{-a.e. in } G \times X.
\]
There exists a set $U_1 \in \mathcal{J}(G)$ such that for some $\alpha \in (G^* \cap \text{supp } \phi) \setminus U_1$ and every $\gamma \in G^*$
\[
\mathcal{J}(G) - \text{ess inf}_{\beta \in \text{supp } \phi} \frac{v(\alpha \beta) + v(\alpha \gamma \beta) + 2v(\alpha)}{|\phi(\alpha) \phi(\beta)|} = 0,
\]
and the function $g$ is $\mathcal{J}(X)$-essentially unbounded, then there exists a multiplicative function $\tilde{\phi} : G \to \mathbb{K},$ a constant $c \in \mathbb{K}^*$ and a function $F : X \to Y$ such that (12) is satisfied and
\[
\phi(\beta) = c\tilde{\phi}(\beta) \quad J(G) \text{-a.e. in } G,
\]
\[
g(x) = F(x) \quad J(X) \text{-a.e. in } X,
\]
for every $C > J(G) - \text{ess inf}_{\beta \in G} v(\beta)$.

**Remark 15.** It is easy to see that if in Theorems 11 and 12 we have $\delta = d_1$ and $\psi = d_2 > 0,$ i.e. the functions $\delta$ and $\psi$ are constants, then from Proposition 10 and the proof of Theorem 11 it follows, that the constant $C$ in the estimation of the difference $f - cF$ may be taken as $C = |d_1|d_2^{-1}.$ Similarly, if in Corollary 14 the function $v = d > 0$ is constant, then we may put $C = d.$

It appears that in our stability results on Pexiderized $\phi$-homogeneity we cannot expect, contrary to the $\phi$-homogeneity case, superstability phenomenon, even if we consider Pexiderized $\phi$-homogeneity equation everywhere in $G \times X.$

**Example 16.** Let us consider $(\mathbb{R}, \cdot, 0)$ as a group with zero and $(\mathbb{R}, \mathbb{R})$ as an $\mathbb{R}$-space. Let $Y$ be a real normed space. Assume that $\phi : \mathbb{R} \to \mathbb{R},$ $\phi(\alpha) = \alpha, \alpha \neq x_0 \in Y$ and $g : \mathbb{R} \to Y,$ $g(x) = xx_0.$ Then $\phi = \text{id}_{\mathbb{R}}$ is a multiplicative function and $g$ is a homogeneous function, i.e.
\[
g(ax) = ag(x) \quad \text{for } (a, x) \in \mathbb{R} \times \mathbb{R}.
\]
Fix $\varepsilon > 0$ and let $a \in Y,$ $\|a\| = \varepsilon.$ Let $f : \mathbb{R} \to Y,$ $f(x) = xx_0 + a.$ Then
\[
\| f(ax) - \phi(\alpha) g(x) \| = \varepsilon \quad \text{for } (\alpha, x) \in \mathbb{R} \times \mathbb{R},
\]
but there exists no multiplicative function $\tilde{\phi} : \mathbb{R} \to \mathbb{R},$ a constant $c \in \mathbb{R}^*$ and a $\tilde{\phi}$-homogeneous function $F : \mathbb{R} \to Y$ satisfying $\phi(\beta) = c\tilde{\phi}(\beta)$ for $\beta \in \mathbb{R}^*$ and $g(x) = F(x),$ $f(x) = cF(x)$ for $x \in \mathbb{R}.$

4. **Stability of Pexiderized multiplicativity**

Assume now that $(G, \cdot, 0)$ is a group with zero, and let $\mathcal{J}(G)$ be a proper linearly invariant ideal in $G.$ Next, assume that $\Psi_{\alpha} : G \times G \to G \times G,$ $\Psi_{\alpha}(\beta_1, \beta_2) = (\alpha \beta_1, \beta_2),$ and let $\mathcal{J}(G \times G)$ be a $(\mathcal{J}(G),$ $\mathcal{J}(G))$-ideal in $G \times G$ such that the family $\{\Psi_{\alpha}\}_{\alpha \in G^*}$ preserves $\mathcal{J}(G \times G).$ From Corollary 14 we derive a result stating stability of Pexiderized version of the equation of multiplicative function almost everywhere. This generalizes results from [1,2,12,13].

**Theorem 17.** Let $v : G \to [0, \infty)$ be given. Assume that functions $f, h, g : G \to \mathbb{K}$ with supp $h \notin \mathcal{J}(G),$ satisfy the inequality
\[
|f(\alpha \beta) - h(\alpha) g(\beta)| \leq v(\alpha) J(G \times G) \text{-a.e. in } G \times G.
\]
There exists a set $U_1 \in \mathcal{J}(G)$ such that for some $\alpha \in (G^* \cap \text{supp } h) \setminus U_1$ and every $\gamma \in G^*$
\[
\mathcal{J}(G) - \text{ess inf}_{\beta \in \text{supp } h} \frac{v(\alpha \beta) + v(\alpha \gamma \beta) + 2v(\alpha)}{|h(\beta)|} = 0,
\]
(16)
and the function $g$ is $\mathcal{J}(G)$-essentially unbounded, then there exists a unique multiplicative function $\tilde{h} : G \to \mathbb{K}$ and unique constants $c_1, c_2 \in \mathbb{K}^*$ such that

$$h(\beta) = c_1 \tilde{h}(\beta) \quad \mathcal{J}(G)\text{-a.e. in } G,$$

$$g(\beta) = c_2 \tilde{h}(\beta) \quad \mathcal{J}(G)\text{-a.e. in } G,$$

$$|f(\beta) - c_1 c_2 \tilde{h}(\beta)| \leq C \quad \mathcal{J}(X)\text{-a.e. in } X,$$

for every $C > \mathcal{J}(G) - \text{ess inf}_{x \in G} v(x)$.

**Proof.** On account of Corollary 14, there exists a multiplicative function $\tilde{h} : G \to \mathbb{K}$, a constant $c_1 \in \mathbb{K}^*$ and a function $F : G \to \mathbb{K}$ such that

$$F(\alpha \beta) = \tilde{h}(\alpha) F(\beta) \quad \text{for } (\alpha, \beta) \in G^* \times G. \quad (17)$$

$$h(\beta) = c_1 \tilde{h}(\beta) \quad \mathcal{J}(G)\text{-a.e. in } G, \quad (18)$$

$$g(\beta) = F(\beta) \quad \mathcal{J}(G)\text{-a.e. in } G, \quad (19)$$

$$\|f(\beta) - c_1 F(\beta)\| \leq C \quad \mathcal{J}(G)\text{-a.e. in } G. \quad (20)$$

Put in (17) $\beta = 1$. Then $F(\alpha) = c_2 \tilde{h}(\alpha)$ for every $\alpha \in G^*$, where $c_2 = F(1)$. Clearly $c_2 \neq 0$, since otherwise $F = 0$ in $G^*$, which contradicts (19) and the assumption that $g$ is $\mathcal{J}(G)$-essentially unbounded. Then

$$g(\beta) = c_2 \tilde{h}(\beta) \quad \mathcal{J}(G)\text{-a.e. in } G,$$

$$\|f(\beta) - c_1 c_2 \tilde{h}(\beta)\| \leq C \quad \mathcal{J}(G)\text{-a.e. in } G. \quad (20)$$

Since supp $h \notin \mathcal{J}(G)$, by (18) we get supp $\tilde{h} \notin \mathcal{J}(G)$, which jointly with Lemma 1 means that $\tilde{h}(G^*) \subset \mathbb{K}^*$. Next, we have assumed that $h$ is $\mathcal{J}(G)$-essentially unbounded, hence from (20) the multiplicative function $\tilde{h}$ is unbounded. Next, suppose that there are unbounded multiplicative functions $\tilde{h} : G \to \mathbb{K}$, $\tilde{h}' : G \to \mathbb{K}$ and constants $c_1, c_1' \in \mathbb{K}^*$ such that

$$c_1 \tilde{h}(\beta) = h(\beta) = c_1' \tilde{h}'(\beta) \quad \mathcal{J}(G)\text{-a.e. in } G.$$

From Lemma 5 we obtain $c_1 = c_1'$ and $\tilde{h}(\beta) = \tilde{h}'(\beta)$ for $\beta \in G^*$. But $\tilde{h}|_{G^*} \neq 1$ and $\tilde{h}'|_{G^*} \neq 1$ ($\tilde{h}$ and $\tilde{h}'$ are unbounded), so by Lemma 1, $\tilde{h}(0) = \tilde{h}'(0) = 0$. Thus $\tilde{h} = \tilde{h}'$. This proves the uniqueness of existence of a constant $c_1$ and an multiplicative function $\tilde{h}$. Then, from (20) we obtain uniqueness of $c_2$. \[\square\]

**Corollary 18.** Fix $\varepsilon > 0$ and let functions $f, h, g : G \to \mathbb{K}$ satisfy

$$|f(\alpha \beta) - h(\alpha) g(\beta)| \leq \varepsilon \quad \mathcal{J}(G \times G)\text{-a.e. in } G \times G.$$

If the functions $g, h$ are $\mathcal{J}(G)$-essentially unbounded, then there exists a unique multiplicative function $\tilde{h} : G \to \mathbb{K}$ and unique constants $c_1, c_2 \in \mathbb{K}^*$ such that

$$h(\beta) = c_1 \tilde{h}(\beta) \quad \mathcal{J}(G)\text{-a.e. in } G,$$

$$g(\beta) = c_2 \tilde{h}(\beta) \quad \mathcal{J}(G)\text{-a.e. in } G,$$

$$|f(\beta) - c_1 c_2 \tilde{h}(\beta)| \leq \varepsilon \quad \mathcal{J}(X)\text{-a.e. in } X.$$

**Proof.** Since $h$ is $\mathcal{J}(G)$-essentially unbounded, so supp $h \notin \mathcal{J}(G)$, and, moreover, (16) is satisfied with a constant function $v = \varepsilon$. The statement we get then by Theorem 17. \[\square\]

**References**