Simple Jordan superalgebras with semisimple even part

M.L. Racine and E.I. Zelmanov

Department of Mathematics, University of Ottawa, Ottawa, Ontario, Canada K1N 6N5
Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, USA

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Finite dimensional simple Jordan superalgebras over algebraically closed fields of characteristic 0 have been classified by Kac [4,9] (see also Kantor [10]). Kac used his classification of Lie superalgebras to obtain his Jordan results. In [13] O. Kühn obtains a classification for unital normal Jordan superalgebras over arbitrary fields of characteristic not 2 or 3, where normal means that the even part possesses a linear form satisfying certain nondegeneracy conditions. Our purpose is to obtain a classification of simple Jordan superalgebras over fields of characteristic different from 2 whose even part is semisimple. Our proof is not as elegant as that of Kac, but since our methods are Jordan theoretic, they are less sensitive to the characteristic of the base field (modular Lie superalgebras have not yet been classified over an algebraically closed field). The main ingredients, Peirce decomposition and representation theory, can be found in [7]. Since the characteristic plays no role in most arguments we obtain the same superalgebras as Kac. However, in characteristic 3, there exists a 12-dimensional (i-exceptional) simple superalgebra having the $3 \times 3$ symmetric matrices as even part and a 21-dimensional (i-exceptional) simple superalgebra having the $3 \times 3$ symmetric matrices with entries in the split quaternions as even part; moreover the 10-dimensional Kac superalgebra is not simple in characteristic 3 but it has a 9-dimensional i-exceptional simple subsuperalgebra which we will call the degenerate Kac algebra. Our results over an algebraically closed field have been announced in [19]. The classification of the simple superalgebras whose even part is not semisimple is obtained in [14]. Applications of Jordan superalgebras can be found in [17].

* Corresponding author.
E-mail address: mracine@science.uottawa.ca (M.L. Racine).

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Introduction

Jordan superalgebras were introduced by Kaplansky [11] and Kac [9]. The interest in modular Jordan superalgebras originated with Kaplansky [11]. The reader is referred to [9] for the chronological details. Let $K$ be a field of characteristic not 2, $\Gamma = \langle 1, g_i | i = 1, 2, \ldots \rangle$ the Grassmann (or exterior) algebra over $K$ on a countable number of generators $g_i$, with $g_i^2 = 0$, $g_ig_j = -g_jg_i$, $i \neq j$. The elements $1, g_1, g_2, \ldots g_i, i_1 < i_2 < \cdots < i_r$, form a $K$-basis of $\Gamma$. Letting $\Gamma_0$ (respectively $\Gamma_1$) be the span of the products of even length (respectively of odd length), $\Gamma$ is the direct sum of its even and odd parts: $\Gamma = \Gamma_0 + \Gamma_1$. If $\mathcal{V}$ is a variety of algebras defined by homogeneous polynomials, a $\mathbb{Z}_2$-graded $K$-algebra $A = A_0 + A_1$ is a $\mathcal{V}$-superalgebra if its Grassmann envelope $\Gamma(A) := A_0 \otimes \Gamma_0 + A_1 \otimes \Gamma_1$ belongs to $\mathcal{V}$. In particular a Jordan superalgebra $J = J_0 + J_1$ satisfies supercommutativity

$$a_\alpha R_\beta b_\gamma = (-1)^{\alpha \beta + \alpha \gamma + \beta \gamma} R_\alpha R_\gamma b_\beta (a_\alpha c_\gamma)$$

where $a_\alpha \in J_\alpha$, $\alpha, \beta \in \mathbb{Z}_2$ and $R$ denotes multiplication on the right. It also satisfies the linearized Jordan identity in operator form [7, p. 34]

$$R_{a_\alpha R_\beta b_\gamma} R_\beta b_\gamma + (-1)^{\alpha \beta + \alpha \gamma + \beta \gamma} R_{a_\alpha R_\beta b_\gamma} R_\beta b_\gamma + (-1)^{\beta \gamma} R_{(a_\alpha c_\gamma) b_\beta}$$

where $a_\alpha \in J_\alpha$, $b_\beta \in J_\beta$ and $c_\gamma \in J_\gamma$. Intuitively if two odd elements are transposed the sign changes. If the characteristic is not 3 then (1) and the first equality of (2) define a Jordan superalgebra. If the characteristic is 3, we also need the Jordan identity for $J_0$. We have included the other equalities in (2), even though they are consequences of the first, as a matter of convenience. We will say that an element of $J$ is homogeneous if it belongs to $J_0$ or $J_1$. By supercommutativity the product of two homogeneous elements is commutative unless they are both odd in which case it anticommutes. As a way of remembering this we will denote the product by a dot in the commutative cases and by brackets in the anticommutative case. Moreover since $J_0$ is a Jordan algebra we denote it $A$ and $J_1$ is an $A$-bimodule which we denote $M$.

Examples. (1) An associative superalgebra is nothing but a $\mathbb{Z}_2$-graded associative algebra. For example, $(n + m) \times (n + m)$ matrices, $M_{n+m}(K)$, can be viewed as an associative superalgebra by taking the diagonal components $M_{n}(K)$ and $M_{m}(K)$ as the even part and the off-diagonal components as the odd part; this is an example of
a simple associative superalgebra. More generally, let $D$ be an associative division algebra. A $D$-superspace is a $\mathbb{Z}_2$-graded left $D$-vector space $V = V_0 \oplus V_1$. The associative algebra $\text{End} V = \text{End}_D V = \text{End}_0 V + \text{End}_1 V$, where $\text{End}_0 V := \{ a \in \text{End} V \mid v_\beta a \in V_\beta + \alpha \}$, is an associative superalgebra. Note that if $\dim V$ is finite and $V' = V'_0 + V'_1$ with $\dim V'_0 = \dim V_1$ and $\dim V'_1 = \dim V_0$ then $\text{End} V'$ is isomorphic to $\text{End} V$. In particular $\mathcal{M}_{n+m}(K) \cong \mathcal{M}_{m+n}(K)$. If $D$ has an involution $\bar{\cdot}$, a hermitian superform is a graded form (i.e., $V = V_0 \perp V_1$)

\[(\cdot, \cdot) : V \times V \to D\]

such that $\Gamma((\cdot, \cdot))$ is hermitian on $\Gamma(V) := \Gamma_0 \otimes V_0 + \Gamma_1 \otimes V_1$, i.e.,

\[
(dv_\alpha, v_\beta) = d(v_\alpha, v_\beta), \quad (v_\alpha, dv_\beta) = (v_\alpha, v_\beta)\bar{d}, \\
(v_\beta, v_\alpha) = (-1)^{\alpha\beta}(v_\alpha, v_\beta), \quad \text{for } d \in D, \; v_i \in V_i.
\]

A superinvolution of an associative superalgebra $B$ is a graded linear map $^* : B \to B$ such that

\[a^{**} = a \quad \text{and} \quad (a_\alpha b_\beta)^* = (-1)^{\alpha\beta}b_\beta^*a_\alpha^*.
\]

A nondegenerate hermitian superform on $V$ induces a superinvolution $^*$ on $\text{End} V$ via

\[(v_\alpha a_\gamma, v_\beta) = (-1)^{\alpha\gamma}(v_\alpha, v_\beta a_\gamma^*), \quad \text{for all } v_\alpha \in V_\alpha, \; v_\beta \in V_\beta.
\]

If $A$ is a simple associative algebra then the associative superalgebra

\[
\left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in A \right\}
\]

is simple as a superalgebra but not as an algebra.

(2) If $B = B_0 + B_1$ is an associative superalgebra then the product defined by $a_\alpha R_{b_\beta} := \frac{1}{2}(a_\alpha b_\beta + (-1)^{\alpha\beta}b_\beta a_\alpha)$ defines a Jordan superalgebra structure on $B$ which we denote by $B^+$. In particular, if $B = \mathcal{M}_{n+m}(K)$ as in example (1), we denote $B^+$ by $\mathcal{M}_{n,m}(K)$. For future reference note that in this case $J = A + M_1$, $A = \mathcal{M}_n(K) \oplus \mathcal{M}_m(K)$, $M = \mathcal{M}_n \times \mathcal{M}_m(K) \oplus \mathcal{M}_{m \times n}(K)$ and the bimodule structure is given by

\[
(a + d).b = \frac{1}{2}(ab + bd), \quad (a + d).c = \frac{1}{2}(ca + dc), \\
a \in \mathcal{M}_n(K), \; d \in \mathcal{M}_m(K), \; b \in \mathcal{M}_n \times \mathcal{M}_m(K), \; c \in \mathcal{M}_{m \times n}(K).
\]

If $B$ is a simple associative superalgebra which is not commutative as an algebra then $B^+$ is a simple Jordan superalgebra [2, Corollary, p. 3755]. Following Kac, we denote by $Q_n(K)$ the Jordan superalgebra

\[
\left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathcal{M}_n(K) \right\}^+
\].
So $Q_n(K)$ is simple if $n > 1$. In this case $A = \mathcal{M}_n(K)^+$, $M = \mathcal{M}_n(K)$, another copy of $\mathcal{M}_n(K)$, and the bimodule structure is given by

$$a \cdot \tilde{b} = \frac{1}{2}(ab + ba), \quad a, b \in \mathcal{M}_n(K).$$

A Jordan superalgebra is said to be special if it is isomorphic to a subsuperalgebra of some $B^+$, $B$ an associative superalgebra.

(3) If an associative superalgebra $B$ has a superinvolution $^*$ then $\mathcal{H}(B, ^*)$, the symmetric elements of $B$, form a Jordan subsuperalgebra of $B^+$. For $B = \mathcal{M}_{n+2m}(K)$ as in example (1), let $^*$ be the superinvolution induced by the superform with matrix

$$\begin{pmatrix} I_n & 0 \\ 0 & S_m \end{pmatrix} \quad \text{where} \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

The simple Jordan superalgebra $\mathcal{H}(B, ^*)$ is called the orthosymplectic superalgebra and is denoted $osp_{n, 2m}(K)$. If $n \neq 0$ then $osp_{n, 2m}(K)$ is simple. $A = \mathcal{H}_n(K) \oplus \mathcal{H}_m(Q)$, $Q$ the split quaternions over $K$, $M = \mathcal{M}_{n \times 2m}(K)$ and the bimodule structure is given by

$$(a + d) \cdot b = \frac{1}{2}(ab + bd), \quad a \in \mathcal{H}_n(K), \quad d \in \mathcal{H}_m(Q), \quad b \in \mathcal{M}_{n \times 2m}(K).$$

(4) For

$$B = \mathcal{M}_2(A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in A \right\},$$

$A$ an associative algebra with involution $^\sim$, let $^*$ be the superinvolution given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \bar{a} & -\bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}.$$ 

Then

$$\mathcal{H}(B, ^*) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2n}(K) \mid d = \bar{a}, \quad \bar{b} = -b, \quad \bar{c} = c \right\}$$

is a subsuperalgebra of $B^+$. If $A = \mathcal{M}_n(K)$ and $^\sim$ is the transpose involution then, following Kac, we denote $\mathcal{H}(B, ^*)$ by $P_n(K)$. For $n > 1$, $P_n(K)$ is simple. So $A = \mathcal{M}_n(K)^+$, $M = \mathcal{S}_n(K) \oplus \mathcal{H}_n(K)$, where $\mathcal{S}_n(K)$ denotes the skewsymmetric matrices, and the bimodule structure is given by

$$a \cdot \tilde{s} = \frac{1}{2}(as + sa'), \quad a \cdot \tilde{h} = \frac{1}{2}(a'h + ha), \quad a \in \mathcal{M}_n(K), \quad s \in \mathcal{S}_n(K), \quad h \in \mathcal{H}_n(K).$$ 

This corresponds to the identification

$$a + \tilde{s} + \tilde{h} \rightarrow \begin{pmatrix} a & s \\ h & a' \end{pmatrix}.$$
However if we had chosen the identification
\[ a + \bar{s} + \bar{h} \rightarrow \begin{pmatrix} a' & s \\ h & a \end{pmatrix} \]
then the bimodule structure is given by
\[ a \bar{s} = \frac{1}{2}(a's + sa), \quad a \bar{h} = \frac{1}{2}(ah + ha'), \quad a \in M_n(K), \quad s \in S_n(K), \quad h \in H_n(K). \]
If \( n \) is even then the Jordan superalgebra obtained by taking \( \bar{\cdot} \) to be the symplectic involution is isomorphic to \( P_n(K) \).

(5) If \( V = V_0 \oplus V_1 \) is a graded \( K \)-vectorspace. For \( \bar{\cdot} = \) the identity map, a hermitian superform \((, \cdot)\) on \( V \) is a bilinear form on \( V = V_0 \perp V_1 \) whose restriction to \( V_0 \) is symmetric and whose restriction to \( V_1 \) is skew-symmetric. Let \( A = K e + V_0, \ M = V_1 \) and \( J = A + M \). Then \[ e.x = x \text{ and } v.w = (v, w)e \text{ for } x \in J, v, w \in V, \text{ define a Jordan superalgebra structure on } J. \text{ We will refer to } J \text{ as the superalgebra of a superform. Note that } V_0, V_1 = [0], J \text{ is simple if and only if the form } (, \cdot) \text{ is nondegenerate.} \]

(6) \( K_3 \), the Kaplansky superalgebra: \( A = Ke + M = Kx + Ky \) with \( e^2 = e, \ e.x = \frac{1}{2}x, \ e.y = \frac{1}{2}y \) and \( [x, y] = e \).

(7) \( D_t, t \in K \). Let \( A = Ke_1 + Ke_2, M = Kx + Ky \) with \( e_1^2 = e_1, e_1.e_2 = 0, e_1.x = \frac{1}{2}x, e_1.y = \frac{1}{2}y \) and \([x, y] = e_1 + te_2 \). The Jordan superalgebra \( D_t \) is simple if and only if \( t \neq 0 \). If \( t = 1 \) then \([x, y] = e_1 + e_2 \) and \( D_t \) is the superalgebra of a superform and conversely. One also checks that \( D_{-1} \) is isomorphic to \( M_{1,1}(K) \).

(8) \( K_{10} \), the Kac superalgebra:
\[
A = \left( Ke + \sum_{1 \leq i, j \leq 4} K v_i \right) \oplus Kf \quad \text{and} \quad M = \sum_{i=1,2} (Kx_i + Ky_i); 
\]
the following basis for the Kac superalgebra \( J = A + M \) is obtained by scaling the basis given in [4]: \( e, v_1, v_2, v_3, v_4, f, x_1, y_1, x_2, y_2, \) where
\[
e^2 = e, \quad e.v_i = v_i, \quad v_1.v_2 = 2e = v_3.v_4, \quad (3)
f^2 = f, \quad f.x_j = \frac{1}{2}x_j, \quad f.y_j = \frac{1}{2}y_j, \quad j = 1, 2, \quad (3')
e.x_j = 1/2x_j, \quad y_1.v_1 = x_2, \quad y_2.v_1 = -x_1, \quad x_1.v_2 = -y_2, \quad x_2.v_2 = y_1, \quad (4)
e.y_j = 1/2y_j, \quad x_2.v_3 = x_1, \quad y_1.v_3 = y_2, \quad x_1.v_4 = x_2, \quad y_2.v_4 = y_1, \quad (4')
[x_i, y_j] = e - 3f, \quad [x_1, x_2] = v_1, \quad [x_1, y_2] = v_3, \quad [x_2, y_1] = v_4, \quad [y_1, y_2] = v_2 \quad (5)\]
and every other product is zero or is obtained by the symmetry or skew-symmetry of one of the above products. One checks that this superalgebra is simple if char. \( K \neq 3 \) and that in characteristic 3, it possesses a simple subsuperalgebra of dimension 9 spanned by \( e, v_i, 1 \leq i \leq 4, x_j, y_j, 1 \leq j \leq 2 \). We denote this superalgebra by \( K_9 \) and refer to it as
the \textit{degenerate Kac superalgebra}. That $K_{10}$ and hence $K_9$ is a Jordan superalgebra can be obtained by Lie methods as in Kac \cite{Kac9} but a direct proof would be desirable.

(9) Denote by $\mathcal{H}_n(K)$ and $S_n(K)$ the symmetric and skew-symmetric $n \times n$ matrices. For $K$ a field of characteristic 3, let $A = \mathcal{H}_3(K)$ and $M = S_3(K) \oplus S_3(K)$, two copies of $S_3(K)$. To extend the Jordan algebra structure on $A$ and $A$-bimodule structure on $M$ to a Jordan superalgebra structure on $J = A + M$ one defines

\[ [S_3(K), S_3(K)] = [S_3(K), S_3(K)] = [0], \]

and for any $a, b \in S_3(K)$,

\[ [\tilde{a}, \tilde{b}] = ab + ba \in \mathcal{H}_3(K) = A. \]

It is relatively easy to see that this superalgebra is simple. That it is Jordan was first checked using computer algebra. Shestakov \cite{Shestakov23} has provided us with another realization for this superalgebra. Let $A_3$ be the Jordan superalgebra of a superform with $\dim V_0 = 0$ and $\dim V_1 = 2$. Since char $K = 3$, $A_3$ is a simple alternative superalgebra \cite{Shestakov22}. This superalgebra has a superinvolution, $\tilde{ae + x} = ae - x$ with $\mathcal{H}(A_3, \sim) = K e$. Let $^* = ^T$ be the superinvolution of $M_3(A_3)$ induced by $\sim$. Since $A_3$ is not associative, the Jordan superalgebra $\mathcal{H}_3(A_3, ^T)$ is $i$-exceptional. That it is isomorphic to $\mathcal{H}_3(K) + S_3(K) \oplus S_3(K)$ can be seen by observing that its Grassmann envelope $\Gamma(\mathcal{H}_3(A_3, ^T)) \cong \mathcal{H}_3(\Gamma(A_3, ^T)).$

(10) Let $B = B_0 + B_1$, with $B_0 = M_2(K), B_1 = K m_1 + K m_2$, where $K$ is a field of characteristic 3. If we define a $B_0$-bimodule structure on $B_1$ by

\[
e_{11} m_1 = m_1, \quad e_{22} m_1 = 0, \quad e_{12} m_1 = m_2, \quad e_{21} m_1 = 0; \]
\[
m_1 e_{11} = 0, \quad m_1 e_{22} = m_1, \quad m_1 e_{12} = -m_2, \quad m_1 e_{21} = 0; \]
\[
e_{11} m_2 = 0, \quad e_{22} m_2 = m_2, \quad e_{12} m_2 = 0, \quad e_{21} m_2 = m_1; \]
\[
m_2 e_{11} = m_2, \quad m_2 e_{22} = 0, \quad m_2 e_{12} = 0, \quad m_2 e_{21} = -m_1, \]

and a multiplication from $B_1 \times B_1$ to $B_0$ by

\[
m_1^2 = -e_{21}, \quad m_2^2 = e_{12}, \quad m_1 m_2 = e_{11}, \quad m_2 m_1 = -e_{22}, \]

then $B$ is a superalternative algebra with superinvolution $(a + m)^* := \tilde{a} - m$, where $\sim$ is the symplectic involution of $B_0$. Then $\mathcal{H}_3(B)$, the symmetric matrices with respect to the $^*$-transpose superinvolution, form a simple Jordan superalgebra which is $i$-exceptional. This example is also due to Shestakov.

Our main result is the following

\textbf{First Classification Theorem.} Let $J = A + M$ be a finite dimensional central simple Jordan superalgebra over an algebraically closed field $K$ of characteristic not 2. If $A$ is semisimple and $M \neq [0]$ then $J$ is isomorphic to one of the following superalgebras.
(i) $K_3$ the Kaplansky superalgebra.
(ii) $K_{10}$ the Kac superalgebra, char. $K \neq 3$, $K_9$ the degenerate Kac superalgebra, char. $K = 3$.
(iii) $D_t$, $t \in K$, $t \neq 0$.
(iv) the superalgebra of a nondegenerate superform.
(v) $M_{n,m}(K)$, $n, m > 0$.
(vi) $osp(n, 2m)$, $n, m > 0$.
(vii) $P_n(K)$, $n > 1$.
(viii) $Q_n(K)$, $n > 1$.
(ix) $H_3(K) + S_3(K) \oplus S_3(K)$, char. $K = 3$, a simple example (9).
(x) $H_3(B)$, char. $K = 3$, a simple example (10).

If $J = A + M$ is a finite dimensional simple Jordan superalgebra over a field $K$ of characteristic not 2 such that $A$ is semisimple then the possibilities for $A$ are known by the structure theory of Jordan algebras. We will show that if $J$ is not the superalgebra of a superform then $M$ is the sum of at most two irreducible summands and the possibilities for these fixed $A$ are known by the representation theory of Jordan algebras. It remains to decide if a bracket $[\cdot, \cdot] : M \times M \to A$ can be defined so as to make $J$ into a simple Jordan superalgebra. The cases where the capacity of $A$ is 3 or less are studied carefully. The cases of greater capacity are patched together using the fact that if $e$ is an idempotent of $A$ then $J \cup e$ is a simple Jordan subsuperalgebra.

Preliminaries

We first recall a few useful facts and definitions. In a Jordan algebra the triple product is defined as

$$\{a \ b \ c\} := bR_aR_c + bR_cR_a - bR_{a,c}, \quad (7)$$

and

$$bU_a := \frac{1}{2}\{a \ a\}. \quad (7')$$

The representation theory of Jordan algebras of degree 2 can be found in [7, Chapter 7]. We recall what we will need. Let $A = J(V, Q) = K1 + V$, the Jordan algebra of a nondegenerate quadratic form with dim $V \geq 2$, $C(V, Q)$, the Clifford algebra of $(V, Q)$. $C(V, Q)$ is the special universal envelope of $J(V, Q)$ and we consider $J(V, Q)$ as embedded in $C(V, Q)$. If $v_1, \ldots, v_n$ is an orthogonal basis of $V$ then the elements $v_\pi := v_{\pi(1)} \ldots v_{\pi(r)}$, $\pi(1) < \ldots < \pi(r)$, $\pi = \{\pi(1), \ldots, \pi(r)\} \subseteq \{1, \ldots, n\}$, with the convention that $v_\emptyset = 1$, form a basis of $C(V, Q)$. For $k$ even, $0 \leq k \leq n$, denote by $W_k$ the subspace of $C(V, Q)$ spanned by all elements $v_{\pi(1)} \ldots v_{\pi(k)}$, $v_{\pi(1)} \ldots v_{\pi(k+1)}$. One checks that in $C(V, Q)$, $W_k \subseteq W_k$ which is therefore an $A$-bimodule.

If $n$ is odd let $V' := K v_0 \perp V$, with $Q(v_0) = 1$, and consider $C(V, Q)$ as embedded in the Clifford algebra $C(V', Q)$. For $k$ odd, $1 \leq k \leq n$, denote by $W_k$ the subspace
of $C(V', Q)$ spanned by all elements $v_0 v_{\pi(1)} \ldots v_{\pi(k)}$, $v_0 v_{\pi(1)} \ldots v_{\pi(k+1)}$, where $\pi$ is as above. Again $W_k, A \subseteq W_k$ and $W_k$ is an $A$-bimodule. Irrespective of the parity,

$$\dim W_k = \binom{n}{k} + \binom{n}{k + 1} = \binom{n + 1}{k + 1}. \tag{8'}$$

The two summands are irreducible of dimension $\binom{n}{k}$, where $k = (n - 1)/2$. In particular if $K$ is algebraically closed, we may choose $v_1, \ldots, v_n$ to be an orthonormal basis and

$$W_k = \begin{cases} W_k(1 - i c) \oplus W_k(1 + i c), & \text{if } n \equiv -1 \mod 4, \\ W_k(1 - c) \oplus W_k(1 + c), & \text{if } n \equiv 1 \mod 4, \end{cases}$$

where $i = \sqrt{-1}$ and $c^2 = \pm 1$ according as $n \equiv \pm 1 \mod 4$.

**Theorem** (Jacobson [7]). Let $A = J(V, Q) = K1 + V$, the Jordan algebra of a nondegenerate quadratic form with $\dim V \geq 2$. Every unital irreducible $A$-bimodule is isomorphic to one of the $W_k$, $0 \leq k \leq n$, or to one of the summands in $(8)$ if $n$ is odd and $c^2 = \rho^2 1$. The only non-unital irreducible $A$-bimodules are isomorphic to the irreducible module of the special universal envelope of $A$, $C(V, Q)$.

We will make extensive use of the following

**Theorem** (Helwig [3]). Let $J$ be a finite-dimensional simple Jordan algebra over a field of characteristic not $2$ and $J_0$, the subspace of elements of trace $0$. Then the only subspaces of $J$ invariant under the action of the Lie algebra of inner derivations of $J$ are $\{0\}$, $K1$, $J_0$ and $J$.

In a Jordan algebra $J$ the inner derivations are those of the form $D_{x, y} := [R_x, R_y]$, $x, y \in J$. As one expects, in a Jordan superalgebra one must introduce a sign change when $x$ and $y$ are both odd elements.

**Lemma 1.** Let $J = A + M$ be a Jordan superalgebra over a field $K$ of characteristic not $2$, $x, y \in M$. Then $D_{x, y} := (R_x)^2$ and its linearization $D_{x, y} := R_x \circ R_y = R_x R_y + R_y R_x$ are inner derivations of $J$.

**Proof.** If $x_t \in M$, then in $\Gamma(J)$, $(R_{\sum x_t} \otimes y)^2 = \sum_{i < j} [R_{x_i}, R_{x_j}] \otimes g_i g_j$, since $g_i^2 = 0$ and $g_i g_j = -g_j g_i$. $D_{x, y}$ is the linearization of $D_{x_t, y}$. \quad \square

Let $J = A + M$ be a Jordan superalgebra over a field $K$ of characteristic not $2$, assume that $A$ has a unit element $e$. Let $M = M_0(e) + M_1/2(e) + M_1(e)$, the Peirce decomposition
of $M$ with respect to $e$ induced by that of $\Gamma(J)$. A few special cases of Eqs. (7) and (2) will play an important role. Let $e$ be an idempotent of a Jordan algebra $B$ such that $B_0(e) = \{0\}$. If $b \in B_1(e)$ and $a, c \in B_{1/2}(e)$, the Peirce relations imply $[abc] \in B_0(e)$ and (7) becomes

$$R_xR_e + bR_eR_d = bR_{a,c}, \quad b \in B_1(e), \ a, c \in B_{1/2}(e) \text{ and } B_0(e) = \{0\}. \quad (7'')$$

Equation QJ30 of [8]

$$8(a,b)^2 = [ba^2b] + [ab^2a] + 2[aba],b$$

(9)

partially linearizes to

$$8(a,b).(a,c) = [ba^2c] + [ab ca] + 2[aba],c + 2[aca],b.$$

If $b, c \in B_{1/2}(e)$, and $a \in B_1(e)$ then, by the Peirce relations, $\{a b a\} = 0, \{a c a\} = 0$. Thus

$$8(a,b).(a,c) = [ab ca] + [ba^2c], \quad a \in B_1(e), \ b, c \in B_{1/2}(e). \quad (9')$$

If $m, x, y \in M_{1/2}(e)$, (2) acting on $e$ becomes

$$e(R_xR_{[y,m]} - R_yR_{[x,m]} + R_mR_{[x,y]}) = e(R_{[y,m]}R_x - R_{[x,m]}R_y + R_{[x,y]}R_m),$$

or

$$\frac{1}{2}(x, [y, m] - y, [x, m] + m, [x, y]) = [y, m], x - [x, m], y + [x, y], m.$$

Therefore

$$m, [x, y] = [x, m], y - [y, m], \quad m, x, y \in M_{1/2}(e), \quad (10)$$

or in operator form

$$mR_xR_y - mR_yR_x = -mR_{[x,y]}, \quad m, x, y \in M_{1/2}(e). \quad (10')$$

**Proposition 2.** If $J = A + M$ is a Jordan superalgebra and $A$ has a unit element $e$ then $M_0 := M_0(e)$ is a superideal of $J$. Moreover if $J$ is simple then $M_0 = \{0\}$ and $[M_{1/2}, M_{1/2}] + M_{1/2}$ is a superideal of $J$. In this case either $M_{1/2} = \{0\}$ or $M_1 = \{0\}$.

**Proof.** Since $A$ is in the Peirce 1-space of $e$, $M_0.A = \{0\}$. By the Peirce relations in $\Gamma(J)$ and hence in $J$, $[M_0, M_0] \subseteq A_0 = \{0\}, [M_0, M_{1/2}] \subseteq A_{1/2} = \{0\}, [M_0, M_1] = \{0\}$ and $M_0$ is a superideal of $J$. If $J$ is simple then $M_0 = \{0\}$ and the Peirce 0-component of $J$ is 0. In that case if $a \in A, x, y \in M_{1/2}$ then (7'') for $\Gamma(J)$ becomes $[a x, y] + [a y, x] = a, [x, y]$. So $[M_{1/2}, M_{1/2}]$ is an ideal of $A$. If $z \in M_1$, again (7'') yields $[z, x], y + [z, y], x = z, [x, y]$. Since $[M_1, M_{1/2}] \subseteq A_{1/2} = \{0\}$, we have $z, [x, y] = 0$ and $[M_{1/2}, M_{1/2}], M_1 = \{0\}$.
Therefore \([M_{1/2}, M_{1/2}] + M_{1/2}\) is a superideal of \(J\). Since \(J\) is simple then either \(M_{1/2} = \{0\}\) or \(M_{1} = \{0\}\). □

The following lemmas will be needed to establish the rather unique role of the Kac superalgebra \(K_{10}\) and \(K_{9}\) (see also [12]).

**Lemma 3.** Let \(J = A + M\) be a central simple Jordan superalgebra over an arbitrary field \(K\) of characteristic not 2 whose even part \(A\), odd part \(M\) and bimodule structure coincide with that of the Kac superalgebra \(K_{10}\), i.e., we have \(K_{10}\) except for the skew-symmetric product from \(M \times M \rightarrow M\). Then \(J\) is isomorphic to \(K_{10}\).

**Proof.** We may assume that \(A\) has a basis \(e, v_1, v_2, v_3, v_4, f\), satisfying (3), (3’) and that \(M\) has a basis \(x_1, y_1, x_2, y_2\), such that the bimodule structure is given by (4). The reader will note that in the following computations no use will be made of (3’). Letting \(a_{\alpha} = y_2, b_{\beta} = v_1 = c_{\gamma}\) in (2) and acting on \(y_1\) yields (in reverse order)

\[
2[y_1, v_1, y_2, v_1] + [y_1, y_2, v_1, v_1] = 2[y_1, v_1, y_2, v_1] - [(v_1, v_1), y_2, y_1]
\]

\[
= -2[y_2, v_1, y_1, v_1] + [(v_1, v_1), y_1, y_2]
\]

\[
= (y_1, y_2, v_1) + [(v_1, v_1), y_1, y_2] - [(y_2, v_1), v_1, y_1]
\]

or, using (4),

\[
2[x_2, -x_1] + 0 = 2[x_2, y_2, v_1] - 0 = 2[x_1, y_1, v_1] + 0
\]

\[
= ([y_1, y_2, v_1] + [x_2, v_1, y_2] - [x_1, v_1, y_1]).v_1
\]

So

\[
2[x_1, x_2] = 2[x_2, y_2, v_1] = 2[x_1, y_1, v_1] = ([y_1, y_2, v_1].v_1)
\]  

(11)

Similarly

\[
2[y_1, y_2] = 2[x_2, y_2, v_2] = 2[x_1, y_1, v_2] = ([x_1, x_2, v_2]).v_2
\]  

(12)

\[
2[x_1, y_2] = 2[x_2, y_2, v_3] = 2[x_1, y_1, v_3] = ([x_2, y_1, v_3]).v_3
\]  

(13)

\[
2[x_2, y_1] = 2[x_2, y_2, v_4] = 2[x_1, y_1, v_4] = ([x_1, y_2, v_4]).v_4
\]  

(14)

Since \(A.v_1 = Ke + Kv_1\), we have \((A.v_1).v_i = Kv_i\) and (11)–(14) imply \([x_1, x_2] \in K v_1, [y_1, y_2] \in K v_2, [x_1, y_2] \in K v_3\) and \([x_2, y_1] \in K v_4\). Again by (11)–(14), \([x_j, y_j].v_i \in K v_i\) and \([x_j, y_j]\) has no \(v_i\) component for \(j = 1, 2\) and \(1 \leq i \leq 4\). Moreover since \([x_1, y_1].v_1 = [x_2, y_2].v_1\) the \(e\) components of \([x_1, y_1]\) and \([x_2, y_2]\) are the same, say

\[
[x_1, y_1] = \alpha e + \beta f \quad \text{and} \quad [x_2, y_2] = \alpha e + \gamma f.
\]
Therefore

\[ [x_1, x_2] = \alpha v_1, \quad [x_1, y_1] = \alpha v_3, \quad [x_2, y_1] = \alpha v_4, \quad [y_1, y_2] = \alpha v_2. \quad (15) \]

The scalar \( \alpha \neq 0 \) for otherwise \( Kf + M \) would be a superideal of \( J \). Finally, by the Jordan superidentity,

\[
[x_1, y_1][x_2, y_2] - [x_1, y_2][x_2, y_1] = \left[ [x_1, x_2], x_2, y_1 \right] + \left[ [y_1, y_2], x_2, x_1 \right].
\]

Using (15), this becomes

\[
(\alpha e + \beta f)(\alpha e + \gamma f) - \alpha^2 v_3 v_4 - \alpha^2 v_1 v_2 = \left( [\alpha e + \beta f], x_2, y_2 \right) - \alpha [v_1, x_2, y_1] + \alpha [v_2, x_2, x_1].
\]

Equating the coefficients of \( e \), \(-3\alpha^2 e + \beta \gamma f = \frac{1}{2}(\alpha + \beta)(\alpha e + \gamma f) - 2\alpha(\alpha e + \beta f)\),

\[-3\alpha^2 e + \beta \gamma f = \left( -\frac{3}{2} \alpha^2 + \frac{1}{2} \alpha \beta \right) e + \left( \frac{1}{2} \alpha \gamma + \frac{1}{2} \beta \gamma - 2\alpha \beta \right) f.\]

Equating the coefficients of \( e \), \(-\frac{3}{2} \alpha^2 = \frac{1}{2} \alpha \beta \) and \( \beta = -3\alpha \). Replacing \( \beta \) by \(-3\alpha \) in the coefficient of \( f \) yields \(-2\alpha \gamma = 6\alpha^2 \) or \( \gamma = -3\alpha \). Summarizing, the product on \( M \) is given by

\[
[x_1, y_1] = \alpha(e - 3f), \quad [x_1, x_2] = \alpha v_1, \quad [x_2, y_1] = \alpha v_4, \quad [x_1, y_2] = \alpha v_3, \quad [y_1, y_2] = \alpha v_2 \quad (5')
\]

and skew-symmetry, where \( \alpha \neq 0 \). Replacing \( v_1 \) by \( \alpha^{-1} v_1, v_2 \) by \( \alpha v_2, x_1 \) by \( \alpha^{-1} x_1 \) and \( x_2 \) by \( \alpha^{-1} x_2 \), one obtains a basis of \( J \) whose product is given by (3), (4) and (5). Hence \( J \) is isomorphic to the Kac algebra. \( \square \)

**Lemma 4.** Let \( J = A + M \) be a central simple Jordan superalgebra over an arbitrary field \( K \) of characteristic 3 whose even part \( A \), odd part \( M \) and bimodule structure coincide with that of the degenerate Kac algebra \( K_9 \). Then \( J \) is isomorphic to \( K_9 \).

**Proof.** We may assume that \( A \) has a basis \( e, v_1, v_2, v_3, v_4 \), satisfying (3) and (4). The argument used in the proof of the preceding lemma yields (11)–(14) in this case also. Arguing as above,

\[
[x_1, y_1] = \alpha e = [x_2, y_2]
\]

and we obtain (15). Again \( \alpha \neq 0 \) and the above substitution yields the standard basis of \( K_9 \). \( \square \)
For $J = A + M$ a finite dimensional simple Jordan superalgebra over an arbitrary field $K$ of characteristic not 2, whose even part $A$ is semisimple, we will try to ascertain the extent to which $A$ and the $A$-bimodule structure of $M$ determine $J$, i.e., determine the bracket $[ , ] : M \times M \to A$.

Nonunital simple Jordan superalgebras

From now on we assume that $M \neq \{0\}$ and that $J$ is simple unless explicitly stated otherwise. First we consider the case $M_1 = \{0\}$. In that case $M = M_{1/2}$ and $J$ is not unital. By Proposition 2, $[M_{1/2}, M_{1/2}] = A$.

**Proposition 5.** Let $J = A + M$ be a finite dimensional simple Jordan superalgebra. If $A$ is semisimple with unit $e$ and $M = M_{1/2}$, then $J$ is a Kaplansky superalgebra or the characteristic is 3 and $J$ is the degenerate Kac algebra.

**Proof.** We first show that $A$ is simple. If not, $A = A' \oplus A''$ and $e = e' + e''$. For $m \in M$, $\frac{1}{2}m = e.m + e'.m + e''m$ and $J = (A' + A'.M) + (A'' + A''.M)$. Let $B$ be the unital hull of $\bar{J}(J)$ with unit $1$ and $f = 1 - e$. Then $\{ f Be' \}, \{ f Be'' \} \subseteq \{ e' Be'' \} = \{0\}$ and $(A' + A'.M)$ is a superideal of $J$. So $A$ is simple.

Write $e = \sum_{i=1}^{n} e_i$, $e_i$ mutually orthogonal primitive idempotents and $M = \sum_{i=1}^{n} M_i := \sum_{i=1}^{n} M.e_i$.

Since $[M, M] = A$, $[M_i, M_j] = K e_i$, and $[M_i, M_j] = (A)_{ij}$. Thus $[ , ]$ induces a skew-symmetric form on $M_i$. If $m \in M_i$ with $[m, M_i] = \{0\}$ then for all $x, y \in M_i$, (10) yields $m.[x, y] = [x, m].y - x.[y, m] = 0$, and since $e_i \in [M_i, M_i]$, $\frac{1}{2}m = m.e_i = 0$. Therefore the form induced by $[ , ]$ on $M_i$ is nondegenerate. Choose $x_i, y_i \in M_i$ such that $[x_i, y_i] = e_i$. Then by (10), for any $z \in M_i$, $\frac{1}{2}z = z.e_i = z.[x_i, z], y_i = [x_i, z], y_i = -z.[y_i, x_i]$. So $x_i, y_i$ span $M_i$ whose dimension must be 2. Therefore $Ke_i + M_i$ is a simple subsuperalgebra of $J$ isomorphic to the Kaplansky superalgebra.

If $x, y, z \in M$ and $a \in A$ then by (7)”, $a[R_x, R_y] = aR_{[x, y]}$ and by (10’), $z[R_x, R_y] = -zR_{[x, y]}$. Since the trace $\text{Tr}_J[R_x, R_y] = 0$, we have $\text{Tr}_A R_{[x, y]} = \text{Tr}_M R_{[x, y]} = 0$.

Consider $J (ij) := \{ e_i A e_j \} + \{ e_j A e_j \} + \{ e_j A e_j \} + M_i + M_j$. It is a subsuperalgebra of $J$.

Since $[M_i, M_j] = \{ e_i A e_j \}$, dim $\{ e_i A e_j \} \leq 4$. Since $[x_i, y_j] = e_i$,

$$\text{Tr}_{(M_i + M_j)} R_{e_i} = \text{Tr}_{\{ e_i A e_j \} + \{ e_j A e_j \} + \{ e_j A e_j \}} R_{e_i},$$

or

$$\frac{1}{2} (\dim M_i) 1_K = \left( \dim \{ e_i A e_j \} + \frac{1}{2} \dim \{ e_i A e_j \} \right) 1_K.$$
Hence \( 1 = 1 + \frac{1}{2} \dim \{ e_i A e_j \} \) in \( K \). Since \( \dim \{ e_i A e_j \} \leq 4 \) we must have \( \dim \{ e_i A e_j \} = 0 \) or \( \text{char } K = 3 \) and \( \dim \{ e_i A e_j \} = 3 \). In the first case \( n = 1 \) and \( J \) is the Kaplansky superalgebra. If \( n > 1 \), we are in the second case, and since the dimensions of the off-diagonal Peirce spaces of a Jordan matrix algebra are powers of 2, we must have \( n = 2 \) and \( A \) is the Jordan algebra of a quadratic form. Hence \( A = (e, v_1, v_2, v_3, v_4) \) with multiplication as in Eq. (3). The special universal envelope \( su(A) \) of \( A \) is a Clifford algebra of dimension 16, that is, \( su(A) = M_4[K] \) since \( K \) is algebraically closed. Since \( M \) has dimension 4, a basis can be chosen such that the bimodule action is as in Eq. (4). By Lemma 4, \( J \) is isomorphic to \( K_9 \). \( \square \)

**Unital simple superalgebras**

From now on we assume that \( J = A + M \) is a finite dimensional simple unital Jordan superalgebra with \( A \) semisimple and \( M \neq \{0\} \). We show next that the above assumptions imply that \( A \) has at most 2 simple components.

**Proposition 6.** Let \( J = A + M \) be a finite dimensional simple unital Jordan superalgebra with \( A \) semisimple. Then \( A \) does not contain 3 central nonzero idempotents.

**Proof.** We first establish an identity. Assume \( e = e_1 + e_2 + e_3, \ e_i \in A, \ e_i^2 = e_i, \) with \( A_{ij} = \{0\} \) when \( i \neq j \). For \( x_{12}, y_{12} \in M_{12}, z_{32} \in M_{32}, \) (2) yields

\[
e_1 R_{x_{12}} R_{y_{12}} R_{z_{32}} \]

\[
e_1 (R_{z_{32}} R_{y_{12}} R_{x_{12}} + R_{[x_{12}, z_{32}]} y_{12} + R_{x_{12}} R_{[y_{12}, z_{32}]} - R_{y_{12}} R_{[x_{12}, z_{32}]} + R_{z_{32}} R_{[x_{12}, y_{12}]}) = 0, \]

(16)

since \( e_1 R_{z_{32}} = 0 \), and \( [x_{12}, z_{32}], [y_{12}, z_{32}] \in A_{13} = \{0\} \).

We claim that \( I := I_J(e_1), \) the superideal of \( J \) generated by \( e_1 \), is \( e_1 R_J R_J \), the span of elements of the form \( e_1 R_J x, x \in J \). Since \( J \) is unital, \( e_1 R_J = e_1 R_J R_c \subset e_1 R_J R_J \).

If \( x \in I \cap J_1(e_1), \) \( x = e_1 R_x \) while if \( y \in I \cap J_1/2(e_1), y = \frac{1}{2} e_1 R_y \) (here the subscripts of \( J \) indicate the respective Peirce components). So we need only consider elements of \( I \cap J_0(e_1) \). If \( e_1 R_x, R_c \in J_0(e_1) \) may assume that \( x, y, z \) are homogeneous. We need only consider \( z \in J_0(e_1) \) since otherwise \( z = \alpha e_1 R_x, \alpha = 1 \) or \( 1/2, \) and \( e_1 R_x R_c = \alpha e_1 R_x R_c R_c \in e_1 R_J R_J \). If \( z \in J_0(e_1), \) by (2),

\[
e_1 R_x R_c R_c = e_1 (\pm R_x R_c R_y \pm R_x R_y R_z \pm R_x R_y R_z \pm R_x R_y R_z), \]

and since \( e_1 R_c = 0, e_1 R_x R_c R_c \in e_1 R_J R_J \), which proves the claim.

Since \( J \) is simple, \( J = e_1 R_J R_J \). Therefore

\[
e_2 \in e_1 R_J R_J U_{e_2} \quad \text{and} \quad e_2 = \sum_i e_1 R_x^{(i)} R_y^{(i)} U_{e_2}, \]

where we may assume that \( x^{(i)}, y^{(i)} \) are homogeneous. Moreover \( x^{(i)}, y^{(i)} \) may be assumed both even or both odd since in the mixed case \( e_1 R_x^{(i)} R_y^{(i)} \notin J_1(e_2) \). If \( x^{(i)}, y^{(i)} \) are both
Proof. In the first case the multiplication on $\mathcal{M}$ even then $e_1 R x \cdot R y = (x^{(i)} y^{(i)}) R + U e_2 = 0$. So $x^{(i)}$, $y^{(i)}$ are both odd and $e_1 R x \cdot R y = [x^{(i)} y^{(i)}] R$. Increasing the number of summands if necessary, we may assume that $x^{(i)} \in M_{1j}(e_1)$ and, since only the 22 component is of interest, that $x^{(i)}$, $y^{(i)} \in M_{12}(e_1)$. So we have

$$\sum_i e_1 R x_i \cdot R y_i = e_2 + a_{11}.$$ \hspace{1cm} \text{Hence}

$$z_{32} = 2(e_2 + a_{11}).z_{32} = \sum_i e_1 R x_i \cdot R y_i R z_{32} = 0,$$

by (16), and $M_{32} = 0$. Similarly $e_3 \in e_2 R M_{32} R M_{32} U e_3$ and $e_3 = 0$. \hspace{1cm} \text{Lemma 7. Let $J = A + M$ be a simple unital Jordan superalgebra with $A = A' \oplus A''$, a direct sum of two simple algebras with respective unit elements $e_1$, $e_2$. Then $M = \{e_1 M e_2\}$.}

\textbf{Proof.} Let $M = M_{11} + M_{12} + M_{22}$, the Peirce decomposition of $M$ with respect to $e_1$, $e_2$. Since $J$ is simple, $M_{12} \neq \{0\}$. By the Peirce relations,

$$[M_{11}, M_{12}], [M_{12}, M_{22}] \subseteq A_{12} = \{0\}, \quad [M_{11}, M_{12} M_{11}] = \{0\}.$$ \hspace{1cm} \text{From these and (7) it follows that}

$$M_{12}, [M_{11}, M_{11}] = \{0\}.$$ \hspace{1cm} \text{The map $\nu : A' \rightarrow \text{End}_K(M_{12})^+$ given by $\nu(a) = 2 R e | M_{12}$ is a homomorphism of Jordan algebras whose kernel is $\{0\}$ since $A'$ is simple and $2 R e | M_{12}$ is the identity. Therefore}

$$[M_{11}, M_{11}] = \{0\},$$ \hspace{1cm} \text{and $M_{11}$ is a superideal of $J$. Hence $M_{11} = \{0\}$. Similarly $M_{22} = \{0\}$ and $M = M_{12}$. \hspace{1cm} \text{Proposition 8. Let $J = A + M$ be a simple unital Jordan superalgebra. If $A = K 1$ then $J$ is the Jordan superalgebra of a superform with $\{0\}$ even part. If $A = K e_1 + K e_2$ then $J \cong D_1$, $t \neq 0$, or the Jordan superalgebra of a superform with a 1-dimensional even part.}}

\textbf{Proof.} In the first case the multiplication on $M$ yields a skew-symmetric bilinear form on $M$ and $J$ is the Jordan superalgebra of a superform with $\{0\}$ even part. If $A = K e_1 + K e_2$ then, by Lemma 7, $M = M_{12} = \{e_1 M e_2\}$ and $J = K e_1 + K e_2 + M_{12}$. Also $[M_{12}, M_{12}] \neq \{0\}$, for otherwise $M_{12}$ would be a superideal of $J$. For $x, y \in M$, writing $[x, y] = (x, y) e_1 + (x, y) e_2$, we obtain skew-symmetric bilinear forms $(,)$, $i = 1, 2$, which cannot both be 0; so renumbering $e_1$, $e_2$ if necessary, we have $[x, y] = e_1 + t e_2$ for some $x, y \in M$ and $t \in K$. If dim $M = 2$ then $t$ cannot be 0, for $K e_1 + K x + K y$ would
then be a superideal of $J$, so $J \cong D_t$. Assume $\dim M > 2$. For arbitrary $x, y, z \in M$, by (2),

$$e_1(R_x R_y R_z - R_z R_y R_x - R_{[x,z]_1} R_z + R_{[x,z]_2} R_y - R_{[y,z]_1} R_x) = 0,$$

or

$$\frac{1}{2}([x,y], z - [z,y], x - [x,z], y - (x,y)_1 z + (x,z)_1 y - (y,z)_1 x) = 0.$$

If $z$ is linearly independent from $x$ and $y$ then the above equation implies $[x,y], z = (x,y)_1 z$ and $\frac{1}{2}((x,y)_1 z + (x,y)_2 z) = (x,y)_1 z$. Thus $(x,y)_2 = (x,y)_1$. Since the radical of $(,),_1$ is a superideal of $J$, $(,),_1$ is nondegenerate and $J$ is the superalgebra of a superform with 1-dimensional even part, $K(e_1 - e_2)$. \hfill \Box

The next three lemmas, Lemmas 10, 11 and 12, are needed to prove.

**Proposition 9.** Let $J = A + M$ be a simple unital Jordan superalgebra and let $M = \sum_{i=1}^r M_i$ be the decomposition of $M$ into a direct sum of irreducible $A$-bimodules. Then either $r \leq 2$ or $J$ is a Jordan superalgebra of a superform.

**Lemma 10.** Let $J = A + M$ be a simple unital Jordan superalgebra such that

$$A(R_M \circ R_M) = \{0\}.$$

Then either $J$ is a superalgebra of a superform or $J \cong D_t$.

**Proof.** Since $A(R_M \circ R_M) = \{0\}$,

$$[x,a,y] = [x,y,a] \quad \forall x, y \in M, \ a \in A. \quad (17)$$

Let $e_1, \ldots, e_m$ be an arbitrary frame in $A$ (i.e., a system of pairwise orthogonal primitive idempotents whose sum is the unit element of $A$) and let $M = \sum M_{ij}$, $A = \sum (A)_{ij}$ be the Peirce decomposition of $M$ and $A$ respectively. Assume that pairs of integers $(i, j), (k, l)$, $1 \leq i, j, k, l \leq m$ can be chosen such that $i \neq k, l$ and let $x$, respectively $y$, be arbitrary elements of $M_{ij}$, respectively $M_{kl}$. By (17) and the Peirce relations,

$$[x,y] = \alpha[x,e_i,y] = \alpha[x,y,e_i] = 0, \quad \text{where } \alpha = 1 \text{ or } 2.$$

Hence

$$[M,M] = \sum [M_{ij}, M_{ij}] \subseteq \sum_{i=1}^m (A)_{ii}.$$

The conclusion is valid for an arbitrary frame. Since in a simple Jordan algebra any two orthogonal primitive idempotents are connected, we have copies of $H_2(K)$ for which $e_{11}$,
$e_{22}$ and $\frac{1}{2}(e_{11} + e_{12} + e_{21} + e_{22}), \frac{1}{2}(e_{11} - e_{12} - e_{21} + e_{22})$ are frames. It follows that

$$[M, M] \subseteq \begin{cases} K \varepsilon & \text{if } A \text{ is simple} \\ Ke_1 + Ke_2 & \text{if } A = A' \oplus A'' - \text{a direct sum of two simple} \\
\text{algebras with unit elements } e_1, e_2 \text{ respectively.} \end{cases}$$

Let us first assume that $A$ is simple. If $a \in A$ and $x, y \in M$, then by (2),

$$x(2R_aR_xa + R_ya^2 - R_1R_2a - 2R_2R_ya) = 0, \quad (18)$$

or

$$[x, y].a^2 - 2[x.a, y].a + \alpha 1 = 0, \quad \text{where } \alpha \in K. \quad (18')$$

If $x, y$ were chosen such that $[x, y] \neq 0$, (18') shows that $a$ is of degree 2 over $K$ and $A$ is the Jordan algebra of a nondegenerate quadratic form, $A = K1 + V$, $\dim V \geq 2$. If $a \in V$ then $a^2 \in K1$ and, by (18'), $[x.a, y].a \in K1$; so $[x.a, y] \neq 0, \forall a \in V, x, y \in M$. Let $M' = M.V$. Then $M'$ is a sub-bimodule of $M$ and $[M', M] = \{0\}$. Hence $M'$ is a superideal of $J$ and $M' = \{0\}$. We have proved that $M.V = \{0\}$ and $[M, M] \subseteq K1$, so $J$ is the superalgebra of a superform.

We consider next the case $A = A' \oplus A''$, a direct sum of two simple algebras with unit elements $e_1, e_2$ respectively. So for arbitrary $x, y \in M$, we have $[x, y] = (x, y)_{11}e_1 + (x, y)_{22}e_2$, where $(x, y)_{11} \in K$. If $(M, M)_{21} = \{0\}$ then $Ke_1 + M$ is a superideal of $J$. We may therefore assume that $(M, M)_{i} \neq \{0\}, i = 1, 2$. Then, arguing as above using (18'), $A'$ is an algebra of degree 2 over $K$, so $A' = Ke_1 + V$. Also if $a \in V$, again by (18'), $(M.a, M)_{1} = \{0\}$. By Lemma 7, $M = [e_1, Me_2]$ and the Peirce specialization $\nu : A' \rightarrow \text{End}_K(M_{12})^+$ given by $\nu(a) = 2R_a |_{M_{12}}$ is a homomorphism of Jordan algebras whose kernel is $\{0\}$ since $A'$ is simple and $2R_{e_2} |_{M_{12}}$ is the identity. If $a \in A$ is invertible then $M.a = M$ and $(M.a, M)_{1} = (M, M)_{1} \neq \{0\}$. Hence $V = \{0\}$ and $A' = Ke_1$. Similarly if $(M, M)_{2} \neq \{0\}$ then $A'' = Ke_2$ and, by Proposition 8, $J$ is either $D_1$ or the superalgebra of a superform. This completes the proof of the lemma. 

**Lemma 11.** Let $J = A + M$ be a simple unital Jordan superalgebra such that the algebra $A$ is simple and the generic trace of the identity element $T(1) \neq 0$. Suppose further that $M = M' \oplus M''$ is the direct sum of $A$-bimodules such that $M''.A_0 = \{0\}$ and $M'.A_0 \neq \{0\}$, where $A_0 = \{a \in A \mid T(a) = 0\}$. Then

1. $[M'.A_0, M''] \subseteq A_0.$
2. $[M'', M'] = \{0\}.$

**Proof.** Since $T(1) \neq 0$, $A = K1 + A_0$. Assume that (1) does not hold. Then there exist elements $m \in M'$, $y \in M''$, $a, b \in A_0$ such that

$$[m.b, y] = 1 + a, \quad m \in M', \quad y \in M'', \quad a, b \in A_0.$$
Multiplying both sides by \( y \) yields

\[
(m.b)(R_y)^2 = y,
\]

since \( a.y = 0 \). By Lemma 1, \( (m.b)(R_y)^2 = (m(R_y)^2).b + m.(b(R_y)^2) \). But \( m(R_y)^2 = [m, y].y \in M'' \), so \( (m(R_y)^2).b = 0 \), and \( (b(R_y)^2) = 0 \), since \( b.y = 0 \). Therefore \( y = 0 \), a contradiction. So (1) must hold.

For arbitrary elements \( m \in M', x, y \in M'' \), \( a \in A_0 \), we have by Eq. (2),

\[
0 = a(R_x R_y R_m - R_m R_x R_y - R_m R_{[x, y]} + R_x R_{[x, m]} - R_y R_{[y, m]}).
\]

By assumption, \( a R_x = 0 = a R_y \); by (1), \( a R_m R_y \in A_0 \) and hence \( a R_m R_y R_x = 0 \). Finally, \([x, m] \in A \), \([x, m].y \in M''\) and thus \( a.([x, m].y) = 0 \). So the remaining term

\[
(a.m).([x, y] = 0, \ m \in M', x, y \in M'', a \in A_0).
\]

By Lemma 1, using (2),

\[
D_{[M'', M''], A_0} \subseteq D_{M', M'', A_0} = \{0\}
\]

since \( M''.A_0 = \{0\} \). But \( A = K1 + A_0 \) implies \( D_{[M'', M''], A} = \{0\} \). Therefore \([M'', M'']\) lies in the center of \( A \) and \([M'', M''] \subseteq K1 \). By assumption, there exist an \( a \in A_0 \) and an \( m \in M' \) such that \( a.m \neq 0 \). So, by (19), \([M'', M''], = \{0\} \), completing the proof of the lemma. □

**Lemma 12.** Let \( J = A + M \) be a simple unital Jordan superalgebra such that \( M = M' \oplus M'' \), a direct sum of \( A \)-bimodules, and \( \tilde{A} = A(R_{M'} \circ R_{M''}) \neq \{0\} \). Then

(1) \( M''.\tilde{A} = \{0\} \).

If in addition the algebra \( A \) is simple then

(2) \( \tilde{A} = A_0 \), \( T(1) \neq 0 \), and

(3) \( A = K1 + V \) is a Jordan algebra of a bilinear form.

**Proof.** For arbitrary elements \( a \in A, x, y \in M' \), we have \( M''.(aD_{x, y}) \subseteq M'' \). By Lemma 1, the operator \( D_{x, y} = R_x \circ R_y \) is a derivation of \( J \). Hence

\[
M''.(aD_{x, y}) \subseteq (M''.a)D_{x, y} + (M''D_{x, y}).a.
\]

But \( MD_{x, y} \subseteq [M, x], y + [M, y], x \subseteq M' \), so \( M''.\tilde{A} \subseteq M' \cap M'' = \{0\} \) and (1) holds.

Assume next that \( A \) is simple. Since \( \tilde{A} \text{Inder}(A) \subseteq \tilde{A} \neq \{0\} \) then \( \tilde{A} = K1, A_0 \) or \( A \). The first and last possibilities are ruled out by (1). Hence \( \tilde{A} = A_0 \) and, again by (1), \( 1 \notin A_0 \). So (2) holds.
To prove (3) we will need the notion of an annihilator in a Jordan algebra. For $S$ a subset of a Jordan algebra $B$, define the annihilator of $S$ in $B$,

$$\text{Ann}_B(S) := \{ x \in B \mid S.x = \{0\}, D_{x,x} = \{0\} \}.$$ 

It was shown in [24, Proposition 2, Lemma 3c] that

(i) $\text{Ann}_B(S)$ is always an inner ideal,
(ii) if $S$ is an ideal of $B$ then $\text{Ann}_B(S)$ is an ideal of $B$,
(iii) if $S$ is an arbitrary subset of $B$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in B$ are arbitrary elements such that $S.x_i = S.y_i = \{0\}$, $1 \leq i \leq n$, and $S(\sum_{i=1}^{n} x_i.y_i) = 0$ then $\sum_{i=1}^{n} x_i.y_i \in \text{Ann}_B(S)$.

Let $B = A + M$, the split null extension of $A$ by $M$. Then (iii) implies that $A_0 \cap A_2^D \subseteq \text{Ann}_B(M)$. Suppose that $A_0 \cap A_2^D \neq \{0\}$. Then $\text{Ann}_B(M) \cap A \neq \{0\}$. By (ii), since $M$ is an ideal of $B$, so is $\text{Ann}_B(M)$. Hence $\text{Ann}_B(M) \supseteq A \ni 1$, a contradiction. Thus $A_0 \cap A_2^D = \{0\}$.

Since this does not hold for $H_3(K)$, the classification of simple finite dimensional Jordan algebras then implies that $A$ is a Jordan algebra of a symmetric bilinear form, completing the proof of the lemma. □

We are now ready to prove Proposition 9.

**Proof of Proposition 9.** Let $M = \bigoplus \sum_{i=1}^{r} M_i$, $r \geq 3$, $M_i$ nonzero irreducible $A$-bimodules. If $A(R_M \circ R_M) = \{0\}$ then, by Lemma 10, $J$ is a superalgebra of a superform or is isomorphic to $D_t$. But the odd part of $D_t$ is the sum of two irreducibles. So $J$ is a superalgebra of a superform.

We therefore assume that

$$A(R_M \circ R_M) \neq \{0\}.$$ 

Without loss of generality, we may assume that

$$A(R_{M_1+M_2} \circ R_{M_1+M_2}) \neq \{0\}.$$ 

Let $M' = M_1 + M_2$, $M'' = \bigoplus \sum_{i=3}^{r} M_i$.

**Case I.** $A$ is simple.

From Lemma 12, it follows that $M''.A_0 = \{0\}$ and $A = K + V$ is a Jordan algebra of a nondegenerate symmetric bilinear form. If $M'.A_0 = \{0\}$ then $A(R_{M'} \circ R_{M'}) = \{0\}$ and we are done by Lemma 10. We may therefore assume that $M'.A_0 \neq \{0\}$. So, by Lemma 11,

$$[M'.A_0, M'] \subseteq A_0, \quad [M'', M''] = \{0\}.$$
We claim

\[ M_1.A_0 = \{0\} \quad \text{or} \quad M_2.A_0 = \{0\}. \]  

(19')

Suppose that \( M_i.A_0 \neq \{0\}, i = 1, 2 \). Then \( A(R_{M_1+M''} \circ R_{M_1+M''}) = \{0\} \), for otherwise, by Lemma 12, we would have \( M_2.A_0 = \{0\} \). In particular,

\[ A(R_{M_1} \circ R_{M''}) = \{0\}. \]

Hence for arbitrary elements \( x \in M_1, a \in A_0 \) and \( y \in M'' \), we have \( [x.a, y] = [x, a.y] = 0 \), since \( a.y = 0 \). Since \( M_1 \) is an irreducible module and \( M_1.A_0 \neq \{0\} \), it follows that \( M_1 = M_1.A_0 \). Thus

\[ [M_1, M''] = [M_1.A_0, M''] = \{0\}. \]

Similarly, \([M_2, M''] = \{0\}\). So \( M'' \) is a superideal of \( J \) which is impossible. Therefore (19') holds.

In what follows we will assume that \( M_2.A_0 = \{0\} \). Denote \( M_2 + M'' \) by \( M''' \). We have \( M = M_1 \oplus M''' \) and \( M''' . A_0 = \{0\} \). Since \( M_1 = M_1.A_0 \), Lemma 11 implies that

\[ [M_1, M'''] \subseteq A_0, \quad [M''', M'''] = \{0\}. \]

Choose arbitrary elements \( m \in M_1, x \in M''' \) and \( a \in A_0 \). Since \( \dim M''' > 1 \), it follows that there exists an element \( y \in M''' \) such that \( x \) and \( y \) are linearly independent. By (2), we have,

\[ R_x R_a R_y - R_y R_a R_x + R_{[x,y]} R_a - R_{[x,y]} R_y - R_{[x,y]} R_a + R_y R_a R_x = 0. \]

In view of \( x.a = y.a = 0 \) and \( [x, y] = 0 \), this equation implies that \( R_x R_y R_a = R_y R_a R_x \). Hence \( ([m, x], a).y = ([m, y], a).x \). Since \( [m, x], a \) and \( [m, y], a \) are scalars and \( x \) and \( y \) are linearly independent, \( [m, x].a = 0 \), i.e., \([M_1, M'''] = \{0\}\). Now \( M''' \) is a superideal in \( J \), which is impossible.

**Case II.** \( A = A' \oplus A'' \), a direct sum of two simple algebras, with \( e_1, e_2 \) their respective unit elements.

By Lemma 7, \( M = \{e_1 M e_2\} \). Since \( \tilde{A} \neq \{0\} \) is invariant under \( \text{Inder}(A) \), it follows that \( \tilde{A} \) contains either one of the elements \( e_1, e_2 \) or one of the subsets \( A'_0 = \{a \in A' \mid T(a) = 0\} \), \( A''_0 = \{a \in A'' \mid T(a) = 0\} \). In all these cases \( \tilde{A} \) contains an element \( a \) which belongs to one of the algebras \( A', A'' \) and is invertible in that algebra. Arguing as before using the Peirce specialization of \( A' \) or \( A'' \) in \( \text{End}_R(M)_+ \), the equality \( M''.a = \{0\} \) implies \( M'' = \{0\} \), a contradiction. This completes the proof of Proposition 9. \( \square \)
Simple Jordan superalgebras with even part isomorphic to $J(V, Q)$

We consider next the $J$’s with $A$ a simple Jordan algebra $J(V, Q)$ of degree 2. Since $K$ is algebraically closed, we may choose $v_1, \ldots, v_n$ an orthonormal basis of $V$. Let $T$ be the elementary abelian group generated by the operators $U_{v_i}$, $i = 1, \ldots, n$. We have $(U_{v_i})^2 = \text{id}$, and $U_{v_i} U_{v_j} = U_{v_j} U_{v_i}$, $i \neq j$; the eigenvalues of $U_{v_i} \in \{\pm 1\}$. $T$ is a subgroup of the automorphism groups of $C(V, Q)$, $\Gamma(J)$ and hence of $J$. The group $T$ acts on $M$ which can then be written as a direct sum of weight spaces. The weight spaces of $K$-bimodules in all irreducible bimodules are 1-dimensional, a contradiction.

Then the Jordan superidentity (2) yields $z$ finds $a$ \[ (v_{\pi(1)} \cdots v_{\pi(k)}) U_{v_j} = \begin{cases} (-1)^k v_{\pi(1)} \cdots v_{\pi(k)}, & \text{if } i \neq \pi, \\ (-1)^{k-1} v_{\pi(1)} \cdots v_{\pi(k)}, & \text{if } i = \pi, \end{cases} \]
all weight spaces for an irreducible bimodule $M$ are 1-dimensional. We are now ready to prove

**Lemma 13.** If $J = A + M$ is simple, $A = J(V, Q)$ a simple Jordan algebra of degree 2 and $M$ an irreducible $A$-bimodule then $\dim V = 3$ and $J \cong Q_2(K)$.

**Proof.** Let $n = \dim V$. Our first aim is to obtain a contradiction when $\dim V > 3$. The following argument will be used several times. Given $x, y \in M$ with $[x, y] \neq 0$, if we can find a $z \in M$ and an element $a \in A$ with

\[ [z, x] = 0, \quad [z, y] = 0, \quad a R_x R_y = 0 = a R_z R_y, \quad \text{but} \quad a R_{[x, y]} R_z \neq 0, \]
then the Jordan superidentity (2) yields

\[ 0 = a (R_x R_z R_y - R_z R_y R_x - R_{[x, z]} R_y - R_y R_{[x, z]} R_y + R_{[x, y]} R_y - R_{[y, z]} R_y) = -a R_{[x, y]} R_z, \]
a contradiction.

We adopt the framework of the preliminaries, that is, we work in $C(V', Q)$ and assume that $\{v_0, v_1, \ldots, v_n\}$ is an orthonormal basis of $V'$. Since $M$ is a unital irreducible bimodule, it is isomorphic to a $W_q$ or a summand of $W_{(n-1)/2}$, $n$ odd, as in (8).

If $[M, M] = [0]$ then $M$ is a superideal. So $[M, M] \neq [0]$. If $x, y$ are arbitrary weight vectors from $M$, since $[x, y]$ belongs to a weight space of $A$, then either (i) $[x, y] \in K 1$ or (ii) $[x, y] \in K v_1$. Moreover, in $C(V, Q), xy$ has the same weight as $[x, y]$.

(i) If $[x, y] \in K 1$ then $x$ and $y$ have the same weight. If $n$ is even, since all weight spaces in all irreducible bimodules are 1-dimensional, $x$ and $y$ are linearly dependent and $[x, y] = 0$. If $n$ is odd, $K 1 + K c$ is a 2-dimensional weight space in $C(V, Q)$. If $x y \in K 1$ then, as in the even case, $x$ and $y$ are linearly dependent and $[x, y] = 0$. The case $x y \in K c$ is more delicate.

Let $x = v_0 v_\pi, y = v_0 v_\tau$, with $[x, y] = \pm 1$ and $xy = \pm c$. Since $xy = \pm c, \pi \cap \tau = \emptyset$ and $\pi \cup \tau = \{1, \ldots, n\}$. Denote by $|\pi|, |\tau|$ the cardinalities of $\pi$ and $\tau$. Exchanging $x$ and $y$ if
necessary, we may assume that $|\pi| < |\tau|$. So $|\tau| = |\pi| + 1$. We must distinguish between two cases:

**Case 1.** $n = 4t - 1$.

Thus $|\pi| = 2t - 1$, $|\tau| = 2t$ and $k = 2t - 1$. Since $n > 3$ we have $n \geq 7$, i.e., $t \geq 2$. Renumbering if necessary, we may take $x = v_0v_1 \ldots v_{2t-1}$, $y = v_0v_2 \ldots v_{4t-1}$. Let $z = v_0v_1v_2 \ldots v_{4t-3}$. Since $zx = \pm v_2 \ldots v_{2t-1}v_{2t} \ldots v_{4t-3}$ does not have the same weight as 1 or a $v_j$, $[z, x] = 0$; similarly, since $zy = \pm v_1v_{4t-2}v_{4t-1}$, $[z, y] = 0$. Letting $a = v_2$, we have $ax = v_0v_1v_3 \ldots v_{2t-1}$, so, comparing weights, $aR_x = 0$; similarly $az = \pm v_2v_1v_{4t-2}v_{4t-1}$, so $aR_y = 0$. Since $aR_{x,y}R_z = \pm aR_z \neq 0$, the argument above yields a contradiction.

**Case 2.** $n = 4t + 1$.

In this case $|\pi| = 2t$, $|\tau| = 2t + 1$, and $k = 2t$ is even. We may take $x = v_1 \ldots v_{2t}$, $y = v_{2t+1} \ldots v_{4t+1}$. Let $z = v_{2t+1} \ldots v_{4t}$. Since $xz = c_{v_{4t+1}}$, $[x, z] = \alpha v_{4t+1}$. To prove that $[x, z] = 0$ we first show that $[x, z]$ is skew-symmetric in $v_1$ and $v_{2t+1}$, that is,

$$[v_1 \ldots v_{2t}, v_{2t+1} \ldots v_{4t}] = -[v_{2t+1}v_2 \ldots v_2, v_1v_{2t+2} \ldots v_{4t}].$$

Denote by $D_{v_i, v_j}$ the inner derivation $[R_{v_i}, R_{v_j}]$ of $J$. Clearly

$$v_iD_{v_i, v_j} = v_j, \quad v_jD_{v_i, v_j} = -v_i, \quad v_kD_{v_i, v_j} = 0, \quad \text{for } k \neq i, j.$$  \ \ \ (20)

Hence $zD_{v_1, v_{4t+1}} = 0$, $xD_{v_1, v_{4t+1}} = v_{4t+1}v_2 \ldots v_{2t}$ and

$$[x, z]D_{v_1, v_{4t+1}} = [v_{4t+1}v_2 \ldots v_{2t}, v_{2t+1} \ldots v_{4t}] = -\alpha v_1.$$ 

Applying $D_{v_{2t+1}, v_1}$ to both sides of the above equation, we obtain

$$[v_{4t+1}v_2 \ldots v_{2t}, v_1v_{2t+2} \ldots v_{4t}] = \alpha v_{2t+1}.$$ 

Finally, applying $D_{v_{4t+1}, v_{2t+1}}$ to both sides of the last equation, we get

$$[v_{2t+1}v_2 \ldots v_{2t}, v_1v_{2t+2} \ldots v_{4t}] = -\alpha v_{4t+1}.$$ 

Thus $[x, z]$ is skew-symmetric in $v_1$, $v_{2t+1}$; similarly it is skew-symmetric in $v_2$, $v_{2t+2}$, in $v_3$, $v_{2t+3}$, \ldots, in $v_{2t}$, $v_{4t}$. Therefore $[x, z] = (-1)^{2t} [z, x] = [z, x]$ and $[x, z] = 0$. Since $yz = \pm v_{4t+1}$ and $y = z, v_{4t+1}$, we have $[y, z] = v_{4t+1}D_2 = 0$. Letting $a = v_1$, $aR_x = 0$, and, since $azy = v_1v_{4t+1}R_zR_y = 0$. Since $aR_{x,y}R_z = \pm aR_z \neq 0$, the argument above yields a contradiction.

(ii) If $[x, y] \in K_{v_j}$, scaling if necessary, we may assume that $[x, y] = v_j$. Since the product of $x$, $y$ in $C(V, Q)$ has the same weight as $[x, y]$, either
\[ xy \in K v_1, \quad \text{or} \quad (21.1) \]
\[ xy \in Kv_{1, \alpha} \text{, with } n \text{ odd}, \quad \text{or} \quad (21.2) \]

(1) If \( xy \in K v_1 \) then \( M \cong W_k \) for otherwise \( xy \) would lie in \( K v_1(1 + \alpha c) \). Let \( x = v_0 v_\pi, y = v_0 v_\tau \), where \( v_0 \) means that \( v_0 \) is present if \( n \) is odd and absent if \( n \) is even. From \( v_\pi v_\tau = \pm v_1 \), follows that one of \( |\pi|, |\tau| \) is odd while the other is even. Assume that \( |\pi| \) is even and \( |\tau| \), odd. If \( k \) is even then \( |\tau| = |\pi| + 1 \) and \( v_\tau = \pm v_\pi v_1 \), hence \( i \notin \pi \) and \( y = \pm x v_\pi v_\tau \). If \( k \) is odd then \( |\pi| = |\tau| + 1 \) and \( v_\tau = \pm v_\pi v_1 \), hence \( i \notin \pi \) and \( x = \pm v_0 v_\pi = \pm (v_0 v_\pi).v_1 = \pm y v_\tau. \) In both cases either \( x = \pm y v_\tau \) if \( y = \pm x v_\pi \). If \( y = \pm x v_\pi \) then \( -v_1 = [y, x] = \pm [x, v_\pi], x = \pm (v_0 D_1) \). But then \( 1D_1 = v_1^2 D_1 = 2v_1(v_\pi D_1) = -2 \), a contradiction. Similarly \( x = \pm y v_\tau \) leads to a contradiction.

(2) If \( xy \in K v_{1, \alpha} \) then \( M \cong W_k \) for otherwise \( xy \) would lie in \( K v_{1, \alpha} \). Since \( x = v_0 v_\pi, y = v_0 v_\tau \), we have \( v_\pi v_\tau = \pm c v_1 \). Then either \( \pi \cap \tau = \emptyset, i \notin \pi, i \notin \tau \), \( |\pi| = |\tau| = (n - 1)/2 \), or \( \pi \cap \tau = \{i\}, |\pi| = |\tau| = (n + 1)/2 \). In both cases there exist integers \( s, t, \) such that \( \pi(s) \neq i, \tau(t) \neq i, \pi(s) \neq \tau(t) \). Let

\[ z = v_0 v_\pi v_\pi(1) \ldots v_\pi(t-1)v_\pi(t) = v_\pi(1) \ldots v_\pi(k) \]

Then \( z x = z v_0 v_\pi = \pm v_\pi(1) v_\pi(t), \) and \( [z, x] = 0 \). Also \( z y = z v_0 v_\tau = \pm c v_\pi v_\pi(1) v_\pi(t) = \pm c v_\pi v_\pi(1) v_\pi(t) \), and \( [z, y] = 0 \). Let \( a = v_1 \). Then \( [a, x, y] \in K 1 \) which, as was shown at the beginning of the proof, implies \( a R_c, R_c = 0 \). Now

\[ v_1 z v_\pi v_\tau v_\pi(1)v_\pi(t) = \pm c v_\pi v_\pi(1)v_\pi(t) \]

Since we assume that \( n > 3 \), then \( a R_2 R_c = 0 \). Finally, \( a R_{x, y} R_c = \lambda_1 R_c = \lambda z \neq 0 \), and, by the argument above, we have a contradiction.

(3) If \( xy \in K v_{1, \alpha} \), \( \alpha = \pm 1, \pi = \sqrt{-1} \), with \( n \) odd, \( M = W_k(1 + \alpha c), k = (n - 1)/2 \), \( x = v_0 v_\pi(1 + \alpha c), y = v_0 v_\tau(1 + \alpha c) \) and \( v_\pi v_\tau \in K v_1 \) or \( v_\pi v_\tau \in K v_{1, \alpha} \). These two cases are treated in a way similar to (21.1) and (21.2) respectively.

It now remains to consider the case \( n = 3 \). In this case \( K 1 + V \cong \mathcal{M}_2[K]^+ \) (the underlying quadratic form is the determinant) and has 3 nonisomorphic \( \mathcal{M}_2[K]^+ \)-bimodules: the regular bimodule \( \mathcal{M}_2[K]^+ \cong W_0 \cong W_2 \), the 3-dimensional bimodule \( \mathcal{H}_2[K] = \{x \in \mathcal{M}_2[K] \mid x^2 = x\} \) with action \( x.a := \frac{1}{2}(x a + a'x), x \in \mathcal{H}_2[K], a \in \mathcal{M}_2[K], \) (this bimodule is isomorphic to \( W_1(1 + \sqrt{-1}c) \)) and the 1-dimensional bimodule \( W_3 \).

To avoid confusion between an element \( a \in A \) and \( a \in M \), denote the latter by \( \tilde{a} \).

We determine the product in \( M = \mathcal{M}_2[K] \). For any \( i, 1 \leq i \leq 3 \), \( [\tilde{v}_1, \tilde{1}] = 0 \), otherwise \( [\tilde{v}_1, \tilde{1}] = 1 \), \( \alpha \neq 0 \). But \( v_1 D_1 = [v_1, \tilde{1}, \tilde{1}] = [\tilde{v}_1, \tilde{1}] \) cannot be \( a v_1, \alpha \neq 0, \) since \( D_1 \) is a derivation. If \( [\tilde{v}_1, \tilde{v}_2] = 0 \), then, using (20), we see that for any \( i, j, [\tilde{v}_1, \tilde{v}_j] = 0 \) and \( [M, M] = 0 \). This is impossible for \( M \) would then be a superideal of \( J \). By weight considerations and scaling if necessary, we may assume that \( [\tilde{v}_1, \tilde{v}_2] = v_3 \). Again applying \( D_{v_2, v_3} \), we see that \( [\tilde{v}_1, \tilde{v}_3] = -v_2 \), and similarly \( [\tilde{v}_2, \tilde{v}_3] = v_1 \).
Consider the superalgebra
\[ \mathcal{M}_2[K] + \mathcal{M}_2[\tilde{K}] := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathcal{M}_2(K) \right\}, \quad [\tilde{a}, \tilde{b}] := \frac{i}{2} [a, b]. \]

The matrices
\[ v_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad v_3 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \]
form an orthonormal basis of \( V := \{ a \in \mathcal{M}_2(K) \mid \text{Tr} a = 0 \} \) and \([\tilde{v}_1, \tilde{v}_2] = v_3, [\tilde{v}_1, \tilde{v}_3] = -v_2 \) and \([\tilde{v}_2, \tilde{v}_3] = v_1 \). The map
\[ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto \begin{pmatrix} a & \rho b \\ \rho b & a \end{pmatrix}, \quad \rho \in K, \quad \rho^2 = i, \]
is an isomorphism onto \( Q_2(k) \).

Now let us suppose that \( M = \mathcal{H}_2[\tilde{K}] \). Arguing as above, \([\tilde{1}, M] = [0, 0]\), and it follows that \( \text{dim } [M, M] \leq 1 \). But \( [M, M] \) is invariant under \( \text{Inder } A \), hence \( [M, M] \) is either \([0] \) or \( K \). Let \( e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in A \). Then \( M = M_1 + M_{1/2} + M_0 \), the Peirce decomposition of \( M \) with respect to \( e \). We have \( \text{dim } M_1 = \text{dim } M_{1/2} = \text{dim } M_0 = 1 \), so the products \([M_i, M_j] \) are \([0] \). From \([M, M] \subseteq K \), it follows that \([M_1, M_{1/2}] = [0] = [M_0, M_{1/2}] \) and \([M, M] = [0] \). Therefore \( M \) is a superideal and \( J \) is not simple.

In the last case, \( \text{dim } M = 1 \), and \( M \) is a superideal. This concludes the proof of the lemma. \( \square \)

We need more information on superalgebras with \( A = J(V, Q) \).

**Lemma 14.** If \( J = A + M \) is a Jordan superalgebra with \( A = J(V, Q) \), the Jordan algebra of a nondegenerate quadratic form \( Q \) on a \( K \)-vectorspace of dimension \( n \geq 2 \), and \( M = M' + M'' \), a direct sum of irreducible \( A \)-bimodules, then either \( [M, M] = [0] \) or at least one of \( M', M'' \) is of dimension 1. (Note that we do not assume that \( J \) is simple.)

**Proof.** We assume that \( \text{dim } M' > 1 \) and \( \text{dim } M'' > 1 \) and will prove that \([M, M] = [0] \). Since \( M' \) and \( M'' \) are irreducible, \( M', V \neq [0] \) and \( M''.V \neq [0] \). We first show that
\[ [M', M'] = [M'', M''] = [0]. \quad (22) \]

If \([M', M'] \neq [0] \) then the superalgebra \( A + M' \) is simple and, by Lemma 13, \( A = \mathcal{M}_2(K) = M' \) and the bracket on \( M' \) is a scalar multiple of the Lie bracket. Let \( D(M', M') \) be the Lie algebra spanned by all derivations \( D_{x,y} = R_x R_y + R_y R_x, \) \( x, y \in M' \). Then \( AD(M', M') = V \). Hence \( M''.(AD(M', M')) = M'', V \neq [0] \). But
\[ M''.(AD(M', M')) \subseteq (M'', A)D(M', M') + (M''D(M', M')) . A \subseteq M' \cap M'' = [0], \]
a contradiction. Therefore \([M', M'] = [0] \) and similarly \([M'', M''] = [0] \).
Similarly, we have
\[[M_{11}^s,M_{12}^r] = 0\]

We show next that $M_{12}^s \neq \{0\}$, $M_{22}^r \neq \{0\}$. Suppose first that $M_{12}^s = \{0\}$. Then $M' = M_{12}$ and $M'.v_0 = M'.(e_1 - e_2) = \{0\}$. Complete $v_0$ to an orthonormal basis of $V$ (i.e., $v_0, v_1, \ldots$), with $v_i^2 = 1$ and $v_i.v_j = 0$ when $i \neq j$). For each $i$ we have $v_0D_{v_0,v_i} = v_i$. Hence

\[M'.v_i = M'.(v_0D_{v_0,v_i}) \subseteq (M'.v_0)D_{v_0,v_i} + (M'D_{v_0,v_i}), v_0 = \{0\},\]

a contradiction. Thus either $M_{11}^s \neq \{0\}$ or $M_{22}^r \neq \{0\}$.

Let $q \in A_{12}$ be such that $q^2 = 1$; such elements exist, for example $v_1, i \geq 1$. Then the quadratic operator $U_q : J \rightarrow J$ is an automorphism and $U_q^2 = \text{Id}$. Since $M_{11}^s U_q = M_{12}^r$, and $M_{22}^r U_q = M_{11}^s$, $M_{11}^s$ and $M_{22}^r$ are both nonzero. Similarly $M_{11}^s \neq \{0\}, M_{22}^r \neq \{0\}$. Finally $M_{12}^r$ and $M_{12}^s$ are also nonzero, for if $M_{12}^s = \{0\}$ then $M'.A_{12} = \{0\}$ and $M_{11}^s, M_{22}^r$ are submodules of $M'$.

For any derivation $D \in D(M', M')$ we have, by (22), $M'D = \{0\}$ and $AD \subseteq [M', M'] = \{0\}$, hence $D : M'' \rightarrow M'$ which sends $m \in M''$ to $mD \in M'$ is a bimodule homomorphism. Since $M''$ is irreducible, if the kernel of the restriction of $D$ to $M''$ is nonzero then $D = 0$.

We have

\[M_{11}^s D(M_{11}^s, M_{12}^r) = M_{11}^s U_{e_1} D(M_{11}^s, M_{12}^r) = M_{11}^s D(M_{11}^s, M_{12}^r) U_{e_1} \subseteq M_{12}^r U_{e_1} = \{0\}.\]

Therefore

\[D(M_{11}^s, M_{12}^r) = \{0\}.\]

Similarly

\[D(M_{22}^r, M_{12}^s) = \{0\}.\]

Finally since $M_{12}^s D(M_{11}^s, M_{22}^r) = \{0\}$ by Peirce considerations, $D(M_{11}^s, M_{22}^r) = \{0\}$ and

\[D(M', M') = \{0\}.\]

We show next that

\[\{M_{12}^r, M_{11}^s\} = \{0\}.\]

By the Peirce relations, the subspace $[M_{12}^r, M_{11}^s]$ lies in $A_{12}$ and it is invariant under $D(A_{12}, A_{12})$. Hence if $[M_{12}^r, M_{11}^s] \neq \{0\}$ then, by (20), $[M_{12}^r, M_{11}^s] = A_{12}$. Since $D(M_{22}^r, M_{12}^s) = \{0\},$

\[\{0\} = M_{11}^s D(M_{22}^r, M_{12}^s) = [M_{11}^s, M_{12}^s] M_{22}^r = A_{12} M_{22}^r.\]
Hence $M'_{22}$ is a sub-bimodule of $M'$, a contradiction. Similarly

$$[M'_{12}, M''_{22}] = [0], \quad [M'_{11}, M''_{12}] = [0], \quad [M'_{22}, M''_{12}] = [0].$$

We have proved that

$$[M', M''] \subseteq A_{11} + A_{22} = K1 + Kv_0.$$

Since, up to multiplication by a scalar, $v_0$ was an arbitrary anisotropic vector, this yields

$$[M, M] = [M', M''] \subseteq K1.$$

We are now ready to prove that $[M, M] = [M', M''] = [0]$. Let $x \in M'$, $z \in M''$ be arbitrary nonzero elements of $M'$ and $M''$. Since $\dim M' > 1$, we can choose $y \in M'$ linearly independent of $x$. Since $D(M', M') = [0]$,

$$0 = zDx, y = [z, x], y - [z, y].x.$$

But $[z, x], [z, y] \in K1$. Therefore $[z, x] = 0$ and $[M'', M'] = [0]$. □

**Proposition 15.** Let $A = J(V, Q)$, the Jordan algebra of a nondegenerate quadratic form $Q$ on a $K$-vectorspace of dimension $n \geq 2$, and let $M = M' \oplus M''$ be a direct sum of two irreducible $A$-bimodules. If $J = A + M$ is a Jordan superalgebra then one of the following holds:

(i) $[M, M] = [0]$,

(ii) $\dim M' = \dim M'' = 1$ and $[M', M''] \subseteq K1$,

(iii) $n = 3$ and $J$ is isomorphic to $P_3(K)$.

**Proof.** If $[M, M] = [0]$, (i) is satisfied. So assume that $[M, M] \neq [0]$. Then, by Lemma 14, we may assume that $\dim M'' = 1$ and $M'' = Km''$. Since $A(Rm'')^2 \subseteq [M'', M''] = [0]$, the restriction of $D_{m''}[M': M' \rightarrow M'']$ is a bimodule homomorphism. Since both bimodules are irreducible, either $M'(Rm'')^2 = [0]$ or $(Rm'')^2 |_{M'}$ is an isomorphism. If $\dim M' > 1$ then $[M', m''].M'' = [0]$ so $[M', m''] \subseteq V$ and we have

$$\dim M' > 1 \Rightarrow [M', m''] \subseteq V. \quad (23)$$

We claim that

$$[M', M''] = [0]. \quad (24)$$

If $[M', M''] \neq [0]$ then $A + M'$ is a simple superalgebra. By Lemma 13, $A \cong M_2(K)^+$ and $M'$ is a regular $M_2(K)$-bimodule. Fix the matrix units in $A$ and consider $M'_{11} = Ke_1', M'_{12} = Ke_1' + Ke_2', M'_{22} = Ke_2'$, the Peirce decomposition of $M'$ with respect to $e_{11}$, $e_{22} \in A$. By (23) and the Peirce relations, we have

$$[M'_{12}, m''] \subseteq (Ke_{11} + Ke_{22}) \cap V = K(e_{11} - e_{22}).$$
Since \( \dim M'_V = 2 \), \( \exists \alpha, \beta \in K \), not both 0, such that \([\alpha e'_1 + \beta e'_2, m''] = 0\). Acting on this equation by the derivation algebra \( D(A, A) \) yields \([V, m''] = 0\) and it remains to determine \([1', m'''\]). By the Peirce relations, \([1', m'''] \subseteq A_{12} = Ke_{12} + Ke_{21}\). But since \([1', m''']\) is annihilated by \( D(A, A) \), \([1', m''']\) is a scalar matrix and, by (23), \([1', m'''] = 0\). Therefore \([M', m'''] = 0\).

We conclude the proof of the claim by verifying that \( J \) with the above bracket operations is not a Jordan superalgebra. On the one hand

\[
m''(R_{e_{12}'} R_{(e_{11} - e_{22})} R_{e_{21}'} - R_{e_{21}'} R_{(e_{11} - e_{22})} R_{e_{12}'} + R_{(e_{12}'}, e_{21}' (e_{11} - e_{22})) = m''R_{(e_{11} - e_{22}), (e_{11} - e_{22})} = m'',
\]

while on the other

\[
m''(R_{e_{12}'} R_{(e_{11} - e_{22})} e_{21}' - R_{e_{21}'} R_{(e_{11} - e_{22})} e_{12}' + R_{e_{11}'}, e_{22} R_{(e_{12}'}, e_{21}'}) = 0.
\]

So the Jordan superidentity does not hold, proving the claim.

Recall that we are assuming

\([M, M] \neq 0\)

and consider the map \( M' \to V \) given by \( m' \mapsto [m', m'''] \). This map is injective, for if \([m', m'''] = 0\) for some \( m' \in M' \) then, by (2), for any \( v \in V \) and \( m'_1 \in M' \),

\[
v(R_{m''} R_{m'_1} R_{m'''} - R_{m''} R_{m'_1} R_{m'''}, R_{[m', m'''], m'_1}) = v(R_{[m', m'_1]} R_{m'''} - R_{[m', m'''], m'_1}) R_{m'''}, R_{[m', m'''], m'_1}) = 0.
\]

Since \( vR_{m''} R_{m'_1} \in [M', M'] = 0 \), \( vR_{m'''} = 0 \) and \([m', m'''] = 0\), the left hand side is 0. Since \([m', m'_1] = 0\), the right hand side reduces to \(-vR_{[m'', m'_1]} R_{m'''}\) and

\[
vR_{[m'', m'_1]} R_{m'''} = 0.
\]

By the arbitrariness of \( v \) and \( m'_1 \) we have

\([V, [m''', M']]. m' = 0\). \hspace{1cm} (25)

If \([m'''', M'] = 0\) then \([M, M] = 0\), contrary to our assumption. So \([m'''', M'] \neq 0\) and since the subspace \([m'''', M']\) is stable under the action of \([R_V, R_V]\) we must have, by (23),

\([m'''', M'] = V\).

In that case, by (25), \( m' = 0 \) and the map \( M' \to V, m' \mapsto [m', m'''] \) is injective. Hence

\[\dim M' \leq n.\]
If \( \dim M' = 1 \) then \([M, M] = [M', M'']\) is annihilated by \([R_V, R_V]\), so \([M, M] \subseteq K1\) and (ii) holds. Assume that \( \dim M' > 1 \). From the classification of the irreducible bimodules of \( J(V, Q) \) it follows that for \( n \geq 2 \) the dimension of a nontrivial irreducible unital bimodule is either \( \binom{n}{k} + \binom{n+1}{k+1} \) or \( \frac{1}{2} \left( \binom{n}{k} + \binom{n}{k+1} \right) \); in the last case \( n \) is odd and \( k = (n - 1)/2 \), so

\[
\binom{n}{k} = \binom{n}{n - (k + 1)} = \binom{n}{k}
\]

and the dimension of the bimodule is \( \binom{n}{k} \), with \( k = (n - 1)/2 \). Since \( \binom{n}{k} > n \) for \( 1 < k < n - 1 \), these dimensions are greater than \( n \) unless we are in the second case and \( n = 3 \), in which case \( A \cong M_2(K)^+ \) with \( Q \) equal the determinant and there is a bimodule isomorphism

\[
\tilde{\cdot}: \mathcal{H}_2(K) \to M', \quad \tilde{h}a = \frac{1}{2} ha + a'^h, \quad h \in \mathcal{H}_2(K), \quad a \in A.
\]

In this case let \( s_{12} = e_{12} - e_{21} \in M_2(K) \). We wish to show that

\[
[\tilde{h}, m''] = a s_{12} h, \quad \forall h \in \mathcal{H}_2(K) \text{ and a fixed } 0 \neq a \in K.
\]

From the Peirce relations we have

\[
\left[ e_{12} + e_{21}, m'' \right] = \alpha e_{11} + \beta e_{22}.
\]

Applying the automorphism \( U_{e_{12} + e_{21}} \) to both sides of this equality, using

\[
e_{12} + e_{21} U_{e_{12} + e_{21}} = e_{12} + e_{21} \quad \text{ and } \quad m'' U_{e_{12} + e_{21}} = -m'\]

we get

\[-(\alpha e_{11} + \beta e_{22}) = (\alpha e_{11} + \beta e_{22}) U_{e_{12} + e_{21}} = \alpha e_{22} + \beta e_{11}\]

which implies

\[\alpha + \beta = 0.\]

Therefore

\[
\left[ e_{12} + e_{21}, m'' \right] = \alpha (e_{11} - e_{22}) = a s_{12} (e_{12} + e_{21}).
\]

Applying the derivation \([R_{e_{11}}, R_{e_{12} + e_{21}}]\) to both sides yields

\[
\left[ e_{11} - e_{22}, m'' \right] = \alpha s_{12} (e_{11} - e_{22}).
\]
It remains to determine \([\bar{1}, m''\)]. By Peirce considerations,
\[
[\bar{1}, m''] = \gamma e_{12} + \delta e_{21}, \quad \gamma, \delta \in K.
\]
By (2),
\[
(e_{11} - e_{22})(R_{e_{12} + e_{21}}R_{m''} - R_{m''}R_{e_{12} + e_{21}}R_{[\bar{1}, m'']})
= (e_{11} - e_{22})(R_{e_{12} + e_{21}}R_{m''} - R_{[\bar{1}, m'']}R_{e_{12} + e_{21}} + R_{[e_{12} + e_{21}, m'']}R_{[\bar{1}, m'']}) \quad (26)
\]
By (24), the first terms on each side of (26) are 0. Since \((e_{11} - e_{22})m'' = 0\), the second term on the left hand side is 0 and since \((e_{11} - e_{22})e_{ij} = 0\), for \(\{i, j\} = \{1, 2\}\), the second term of the right hand side of (26) is also 0. Therefore (26) reduces to
\[
-(e_{11} - e_{22}).\left([\bar{1}, m''], e_{12} + e_{21}\right) = ((e_{11} - e_{22}).\left([e_{12} + e_{21}, m'']\right)).\bar{1}.
\]
Which in turn equals
\[
-(e_{11} - e_{22}).\left((\gamma e_{12} + \delta e_{21}).e_{12} + e_{21}\right) = ((e_{11} - e_{22}).\alpha(e_{11} - e_{22})).\bar{1}.
\]
This yields
\[
\gamma e_{22} - \delta e_{11} = \alpha \bar{1}.
\]
Therefore \(\gamma = \alpha = -\delta\) and
\[
[\bar{1}, m''] = \alpha(e_{12} - e_{21}) = \alpha s_{12} 1.
\]
Identifying \(m''\) with \(-\alpha s_{12}\), we see that \(J \cong P_2(K)\) which completes the proof of the proposition. \(\square\)

Proposition 9, Lemma 13 and Proposition 15 yield

**Proposition 16.** Let \(J = A + M\) be a simple unital Jordan superalgebra with \(A = J(V, Q)\), the Jordan algebra of a nondegenerate quadratic form on a vector space \(V\) of dimension \(n \geq 2\). Then one of the following holds:

(i) \(J\) is the superalgebra of a superform (in which case \(M.V = \{0\}\) and \([M, M] \subseteq K 1\)), i.e., \([, ]\) is a skew-symmetric form on \(M\),
(ii) \(n = 3\) and \(J\) is isomorphic to \(Q_2(K)\),
(iii) \(n = 3\) and \(J\) is isomorphic to \(P_2(K)\).
Simple Jordan superalgebras with even part isomorphic to \( J(V, Q) \oplus Kf \)

We consider next superalgebras where the even part \( A = J(V, Q) \oplus Kf \) with \( J(V, Q) \) central simple over \( K \). As above if we fix an element \( v_0 \in V \) with \( Q(v_0) = 1 \) and denote the unit element of \( J(V, Q) \) by \( e \), then \( e_1 = \frac{1}{2}(e + v_0) \), \( e_2 = \frac{1}{2}(e - v_0) \) are orthogonal idempotents of \( J(V, Q) \) and \( e = e_1 + e_2 \). The Peirce half-space of \( J(V, Q) \) with respect to \( e_1, e_2 \) is \( V_{1/2} = \{ v \in V \mid T(v_0, v) = 0 \} = V_0^+ \), where \( T \) is the trace form of \( Q \).

Let \( J = A + M \) be a simple unital Jordan superalgebra. By Lemma 7,

\[
M = \{ eMf \} = M_1 + M_2, \quad \text{where } M_i = \{ e_iMf \}, \quad i = 1, 2.
\]

If \( v \in V_{1/2} \) then, by the Peirce relations, \( M_i.v \subseteq M_j \), \( \{ i, j \} = \{ 1, 2 \} \), and for \( x \in M \), \( xU_v = 0 \). If \( Q(v) = 1 \) then \( (v + f)^2 = e + f = 1 \) and \( (U_{v+f})^2 = U_1 \). Therefore \( U_{v+f} \) induces an automorphism of order 2 of \( \Gamma(J) \) and hence of \( J \) itself. Now \( U_{v+f} = U_v + U_v.f + U_f \) and \( U_{v+f} \) acts on \( M \) as \( U_{v,f} = 2R_v \). Therefore

\[
(x.v).v = \frac{1}{4} x, \quad \forall x \in M \forall v \in V_{1/2} \text{ with } Q(v) = 1. \tag{27}
\]

Moreover \( U_{v+f} \) interchanges \( e_1 \) and \( e_2 \) and therefore \( M_1 \) and \( M_2 \). So

\[
M_1.v = M_2, \quad M_2.v = M_1, \quad \forall \text{ anisotropic } v \in V_{1/2}. \tag{28}
\]

The next lemma will allow us to apply some of our previous results to the task at hand.

**Lemma 17.** Let \( J = A + M \) be a simple unital Jordan superalgebra with \( A = J(V, Q) \oplus Kf \), \( J(V, Q) \) the Jordan algebra of a nondegenerate quadratic form \( Q \) on a \( K \)-vectorspace of dimension \( n \geq 2 \), \( e = e_1 + e_2 \) the unit element of \( J(V, Q) \). Then the subsuperalgebra \( J_1 := (Ke_1 + Kf) + M_1 \), where \( M_1 := M.e_1 \), is simple.

**Proof.** Fix \( v \in V_{1/2} \) with \( Q(v) = 1 \). For arbitrary elements \( x, y \in M_i, i = 1, 2 \), we have

\[
[x, y]U_{v+f} = [xU_{v+f}, yU_{v+f}],
\]

or

\[
[x.v, y.v] = \frac{1}{4}[x, y]U_v + \frac{1}{4}[x, y].f. \tag{29}
\]

Let us first assume that \( [M_1, M_1] \subseteq Ke_1 \). Then, acting by \( U_{v+f} \), \( [M_2, M_2] \subseteq Ke_2 \). And since \( [M_1, M_2] \subseteq A_{1/2} = V_{1/2} \), \( (Ke + V) + M \) is a superideal of \( J \), contradicting the simplicity of \( J \).

Assume next that \( [M_1, M_1] \subseteq Kf \). Then acting by \( U_{v+f} \), \( [M_2, M_2] \subseteq Kf \) and \( [M, M] \subseteq V_{1/2} + Kf \). But the subspace \( [M, M].e \) of \( V_{1/2} \) is invariant under \( \text{Der}(V, V) = \text{Inder}(J(V, Q)) \). Hence either \( [M, M].e = V \) or \( [M, M].e = 0 \). Since we can’t have
the first, we must have \([M, M] \subseteq Kf\). But then \(Kf + M\) is a superideal of \(J\), again a contradiction.

Finally let \(I\) be a nonzero superideal of \(J_1\). If \(I \cap A \neq \{0\}\) then either \(e_1 \in I\) or \(f \in I\). In both cases \(M_1 \subseteq I\) which implies \([M_1, M_1] \subseteq I\). But we have shown that \([M_1, M_1]\) is not contained in \(Ke_1\) or \(Kf\), so \(I = J_1\). If \(I \cap A = \{0\}\) then \(I \subseteq M_1\). In this case \([I, M_1] \subseteq I \cap A = \{0\}\) and we will construct a proper superideal of \(J\).

Let \(w\) be an arbitrary element of \(V_{1/2}\) with \(Q(w) = 1\). Let \(x \in I, m \in M_2\) be arbitrary elements of \(I\) and \(M_2\) respectively. By (27),

\[ [x, m].w = 4[x, (m.w).w].w. \tag{30} \]

Denoting \(m.w\) by \(m'\) we have \(m' \in M_1\), so \([x, m'] = 0\) and by (2),

\[ 2[x, m'.w].w = -xR_wR_{m'} + xR_{m'}R_wR_m + [x, m']R_{m'}R_w + [x, (m'.w).w].w. \tag{31} \]

But by (27) and (28),

\[ [x, w^2, m'] = \frac{1}{2}[x, m'] = 0, \]

\[ [(x.w).w, m'] = \frac{1}{4}[x, m'] = 0, \tag{32} \]

\[ [x, (m'.w).w] \in [x, M_1] = \{0\}. \]

By (30), (31) and (32), \([x, m].w = 0\) and, since \(V_{1/2}\) is spanned by vectors \(w\) with \(Q(w) = 1\), \([x, M_2], V_{1/2} = \{0\}\). This implies that \([x, M_2] = \{0\}\) and \([x, M] = \{0\}\). We show next that

\[ \{z \in M \mid [z, M] = \{0\}\} \]

is an \(A\)-subbimodule of \(M\). This implies that \(\{z \in M \mid [z, M] = \{0\}\}\) is a superideal of \(J\).

It is easy to see that \(\{z \in M \mid [z, M] = \{0\}\}\) is Peirce homogeneous. Therefore it suffices to consider \(x \in M_1, [x, M] = \{0\}\) and let \(w\) be an arbitrary element of \(V_{1/2}\) with \(Q(w) = 1\). As we have seen earlier, to prove that \([x, w, M] = \{0\}\) it suffices to show that \([x, w, M_2] = \{0\}\), then use (29) to get \([x, w, M_1] = \{0\}\). But for an arbitrary element \(y \in M_2\), by (27) and (29),

\[ [x, w, y] = 4[x, (w.y).w].w = [x, y.w]U_w + [x, y.w].f = 0, \]

since \([x, M] = \{0\}\). Therefore \(\{z \in M \mid [z, M] = \{0\}\}\) is a proper superideal of \(J\), a contradiction, and the lemma is proved. \(\square\)

While the above proof does not work in general (in particular if \(K\) has only 3 elements), extending the base field shows that the results hold over an arbitrary field of characteristic not 2.
We want to show that, for \( J \) as in Lemma 17, the subsuperalgebra \( J_1 \) determines \( J \). Now let \([ , ] : M_1 \times M_1 \to K e_1 + K f\) be an arbitrary skew-symmetric bilinear bracket on \( M_1 \), that is,
\[
[x, y] = \lambda(x, y)e_1 + \mu(x, y)f, \quad x, y \in M_1,
\]
where \( \lambda(, ) , \mu(, ) : M_1 \times M_1 \to K \) are skew-symmetric bilinear forms. Using the relationships we have obtained above we will extend \([ , ]\) to a bracket on \( M \) without worrying, for the time being, whether this extension yields a Jordan superalgebra structure on \( A + M \).

Fix \( v, e_1, e_2, M_1 \) and \( M_2 \) as above; by (23), \( M_2 = M_1, v \) and, in accordance with (29), we extend \([ , ]\) to \( M_2 \times M_2 \) by defining
\[
[x.v, y.v] := \frac{1}{4} [x, y]U_v + \frac{1}{4} [x, y].f \quad \forall x, y \in M_1. \tag{33}
\]
It remains only to define \([x, m]\) for arbitrary \( x \in M_1, m \in M_2 \). Since \([M_1, M_2]\) should lie in \( V_{1/2} \), to define \([x, z]\) it suffices to define the functional
\[
V_{1/2} \to Ke, \quad \text{given by } V_{1/2} \ni w \mapsto [x, m].w.
\]
If \( Q(w) = 1 \) then, by (27), \( m = 4(m.w).w \). Letting \( m' = m.w \in M_1 \), by (30), \([x, m].w = 4[x, (m.w).w].w = 4[x, m'.w].w \). Since, by (27),
\[
[x.w^2, m'] = \frac{1}{2} [x, m'],
\]
\[
[(x.w), m'] = \frac{1}{4} [x, m'],
\]
\[
[x, (m'.w).w] = \frac{1}{4} [x, m'],
\]
Eq. (31) becomes
\[
2[x, m'.w].w = ([x, m'].w).w,
\]
or, by (27),
\[
[x, m].w = 2(\lambda(x, m.w)e_1 + \mu(x, m.w)f).w
\]
\[
= \lambda(x, m.w)e.
\]
So to be compatible with (31) we define
\[
[x, m].w := \lambda(x, m.w)e, \quad x \in M_1, \ m \in M_2, \ w \in V_{1/2}. \tag{34}
\]
The following lemma will be used to conclude that a superalgebra must be one of the known ones.
Lemma 18 (Uniqueness Lemma). Let \( J = A + M \) be a unital Jordan superalgebra with \( A = J(V, Q) \oplus Kf, \) \( J(V, Q) \) the Jordan algebra of a nondegenerate quadratic form \( Q \) on a \( K \)-vectorspace of dimension \( n \geq 2 \), \( e = e_1 + e_2 \) the unit element of \( J(V, Q) \), \( M_1 := M.e_1 \) and \( v \) a fixed element of \( V_{1/2} \) with \( Q(v) = 1 \). Let \( [\cdot,\cdot]_1 \) and \( [\cdot,\cdot]_2 \) be two skew-symmetric bilinear brackets on \( M_1 \) such that for any \( x, y \in M_1 \) we have

\[
[x, y]_2 - [x, y]_1 = (x, y)f,
\]

where \( (\cdot,\cdot) \) is a (necessarily skew-symmetric) bilinear form on \( M_1 \), and the extensions of both \( [\cdot,\cdot]_1 \) and \( [\cdot,\cdot]_2 \) defined by (33) and (34) are Jordan. Then \( [\cdot,\cdot]_1 = [\cdot,\cdot]_2 \).

Proof. Denote by \( [\cdot,\cdot]_i \) the extension of \( [\cdot,\cdot]_i \), \( i = 1, 2 \). By (34) and (35), \( [\cdot,\cdot]_1 \) and \( [\cdot,\cdot]_2 \) coincide on \( M_1 \times M_2 \). Choose arbitrary elements \( x_1, y_1 \in M_1, z_2 \in M_2, w \in V_{1/2} \) and consider the Jordan identities for \( [\cdot,\cdot]_1 \) and \( [\cdot,\cdot]_2 \).

\[
[[z_2, x_1]_2, y_1]_2 - [z_2, [x_1, y_1]_2]_2 = [[z_2, x_1]_2, (y_1, w)]_2 - [z_2, [x_1, (y_1, w)]_2]_2 \tag{36}
\]

\[
[[z_2, x_1]_1, y_1]_2 - [z_2, [x_1, y_1]_1]_2 = [[z_2, x_1]_1, (y_1, w)]_2 - [z_2, [x_1, (y_1, w)]_1]_2 \tag{37}
\]

Subtracting (37) from (36) and taking into account the fact that \( [\cdot,\cdot]_1 \) and \( [\cdot,\cdot]_2 \) coincide on \( M_1 \times M_2 \) we get

\[
(z_2, w, y_1)x_1 = (x_1, y_1)(z_2, w). \tag{38}
\]

If we assume that \( Q(w) = 1 \) and \( z_2 = y_1, w \), then the left-hand side of (38) becomes

\[
((y_1, w), w, y_1)x_1 = \frac{1}{4}(y_1, y_1)x_1 \quad \text{by (27)},
\]

which is 0 by the skew-symmetry of \( (\cdot,\cdot) \). This implies that \( (x_1, y_1) = 0 \) for all \( x_1, y_1 \in M_1 \) and completes the proof of the lemma. \( \square \)

Before we try to use the Uniqueness Lemma, we need to obtain finer information on simple Jordan superalgebras with even part \( A = J(V, Q) \oplus Kf, \) \( Q, \) a nondegenerate quadratic form on a space \( V \) of dimension \( n \geq 2 \) over \( K \). Let us summarize this information for the ones that appear in the Classification Theorem. In all cases, having chosen a primitive idempotent \( e_1 \in J(V, Q), \) we let \( J_1 = (K e_1 + Kf) + M_1, \)

\[
\text{osp}(2, 2) = \mathcal{H}_2(K) \oplus Kf + M, \quad Q \text{ is the determinant on the trace 0 elements of } \mathcal{H}_2(K),
\]

\[
n = 2, \quad M = M' \oplus M'',
\]

\[
M' = K(e_{13} - e_{41}) + K(e_{23} - e_{42}), \quad M'' = K(e_{14} + e_{31}) + K(e_{24} + e_{32}),
\]

corresponding to the left and right regular representations of \( \mathcal{H}_2(K) \subset \mathcal{M}_2(K) \). For \( e_1 = e_{11}, M_1 = K(e_{13} - e_{41}) + K(e_{14} + Ke_{31}) \) and \( J_1 \cong D_{-1/2} \).
such that $(,) \in M_2(\mathbb{K})$ and $M = M' \oplus M''$, $M' = Ke_{31} + Ke_{32}$, $M'' = Ke_{13} + Ke_{23}$, corresponding to the right and left regular representations of $M_2(\mathbb{K})$. For $e_1 = e_{11}$, $M_1 = Ke_{13} + Ke_{31}$ and $J_1 \cong D_{-1}$.

$K_{10} = J(V, Q) \oplus Kf + M$. $Q$ is a nondegenerate quadratic form on a space $V$ of dimension $n = 4$, $M$ is the irreducible bimodule of $C(V, Q)$, the special universal envelope of $J(V, Q)$, $(K$ is assumed algebraically closed of characteristic not 3 so $C(V, Q) \cong M_4(K)$). Consider the primitive idempotent $e_1 = \frac{1}{2}(2e + v_1 + v_2)$. Then $J_1 = Ke_1 + Kf + K(x_1 - y_2) + K(x_2 + y_1)$ and $J_1 \cong D_{-3/2}$.

$osp(1, 4) = Kf \oplus H_2(Q) + M$, where $\mathbb{Q}$ are the split quaternions and $Q$ is the generic norm on the trace 0 elements of $H_2(Q)$, $n = 5$.

$M = Ke_{12} - e_{31} + Ke_{13} + e_{21} + Ke_{14} - e_{51} + Ke_{15} + e_{41}$

corresponding to the right regular representation of $H_2(Q) \subset M_4(K)$. For $e_1 = e_{22} + e_{33}$, $M_1 = Ke_{12} - e_{31} + Ke_{13} + e_{21}$ and $J_1 \cong D_{-1/2}$.

It follows from Lemma 17 and Proposition 8 that $J_1 = (Ke_1 + Kf) + M_1$ is either of type $D_r$ or a Jordan superalgebra of a superform. Our aim is to prove that the second case is impossible. For this purpose, we will use the elementary representation theory of $sl_2$. Let $dim V = n$ and assume that $J_1$ is a superalgebra of a superform, that is,

$$[x, y] = (x, y)(e_1 + f), \quad \text{for } x, y \in M_1$$

and $(\cdot, \cdot)$ a nondegenerate skew-symmetric bilinear form on $M_1$. If $x, y$ are chosen in $M_1$ such that $(x, y) = 1$, then $[x, y] = e_1 + f$. Consider the derivations

$$X = (R_x)^2, \quad Y = (R_y)^2, \quad H = R_xR_y + R_yR_x.$$  \hspace{1cm} (39)

One checks that

$$xY = [x, y]y = y, \quad xH = [x, y]x = x, \quad yH = [y, x]y = -y.$$  \hspace{1cm} (40)

Hence

$$[X, Y] = R_xYR_x + R_yR_y = H, \quad [X, H] = R_xHR_y + R_yH = 2X,$$

$$[Y, H] = -2Y,$$  \hspace{1cm} (41)

and we obtain a standard basis for $sl_2(K)$.

**Lemma 19.** Let $J = A + M$ be a simple unit Jordan superalgebra with $A = J(V, Q) \oplus Kf$, $J(V, Q)$ the Jordan algebra of a nondegenerate quadratic form $Q$ on a $K$-vector-space of dimension $n \geq 2$, $e = e_1 + e_2$ the unit element of $J(V, Q)$. Assume that the subsuperalgebra $J_1 := (Ke_1 + Kf) + M_1$, where $M_1 := Me_1$, is a Jordan superalgebra of a superform, that is, $[M_1, M_1] \subseteq Ke_1 + f$. Let $x, y \in M_1$ such that $[x, y] = e_1 + f$
and denote by $X$ and $Y$ the derivations $(R_x)^2$ and $(R_y)^2$. If there exists a $v \in V_{1/2}$ such that $v^2 = e$ and $vX = vY = 0$ then $\text{char} \ K = 3$ and $n \leq 7$.

**Proof.** Let $w = [x, y.v] - v$ and consider the subspace

$$S = S_0 + S_1 = (Ke_1 + Ke_2 + K v + K w + K f) + (K x + K y + K x.v + K y.v).$$

Before we check that it is a subsuperalgebra, observe that

$$[x, y.v] = -[y, x.v].$$

Indeed, since $H = [X, Y]$ and $vX = vY = 0$,

$$vH = 0.$$  

But $H = R_x R_y + R_y R_x$, so $0 = -vH = [x, y.v] + [y, x.v]$. Furthermore, since $X$ is a derivation such that $yX = x$ and $vX = 0$, $[x, y.v]x = ([x, y].x).v = x.v$, so $w.x = 0$. Similarly $w.y = 0$. By (2) and (27),

$$2[x, y.v].v = -xR_x R_y + xR_y R_x + [x, y]R_y R_v + [x, (y.v).v]$$

$$= -\frac{1}{2}[x, y] + \frac{1}{4}[x, y] + (e_1 + f)R_y R_v + \frac{1}{4}[x, y]$$

$$= \frac{1}{2}vR_v = \frac{1}{2}e.$$

Therefore

$$[x, y.v].v = \frac{1}{4}e.$$  

(44)

By (42),

$$[x, y.v]R_{x,v} = -yR_{x,v} R_{x,v}.$$  

By (2),

$$y(R_{x,v} R_{x,v} - R_x R_{x,v}, v + R_v R_{v,v,x}) = y(R_{x,v} R_v R_x - R_v R_{x,v} R_x + R_{v,v,v,x} v).$$

So by (27), (42) and $vX = 0$,

$$yR_{x,v} R_{x,v} + [x, y]R_{x,v} = -([x, y.v].x + ([x, y].v).x).v.$$

By (44) and (27),

$$yR_{x,v} R_{x,v} + \frac{1}{4}x = -\frac{1}{8}x + \frac{1}{8}x.$$
and by (42) and (27),

\[ [x, y.v] R_{x,v} = \frac{1}{4} x = v.(x.v). \]

So again \( w.(x.v) = 0 \) and similarly \( w.(y.v) = 0 \). From (27), (29) and these formulas it follows that \( S \) is a subsuperalgebra. Since \( w \in V_{1/2} \) then \( w^2 \in K e \). Since \( u \mapsto 2R_u \) is a Jordan homomorphism of \( J(V, Q) \to \text{End}(M)^+ \) then \( w.x = 0 \) implies

\[ w^2 = [x, y.v]^2 - 2v.[x, y.v] + e = 0. \]

So by (44),

\[ [x, y.v]^2 = -\frac{1}{2} e. \]

By (44), \( w.v = [x, y.v]v - v - \frac{1}{2} e \). If \( \text{char} \ K \neq 3 \) then this is nonzero and the product on \( K v + K w \) is determined by a nondegenerate quadratic form. So \( S_0 \cong M_2(K)^+ \oplus Kf \). Since \( S_1 \) is a nonunital \( M_2(K)^+ \)-bimodule of dimension 4, this determines the \( S_0 \)-bimodule structure of \( S_1 \): \( S_1 \) is the sum of two irreducible \( S_0 \)-bimodules. By the Uniqueness Lemma,

\[ S \cong M_{2,1}(K) = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}. \]

But \( \{e_1 S_1.f\} = Ke_{13} + Ke_{31}, \ [e_{13}, e_{31}] = e_{11} - e_{33} \), so \( (Ke_{1} + Kf) + \{e_1 S_1.f\} \cong D_{-1} \), contradicting our assumption on \( J_1 \). We interrupt the proof for the following remark.

**Remark.** If \( \text{char} \ K = 3 \) then \( K([x, y.v] - v) \) is a 1-dimensional superideal of \( S \), \( S/K([x, y.v] - v) \cong osp_{2,2}(K) \) and the corresponding \( (e_1 + f) \)-Peirce component is of type \( D_{-1/2} \), but \(-1/2 = 1\), so this does not give us any contradiction with \([M_1, M_1] \subseteq Ke_1 + Kf\).

Assume now that the characteristic is 3 and consider the orthogonal complement \( Z := [x, y]^\perp \) in \( M_1 \); \( M_1 = Kx + Ky + Z \). Since \( M_2 = M_1.v \) it follows that \( M_2 = Kx.v + Ky.v + Z.v \). If we suppose further that there exists a nonzero element \( z \in Z \) such that \([x, z] = 0\), then, by (2),

\[ x(R_{y,z}R_z - R_{z,v}R_y + R_{y,z}R_v) = x(R_{y,z}R_{z,v} - R_{z,v}R_y + R_yR_{y,z}). \]

Since \([y, z] = [x, z] = 0\), we have

\[ [x, y.v]z - [x, z.v]y = [x, y].(z.v). \]
But we assumed that \([x, z.v] = 0\), so

\[ [x, y.v].z = [x, y].(z.v) = (e_1 + f).z.v = f.(z.v) = 1/2 z.v. \]

Thus

\[ \left( [x, y.v] - \frac{1}{2} v \right) z = 0. \]

By the same argument used to show that \(([x, y.v] - v)^2 = 0\), we have

\[ \left( [x, y.v] - \frac{1}{2} v \right)^2 = 0, \]

which is impossible by (44) and (45). Hence the operator

\[ R_v R_x \]

is nonsingular on \(Z\).

Denote \(\text{dim}\,Z\) by \(s\) and let \(V_{1/2}(\alpha)\) be the root space of \(V_{1/2}\) with respect to \(H\) corresponding to the root \(\alpha\). By (40), (43) and \(Z = \{x, y\}\),

\[ ZR_v R_x \subseteq V_{1/2}(1) \quad \text{and} \quad ZR_v R_y \subseteq V_{1/2}(-1), \]

\(\dim V_{1/2}(\pm1) \geq s\) and \(\dim V_{1/2} \geq 2s + 1\) (the last dimension coming from \(v\)). By (28), \(\dim M = 4 + 2s\) and \(n - 1 = \dim V_{1/2}\), so \(n \geq 2s + 2\) and \(n + 2 \geq \dim M\). Since \(M\) is a half-unital bimodule it is the direct sum of regular modules of \(C(V, Q)\). If \(n = 2k\), then \(\dim M \geq 2^k\), \(2k + 2 \geq 2^k\), which implies \(k \leq 3\). If \(n = 2k + 1\) then \(\dim M \geq 2^k\), \(2k + 3 \geq 2^k\) and again \(k \leq 3\). This completes the proof of the lemma. \(\square\)

**Lemma 20.** Let \(J, A, M, M_1, x, y, v\) be as in Lemma 19 and \(\text{char}\,K = 3\). Then \(n \neq 6\).

**Proof.** Assume that \(n = 6\) and recall that \(x, y \in M_1; v \in V_{1/2}\) with \(v^2 = e, vX = vY = vH = 0\). Let us consider the orthogonal complement \(\{x, y\}^\perp\) in \(M_1\) and choose elements \(z, t \in \{x, y\}^\perp\) such that \([z, t] = e_1 + f\). In the proof of the previous lemma we have shown that the root space decomposition of \(V_{1/2}\) with respect to \(H\) is given by

\[ V_{1/2} = V_{1/2}(-1) + V_{1/2}(1) + V_{1/2}(0), \]

where

\[ V_{1/2}(-1) = K[z, v, y] + K[t, v, y], \]

\[ V_{1/2}(1) = K[z, v, x] + K[t, v, x], \]

\[ V_{1/2}(0) = Kv. \]
Since $z, t \in \{x, y\}^\perp$, we have $zH = iH = 0$ and it follows that $Kv$ is the 1-dimensional module over $sl_2 = K(R_z)^2 + K(R_y)^2 + K(R_zR_y + R_yR_z)$. Hence

$$v(R_z)^2 = v(R_y)^2 = 0.$$  

Now we have the following basis of $J$:

- basis of $M_1$: $x, y, z, t$;
- basis of $M_2$: $x.v, y.v, z.v, t.v$;
- basis of $A$: $e_1, e_2, v, [x, z.v], [x, t.v], [y, z.v], [y, t.v], f$.

We wish to determine enough products in this basis to obtain a contradiction. The element $[x, y.v]$ has weight 0 with respect to $H$ since the respective weights of $x$, $y$ and $v$ are 1, $-1$ and 0. Hence

$$[x, y.v] = \alpha v. \quad (47)$$

Since $\text{char } K = 3$, by (44), $[x, y.v].v = e$. Multiplying both sides of (47) by $v$ and using $v^2 = e$, we have $e = \alpha e$ which implies $\alpha = 1$, so

$$[x, y.v] = v. \quad (48)$$

Similarly

$$[z, t.v] = v. \quad (49)$$

By (29), using $\text{char } K = 3$,

$$[x.v, y.v] = e_2 + f,$$

$$[z.v, t.v] = e_2 + f. \quad (50)$$

Again by weight considerations (this time with respect to $R_zR_y + R_yR_z$),

$$x.[y, z.v] = \beta z.v.$$  

Applying $R_x$ to both sides, using $vX = 0$ and $z \in \{x, y\}^\perp$, we get

$$[y, x].[x, z.v] = \beta [z.v, x],$$

$$-[x, z.v] = \beta [z.v, x].$$

So $\beta = 1$ and

$$x.[y, z.v] = z.v.$$
This implies
\[ y[x, z.v] = -x[y, z.v] = -z.v. \]

Similarly
\[ z[x, t.v] = x.v, \]
\[ t[x, z.v] = -x.v. \]  (51)

To show
\[ [z, x.v] = [x, z.v], \]
\[ [z, y.v] = [y, z.v], \]
\[ [t, x.v] = [x, t.v], \]
\[ [t, y.v] = [y, t.v], \]  (52)

by symmetry, it suffices to show the first equation. Since the characteristic is 3, by (27),
\[ (z.v).v = \frac{1}{4}z = z, \]  so
\[ [z, x.v] = [(z.v).v, x.v]. \]

By (2),
\[ 2[(z.v).v, x.v] + [z.v, x]v^2 = 2[z.v, x.v].v + [(z.v).v^2, x]. \]

By (29),
\[ [z.v, x.v] = 0 \quad (\text{since } [z, x] = 0), \]
\[ [z.v, x].v^2 = [z.v, x].e = [z.v, x] \quad (\text{since } [z.v, x] \in V_{1/2}), \]
\[ [(z.v).v^2, x] = [(z.v).e, x] = \frac{1}{2}[z.v, x] \quad (\text{since } z.v \in M_2). \]

Hence
\[ 2[(z.v).v, x.v] + [z.v, x] = \frac{1}{2}[z.v, x], \]
\[ 2[z.v, x.v] = -\frac{1}{2}[z.v, x], \]
\[ [z, x.v] = -[z.v, x] = [x, z.v]. \]

By (2),
\[ x(2R_e, v R_e + R_e, v R_e) = x(R_e, v R_e + R_e, v R_e + R_{(t,v), e}). \]
\[ 2[x, t.v], v + [x, t] = [x, t].v, v + [x, t] + [x, t], \quad \text{by (27)} \]
and since \([x, t] = 0\)

\([x, t.v].v = 0\).

Let us compute \((z.v)[x, t.v]\) in two different ways. Since \(a \mapsto 2R_a \mid_M\) is a Jordan homomorphism of \(A\) into \(\text{End} M^+\) and, since \(v.[x, t.v] = 0\), we have

\[
(z.v)[x, t.v] = (z.[x, t.v]).v
\]

by (52)

\[
= (x.v).v
\]

by (51)

\[
= x
\]

by (27)

On the other hand,

\[
(z.v).[x, t.v] = -(t.v).[z.v].
\]

By (2),

\[
(t.v)(R_x R_{z,v} - R_z R_{x,v} + R_v R_{[t, z,]}(x)) = (t.v)(R_{[x, z]}R_v + R_{x,v} R_z - R_{z,v} R_x).
\]

Since \([x, z] = 0\), this implies

\[
(t.v)R_x R_{z,v} - [t.v, z].(x.v) = [t.v, x.v].z - [t.v, z.v].x.
\]

By (49), \([t.v, z] = -v\), and, by (29), \([t, x] = 0\) implies \([t.v, x.v] = 0\). Hence, by (27) and (50),

\[
(t.v)R_x R_{z,v} + x = \frac{1}{2}x,
\]

\[
(t.v)R_x R_{z,v} = -\frac{1}{2}x = x,
\]

\[
(z.v).[x, t.v] = -x,
\]

a contradiction. \(\Box\)

**Lemma 21.** Let \(J = A + M\) be a simple unital Jordan superalgebra with \(A = J(V, Q) \oplus Kf, J(V, Q)\), the Jordan algebra of a nondegenerate quadratic form \(Q\) on a \(K\)-vector-space of dimension \(n \geq 2\), \(e = e_1 + e_2\), the unit element of \(J(V, Q)\). Assume that the subsuperalgebra \(J_1 := (Ke_1 + Kf) + M_1\), where \(M_1 := M.e_1\), is a Jordan superalgebra of a superform and that \(M_1\) contains a nonzero element \(0 \neq x \in M_1\) such that \(V_{1/2}(R_x)^2 = \{0\}\) then the characteristic is 3, and \(J \cong \mathfrak{osp}_{2,2}(K)\).
Proof. As we did earlier, choose an element \( y \in M_1 \) such that \([x, y] = e_1 + f\) and let

\[
X = R^2_x, \quad Y = R^2_y, \quad H = R_x R_y + R_y R_x.
\]

Observe that \( e_2 X = [e_2, x, x] = 0 \) since \( e_2 x = 0 \), \( e_1 X = [e_1, x, x] = \frac{1}{2}[x, x] = 0 \) and similarly \( f X = 0 \). Therefore \( AX = [X, Y] = e_1 X = \frac{1}{2}[x, x] = 0 \). Let \( Y = -\frac{1}{2}[Y, H] \), it follows that \( V_{1/2} H = [0] = V_{1/2} Y \). Then we are in the situation of Lemmas 18, 19, and 20, which imply that \( \text{char.} K = 3 \) and, by (46), for any nonzero element \( z \in [x, y]^1 \) and any invertible element \( v \in V_{1/2} \), the element \( [x, z, v] \) is nonzero. But \( [x, z, v] \) has weight 1 with respect to \( H \) whereas \( V_{1/2} H = [0] \). Hence \( [x, y]^1 = [0] \), so \( \dim M_1 = 2 \) and \( \dim M = 4 \). This means that \( n \leq 5 \) and for each \( n \) the bimodule structure of \( M \) over \( A \) is uniquely determined.

By the Uniqueness Lemma, either \( J \cong osp_{1,4}(K) \), for \( n \neq 5 \), or \( J \) is the Kac superalgebra, for \( n = 4 \), or \( J \cong M_{2,1}(K) \), for \( n = 3 \), or \( J \cong osp_{2,2}(K) \), for \( n = 2 \).

For \( M_{2,1}(K) \) and for the Kac superalgebra, the subsuperalgebra \( J_1 \) is not a superalgebra of a superform. While for \( osp_{1,4}(K) \), one can easily check that there does not exist an element \( x \in M_1 \) satisfying the hypotheses. \( \square \)

Lemma 22. Let \( J \) be an arbitrary Jordan superalgebra such that \( A = J(V, Q) \oplus K f \). If some \( x \in M, v \in V \) satisfy \( v^2 = 0, [x, v, x] = 0 \) then \( V(R_{x,v})^2 = [0] \).

Proof. For an arbitrary element \( w \in V \), by (2),

\[
w(R_{x,v} R_{x,v} + R_{(x,v),v} R_{x,v} + R_{[x,v,x]} R_v) = w(R_{x,v} R_{x,v} - R_x R_y R_{x,v} + R_{[x,v,x]} v).
\]

We will consider each term of (53). The by now familiar map \( a \mapsto 2R_a \in \text{End}(M)^+ \) yields

\[
(x,v) v = \frac{1}{2} x v^2 = 0.
\]

Now

\[
w R_{x,v} R_{x,v} = [(x,v) w, v, x]
\]

and, again by (2),

\[
2R_{x,v} R_{v} + R_{w,v} v^2 = 2R_{v,w} R_{v} + R_{w,v} R_{w}.
\]

Hence

\[
R_{x,v} R_{v} = R_{v,w} R_{v}
\]

and, since \([x,v,x] \) is assumed to be 0,

\[
[(x,v) w, v, x] = \frac{1}{2} Q(v,w)[x,v,x] = 0.
\]
Furthermore,

\[ 2R_v R_{x,v} + R_{x,v}^2 = R_{x,v} R_{x,v} + R_{x,v} R_{x,v} + R_{(x,v),v}. \]

Since \( v^2 = 0 \) and \( (x,v) v = 0 \),

\[ 2R_v R_{x,v} = R_{x,v} R_{x,v} + R_{x,v} R_{x,v}, \]

so

\[ wR_x R_{x,v} = \frac{1}{2} wR_x R_{x,v} R_{x,v} + \frac{1}{2} wR_x R_{x,v} R_{x,v}. \]

But

\[ wR_x R_{x,v} = \frac{1}{2} wR_x R_{x,v} = 0 \]

and since \( (R_x)^2 \) is a derivation,

\[
w(R_x)^2.v = (w.v)(R_x)^2 - w.(v(R_x)^2) \\
= Q(w, v)eR_x^2 - w.[x.v, x] = 0.
\]

Therefore all terms of (53), other than \( wR_x,v R_{x,v} \), vanish, proving the lemma. \( \square \)

Combining Lemmas 21 and 22, yields the following corollary.

**Corollary 23.** If \( J = A + M \) is a simple Jordan superalgebra with \( A = J(V, Q) \oplus Kf \) such that \( J_1 \) is a superalgebra of a superform then either \( J \cong osp_{2,2}(K) \) and the characteristic is 3 or, for \( v \in V \) and \( x \in M \), \( v^2 = [x,v, x] = 0 \) implies \( x.v = 0 \).

**Lemma 24.** Let \( J = A + M \) be a simple Jordan superalgebra with \( A = J(V, Q) \oplus Kf \), \( J(V, Q) \), the Jordan algebra of a nondegenerate quadratic form \( Q \) on a \( K \)-vectorspace of dimension \( n \geq 2 \), \( e = e_1 + e_2 \), the unit element of \( J(V, Q) \). If \( J_1 = (Kf + Kf) + M_1 \) is a superalgebra of a superform and if there exist elements \( x, y \in M_1 \), \( v \in V_{1/2} \) such that

\[
[x, y] = e_1 + f, \quad v^2 = e, \quad vR_x^2 = vR_y^2 = 0,
\]

then the characteristic is 3, and \( J \cong osp_{2,2}(K) \).

**Proof.** By Lemma 19, \( \text{char}.K = 3 \) and \( n \leq 7 \) and, by Lemma 20, \( n \neq 6 \). Let us first consider the case \( n = 7 \). Then \( \dim M_1 = \dim M_2 = 4 \). As above let \( \{x, y\} = Kz + Kr \); \( \{x, y, z, t\} \) is a basis of \( M_1 \) and \( \{x,v, y,v, z,v, t,v\} \) is a basis of \( M_2 \). Let \( V_{1/2} = \sum V_{1/2}(\alpha) \) be the root space decomposition of \( V_{1/2} \) with respect to \( H \). Since \( V_{1/2} = [M_1, M_2] \), it follows from (40) and (43) that \( V_{1/2}(0) \) is the linear span of \( [x, y,v], [y, x,v], [z, z,v], \)
\[ [z, t.v], [t, z.v] \text{ and } [t, t.v]. \] Since the derivation \( X = (R_x)^2 \) annihilates \( v, x, z \) and \( y \), then each of the elements in the spanning set is annihilated by \( X \). Therefore

\[ V_{1/2}(0)(R_x)^2 = \{0\}. \]

Since the root subspaces \( V_{1/2}(\alpha), V_{1/2}(-\alpha), \alpha \neq 0, \) are dual with respect to \( Q(, ) \) and \( \dim V_{1/2} = n - 1 \) is even, it follows that \( \dim V_{1/2}(0) \) is even. Hence \( V_{1/2}(0) \) has a basis which consists of nilpotent elements. By Corollary 23, either \( J \cong \text{osp}_{2,2}(K) \) or \( V_{1/2}(0).x = \{0\} \). But \( V_{1/2}(0) \supseteq v \) an invertible element, so \( v.x \neq 0 \). Hence \( J \cong \text{osp}_{2,2}(K) \), which excludes the case \( n = 7 \).

For \( n \leq 5 \), it follows from the Uniqueness Lemma that \( J \) is isomorphic to one of \( \text{osp}_{2,2}(K), \text{M}_{2,1}(K) \), the Kac superalgebra or \( \text{osp}_{1,4}(K) \). Again the \( J_1 \) of \( \text{M}_{2,1}(K) \) or of the Kac superalgebra is not a superalgebra of a superform and we claim that \( \text{osp}(1,4) \) does not contain elements \( x, y, v \) satisfying the hypotheses of the lemma.

Let us show that \( \text{osp}_{1,4}(K) \) does not contain suitable \( x, y, v \). Assume the contrary. Then \( M_1 = K.x + K.y, M_2 = K.x.v + K.y.v \) and \( V_{1/2} = [M_1, M_2] \) is spanned by \( [x, x.v], [x, y.v], [y, x.v] \) and \([y, y.v] \) which must be linearly independent since \( \dim V_{1/2} = n - 1 = 4 \). Thus \([x, x.v] \) cannot be equal to zero. \( \square \)

**Lemma 25.** Let \( J = A + M \) be a simple Jordan superalgebra with \( A = J(V, Q) \oplus Kf, J(V, Q), \) the Jordan algebra of a nondegenerate quadratic form \( Q \) on a \( K \)-vector space of dimension \( n \geq 2, e = e_1 + e_2, \) the unit element of \( J(V, Q) \). If \( J_1 \) is the superalgebra of a superform and \( J \not\cong \text{osp}_{2,2}(K) \) then \( M_1 \) contains an element, \( m \) such that \( M_2 = V_{1/2}.m \).

**Proof.** By the Peirce relations, \( V_{1/2}.m \subseteq M_2 \) for \( m \in M_1 \), and we need only find an \( m \in M_1 \) such that \( M_2 \subseteq V_{1/2}.m \). Suppose that there exist elements \( w \in V_{1/2} \) and \( x \in M_1 \) such that \( (wX)^2 \neq 0 \), where \( X = (R_x)^2 \). Then, by (28) and the fact that \( X \) is a derivation,

\[ M_2 = M_1.(wX) \subseteq (M_1.w)X + M_1.X.w. \]

By the Peirce relations,

\[ (M_1.w)X \subseteq M_2X = [M_2, x].x \subseteq V_{1/2}.x, \]

and

\[ M_1.X.w = ([M_1, x].x).w \subseteq ([Ke_1 + Kf].x).w \subseteq K.w.x \subseteq V_{1/2}.x. \]

We may therefore assume that for arbitrary elements \( w \in V_{1/2}, x \in M_1 \) we have

\[ (wX)^2 = 0, \]

which implies

\[ Q(V_{1/2}X, V_{1/2}X) = \{0\} \]
and

\[ Q(V_{1/2}X^2, V_{1/2}) = Q(V_{1/2}X, V_{1/2}X) = \{0\}. \]

From the nondegeneracy of \( Q(, ) \) on \( V_{1/2} \) it follows that

\[ V_{1/2}X^2 = \{0\}. \]  \hspace{1cm} (55)

Choose elements \( x, y \in M_1 \) such that \([x, y] = e_1 + f \) and denote \( Y = (R_s)^2 \), \( H = [X, Y] \) and let

\[ V_{1/2} = \sum \alpha V_{1/2}(\alpha) \]

be the root space decomposition of \( V_{1/2} \) with respect to \( H \). For an arbitrary element \( w \in V_{1/2} \) we have by (55),

\[ 0 = wYX^2 = -wHX - wXH. \]  \hspace{1cm} (56)

On the other hand \( XH - HX = 2X \), so

\[ wXH - wHX = 2wX, \]

which, together with (56), implies

\[ wXH = wX. \]  \hspace{1cm} (57)

Let \( w \in V_{1/2}(\lambda) \). Then \( wX \in V_{1/2}(\lambda + 2) \). If \( wX \neq 0 \) then \( \lambda + 2 = 1 \) and \( \lambda = -1 \). Similarly, if \( w \in V_{1/2}(\lambda) \) and \( wY \neq 0 \) then \( \lambda = 1 \). Hence \( \sum_{\lambda \neq \pm 1} V_{1/2}(\lambda) \) is annihilated by \( X \) and \( Y \) and hence by \( H = [X, Y] \). If \( V_{1/2}(0) \neq \{0\} \) then \( V_{1/2}(0) \) contains an invertible element. Since \( J \not\cong \text{osp}_{2,2}(K) \), this is ruled out by Lemma 24, and

\[ V_{1/2} = V_{1/2}(-1) + V_{1/2}(1). \]

Arguing as above, using (57),

\[ V_{1/2}(1)X \subseteq V_{1/2}(3). \]

But \( 3 \neq \pm 1 \), since \( \text{char } K \neq 2 \). Hence

\[ V_{1/2}(1)X = \{0\}. \]

Since \( Q(V_{1/2}(1), V_{1/2}(1)) = \{0\} \), Corollary 23 yields

\[ V_{1/2}(1)x = \{0\} \]
and similarly

\[ V_{1/2}(-1).y = \{0\}. \]

We show next that \( M_2 = V_{1/2}.x + V_{1/2}.y \). Since \( V_{1/2}(\pm 1) \) are dual with respect to \( Q \) there exists an element \( w \in V_{1/2} \) such that \( wH \) is invertible in \( Ke + V \). Arguing as above,

\[ M_2 = M_1.(wH) \subseteq (M_1.w)H + (M_1H).w. \]

By (39),

\[ (M_1.w)H \subseteq M_2H \subseteq [M_2, x].y + [M_2, y].x \subseteq V_{1/2}.y + V_{1/2}.x, \]

\[ (M_1H).w \subseteq ([M_1, x].y).w + ([M_1, y].x).w \subseteq Kw.y + Kw.x. \]

But

\[ V_{1/2}.x + V_{1/2}.y = V_{1/2}(-1).x + V_{1/2}(1).y = V_{1/2}.(x + y). \]

Taking \( m = x + y \) we have the desired element of \( M_1 \). □

**RemarK.** Actually we proved that for arbitrary elements \( x, y \in M_1 \) such that \([x, y] = e_1 + f \) there exists an element \( u \in Kx + Ky \) such that \( M_2 = V_{1/2}.u \).

**Corollary 26.** Let \( J = A + M \) be a simple Jordan superalgebra with \( A = J(V, Q) \oplus Kf \), \( J(V, Q) \), the Jordan algebra of a nondegenerate quadratic form \( Q \) on a \( K \)-vectorspace \( V \) of dimension \( n \geq 2 \), \( e = e_1 + e_2 \), the unit element of \( J(V, Q) \). If \( J_1 \) is the superalgebra of a superform and \( J \not\cong \mathfrak{osp}_{2,2}(K) \), then \( n \leq 9 \) or \( n = 9 \).

**Proof.** If \( n = 2k \) then \( C(V, Q) \) has dimension \( 2^{2k} \) and the half unital bimodule \( M \) has dimension \( \geq 2^{k} \). By Lemma 25, its dimension is \( \leq 2 \dim V_{1/2} \). Hence

\[ 2^k \leq \dim M \leq 2(2k - 1) \]  \( (58) \)

and \( n \leq 6 \).

If \( n = 2k + 1 \) then arguing as above

\[ 2^k \leq \dim M \leq 4k. \]  \( (58') \)

So \( k \leq 4 \) and \( n \leq 9 \). □

**Lemma 27.** Let \( J = A + M \) be a simple Jordan superalgebra with \( A = J(V, Q) \oplus Kf \), \( J(V, Q) \), the Jordan algebra of a nondegenerate quadratic form \( Q \) on a \( K \)-vectorspace of dimension \( n \geq 2 \), \( e = e_1 + e_2 \), the unit element of \( J(V, Q) \). If \( J_1 \) is a superalgebra of a superform then the characteristic is 3 and either \( J \cong \mathfrak{osp}_{2,2}(K) \) or \( J \cong \mathfrak{osp}_{1,4}(K) \) or the characteristic is 5 and \( J \cong K_{10} \).
Proof. Assume that $J \not\cong osp_{2,2}(K)$, $J \not\cong osp_{1,4}(K)$. By Corollary 26, $n \leq 7$ or $n = 9$. We will eventually consider each case separately. Inequalities (58) and (58') yield Table 1, where $[m]$ is the greatest integer in $m$. The second column is the dimension of the irreducible half-unital $J(V, Q)$-bimodule, and hence a lower bound on the dimension of $M$, while the third is an upper bound on the dimension of $M$. By Proposition 9, $M$ is irreducible or the direct sum of two irreducibles. From the above table we may conclude that $M$ is irreducible when $n = 9$, 7, 6, 4 and 2. In all cases, by Lemma 25, we may choose an element $x \in M_1$ such that

$$M_2 = V_{1/2}R_x$$

and find a $y \in M_1$ such that $[x, y] = e_1 + f$. As before let

$$X = (R_x)^2, \quad Y = (R_y)^2, \quad H = [X, Y],$$

and let

$$V_{1/2} = \sum_{\alpha} V_{1/2}(\alpha), \quad M_i = \sum_{\alpha} M_i(\alpha)$$

be the root space decompositions of $V_{1/2}$ and $M_i$ with respect to $H$. We will make repeated use of the fact that $V_{1/2}(\alpha)$ and $V_{1/2}(-\alpha)$ are dual with respect to $Q$. If $V_{1/2}(0) \neq \{0\}$ then $V_{1/2}(0)$ contains an invertible element $v$. This implies

$$M_2(\alpha) = M_1(\alpha)v, \quad \text{for any } \alpha,$$

and hence,

$$\dim .M_2(\alpha) = \dim .M_1(\alpha). \tag{60}$$

By (59) and weight considerations,

$$M_2(0) \subseteq V_{1/2}(-1)R_x. \tag{61}$$
Assume that $n = 9$. Then $M$ is an irreducible $A$-bimodule and $\dim M = 16$. If $V_{1/2}(0) \neq \{0\}$ then, by (60), $\dim M_2(0) = \dim M_1(0) = 6$, the dimension of the orthogonal complement $\{x, y\}^\perp$, and, by (61),

\[ \dim V_{1/2}(-1) \geq 6. \]

The subspaces $V_{1/2}(-1), V_{1/2}(1)$ are dual with respect to $Q$ so

\[ \dim V_{1/2}(1) \geq 6. \]

This is impossible since the dimension of the whole of $V_{1/2}$ is 8.

We may therefore assume that $V_{1/2}(0) = \{0\}$. We wish to show that $X$ is nonsingular on $V_{1/2}$. Since $M_2 = V_{1/2}R_x$, and $\dim M_2 = 8 = \dim V_{1/2}$ it follows that $R_\alpha : V_{1/2} \to M_2$ is nonsingular. Now $\ker X |_{V_{1/2}} = \sum_\alpha (\ker X |_{V_{1/2}} \cap V_{1/2}(\alpha))$. Since $v^2 = 0$ for $v \in V(\alpha)$, $\alpha \neq 0$, by Corollary 23, we have $(\ker X |_{V_{1/2}})R_x = \{0\}$. Hence $\ker X |_{V_{1/2}} = \{0\}$. So, for any root $\alpha$,

\[ \dim V_{1/2}(\alpha) \leq \dim V_{1/2}(\alpha + 2) \leq \cdots \leq \dim V_{1/2}(\alpha + 2(p - 1)) \leq \dim V_{1/2}(\alpha), \]

where $p$ is the characteristic of $K$. Hence $\dim V_{1/2}$ is a multiple of an odd prime $p$, a contradiction. So $n \neq 9$.

Assume that $n = 7$. Then $M$ is irreducible of dimension 8. So $\dim M_1 = \dim M_2 = 4$. Again, by Lemma 25, we may choose $x, y \in M_1$ such that $M_2 = V_{1/2}R_x$ and $[x, y] = e_1 + f$. Consider the orthogonal complement $\{x, y\}^\perp$ of $x, y$ in $M_1$. From the proof of the previous lemma, there exists a $z \in \{x, y\}^\perp$ such that $M_2 = V_{1/2}R_z$. Choose $t \in \{x, y\}^\perp$ such that $[z, t] = e_1 + f$. In other words, the pair $z, t$ enjoys the same properties as $x, y$. Let

\[ H = [X, Y], \quad Z = (R_z)^2, \quad T = (R_t)^2, \quad L = [Z, T] \]

and

\[ V_{1/2} = \sum_\alpha V_{1/2}(\alpha), \quad M_i = \sum_\alpha M_i(\alpha) \]

be the root decompositions of $V_{1/2}$ and $M_i$ with respect to $H$.

If $V_{1/2}(0) \neq \{0\}$ then, arguing as in the previous case, 

\[ \dim M_1(0) = \dim M_2(0) = 2, \quad \dim M_1(\pm 1) = \dim M_2(\pm 1) = 1. \]

Since $M_2 = V_{1/2}R_x$, weight considerations yield $V_{1/2}(-1)R_x = M_2(0)$, which implies that $\dim V_{1/2}(-1) \geq 2$. Since $V_{1/2}(-1)$ and $V_{1/2}(1)$ are dual with respect to $Q$ and $\dim V_{1/2} = 6$, we have

\[ \dim V_{1/2}(-1) = \dim V_{1/2}(1) = \dim V_{1/2}(0) = 2. \]
and $R_\alpha : V_{1/2}(-1) \to M_2(0)$ is nonsingular.

Now $V_{1/2}(-1), V_{1/2}(1)$ and $V_{1/2}(0)$ are all invariant under $Z, T$ and $L$. If $W$ is a 2-dimensional $sl_2(K)$-module (the characteristic is arbitrary but odd) then either $W|_{sl_2(K)} = \{0\}$ or $W = W(-1) + W(1)$, where $W(\alpha)$ is a root space with respect to $L$. Since $R_\alpha : V_{1/2}(-1) \to M_2(0)$ is nonsingular, by Corollary 23, $X : V_{1/2}(-1) \to V_{1/2}(1)$ is nonsingular. Moreover this is an isomorphism of $\{Z, T, L\}$-modules. Therefore $V_{1/2}(\pm 1) [Z, T, L] \neq \{0\}$, for otherwise the 0-root space of $L$ in $V_{1/2}$ would have dimension $\geq 4$ which is impossible (interchange the role of $L$ and $H$).

If $V_{1/2}(0)[Z, T, L]$ were $\{0\}$, since $V_{1/2}(0)$ contains an invertible element we would be in the situation of Lemma 21 for $Z, T, L$, contradicting our assumption that $J \ncong osp_{2,2}(K)$. We may therefore assume that $V_{1/2}(0)$ is also not a trivial $\{Z, T, L\}$-module. Let $V_{1/2} = \sum_\alpha V_{1/2}^{(L)}(\alpha)$ be the root decomposition of $V_{1/2}$ with respect to $L$. We have shown that

$$\dim V_{1/2}^{(L)}(1) = \dim V_{1/2}^{(L)}(-1) = 3.$$  

But $R_\alpha$ maps $V_{1/2}^{(L)}(1)$ to $M_2^{(L)}(2)$ whose dimension is at most 1 and $V_{1/2}^{(L)}(-1)$ to $M_2^{(L)}(0)$ whose dimension is 2. Thus $\dim V_{1/2}R_\alpha \leq 3$, a contradiction.

We may therefore assume that $V_{1/2}(0) = \{0\}$. If $\dim V_{1/2}(\alpha) = 1$ for some $\alpha$ then $V_{1/2}(\alpha)$ is a 1-dimensional $\{Z, T, L\}$-module; so, $V_{1/2}(\alpha)L = \{0\}$ and $V_{1/2}^{(L)}(0) \neq \{0\}$ which is ruled out by interchanging the role of $H$ and $L$. If all root subspaces are 2-dimensional then there are 3 of them, so one must be the 0-space. This is impossible since we have $V_{1/2}(0) = \{0\}$. If $V_{1/2} = V_{1/2}(\alpha) + V_{1/2}(-\alpha)$, $\dim V_{1/2}(\pm \alpha) = 3$, a case which we have already shown leads to a contradiction.

Assume that $n = 6$. So $M$ is irreducible of dimension 8 and dimension $M_i = 4$, $i = 1, 2$. Let $M_1 = Kx + Ky + Kz + Kt$, as above, and $Z, T, L$ as in (62). Since all $V_{1/2}(\alpha), V_{1/2}(-\alpha)$, $\alpha \neq 0$, are dual, the dimension of $V_{1/2}(0)$ must be odd. By (60), $\dim M_2(0) = \dim M_1(0) = \dim(x, y)^\perp = 2$. So $\dim M_2(\pm 1) = 1$, and $\dim V_{1/2}(1) \geq 2$, by (61). But $\dim V_{1/2} = 5$. So

$$\dim V_{1/2}(\pm 1) = 2, \quad \dim V_{1/2}(0) = 1.$$  

Let $0 \neq v \in V_{1/2}(0)$. Then $v^2 \neq 0$, for otherwise $Q(v, V_{1/2}) = \{0\}$. Now $V_{1/2}(0)$ is a 1-dimensional $\{Z, T, L\}$-module, so $vZ = vT = 0$, and the hypotheses of Lemma 21 are satisfied for $z, t \in M_1$ and $v \in V_{1/2}$. Therefore $J \cong osp_{2,2}(K)$, a contradiction.

Assume that $n = 5$. Then $M$ is either an irreducible $A$-bimodule or it is the sum of two irreducible $A$-bimodules. If $M$ is irreducible then, by the Uniqueness Lemma, $J \cong osp_{1,4}(K)$.

If $M$ is the sum of two irreducible bimodules then $\dim M = 8$. Again we may suppose that $M_1$ has a basis $x, y, z, t$ such that $[z, t] = \{x, y\}^\perp, [x, y] = [z, t] = e_1 + f$ and $M_2 = V_{1/2}R_{\gamma} = V_{1/2}R_{\zeta}$. Since $\dim V_{1/2} = 4 = \dim M_2, R_\gamma : V_{1/2} \to M_2$ and $R_{\zeta} : V_{1/2} \to M_2$ are both nonsingular. Suppose that the root space with respect to $H$, $V_{1/2}(0) \neq \{0\}$. Then, since $\dim V_{1/2}(0)$ must be even, $\dim V_{1/2}(0) \geq 2$. But then $\dim M_2(1) = \dim M_1(1) = 1$ and $V_{1/2}(0)R_{\zeta} \subseteq M_2(1)$, a contradiction.
So we may take $V_{1/2}(0) = \{0\}$. If one of the root subspaces $V_{1/2}(\alpha)$ is 1-dimensional then this subspace is annihilated by $Z = (R_z)^2$, $T = (R_t)^2$ and $L = [Z, T]$. So $V_{1/2}(0) \neq \{0\}$ and we are in the previous case. The only remaining possibility is that $V_{1/2} = V_{1/2}(-1) + V_{1/2}(1)$. dim.$V_{1/2}(-1) = \dim V_{1/2}(1) = 2$. But in this case $V_{1/2}(1) R_x \subseteq M_2(2)$. Since dim.$M_2(2) \leq 1$, this implies that $\ker R_x \neq \{0\}$, a contradiction.

Assume that $n = 4$. $M$ is irreducible and, by the Uniqueness Lemma, $J$ is isomorphic to the Kac superalgebra. But in that case $J_1 \cong D_{-3/2}$ which is not a superalgebra of a superform unless $−3/2 = 1$, that is, unless the characteristic is 5.

Assume that $n = 3$. Here the dimension of the irreducible $A$-bimodule is 2, so dim.$M_1 = \dim M_2 = 1$ and $J_1$ cannot be simple. Hence $M$ is the sum of two irreducible bimodules. By the Uniqueness Lemma, $J \cong M_{2,1}(K)$. But then $J_1$ is not a superalgebra of a superform.

Assume finally that $n = 2$. As in the previous case, $M$ must be the sum of two irreducible $A$-bimodules and by the Uniqueness Lemma $J \cong osp_{2,2}(K)$. This finishes the proof of the lemma. \(\square\)

We are finally able to classify the simple Jordan superalgebras with even part isomorphic to $J(V, Q) \oplus K f$.

**Proposition 28.** Let $J = A + M$ be a simple Jordan superalgebra with $A = J(V, Q) \oplus K f$, $Q$, a nondegenerate quadratic form on a vectorspace $V$. Then $J$ is isomorphic to one of the following: $osp_{2,2}(K)$, $M_{2,1}(K)$, the Kac superalgebra (in this case char.$K \neq 3$), $osp_{1,4}(K)$.

**Proof.** By Lemma 17, the subalgebra $J_1 = (K e_1 + K f) + M_1$ is simple. By Proposition 8, either $J_1 \cong D_t$, $t \neq 0, 1$, or $J_1$ is a superalgebra of a superform. If $J_1 \cong D_t$, by Lemma 27, char.$K = 3$ and $J$ is isomorphic to one of $osp_{2,2}(K)$, $osp_{1,4}(K)$. So let us assume that $J_1 \cong D_t$. Then $\dim M = 4$ and $n = \dim V = 2, 3, 4, 5$. In each of these cases the superalgebra in question exists and is unique by the Uniqueness Lemma: $osp_{2,2}(K)$, $M_{2,1}(K)$, $K_{10}$, $osp_{1,4}(K)$. \(\square\)

**Simple Jordan superalgebras with even part of capacity \(\geq 3\)**

In this section we complete the proof of the First Classification Theorem. This will be done by piecing together the results of the previous sections. The next proposition is a key step in that direction.

If $A$ is a simple Jordan algebra and $e \in A$, an idempotent, then $J U_e$ is also simple [15]. The following proposition establishes that the Peirce-1 space of a finite-dimensional simple Jordan superalgebra with semisimple even part is simple. In view of McRimmon’s result for Jordan algebras [15], it makes sense to ask if this is true for any simple Jordan superalgebra.

**Proposition 29.** Let $J = A + M$ be a simple finite-dimensional unital Jordan superalgebra over a field $K$ with $A$ semisimple, $e \in A$, an idempotent. Then $J U_e$ is a simple superalgebra.
Proof. It suffices to prove the proposition for $K$ algebraically closed. Assume first that the algebra $A$ is simple. Let $e_1, \ldots, e_n$ be a frame of $A$. Without loss of generality, we may assume that $e = e_1 + \cdots + e_r$, $r < n$, and that $JU_{e+e_{r+1}}$ is simple (otherwise take $e + e_{r+1}$ instead of $e$). So let us assume that $e = e_1 + \cdots + e_{n-1}$.

Since the proposition is true for superalgebras of capacity 2, as follows from our classification, we may assume that $n \geq 3$. Then, by Proposition 9, the bimodule $M$ is either irreducible or a sum of two irreducible bimodules.

If $M$ is irreducible then from the classification of irreducible bimodules over simple Jordan algebras [7], it follows that $MU_e$ is an irreducible bimodule over $AU_e$. Thus if the subalgebra $AU_e + M_U$ is not simple then $[MU_e, MU_e] = [0]$. If $M = M' \oplus M''$ is the direct sum of two irreducible $A$-bimodules $M'$, $M''$, then, since we are assuming that $A$ is simple, $AU_{e_i + e_j}$ is the Jordan algebra of a nondegenerate quadratic form on a space of dimension $\geq 2$ and by Proposition 15, $[M', M'] = [M'', M'']$. Again $AU_e + MU_e = AU_e + M'U_e \oplus M''U_e$ is not simple if and only if $[M'U_e, M''U_e] = [0]$.

In both cases

$$[MU_e, MU_e] = [0]$$

and in particular

$$[MU_{e_1 + e_2}, MU_{e_1 + e_2}] = [0].$$

(63)

For arbitrary $i \neq j$, $1 \leq i, j \leq n$, there exists an element $h_{ij} \in \{e_iAe_j\} + \sum_{k \neq i, j} e_k$ such that $h_{ij}^2 = 1, e_i U_{h_{ij}} = e_i, e_j U_{h_{ij}} = e_j$ and $e_k U_{h_{ij}} = e_k$, for $k \neq i, j$. By [7, Corollary, p. 245], $U_{h_{ij}} \in \text{Aut} J$. Applying the automorphisms $U_{h_{ij}}$ to (63) we get

$$[MU_{e_i + e_j}, MU_{e_i + e_j}] = [0],$$

for arbitrary $i, j$. Hence

$$[M, M] \subseteq \sum_{i \neq j} \{e_iAe_j\}.$$

But $[M, M]$ is closed under Inder $A$, which yields a contradiction.

Now let $A$ be the sum of two simple algebras, $A = A' \oplus A''$; Let $e_1, \ldots, e_n$ be a frame of $A'$ and $f_1, \ldots, f_m$ be a frame of $A''$. Then $1 = \sum_{i=1}^n e_i + \sum_{j=1}^m f_j$ and, by Lemma 7, $M = \sum M_{ij}$, where $M_{ij} = \{e_iMf_j\}$, $1 \leq i \leq n, 1 \leq j \leq m$; $A' = \sum_{1 \leq i, j \leq n} A'_{ij}$, $A'' = \sum_{1 \leq i, j \leq m} A''_{ij}$. We have assumed that the capacity is $\geq 3$, so without loss of generality, we may assume that $m \geq 2$.

Suppose that we have an element $x_{ij} \in M_{ij}$ with $[x_{ij}, M_{ij}] = [0]$. We show that, in that case, $[x_{ij}, M] = [0]$. We start by showing that $[x_{ij}, M_{ik}] = [0]$. Clearly, $[x_{ij}, M_{ik}] \subseteq A'_{jk}$. The subspace $A'_{jk}$ is spanned by elements which are invertible in $A'U_{f_j + f_k}$, Hence, to prove that $[x_{ij}, M_{ik}] = [0]$ it suffices to show that

$$[x_{ij}, M_{ik}]h_{jk} = [0],$$
for any element \( h_{jk} \in A'_{jk} \) such that \( h_{jk}^2 = f_j + f_k \). Since \( a \mapsto 2R_a \mid_M \) is a Jordan homomorphism of \( A' \) into \( \text{End} \ M^+ \) we have
\[
M_{ik} = M_{ij} \cdot h_{jk}.
\]
Now for an arbitrary element \( m_{ij} \in M_{ij} \) we have, by (2),
\[
2[x_{ij}, m_{ij}] \cdot h_{jk} = [x_{ij}, (h_{jk}^2), m_{ij}]
\]
\[
\quad = [(x_{ij} \cdot h_{jk}) \cdot h_{jk}, m_{ij}] + [(x_{ij} \cdot m_{ij} \cdot h_{jk}) \cdot h_{jk} + [x_{ij}, (m_{ij} \cdot h_{jk}) \cdot h_{jk}].
\]
Since \([x_{ij}, M_{ij}] = \{0\}\), this implies that
\[
[x_{ij}, m_{ij} \cdot h_{jk}] \cdot h_{jk} = 0.
\]
Thus
\[
[x_{ij}, M_{ik}] \cdot h_{jk} = \{0\} \quad \text{and} \quad [x_{ij}, M_{jk}] = \{0\}.
\]
Similarly
\[
[x_{ij}, M_{ij}] = \{0\} \quad \text{and} \quad [x_{ij}, M] = \{0\}.
\]
We have shown that
\[
x_{ij} \in M_{ij}, \ [x_{ij}, M_{ij}] = \{0\} \quad \Rightarrow \quad [x_{ij}, M] = \{0\}. \tag{64}
\]
We prove next that
\[
x_{ij} \in M_{ij}, \ [x_{ij}, M] = \{0\} \quad \Rightarrow \quad x_{ij} = 0. \tag{65}
\]
Let \( h_{jl} \) be an element of \( A'_{jl} U_{f_j, f_l} \) such that \( h_{jl}^2 = f_j + f_l \). We will prove that
\[
[x_{ij}, h_{jl}] \cdot M = \{0\}.
\]
Since \( x_{ij} \cdot h_{jl} \in M_{jl} \), by (64), it suffices to show that \([x_{ij} \cdot h_{jl}, M_{jl}] = \{0\}\). But \( M_{jl} = M_{ij} \cdot h_{jl} \), so by (2),
\[
[x_{ij}, h_{jl}, M_{ij} \cdot h_{jl}] \subseteq [x_{ij}, M_{ij}] \cdot h_{jl}^2 + [x_{ij}, M_{ij} \cdot h_{jl}] \cdot h_{jl} + [x_{ij}, h_{jl}^2, M_{ij}] = \{0\}.
\]
A similar argument works for \( A' \) provided its capacity is greater than 1. Now let \((x_{ij})_A\) be the sub-bimodule of \( M \) generated by \( x_{ij} \). We have proved that
\[
[(x_{ij})_A, M] = \{0\}.
\]
Thus \((x_{ij})_A\) is a superideal of \( J \), which implies that \( x_{ij} = 0 \).
Let us show next that $[M_{11}, M_{11}]$ is not contained in $A'$. Assume that $[M_{11}, M_{11}] \subseteq A'_{11}$.
We will show that then $[M_{ij}, M_{ij}] \subseteq A'_{ii}$ for any $i, j$. If $1 < i \leq n$, choose an element $q_{11} \in A'U_{e_1,e_1}$ such that $q_{11}^2 = e_1 + e_i$. Then $M_{11} = q_{11}M_{11}$. By (9') and the Peirce relations,

$$[M_{11}, M_{11}] = [q_{11}, M_{11}, q_{11}, M_{11}] \subseteq [q_{11}[M_{11}, M_{11}]q_{11}] + [M_{11}q_{11}^2, M_{11}] \subseteq A'_{ii},$$

since

$$[M_{11}q_{11}^2, M_{11}] = [M_{11}e_1 + e_i, M_{11}] = [M_{11}e_1, M_{11}],$$

the projection of $[M_{11}, M_{11}]$ on $A''$, which is $\{0\}$ by assumption. Similarly, if $h_{1j} \in A''U_{f_1,f_j}$, with $h_{1j}^2 = f_1 + f_j$, then

$$[M_{ij}, M_{ij}] = [M_{1j}, h_{1j}, M_{1j}, h_{1j}] \subseteq A'_{ij}.$$  

Thus all $[M_{ij}, M_{ij}]$ are contained in $A'$. Finally, for $j \neq i \in \{1, \ldots, m\}$, we may write $M_{ij} = M_{ij,h_{ij}}$ for $h_{ij} \in A''U_{f_j+f_i}$ with $h_{ij}^2 = f_j + f_i$ and

$$[M_{ij}, M_{ij}] = [M_{ij}, M_{ij}, h_{ij}].$$

For $x, y \in M_{ij}$, (2) yields

$$2[x, y; h_{ij}, h_{ij}] = -[x, h_{ij}^2, y] + [(x, h_{ij}), h_{ij}, y]$$

$$+ [(x, y), h_{ij}, h_{ij}] + [x, (y, h_{ij}), h_{ij}] \in [M_{ij}, M_{ij}] \subseteq A'.$$

But $[x, y; h_{ij}, h_{ij}] \in [M_{ij}, M_{ij}] \subseteq A'_{ij}''$ and $[x, y; h_{ij}, h_{ij}] \in A'_{ij}'' + A''_{ij}''$. Therefore $[x, y; h_{ij}, h_{ij}] = 0$ and $[x, y; h_{ij}, h_{ij}] = 0$ since $h_{ij}$ is invertible in $A''U_{f_j+f_i}$. This shows that $[M_{ij}, M_{ij}] = \{0\}$ so that $[M, M] \subseteq A'$. But if $[M, M] \subseteq A'$ then $A' + M$ is a superideal in $J$, contradicting the simplicity of $J$. Therefore $[M_{11}, M_{11}] \not\subseteq A'_{11}$ or equivalently $[M_{11}, M_{11}] \not\subseteq A'$. Similarly, $[M_{11}, M_{11}] \not\subseteq A''$.

We can now complete the proof of the proposition. Without loss of generality we may assume that

$$e = e_1 + \cdots + e_r + f_1 + \cdots + f_s, \quad 1 \leq r \leq n, \ 1 \leq s \leq m.$$  

Let $I$ be a superideal in $AU_e + MU_e$. If $I \cap AU_e \neq \{0\}$ then either $I \ni e_1, \ldots, e_r$ or $I \ni f_1, \ldots, f_s$. In both cases $M_{11} \subseteq I$. Since $[M_{11}, M_{11}] \subseteq A' \lor A''$, $I \cap A'U_e \neq \{0\}$ and $I \cap A''U_e \neq \{0\}$. Hence $e \in I$ which then equals $AU_e + MU_e$.

If $I \cap AU_e = \{0\}$, then $I \subseteq MU_e$ and $[I, MU_e] = \{0\}$. If $I \neq \{0\}$ then there exist $i, j, 1 \leq i \leq r, 1 \leq j \leq s$ such that $IU_{e_i,f_j} \neq \{0\}, IU_{e_i,f_j} \subseteq M_{ij} \subseteq MU_e$, which contradicts (65). This completes the proof of the proposition. \( \square \)

Now let us classify the simple Jordan superalgebras with simple even parts of capacity $n \geq 3$. Every simple Jordan algebra is one of
(I) $\mathcal{H}_n(K)$ (symmetric $n \times n$ matrices),

(II) $\mathcal{H}_n(Q)$ (Hermitian $n \times n$ matrices over the split quaternions),

(III) $\mathcal{H}_3(O)$ (Hermitian $3 \times 3$ matrices over the split octonions),

(IV) $\mathcal{M}_n(K)^+$ ($n \times n$ matrices).

(I) $A = \mathcal{H}_n(K)$, $n \geq 3$. We will show that there are no simple superalgebras with such an $A$ unless $n = 3$ and char. $K = 3$. The irreducible unital $A$-bimodules are: the regular bimodule $\mathcal{H}_n(K)$ and the skew-symmetric matrices $S_n(K)$. Denote by $e_{ij}$ the usual matrix units.

By Proposition 29, the superalgebra $JU_{e_{11} + e_{22}}$ is simple. Since $AU_{e_{11} + e_{22}} \cong \mathcal{H}_2(K) \cong J(V, Q)$, with dim. $V = 2$, Proposition 16 implies that $JU_{e_{11} + e_{22}}$ is a superalgebra of a superform. The elements $e_{11} - e_{22}$, $e_{12} + e_{21} \in \mathcal{H}_2(K)$ have trace 0. Hence

$$MU_{e_{11} + e_{22}}:(e_{11} - e_{22}) = MU_{e_{11} + e_{22}}:(e_{12} + e_{21}) = [0].$$

This rules out the presence of regular bimodules $\mathcal{H}_n(K)$ in the decomposition of $M$. Hence either $M \cong S_n(K)$ or $M \cong S_n(K) \oplus S_n(K)$. If $M \cong S_n(K)$ then dim. $KMU_{e_{11} + e_{22}} = 1$ and $[[e_{11}M_{e_{22}}], [e_{11}M_{e_{22}}]] = [0]$. This implies that matrices belonging to $[M, M]$ have only zeros on the diagonal. But this cannot be since $[M, M] \neq [0]$ is invariant under Inder $A$.

Assume that $M$ is the direct sum of two copies $S_n(K)$. As above the product of a copy of $S_n(K)$ with itself must be $[0]$.

$$M = S_n(K) \oplus S_n(K), \quad [[S_n(K), S_n(K)], [S_n(K), S_n(K)]] = [0].$$

We show next that for any $a, b \in S_n(K)$ we have

$$[a, b] = ab + ba \in \mathcal{H}_n(K) = A. \quad (66)$$

Denote $e_{ij} - e_{ji} \in S_n(K)$ by $s_{ij}$ and recall that in $S_n(K)$ the Lie product $[s_{ij}, s_{jk}] = s_{ik}$, for distinct $i, j, k$. If $[[s_{ij} , s_{ij}], s_{ik}] = 0$ then, applying the automorphisms $U_{e_{ij} + e_{ji} + \sum_{k\neq i, j} e_{kk}}$, we get $[[s_{ij} , s_{ij}], s_{ik}] = 0$, for any $i, j$ and again any matrix from $[M, M]$ has zero diagonal, which leads to a contradiction.

If $[[s_{ij} , s_{ij}], s_{ik}] \neq 0$, since $JU_{e_{11} + e_{22}}$ is a superalgebra of a superform, we can normalize the bracket $[,]$ so that

$$[[s_{12}, s_{12}] = -2(e_{11} + e_{22}) = 2s_{12}^2.$$  

Applying the automorphisms $U_{e_{ij} + e_{ji} + \sum_{k\neq i, j} e_{kk}}$ we get

$$[[s_{ij}, s_{ij}] = [s_{ii} - s_{jj}] = 2(s_{ij}^2),$$

for any $i, j, 1 \leq i, j \leq n$. By the Peirce relations, $[s_{12}, s_{12}] \in K(e_{23} + e_{32})$. Hence

$$[[s_{12}, s_{12}] = [s_{12}, s_{13}]U_{e_{23} + e_{32} + \sum_{k \neq 2, 3} e_{kk}} = [s_{13}, s_{12}].$$
Consider the inner derivation $D := 4[R_{e_{22}}, R_{e_{23} + e_{32}}]$ of $J$. Since
\[ s_{12}D = s_{13}, \quad s_{13}D = s_{12} \] we have
\[ (e_{11} + e_{22})D = e_{23} + e_{32}. \]
we have
\[ -2(e_{23} + e_{32}) = [s_{12}, s_{13}D] = [s_{12}, s_{13}] + [s_{12}, s_{12}D] = [s_{13}, s_{12}] + [s_{12}, s_{13}] \]
so $[s_{12}, s_{13}] = -(e_{23} + e_{32}) = s_{12}s_{13} + s_{13}s_{12}$, which proves (67) by the arbitrariness of the indices.

Let us prove that $A_{n}(K) + S_{n}(K) \oplus S_{n}(K)$ is a not Jordan superalgebra if $n \geq 4$ or $n = 3$ and char. $K \neq 3$. Choose $h \in A_{n}(K); a, b, c \in S_{n}(K)$. By (2) and (66),
\[ h(R_{a}R_{b}R_{c} - R_{b}R_{c}R_{a}) = h(R_{a}, h)R_{b} + R_{b}(c, h)R_{a}. \]
This is equivalent to
\[ ((h, a).c).b - (h, b).c, a = (h, (a, c)).b - (h, (b, c)).a, \]
or
\[ ((h, a).c - h, (a, c)).b = (h, b).c - h, (b, c)).a, \]
or, in terms of the associative multiplication in $A_{n}(K)$, to
\[ [h, c], a . b = [h, c].b, a, \]
where $[\cdot, \cdot]$ is the commutator in $A_{n}(K)$. Thus for arbitrary elements $d \in [A_{n}(K), S_{n}(K)]$, $a, b \in S_{n}(K)$ we have
\[ [d, a].b = [d, b].a. \]

Let $n \geq 4$. For $d = e_{12} + e_{23} = [e_{11}, s_{12}] \in [A_{n}(K), S_{n}(K)]$, $a = s_{23}$ and $b = s_{34}$, we have $[d, a].b = \frac{1}{2}s_{14} \neq 0$ whereas $[d, b] = 0$.

Now let $n = 3$ but char. $K \neq 3$. For $d = e_{11} - e_{22} = \frac{1}{2}[e_{12} + e_{21}, s_{23}]$, $a = s_{13}$ and $b = s_{12}$, we have $[d, a].b = \frac{1}{2}s_{32} = \frac{1}{2}(e_{32} - e_{23})$ whereas $[d, b].a = e_{23} - e_{32}$, a contradiction.

If $n = 3$ and the characteristic is 3, by (66) and (67), we have the superalgebra described in example (9).

(II) $A = A_{n}(Q)$, $n \geq 3$. The only simple superalgebra with such $A$ occurs in characteristic 3 and is described in example (10). Indeed, the irreducible unital $A$-bimodules can be obtained from the Jordan admissible alternative bimodules with involution for the split quaternion algebra $(Q, \cdot)$ [7, Chapter VII, Section 4]. For the split quaternions, these are the regular bimodule $regQ$ and the negative regular bimodule $-regQ$ which are Jordan admissible for $n \geq 3$ and correspond to the regular bimodule
$H_n(Q)$ and the skew-hermitian $n \times n$ matrices over $Q$, $S_n(Q)$. When $n = 3$, one must also consider the Cayley bimodule. But for the split quaternions the Cayley bimodule is not irreducible [7, case IIIa, p. 283]. It is the sum of two irreducible bimodules, each of dimension 2. Only the first is Jordan admissible. It corresponds to $e_{11}Q$, identifying $Q$ with $M_2(K)$. For lack of a better name we will call this bimodule $\text{halfcay}$.

We start by showing that the first two bimodules cannot occur. Since $AU_{e_{11} + e_{22}} \cong H_2(Q)$ which is the Jordan algebra of a quadratic form (the Pfaffian) defined on a space of dimension 5 (the trace 0 elements of $H_2(Q)$), Proposition 16 implies that $JU_{e_{11} + e_{22}}$ is a superalgebra of a superform. Hence, for an arbitrary $b \in Q$, we have

$$MU_{e_{11} + e_{22}}(b_{12} + \bar{b}_{21}) = [0].$$

But, for any $a \in Q,$

$$(a_{12} - \bar{a}_{21}) \in S_n(Q)U_{e_{11} + e_{22}},$$

and

$$\begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \neq 0, \quad \text{for some } a, b \in Q.$$

Similarly,

$$H_n(Q)U_{e_{11} + e_{22}}(b_{12} + \bar{b}_{21}) \neq [0], \quad \text{for } b \neq 0.$$

So neither $H_n(Q)$ nor $S_n(Q)$ can be a component of $M$.

We are left with $A = H_3(Q)$. All irreducible components of the bimodule $M$ other than the Jordan admissible summand of the Cayley bimodule, $\text{halfcay}$, were ruled out in the argument above. So $J = H_3(B)$. $B = B_0 + B_1$ be a simple unital alternative superalgebra such that $B_0 = M_2(K)$. From the Classification Theorem of Shestakov and Zel’manov [22, Theorem 2], it follows that char $K = 3$. It was shown by Shestakov [23, Corollary A] that $B$ is the 6-dimensional superalgebra of example (10). So $J \cong H_3(B)$ as in example (10).

(III) $A = H_3(O)$. Consider the Grassmann envelope $\Gamma(J)$ of $J$. Identifying $H_3(O)$ with $H_3(O) \otimes 1 \subset J_0 \otimes I_0$, we conclude, by Jacobson’s Kronecker Factorization Theorem [5, Theorem 3], that $\Gamma(J) \cong H_3(O) \otimes Z$, where $Z$, the centralizer of $H_3(O)$ in $\Gamma(J)$, is a commutative associative algebra. Since the centralizer of $H_3(O)$ in the Grassmann envelope of $J$ is the Grassmann envelope of the centralizer of $H_3(O)$ in $J$, $J \cong H_3(O) \otimes (K + B_1)$, where $K + B_1$ is a supercommutative associative superalgebra and thus $J$ is not simple unless $B_1 = [0]$.

(IV) $A = M_\alpha(K)^{\pm}$, $n \geq 3$. We will show that in this case either

$$J \cong Q_\alpha(K) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in M_\alpha(K) \right\}$$
or
\[
J \cong P_n(K) = \left\{ \begin{pmatrix} a & b \\ c & a' \end{pmatrix} \mid a \in M_n(K), \ b \in S_n(C), \ c \in \mathcal{H}_n(C) \right\}.
\]

The irreducible \(A\)-bimodules are: the regular bimodule \(M_n(K)\), the symmetric matrices \(\mathcal{H}_n(K)\) (with action \(h.a := \frac{1}{2}(ha + a'h)\) or \(h.a := \frac{1}{2}(ha' + ah)\), for \(h \in \mathcal{H}_n(K)\), \(a \in M_n(K)\)), the skew-symmetric matrices \(S_n(K)\) (with action \(s.a := \frac{1}{2}(sa + a's)\) or \(s.a := \frac{1}{2}(sa' + as)\), for \(s \in S_n(K)\), \(a \in M_n(K)\)). Now \(AU_{e_{11} + e_{22}} \cong M_2(K)^+\) and, by Propositions 16 and 29, \(JU_{e_{11} + e_{22}}\) is either a superalgebra of a superform or is isomorphic to \(Q_2(K)\) or \(P_2(K)\). If \(JU_{e_{11} + e_{22}}\) is a superalgebra of a superform then \(MU_{e_{11} + e_{22}}M_2(K)_{\text{0}} = \{0\}\), where \(M_2(K)_{\text{0}}\) denotes the elements of \(M_2(K)\) of trace 0. But, for any of the three irreducible \(A\)-bimodules above, the action of the operator \(U_{e_{11} + e_{22}}RK_{e_{12} + k_{e_{21} + k(e_{11} - e_{22})}}\) is nontrivial. Hence
\[
JU_{e_{11} + e_{22}} \cong Q_2(K) \text{ or } P_2(K).
\]

In the first case, the irreducibility of \(MU_{e_{11} + e_{22}}\) over \(AU_{e_{11} + e_{22}}\) together with the, by now familiar, automorphism argument which allows us to transport this action to other components of \(J\) imply that \(M\) is irreducible over \(A\) and, moreover, \(M\) is the regular bimodule. Let \(\sim: A \to \overline{A} = M\) be a bimodule isomorphism. We may assume that \([\overline{e}_{ij}, \overline{e}_{kl}] = e_{ij}e_{kl} - e_{kl}e_{ij}\) for \(1 \leq i, j, k, \ell \leq 2\). From this, by applying the automorphisms \(U_{e_{ij} + e_{kl} + \sum_{s \neq i, j} e_{sk}}\), one obtains \([\overline{e}_{ij}, \overline{e}_{kl}] = e_{ij}e_{kl} - e_{kl}e_{ij}\) whenever \([i, j, k, \ell]\) has no more than 2 elements.

We wish to show that \([\overline{e}_{12}, \overline{e}_{23}] = e_{13}\). By considering the Peirce decomposition, we get
\[
[\overline{e}_{12}, \overline{e}_{23}] = \alpha e_{13} + \beta e_{31}, \quad \text{for some } \alpha, \beta \in K. \tag{68}
\]

We show first that the derivation \((R_{e_{23}})^2\) is trivial. Since every derivation of \(M_n(K)^+\) is inner and since the derivation \((R_{e_{23}})^2\) preserves the Peirce (i.e., matrix) structure, the derivation \((R_{e_{23}})^2\) corresponds to commutation with some diagonal matrix \(d = \text{diag}(d_1, \ldots, d_n)\). If \(d_i \neq d_j\), for some \(i, j \in \{1, \ldots, n\} \setminus \{2, 3\}\) then \([e_{ij}, d] = (d_i - d_j)e_{ij} \neq 0\) whereas \(e_{13}(R_{e_{23}})^2 = 0\). If \(d_2 \neq d_3\) then \([e_{23}, d] \neq 0\), which is also a contradiction. Finally, if \(d_1 \neq d_3\) then \([e_{13}, d] \neq 0\) whereas \(e_{13}(R_{e_{23}})^2 = 0\). Hence \(d\) is a scalar matrix and \((R_{e_{23}})^2 = 0\).

Acting on (68) by \(R_{e_{23}}\), we obtain
\[
0 = \overline{e}_{12}(R_{e_{23}})^2 = \beta \overline{e}_{21}.
\]

Hence \(\beta = 0\). The inner derivation \([R_{e_{23}}, R_{e_{22}}]\) corresponds to commutation by \(e_{32}\). Applying it to (68) yields
\[
[\overline{e}_{12}, \overline{e}_{22}] = \alpha e_{12},
\]
which implies that \(\alpha = 1\). So \([\overline{e}_{12}, \overline{e}_{23}] = e_{13}\) and, as usual, acting on this equation by automorphisms shows that \([\overline{e}_{ij}, \overline{e}_{jk}] = e_{ik}\) and \(J \cong Q_n(K)\).
Assume next that $JU_{e_{11} + e_{22}} \cong P_2(K)$. This forces $M$ to be isomorphic to $N_n(K)$. If $[\mathcal{H}_n(K), \mathcal{H}_n(K)] \neq \{0\}$ or $[\mathcal{S}_n(K), \mathcal{S}_n(K)] \neq \{0\}$ then either $A + \mathcal{H}_n(K)$ or $A + \mathcal{S}_n(K)$ is a simple superalgebra which contradicts Proposition 16. Thus

$$[\mathcal{H}_n(K), \mathcal{H}_n(K)] = [\mathcal{S}_n(K), \mathcal{S}_n(K)] = \{0\}.$$ 

For $h_{ij} = e_{ij} + e_{ji}$ and $s_{kl} = e_{kl} - e_{lk}$, by Proposition 16, we have

$$[h_{ij}, s_{kl}] = -s_{kl} h_{ij}, \quad \text{whenever } 1 \leq i, j, k, \ell \leq 2.$$ 

Applying the derivations $D_{ij} = [R_{e_{ij}}, R_{h_{ij}}]$ we see that the last equation holds whenever $[i, j, k, \ell]$ is a 2-element set. In particular

$$[e_{22}, s_{23}] = -s_{23} e_{22}.$$ 

Applying $D_{12}$ to both sides yields

$$[h_{12}, s_{23}] = -s_{23} h_{12}.$$ 

Arguing as above, this equation holds for any choice of three indices and $J$ is of type $P_n(K)$. This completes case (IV).

We must now consider the remaining cases where $A$ is not simple:

(V) $A = \mathcal{H}_n(K) \oplus \mathcal{H}_m(Q)$,

(VI) $A = M_n(K)^+ \oplus M_m(K)^+$,

(VII) $A = \mathcal{H}_n(Q) \oplus \mathcal{H}_m(Q)$,

(VIII) $A = \mathcal{H}_n(K) \oplus M_m(K)^+$,

(IX) $A = \mathcal{H}_n(K) \oplus \mathcal{H}_m(K)$,

(X) $A = M_n(K)^+ \oplus \mathcal{H}_m(Q)$.

We have deliberately ignored blocks $\mathcal{H}_3(Q)$ and $J(V, Q)$ because

(i) the $\mathcal{H}_3(Q)$'s have no nontrivial special representations,

(ii) if $A = A' \oplus J(V, Q)$, taking $e'$ a primitive idempotent of $A'$ and $e''$ the unit element of $J(V, Q)$, then $JU_{e'+e''}$ is a simple Jordan superalgebra having $Ke' \oplus J(V, Q)$ as its even part. From the classification of such superalgebras it follows that $J(V, Q)$ is isomorphic to one of the following algebras: $K, \mathcal{H}_2(K), M_2(K)^+$, $\mathcal{H}_2(Q)$.

(V) $A = \mathcal{H}_n(K) \oplus \mathcal{H}_m(Q)$. We aim to prove that $J \cong \mathfrak{osp}_{n, 2m}(K)$. Let $e_1', \ldots, e_m'$; $e_1'', \ldots, e_m''$ be frames of $\mathcal{H}_n(K)$, $\mathcal{H}_m(Q)$ respectively. The subsuperalgebra $JU_{e'_1 + e'_2 + e''_2}$ has $Ke_1' \oplus \mathcal{H}_2(Q)$ as even part and hence $JU_{e'_1 + e'_2 + e''_2} \cong \mathfrak{osp}_{1, 4}(K)$ and $\dim JU_{e'_1 + e'_2 + e''_2} = 4$. This rules out the case when $M$ is a sum of two irreducible bimodules over $A$. Therefore $M$ is an irreducible $A$-bimodule. If $m = 1$ and $n > 1$ then we repeat this argument with $U_{e'_1 + e'_2 + e''_1}$. The case $n = 1$, $m = 1$ was treated in Proposition 28.
By the Uniqueness Lemma, the brackets on \( MU_{\epsilon_1, \epsilon_2} + e''_1 \) are homotetic to those in \( osp_{1,4}(K) \) while the brackets on \( MU_{\epsilon_1, \epsilon_2} + e''_2 \) are homotetic to those in \( osp_{2,2}(K) \).

Multiplying by a suitable scalar, we may assume that the brackets on \( M_{ij} = OU_{\epsilon_1, \epsilon_2} + e''_1 \) coincide with the brackets on the corresponding Peirce component of \( osp(n, 2m) \).

Therefore so do the brackets on \( M_{ij} \) (consider the brackets on \( MU_{\epsilon_1, \epsilon_2} + e''_1 \)), hence so do the brackets on \( M_{ij} \) (consider the brackets on \( MU_{\epsilon_1, \epsilon_2} + e''_1 \)). In fact so do the brackets \( [M_{ij}, M_{kl}] \) (consider \( MU_{\epsilon_1, \epsilon_2} + e''_1 \)) and \( [M_{ij}, M_{kl}] \) (consider \( MU_{\epsilon_1, \epsilon_2} + e''_1 \)). Thus the brackets on \([M, M]\) coincide with those of \( osp_n, 2m(K) \).

(VI) \( A = M_n(K)^+ \oplus M_m(K)^+ \), \( (n, m) \neq (1, 1) \). Let \( e'_1, e'_2 \) be a frame of the first summand and \( e'' \) the unit element of the second summand. By Proposition 28, we have \( Ju_{e'_1, e''} \cong osp_{1,4}(K) \), hence \( \dim MU_{e'_1, e''} = 4 \) and \( \dim M = 8 \). However the dimension of the irreducible \( H_2(Q) \oplus H_2(K) \)-bimodule is equal to 16, a contradiction.

(VII) \( A = H_n(Q) \oplus H_m(Q) \), \( n > 1, m > 1 \). To prove that there are no simple Jordan superalgebras having \( A \) as an element part, it is sufficient to do it in the case \( A = H_2(Q) \oplus H_2(Q) \). Let \( e'_1, e'_2 \) be a frame of the first summand and \( e'' \) the unit element of the second summand. By Proposition 28, we have \( Ju_{e'_1, e''} \cong osp_{1,4}(K) \), hence \( \dim MU_{e'_1, e''} = 4 \) and \( \dim M = 8 \). The dimension of the irreducible \( H_2(K) \oplus H_2(K) \)-bimodule is 4. Thus \( M \) is a sum of two irreducible \( A \)-bimodules. The rest of the proof follows that of case V.

(VIII) \( A = H_n(K) \oplus M_m(K)^+ \), \( n > 1, m > 1 \). Again to prove that there are no simple Jordan superalgebras having \( A \) as an element part, it is sufficient to do it in the case \( A = H_2(K) \oplus M_2(K)^+ \). Let \( e_1, e_2 \) be a frame of \( H_2(K) \) and \( f_1, f_2 \) a frame of \( M_2(K)^+ \).

By Proposition 28, \( Ju_{e_1, f_1} \cong M_{1,2}(K) \), so \( \dim MU_{e_1, f_1} = 4 \) and \( \dim M = 8 \). The dimension of the irreducible \( H_2(K) \oplus M_2(K)^+ \)-bimodule is 4. Thus \( M \) is a sum of two irreducible bimodules

\[ M = M' \oplus M'', \quad [M', M'] = [M'', M''] = [0]. \]

By Proposition 28, we have

\[ Ju_{e_1, e_2} \cong osp_{2,1}(K), \]

hence

\[ [e_1 M f_1], [e_2 M f_1] \ni e_{12} + e_{21}. \]

This means that there exist elements \( m' \in M', m'' \in M'' \) such that

\[ [e_1 m' f_1], [e_2 m'' f_1] = e_{12} + e_{21} \quad \text{or} \quad [e_1 m'' f_1], [e_2 m' f_1] = e_{12} + e_{21}. \]

Since the two cases are symmetrical we treat only the first one. In that case,

\[ \{e_1 m' f_1\}(R_{e_2 m'' f_1})^2 = (e_{12} + e_{21})\{e_2 m'' f_1\} \neq 0, \]
indeed, $e_{12} + e_{21}$ is invertible in $\mathcal{H}(K)$ and $M''$ is a special $\mathcal{H}(K)$-bimodule. The derivation $D = (R_{e_{12}m''f_1})^2$ acts trivially on $A$. Therefore $D : M' \to M'$ is an $A$-bimodule homomorphism. By Schur’s Lemma, $D$ is either 0 or one-to-one. But

$$\{e_1M'f_1\}D = 0.$$  

Hence $D = 0$, a contradiction.

(IX) $A = \mathcal{H}_e(K) \oplus \mathcal{H}_m(K)$. Arguing exactly as in the previous case, on sees that there are no simple Jordan superalgebra with $A$ as even part.

(X) $A = \mathcal{M}_e(K)^+ \oplus \mathcal{H}_m(\mathbb{Q})$. Again we show that there are no simple Jordan superalgebra with $A$ as even part. For the special case $A = \mathcal{M}_2(K)^+ \oplus \mathcal{H}_2(\mathbb{Q})$, let $e_1$, $e_2$ be a frame in $\mathcal{M}_2(K)^+$ and $f_1$, $f_2$ a frame in $\mathcal{H}_2(\mathbb{Q})$. Arguing as above, we conclude that $JU_{e_1+f_1+f_2} \cong \mathfrak{osp}_{2,4}(K)$ which implies that dim $(e_1, M) = 4$. Hence dim $M = 8$ and $M$ is an irreducible $A$-bimodule. The Jordan algebra $B = \mathcal{H}_2(K) \oplus \mathcal{H}_2(\mathbb{Q})$ is naturally embedded in $A$ and $M$ is an irreducible $B$-module. There is a bracket $(\cdot, \cdot)$ on $M$, $(\cdot, \cdot) : M \times M \to B$ which gives $B + M$ a superalgebra structure isomorphic to $\mathfrak{osp}_{2,4}(K)$. Let us denote the bracket on $M$ which corresponds to the superalgebra $J = A + M$ by $[\cdot, \cdot]$. We now have two brackets on $M$ which are proportional on $e_1, M$ by the Uniqueness Lemma. Hence multiplying $(\cdot, \cdot)$ by a suitable scalar we may assume that $[\cdot, \cdot]$ and $(\cdot, \cdot)$ coincide on $e_1, M$. We claim that $[\cdot, \cdot]$ and $(\cdot, \cdot)$ must also coincide on $e_2, M$. Let $h_{12} = e_{12} + e_{21} \in \mathcal{H}_2(K)$. Then $e_2, M = h_{12}, (e_1, M)$. For arbitrary elements $m, m' \in e_1, M$, we have, by (2),

$$2[m', h_{12}, m, h_{12}] + [m', m]h_{12}^2 = [(m', h_{12}), h_{12}, m] + ([m', m], h_{12})h_{12} + [m', (m, h_{12})]h_{12}.$$  

Since the same holds with $[\cdot, \cdot]$ replaced by $(\cdot, \cdot)$, we must have

$$[m', h_{12}, m, h_{12}] = [m', h_{12}, m, h_{12}].$$  

If $m \in \{e_1Mf_1\}$ and $m' \in \{e_2MFf_1\}$ then $(m, m') = \alpha h_{12}, \alpha \in K$ and $[m, m'] = \beta h_{12} + \gamma k_{12}$, where $k_{12} = e_{12} - e_{21}$, $\beta, \gamma \in K$. We wish to show that $\alpha = \beta$. Since $h_{12}, k_{12} = 0$, $[m, m'], h_{12} = \beta(e_1 + e_2)$. But

$$m' = 4(m', h_{12})h_{12} = 4m''h_{12},$$  

where $m'' = m'. h_{12} \in \{e_1Mf_1\}$. Now

$$[m, m']h_{12} = 4[m, m' h_{12}]h_{12},$$  

but, by (2),

$$2[m, m'' h_{12}, h_{12}] + [m, h_{12}^2, m''] = [(m, h_{12}), h_{12}, m''] + ([m, m''], h_{12})h_{12} + [m, (m'', h_{12}), h_{12}].$$
The Peirce relations and the fact that the brackets agree on \([e_1 M f_1]\) applied to the above equation yield

\[
2[m, m''.h_{12}], h_{12} = 2[m, m''.h_{12}], h_{12}.
\]

So \(\alpha = \beta\).

We can now define a bilinear form \(g(,): [e_1 M f_1] \times [e_2 M f_1] \rightarrow K\) via

\[
[m, m'] - \langle m, m' \rangle = g(m, m')k_{12}, \quad m \in [e_1 M f_1], \quad m' \in [e_2 M f_1].
\]

If \(n \in [e_1 M f_1]\) then, again by (2),

\[
h_{12}(R_{[m, m']}R_n - R_{[m, n]}R_{m'} + R_{[m', n]}R_m) = h_{12}(R_mR_{[m', n]} - R_{m'}R_{[m, n]} + R_nR_{[m, m']})
\]

and

\[
h_{12}(R_{[m, m']}R_n - R_{(m, n)}R_{m'} + R_{(m', n)}R_m)
= h_{12}(R_mR_{(m', n)} - R_{m'}R_{(m, n)} + R_nR_{(m, m')}).
\]

But \([m, m']\) (respectively \([n, m']\)) differs from \(\langle m, m' \rangle\) (respectively \(\langle n, m' \rangle\)) by a multiple of \(k_{12}\), and \(h_{12}.k_{12} = 0\). Hence the left-hand sides of the two equalities above are equal while the right-hand sides differ by

\[
-g(n, m')(h_{12}.m).k_{12} + g(m, m')(h_{12}.n).k_{12}.
\]

Hence,

\[
g(m, m')(h_{12}.n).k_{12} = g(n, m')(h_{12}.m).k_{12}.
\]

Since the elements \(h_{12}, k_{12}\) are invertible in \(M_2(K)^{+}\), the operator \(R_{h_{12}}R_{k_{12}}\) is nonsingular on \(M\). Therefore

\[
g(m, m')n = g(n, m')m.
\]

We have \(\dim [e_1 M f_1] = 2\). Let \(n, m\) be a basis of \([e_1 M f_1]\). Then

\[
g(m, m') = g(n, m') = 0.
\]

We have proved that

\[
g([e_1 M f_1], [e_2 M f_1]) = \{0\}.
\]
Similarly we can prove that \([\cdot,\cdot]\) and \(\langle\cdot,\cdot\rangle\) coincide on \([e_1 Mf_2] \times [e_2 Mf_2]\). This implies that \([\cdot,\cdot]\) and \(\langle\cdot,\cdot\rangle\) coincide on \(M\). So \(B + M\) is a subalgebra of \(J = A + M\). But

\[
\left( M_2(K)^+ + Kf_1 \right) + M.f_1 \cong M_{2,1}(K).
\]

Hence

\[
k_{12} \in \left( [e_1 Mf_1], [e_2 Mf_1] \right),
\]
a contradiction. This completes the proof of the first classification theorem.

**Forms of simple Jordan superalgebras with semisimple even part**

If \(J = A + M\) is a finite dimensional simple Jordan superalgebra with semisimple even part over an arbitrary field \(K\) of characteristic not 2 then \(J\) is a form of one of the superalgebras in our First Classification Theorem. The forms of finite dimensional simple Jordan algebras are known as are their bimodules [7]. Note that, with the exception of \(D_4\), the superalgebras in the First Classification Theorem have integral structure constants for a suitably chosen basis. In any case, we will refer to the superalgebras listed in the First Classification Theorem as the *split simple finite dimensional Jordan superalgebras with semisimple even part*. We first consider the small capacity cases.

One checks that up to isomorphism there is only one form of \(K_3, K_3\) itself. Forms of the superalgebra of a nondegenerate superform are themselves superalgebras of a nondegenerate superform.

Recall that, for \(t \in K^\times\), \(D_t = A + M, A = Ke_1 + Ke_2, e_i\) orthogonal idempotents, \(M = Kx + Ky\) with \(e_i[x] = \frac{1}{2}x, e_i[y] = \frac{1}{2}y, [x, y] = e_1 + te_2\). Note that exchanging \(e_1\) and \(e_2\) and scaling \(x\) shows that \(D_t \cong D_{-t}\). This is the only isomorphism between \(D_t\) and \(D_y\).

Since forms of \(Ke_1 + Ke_2\) are 2-dimensional composition algebras \(C\), forms of \(D_t\) are of the form \(J(C + M, M = Kx + Ky)\) and, since \(J\) is simple, \([x, y] = v\), an invertible element of \(C\). We will denote \(J\) by \(J(C, v)\) and let \(^\sim\) denote the canonical involution of \(C\). If \(C\) is not split then \(C = K[z]\), a quadratic field extension of \(K\). Since char \(K \neq 2\), we may assume that \(z^2 = \delta 1, \delta \in K \setminus K^2\). Therefore \(M\) is a unital bimodule and \(z.M = [0]\).

We may split \(J(C, v)\) by tensoring with \(K[z]\). One can check that, as a superalgebra over \(K[z]\), \(J(C, v) \otimes_K K[z] \cong D_{v-1}\). If \(C\) is split, \(C = Ke_1 + Ke_2\) and \(v = ae_1 + \beta e_2\) with \(a\beta \neq 0\), scaling \(x\) we may assume that \([x, y] = e_1 + \alpha^{-1} \beta e_2\); so \(J(C, v) \cong D_{v-1}\) over \(K\).

We wish to determine \(J(C, v)\) and \(J(C', v')\) are isomorphic. Since \(J(C, v) \cong J(C', v')\) implies that \(C \cong C'\), we may assume that \(C' = C\). Since \([ax + \beta y, yx + \delta y] = (a\delta - \beta \gamma)[x, y]\), we see that \(J(C, v) \cong J(C, \lambda v)\) for any \(\lambda \in K^\times\) and that these are all the isomorphisms which extend the identity map on \(C\). If the restriction of an isomorphism \(\phi: J(C, v) \mapsto J(C, v')\) to \(C\) is not the identity then it must be \(\sim\) and we have \(J(C, v) \cong J(C, \sim v)\).

Let \(e, v_1, v_2, v_3, v_4, f, x_1, y_1, x_2, y_2\) be the basis of the Kac algebra given in example (8). The even part is the direct sum of \(Kf\) and \(J(V, Q)\), the Jordan algebra of the quadratic form \(Q\) which is nondegenerate of Witt index 2 on a four dimensional space \(V\).
In fact up to the scalar 2, \( v_1, v_2 \) and \( v_3, v_4 \) are hyperbolic pairs. The odd part \( M \) is a simple bimodule of \( J(V, Q) \) which, by the representation theory of \( J(V, Q) \), is a module of the special universal envelope of \( J(V, Q) \), \( C(V, Q) \), which is isomorphic to \( \mathcal{M}_4(K) \) since \( V \) is the sum of two hyperbolic planes. Since the structure constants are half-integers the above products make sense over any field of characteristic not 2 (in fact any commutative ring containing 1/2).

We define a superalgebra \( K_{10}(d) \), \( d \in K^\times \) with even part \( A = Ke + Ku_1 + Ku_2 + Ku_3 + Ku_4 + Kf \) and odd part \( M = Kz_1 + Kz_2 + Kz_3 + Kz_4 \). We will show that they are the forms of \( K_{10} \) over \( \overline{K} \). The product is given by

\[
e^2 = e, \quad e.u_i = u_i, \quad u_1^2 = 4e, \quad u_2^2 = -4de, \quad u_3.u_4 = 2e,
\]

\[
f^2 = f, \quad f.z_j = \frac{1}{2}z_j = e.z_j, \quad j = 1, 2, \quad z_1.u_1 = z_1, \quad z_2.u_1 = -z_2, \quad z_3.u_1 = -z_3, \quad z_4.u_1 = z_4,
\]

\[
z_1.u_2 = -z_2, \quad z_2.u_2 = d z_1, \quad z_3.u_2 = z_4, \quad z_4.u_2 = -d z_3, \quad z_3.u_3 = z_1, \quad z_4.u_3 = z_2, \quad z_1.u_4 = z_3, \quad z_2.u_4 = z_4,
\]

\[
[z_1, z_2] = -2u_3, \quad [z_3, z_4] = -2u_4, \quad [z_1, z_3] = d^{-1}u_2, \quad [z_2, z_4] = u_2,
\]

\[
[z_1, z_4] = -u_1 - 2(e - 3f), \quad [z_2, z_3] = -u_1 + 2(e - 3f).
\]

All other products are zero or obtained by symmetry or skew-symmetry.

**Proposition 30.** Let \( K \) be an arbitrary field of characteristic not 2. The superalgebras \( K_{10}(d) \), \( d \in K^\times \) are the forms of the Kac superalgebra over \( \overline{K} \). \( K_{10}(d) \cong K_{10}(d') \) if and only if \( d = d' \in K^\times / K^\times_2 \).

**Proof.** We show first that \( K_{10}(d) \) is a form of \( K_{10} \) and, in particular, a Jordan superalgebra. Let \( \rho \in \overline{K} \) be a square root of \( d \), \( \rho^2 = d \). One checks that the following elements of \( K_{10}(d)_{K(\rho)} \) form a basis with product as in (3), (3'), (4) and (5):

\[
v_1 = \frac{1}{2}(u_1 + \rho^{-1}u_2), \quad v_2 = \frac{1}{2}(u_1 - \rho^{-1}u_2), \quad v_3 = \rho^{-1}u_3, \quad v_4 = \rho u_4,
\]

\[
x_1 = \frac{1}{2}(z_1 - \rho^{-1}z_2), \quad y_1 = -\frac{1}{2}\rho(z_3 + z_4),
\]

\[
x_2 = \frac{1}{2}(\rho z_3 - z_4), \quad y_2 = -\frac{1}{2}(z_1 + \rho^{-1}z_2).
\]

Hence \( K_{10}(d) \) is a form of \( K_{10} \) over \( \overline{K} \).

Let \( J = A + M \) be a form of the Kac algebra over \( K \). Then \( A = J(V, Q) \oplus Kf \), where \( J(V, Q) \), the algebra of a nondegenerate quadratic form \( Q \) with unit element \( e \), is five dimensional, \( Kf \) is one dimensional and \( M \) is a four dimensional simple \( J(V, Q) \)-bimodule. The special universal envelope of \( J(V, Q) \) is a sixteen dimensional simple Clifford algebra \( C(V, Q) \) which must therefore be a division algebra or \( 2 \times 2 \) matrices.
with entries in a quaternion division algebra or $4 \times 4$ matrices with entries in $K$. However only the last algebra has a four dimensional simple module. Therefore $C(V, Q)$ is split and the bimodule structure of $M$ is determined.

Let $d$ be the discriminant of $Q$. Since $Q$ is isotropic if and only if $C(V, Q)_{K(\sqrt{d})}$ is split [21, Theorem 14.3, p. 88], $Q \cong (1, -1, \alpha, \beta)$ and $C(V, Q) \cong (1, -1) \otimes (-\alpha, -\beta) = \mathcal{M}_2(K) \otimes (-\alpha, -\beta) = \mathcal{M}_2((-\alpha, -\beta))$, where $(\alpha, \beta)$ denotes the quaternion algebra with basis 1, $a$, $b$, $ab$ such that $a^2 = \alpha$, $b^2 = \beta$ and $ba = -ab$. But since $C(V, Q)$ is split, $(-\alpha, -\beta) \cong \mathcal{M}_2(K)$ and $(-\alpha, -\beta)$ is isometric to $(1, \lambda)$ [18, 57; 10 (1)]. Therefore $(V, Q) \cong (1, -d) \perp H$, where $H$ is a hyperbolic plane and $d$ is the discriminant of $Q$. We may therefore choose bases $[u_i]_{1 \leq i \leq 4}$ of $V$ and $[z_i]_{1 \leq i \leq 4}$ of $M$ such that the product on $M$ is given by (68) and the action of $J(V, Q)$ on $M$ is given by (71).

Let $e_1 = \frac{1}{4}(2e + u_1)$, $e_2 = \frac{1}{4}(2e - u_1)$. One verifies that $e_i$ are idempotents. By Lemma 17, $J_1 = Ke_1 + Kf + M.e_1$ is simple and, by the Uniqueness Lemma, it is a form of $D_{3/2}$ and determines the structure of $J$. Since $Ke_1 + Kf$ is split, $J_1$ is itself a $D_3, t \in K^\times$. So $t = -3/2$. Now $M_1 := M.e_1 = Kz_1 + Kz_4$ and $[z_1, z_4] = -\lambda(e_1 + \frac{1}{2}f)$, for some scalar $\lambda \in K^\times$. So $[z_1, z_4] = \lambda(\frac{1}{2}(2e + u_1) + [-\frac{1}{2}f]) = \frac{\lambda}{2}(u_1 + 2(e - 3f))$. Replacing $u_3$ by $\frac{\lambda}{2}u_3$, $u_4$ by $\frac{\lambda}{2}u_4$, $z_1$ by $\frac{\lambda}{2}z_1$ and $z_4$ by $\frac{\lambda}{2}z_4$, we obtain a new basis of $J$ for which (69) and (71) hold and $[z_1, z_4] = -u_1 - 2(e + 3f)$. By the Uniqueness Lemma, (72) must also hold and $J$ is isomorphic to $K_{10}(d)$.

We have just proved that a form $J = A + M$ of $K_{10}$ is determined up to isomorphism by $A = J(V, Q) + Kf$ and hence by the discriminant $d$ of $Q$ since it determines the admissible $J(V, Q)$’s. □

For a slightly different proof, see [1].

Consider next the superalgebra of example (9). Since associative division algebras of degree 3 do not afford an involution of the first kind, forms of $\mathcal{H}_3(K)$ are $\mathcal{H}_3(K, *)$ for an involution of $\mathcal{M}_3(K)$. So if $J = A + M$ with $A = \mathcal{H}_3(K)$, $K$ a field of characteristic 3, then $M = \mathcal{S}_3(K, *) \oplus \mathcal{S}_3(K, *).$ Arguing as in case (I) of the previous section, we may scale the product on $M$ so that $[\mathcal{F}a, b] = ab + ba, a, b \in \mathcal{S}_3(K, *)$.

Let $J$ be a $K$-form of $\mathcal{H}_3(B) = A + M$ over a field $K$ of characteristic 3, as in example (10), $A = \mathcal{H}_3(M_2(K))$. Observe that $K$-forms of $A$ are split since division quaternion algebras do not have a Jordan admissible bimodule of dimension 2 [7, case III, p. 283]. Thus the even part of $J$ is $\mathcal{H}_3(M_2(K))$ and the odd part $\mathcal{H}_3(\text{half Cay})$ with the same bimodule structure as in example (10). Arguing as in the proof of Eq. (67), we can show that the product on the odd part agrees with that of $\mathcal{H}_3(B)$.

Since a Jordan superalgebra of the form $A^+$, $A$ an associative superalgebra, is isomorphic to the superalgebra of symmetric elements $\mathcal{H}(A \oplus A^{op}, *)$, where $A^{op}$ is the opposite superalgebra of $A$, i.e., $a_a^{op}b_\beta := (-1)^{a_\beta b_\gamma}a_\alpha$, and $*$, the exchange superinvolution, the remaining families of simple Jordan superalgebras in the First Classification Theorem, namely $\mathcal{M}_{n,m}(K), os\mathcal{P}_{n,2m}(K), n, m \geq 1, P_n(K)$ and $Q_n(K), n \geq 2$, are all of the form $\mathcal{H}(B, *), (B, *)$ a simple associative superalgebra with superinvolution. We wish to show that the same is true of their forms.

Since $\mathcal{M}_{1,1}(K) \cong D_{-1}$ its forms are already determined. We consider next the split superalgebras $P_2(K)$ and $Q_2(K)$. 
The even part of $P_2(\mathcal{K})$ and of $Q_2(\mathcal{K})$ is isomorphic to $\mathcal{M}_2(\mathcal{K})^\perp$. Since $\mathcal{M}_2(\mathcal{K})^\perp$ is the Jordan algebra of a quadratic form, the determinant, the $K$-forms of $\mathcal{M}_2(\mathcal{K})^\perp$ are the Jordan algebras $J(V, Q)$, $Q$ a nondegenerate quadratic form on a $K$-vector space $V$ of dimension 3. We consider $J(V, Q)$ as embedded in its special universal envelope, the Clifford algebra $C(V, Q)$. Let $v_1, v_2, v_3$ be an orthogonal basis of $V$. Since $V$ is of dimension 3, the center of $C(V, Q)$ is $K[c]$, where $c = v_1v_2v_3$. Moreover $c^2 = \delta 1 \in K$, where $\delta \in K/K^2$ is the discriminant of $Q$. $C(V, Q)$ is a quaternion algebra $Q$ over $K[c]$. Let $*$ be the main involution of $C(V, Q)$. Then $c^* = -c$ so $*$ is of the second kind and $J(V, Q) = \mathcal{H}(Q, *)$. Moreover $C(V, Q)$ is isomorphic to the tensor product of $K[c] \otimes_K Q'$, where $Q'$ is a quaternion algebra with center $K$ [7, Theorem VII 2]. The main involution is $* |_{K[c]} \otimes_\mathbb{R}^*$, where $\tilde{\cdot}$ is the canonical involution of $Q'$.

Let $J = A + M$ be a $K$-form of $Q_2(\mathcal{K})$. So $A = \mathcal{H}(Q, c)$ and $M$ is the regular bimodule. As in the proof of Lemma 13, we let $M = \tilde{A}$. Since this holds for $Q_2(\mathcal{K})$, $[1, \tilde{A}] = \{0\}$. If $a, b \in Q_0$ are orthogonal with respect to the norm form of $Q'$ then $ab \in Q_0$ is orthogonal to $a$ and $b$ and, since this holds in $Q_2(\mathcal{K})$, $[\tilde{a}, \tilde{b}]$ is a scalar multiple $\lambda$ of $ab$. Arguing as in the proof of Lemma 13, if we scale the bracket on $M$ so that $[\tilde{a}, \tilde{b}] = ab$, we have $[\tilde{x}, \tilde{y}] = \frac{1}{2}(xy - yx)$ for all $\tilde{x}, \tilde{y} \in M$. Therefore $J \cong \mathcal{H}(A, *)$, where $A = Q + Qu$, $u$ an odd symmetric central element of $A$ with $u^2 = \lambda$.

If $J = A + M$ is a $K$-form of $P_2(\mathcal{K})$, again $A$ may be viewed as $\mathcal{H}(Q, c)$ as above but $M$ is the sum of a one-dimensional and a three-dimensional bimodule (if it were irreducible then, by Lemma 13, $J$ would be a form of $Q_2(\mathcal{K})$). As was noted in the proof of Proposition 15, $J(V, Q)$ has a bimodule of dimension 3 if and only if $c^2 = \rho^2 1$, $\rho \in K$. Therefore the center of $Q$ is isomorphic to $K \otimes K$ so $Q = Q' \oplus Q''$, where $Q'$ is a quaternion algebra over $K$. The involution $*$ on $Q$ is given by $(a, b)^* = (b, \tilde{a})$, where $\tilde{\cdot}$ is an involution of the first kind of $Q'$. We may extend $*$ to $B = \mathcal{M}_2(Q')$ by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ \tilde{c} & \tilde{a} \end{pmatrix}.
\]

Then \[\mathcal{H}(B, *) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2n}(K) \mid d = \tilde{a}, \tilde{b} = -b, \tilde{c} = c \right\}\]
is a $K$-form of $P_2(\mathcal{K})$. Arguing as in the proof of Proposition 15, we see that, up to isomorphism, any $K$-form of $P_2(\mathcal{K})$ is of this form.

Extending the definition for algebras, we say that a special Jordan (super)algebra $J$ is reflexive if it is closed under tetrads, that is, whenever $J \subseteq A^+$, $A$ an associative (super)algebra, then, for arbitrary elements $a_\alpha, b_\beta, c_\gamma, d_\delta \in J$, the tetrad

\[\left\{ a_\alpha b_\beta c_\gamma d_\delta := a_\alpha b_\beta c_\gamma d_\delta + (-1)^{a_\beta + a_\gamma + a_\delta + b_\gamma + b_\delta + c_\delta} d_\delta c_\gamma b_\beta a_\alpha \in J.\right\}
\]

A homogeneous element $a_\alpha$ of the special Jordan superalgebra $J$ is said to be a tetrad-eater if any tetrad of homogeneous elements of $J$ with $a_\alpha$ as one of its entries is necessarily an element of $J$. The tetrad-eaters form a subsuperalgebra which is invariant under inner
derivations [16]. By extension a polynomial which always evaluates to a tetrad-eater will also be called a tetrad-eater.

Lemma 31. Let \( J = A + M \) be a simple unital finite dimensional special Jordan superalgebra over the field \( K \). If \( A = A' \oplus A'' \), a direct sum of simple Jordan algebras at least one of whose summand is of dimension greater than 1, then \( J \) is reflexive.

Proof. Extending the base field if necessary, we may assume that \( K \) is algebraically closed. Assume further that \( \dim_K A'' \geq 2 \). Since the characteristic of \( K \) is different from 2, to show that \( J \) is reflexive, arguing as in [16], it suffices to show that a tetrad-eater does not vanish on \( J \). If the even part contains a subalgebra isomorphic to \( H_3(K) \) then we may refer to [16,24] where it is shown that this is indeed the case. Thus if \( A'' \) is of capacity 3 or more then the result for algebras extends to superalgebras.

So assume that \( A'' \) is of capacity 2. Let \( e_1 \) be the identity element of \( A' \) and \( e_2, e_3 \) orthogonal idempotents of \( A'' \). The algebra \( A'' = K(e_2 + e_3) + V \) is the algebra of a non-degenerate quadratic form on a vector space \( V \), \( \dim_K V \geq 2 \). We may choose an element \( v \in \{ e_2 A'' e_3 \} \subseteq V \) such that \( v^2 = e_2 + e_3 \). By Lemma 7,

\[
M = \{ e_1, M, e_2 + e_3 \} = M_{12} + M_{13}, \quad M_{12} = \{ e_1, M, e_2 \}, \quad M_{13} = \{ e_1, M, e_3 \}.
\]

We claim that there exist elements \( x_{12}, y_{12} \in M_{12} \) such that \([x_{12}, y_{12}] \cdot e_2 \neq 0 \) or elements \( x_{13}, y_{13} \in M_{13} \) such that \([x_{13}, y_{13}] \cdot e_3 \neq 0 \). For otherwise the projection of \([M, M] \) on \( A'' \), is \( D(A'', A'') \)-invariant. Since \( V \) does not contain proper \( D(A'', A'') \)-invariant subspaces and \([e_2 A'' e_3] \neq V \), it follows that \([M, M] \subseteq A' \) and \( A' + M \) is a superideal of \( J \), a contradiction. We may therefore suppose that \( x_{12}, y_{12} \in M_{12} \) with \([x_{12}, y_{12}] = a_1 + a_2, a_1 \in A', 0 \neq a_2 \in A'' \).

The polynomial

\[
p_{12}(x, y, z) := (z D^2_{x,y})^2 D_{x,y} = z D^2_{x,y} \circ z D^3_{x,y}
\]

while not a hearty tetrad-eater is nonetheless a tetrad-eater [16, §14.1], [24]. Passing to the Grassmann envelope of the free special Jordan superalgebra, we see that if \( x \) and \( y \) are even variables and \( z_1 \) and \( z_2 \) are odd variables then the polynomial

\[
p_{12}(x, y; z_1, z_2) := [z_1 D^2_{x,y}, z_2 D^2_{x,y}] D_{x,y}
\]

is a tetrad-eater in any special Jordan superalgebra.

We wish to evaluate \( p_{12}(v, e_3, x_{12}, y_{12}) \). We have

\[
 x_{12} D_{v,e_3} = \frac{1}{2} x_{12} R_v,
\]

\[
 x_{12} D_{v,e_3}^2 = -\frac{1}{4} x_{12} (R_v)^2 = -\frac{1}{16} x_{12} R_v^2 = -\frac{1}{16} x_{12}.
\]
Similarly $y_{12}D_{v,e_3}^2 = -\frac{1}{16}y_{12}$ and
\[
p_{12}(v, e_3, x_{12}, y_{12}) = 2^{-8}(a_1 + a_2)D_{v,e_3} = 2^{-8}(a_2 \cdot v) \cdot e_3 = 2^{-8}a_2 \cdot v = w.
\]
Note that $a_2 \cdot v \neq 0$ for otherwise $a_2(R_v)^2 = 0$. However
\[
a_2(R_v)^2 = \frac{1}{2}(a_2U(v) + a_2 \cdot v^2) \in A''U_{e_3} + \frac{1}{2}a_2 \neq 0.
\]
Since the tetrad-eaters form a subalgebra which is invariant with respect to derivations, it follows that all elements from $V$ are tetrad-eaters and $e_2 + e_3$ is a tetrad-eater. If the capacity of $A'$ is greater or equal 2 then, as above $A'$ consists of tetrad-eaters. If $A' = Ke_1$ then $e_1 = 1 - (e_2 + e_3)$ is a tetrad-eater and, finally, any element $z_{1i} \in M_{1i}$, $i = 2, 3$, can be written $z_{1i} = 4e_1D(z_{1i}, e_i)$ and is a tetrad-eater. Thus $J$ is composed entirely of tetrad-eaters and so is reflexive.

\textbf{Corollary 32.} If $J$ is a simple finite dimensional Jordan superalgebra over a field $K$ of characteristic not 2 such that $J$ is a form of $\mathcal{H}(B, \ast)$ an associative superalgebra with superinvolution over $\overline{K}$ the algebraic closure of $K$ then $J = \mathcal{H}(A, \ast)$ for $(A, \ast)$ a finite dimensional simple associative superalgebra with superinvolution $\ast$ over $K$.

\textbf{Proof.} We have established the result for $M_{1,1}(\overline{K})$, $P_2(\overline{K})$ and $P_2(\overline{K})$, separately. Lemma 31 applies to all other special $\overline{K}$-superalgebras of the form $\mathcal{H}(B, \ast)$ in our classification.

If $(A, \ast)$ is a central simple associative superalgebra with superinvolution over $K$ then $\mathcal{H}(A, \ast)$ is a central simple Jordan superalgebra unless $A_0$ is commutative [2, Theorem 2]. The simple finite dimensional associative superalgebras with superinvolution over an arbitrary field $K$ were classified in [20]. It remains to determine what they become under extension to $\overline{K}$. We first recall results of [20] for fields of characteristic not 2 which we need to determine forms.

Finite dimensional central simple associative superalgebras over $K$ are isomorphic to $\text{End} V \cong M_n(D)$, where $D = D_0 + D_1$ is a finite dimensional associative division superalgebra, i.e., all nonzero elements of $D_\alpha$, $\alpha = 0, 1$, are invertible, and $V = V_0 + V_1$ is an $n$-dimensional $D$-superspace. If $D_1 = \{0\}$, the grading of $M_n(D)$ is induced by that of $V = V_0 + V_1$, $A = M_{p+q}(D)$, $p = \dim_D V_0$, $q = \dim_D V_{a+1}$, so $p + q$ is a nontrivial decomposition of $n$. While if $D_1 \neq \{0\}$ then the grading of $M_n(D)$ is given by $M_n(D)_\alpha := M_n(D_\alpha)$.

\textbf{Division Superalgebra Theorem [20].} If $D = D_0 + D_1$ is a finite dimensional central associative division superalgebra over $K$ then exactly one of the following holds where throughout $E$ denotes a finite dimensional central division algebra over $K$.

(i) $D = D_0 = E$, and $D_1 = \{0\}$.
(ii) $D = E \otimes_K K[u]$, $a^2 = \lambda \in K^\times$, $D_0 = E \otimes K1$, $D_1 = E \otimes Ku$,
(iii) $\mathcal{D} = \mathcal{E}$ or $\mathcal{M}_2(\mathcal{E})$, $u \in \mathcal{D}$ such that $u^2 = \lambda \in K/K^2$, $\mathcal{D}_0 = C_{\mathcal{D}}(u)$, $\mathcal{D}_1 = S_{\mathcal{D}}(u)$, where $C_{\mathcal{D}}(u) = \{d \in \mathcal{D} \mid du = ud\}$, $S_{\mathcal{D}}(u) = \{d \in \mathcal{D} \mid du = -ud\}$, moreover, in the second case, $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $K[u]$ does not embed in $\mathcal{E}$.

Tensoring with the algebraic closure $\bar{K}$ of $K$ and letting $d$ be the degree of the division algebra $\mathcal{D}_0$ over its center, we obtain

(i) $\mathcal{D} \otimes \bar{K} \cong M_d(\bar{K})$, with the trivial grading,
(ii) $\mathcal{D} \otimes \bar{K} \cong M_d(\bar{K}) \otimes \bar{K}[u]$, with $u^2 = 1$,
(iii) $\mathcal{D} \otimes \bar{K} \cong M_{d+d}(\bar{K})$.

Example. The real division superalgebras are

(i) The reals $\mathbb{R}$, the complexes $\mathbb{C}$ and the Hamiltonian quaternions $\mathbb{H}$ with the trivial grading.
(ii) For a fixed even part $\mathcal{D}_0 = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, we have two choices of $\mathcal{D}_1 = \mathcal{D}_0 u$ corresponding to $u^2 = \pm 1$. Strictly speaking $\mathbb{C} + \mathbb{Cu}$ is a complex superalgebra and $u$ can be chosen with $u^2 = 1$ but we will need it when we consider superinvolutions.
(iii) Since $\mathbb{R}[u]$, the center of $\mathcal{D}_0$, is a quadratic extension of $\mathbb{R}$, it must be $\mathbb{C}$. Hence $\mathcal{D}_0 = \mathbb{C}$ and $\mathcal{D}$ is of dimension 4 over $\mathbb{R}$. So $\mathcal{D}$ is a quaternion algebra, $\mathcal{M}_2(\mathbb{R})$ or $\mathbb{H}$. In the split case, we may take $u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, while in the division case we may take $u = i$, where $\{1, i, j, k\}$ is the standard basis of $\mathbb{H}$. The odd part $\mathcal{D}_1 = S_{\mathcal{D}}(u) = \mathcal{D}_0 u$ with $v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $j$, as the case may be. We may view $\mathcal{D}$ as being obtained from $\mathbb{C}$ by the Cayley–Dickson process. So $\mathcal{D}$ must be $\mathcal{M}_2(\mathbb{R})$ or $\mathbb{H}$.

If $(A, \ast)$ is a simple associative superalgebra with superinvolution then either $A = B \oplus B^*$ with $B$ a simple associative superalgebra or $A$ is a simple associative superalgebra.

In both cases, the underlying division superalgebra $\mathcal{D}$ has a superinvolution and $\ast$ is induced by a nondegenerate hermitian superform on an appropriate superspace $V$.

If $A$ is not simple as a super-ring $A = B \oplus B^*$ with $B \cong M_{n_d}(\mathcal{D})$ a simple super-ring and $d$ equals the degree of $\mathcal{D}_0$ then $\mathcal{H}(A, \ast) \cong B^+$, a form of

(i) $M_{p,q}(\bar{K})$ if $\mathcal{D}$ is as in (i), where $p = d(dim V_d)$, $q = d(dim V_{d+1})$.
(ii) $Q_{nd}(\bar{K})$ if $\mathcal{D}$ is as in (ii),
(iii) $M_{nd,nd}(\bar{K})$ if $\mathcal{D}$ is as in (iii).

We now assume that $A$ is simple. So $A = M_n(\mathcal{D})$ and we deal with each possibility for $\mathcal{D}$ separately. If $\mathcal{D}_1 = \{0\}$, assume first that $(\mathcal{A}_0, \ast|_{\mathcal{A}_0})$ is simple (as an algebra with involution) then $n = 2p$, $A = M_2(A)$, $A = M_p(\mathcal{D})$ is a simple associative algebra with involution $\overline{\ast}$ and the superinvolution $\ast$ is given by

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\ast = \begin{pmatrix} \overline{a} & -\mu \overline{b} \\ \overline{\mu} \overline{c} & \overline{a} \end{pmatrix},
$$

where $\mu \in K$, such that $\mu \overline{\mu} = 1$ [20, Proposition 13]. Recall that an involution is of the first or second kind according as it fixes the center of $A$ or not. If $\overline{\ast}$ is of the first kind, as
was noted in [20, Proposition 13], \( \mu \) can be chosen equal to 1. If \( \gamma \) is of the second kind, then, by Hilbert’s Theorem 90, \( \mu = \gamma \gamma^{-1} \) for some \( \gamma \) in the center of \( \mathcal{A} \). Identifying

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a & \gamma b \\ \gamma^{-1} c & d \end{pmatrix},
\]

we see that in that case also \( \mu \) may be chosen equal to 1. Therefore in both cases

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ \tilde{c} & \tilde{a} \end{pmatrix}
\]

and

\[
\mathcal{H}(\mathcal{A}, \ast) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \in \mathcal{A}, b \in \mathcal{S}(\mathcal{A}, \gamma) \right\}.
\]

(I) If \( \gamma \) is of the first kind then \( \mathcal{H}(\mathcal{A}, \ast) \) is a form of \( \mathcal{P}_n(\overline{K}) \), where \( \dim_K A = n^2 \) while if \( \gamma \) is of the second kind then \( \mathcal{H}(\mathcal{A}, \ast) \) is a form of \( \mathcal{M}_{n \times n}(\overline{K}) \), where \( \dim_K A = n^2 \).

(1) If \( (\mathcal{A}_0, \ast|_{\mathcal{A}_0}) \) is not simple (as an algebra with involution) then \( \mathcal{A} = \mathcal{M}_{p+q}(\mathcal{D}) \). Letting \( \mathcal{A}_1 = \mathcal{M}_p(\mathcal{D}) \) and \( \mathcal{A}_2 = \mathcal{M}_q(\mathcal{D}) \), \( (\mathcal{A}_1, \ast|_{\mathcal{A}_1}) \) and \( (\mathcal{A}_2, \ast|_{\mathcal{A}_2}) \) are of the same kind and if they are of the first kind then one is of orthogonal type and the other of symplectic type [20, Proposition 14]. In the latter case we have a form of \( \mathcal{osp}_{n,2m}(\overline{K}) \), where \( n \) and \( 2m \) are the square roots of the dimension of the appropriate \( \mathcal{A}_1 \)'s. If \( \ast \) is of the second kind then \( \mathcal{H}(\mathcal{A}, \ast) \) is a form of \( \mathcal{M}_{n \times m}(\overline{K}) \), where \( \dim_K A_1 = 2n^2 \) and \( \dim_K A_2 = 2m^2 \).

We consider next \( \mathcal{A} = \mathcal{M}_n(\mathcal{D}) \), with \( \mathcal{D}_1 \neq [0] \), and denote by \( \gamma \) the restriction \( \ast|_{\mathcal{A}_0} \).

We also identify \( \mathcal{D} \) with \( \mathcal{D} \) multiples of the identity matrix.

(2) In case (ii), we may choose \( u \) such that \( u^* = u, \lambda = -\lambda \) [20, Proposition 9] and

\[
(a + bu)^* = \tilde{a} + \tilde{b} u, \quad a, b \in \mathcal{D}_0.
\]

In particular \( \gamma \) is of the second kind. Since we are classifying Jordan superalgebras over \( K \), we will assume that \( \mathcal{K} \) is the symmetric part of the center of \( \mathcal{D}_0 \), i.e., the center of \( \mathcal{D}_0 \) is \( K[\lambda] \). In this way \( \mathcal{H}(\mathcal{A}, \ast) = \mathcal{H}(\mathcal{A}_0, \gamma) + \mathcal{H}(\mathcal{A}_0, \gamma) u \) is a Jordan superalgebra over \( K \).

This is a form of \( \mathcal{Q}_n(\mathcal{K}) \), where \( \dim_K A_0 = 2n^2 \).

(III) Let \( \mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 \) be a division superalgebra as in case (iii). If \( \ast \) is a superinvolution, let \( \gamma \) be the restriction of \( \ast \) to \( \mathcal{D}_0 \). Then there exists a nonzero \( v \in \mathcal{D}_1 \) such that \( v^* = v, v^2 = d \in \mathcal{D}_0 \) with \( d = -d \). Moreover if for \( a \in \mathcal{D}_0, a^\phi := va \gamma^{-1} \) is an outer automorphism whose restriction to the center of \( \mathcal{D}_0 \) is of order 2. The map \( \ast \phi \) is an involution,

\[
(a + bv)^* = \tilde{a} + \tilde{b} \phi v, \quad a, b \in \mathcal{D}_0
\]

and

\[
\mathcal{H}(\mathcal{D}, \ast) = \mathcal{H}(\mathcal{D}_0, \gamma) + \mathcal{H}(\mathcal{D}_0, \gamma \phi) v.
\]

If \( \gamma \) is of the first kind then \( \ast \phi \) is of the second kind and vice versa.
Assume first that $\sim$ is of the first kind. If $E = H(K, \sim)$ then $D_0 \otimes_E K \cong B \oplus B$ and the involution $\sim \otimes 1$ is the identity on the center of $B \oplus B$. Therefore $\sim \otimes 1$ stabilises each summand of $B \oplus B$ and $(D \otimes_E K, \ast \otimes 1)$ is as in (I') above. So one of the summand of $B \oplus B$ has an orthogonal involution and the other a symplectic involution. Thus $\dim_K H(B \oplus B, \sim \otimes 1) = n(n + 1)/2 + n/(n - 1)/2 = n^2$. This is impossible since $\dim_K H(D_0 \otimes_E K, \sim \otimes 1) = 2 \dim_K H(D_0, \sim)$.

Now if $\sim$ is of the second kind then

$$a + bv^{-1} \mapsto a^\theta + bv^{-1}, \quad a, b \in D_0,$$

is another superinvolution on $D$ whose restriction to $D_0$ is of the first kind and will lead to the contradiction above.

We are now ready to summarize our results concerning forms in the

**Second Classification Theorem.** Let $J = A + M$ be a finite dimensional central simple Jordan superalgebra over an arbitrary field $K$ of characteristic not 2. If $A$ is semisimple and $M \neq \{0\}$ then $J$ is a form of one of the superalgebras listed in the first classification theorem. These can be described as follows.

(i) All forms of $K_3$, the Kaplansky superalgebra, are split.

(ii) All forms of the Kac superalgebras, $K_{10}$, char. $K \neq 3$, are of the form $K_{10}(d)$ and forms of $K_9$, char. $K = 3$, are subsuperalgebras of codimension 1 in $K_{10}(d)$.

(iii) Forms of $D_{l+1}$ are isomorphic to $J(C, v), C$ a composition algebra of dimension 2 over $K$, $v \in C^\times$.

(iv) The superalgebra of a nondegenerate superform.

(v) Forms of $M_{n+m}(K)$ are isomorphic to $M_{p+q}(D)^+$ with $D$ a division superalgebra as in (i) or (iii), or to $H(A, \ast)$, where $A \cong M_{p+q}(D)$ with $D$ a division superalgebra as in (i), $\ast$ of the second kind.

(vi) Forms of $osp(n, 2m)$ are isomorphic to $H(A, \ast)$, where $A \cong M_{p+q}(D)$ with $D$ a division superalgebra as in (i), $\ast$ of the first kind and $H(A_0, \ast)$ not simple.

(vii) Forms of $P_n$ are isomorphic to $H(A, \ast)$, where $A \cong M_k(D)$ with $D$ a division superalgebra as in (i) with $\ast$ of the first kind and $(A_0, \ast | A_0)$ simple.

(viii) Forms of $Q_{n+1}$ are isomorphic to $A^+$ or $H(A, \ast)$, where $A \cong M_k(D)$ with $D$ a division superalgebra as in (ii).

(ix) $H(B, \ast) + \overline{S}(B, \ast) \oplus \overline{S}(B, \ast)$, $(B, \ast)$ a central simple associative algebra of degree 3 with involution of the first kind over a field $K$ of characteristic 3.

(x) All forms of $H_3(B)$, char. $K = 3$, are split.

This result can be stated in a more appealing way.

**Theorem.** Let $J = A + M$ be a finite dimensional central simple Jordan superalgebra over an arbitrary field $K$ of characteristic not 2. If $A$ is semisimple and $M \neq \{0\}$ then $J$ is isomorphic to one of the following superalgebras:

(i) $K_3$, the Kaplansky superalgebra.
(ii) a Kac superalgebra, $K(d)$ if char $K \neq 3$, or a subsuperalgebra of codimension 1 in $K(d)$ if char $K = 3$.

(iii) $J(C, v)$ a composition algebra of dimension 2 over $K$, $v \in C^\times$.

(iv) a superalgebra of a nondegenerate superform.

(v) $\mathcal{H}(A, \ast)$, $(A, \ast)$ a simple finite dimensional associative superalgebra with superinvolution with $A_1 \neq \{0\}$.

(vi) $\mathcal{H}(B, \ast) + \overline{S}(B, \ast) \oplus \overline{S}(B, \ast)$, $(B, \ast)$ a central simple associative algebra of degree 3 with involution of the first kind over a field $K$ of characteristic 3.

(vii) $\mathcal{H}_3(B)$, char $K = 3$, are split.

Example. We end by applying this classification to obtain the real forms of the finite dimensional complex simple Jordan superalgebras with semisimple even part. We assume as known the classification of simple real associative algebras with involution as can be found for example in [6, Chapter X, Section 7].

(i) The only real form of $K_3$, the Kaplansky superalgebra, is the split one.

(ii) Besides the split Kac superalgebra, $K_{10}$, there is a nonsplit one $K_{10}(-1)$ corresponding the quadratic form with Witt index 1 and discriminant $-1$.

(iii) Real forms of $D_t$, $t \in C$, exist if and only if $t \in R$ or the complex norm of $t \in C$, $n(t) := it$, equals 1. In the first case we have a real $D_t$ while in the second case $J(C, v)$, $v \in C$ with $v \bar{v}^{-1} = t$ is a real form of $D_t$. For $t \in R^+$, since $D_t \cong D_{t^{-1}}$ we have non-isomorphic $D_t$, $-1 < t < 0$, $0 < t \leq 1$. For $t \in C$ with $n(t) = 1$, we have $t = v \bar{v}^{-1}$. For $\lambda \in R$, $n(\lambda v) = \lambda^2 n(v)$, we need only consider $J(C, v)$ with $n(v) = 1$ and since $J(C, v) \cong J(C, \bar{v})$ we have non-isomorphic $J(C, v)$, $n(v) = 1$, $v$ in the upper half plane, i.e., $v - \bar{v} \geq 0$.

(iv) The real forms of a superalgebra of a nondegenerate complex superform correspond to real superforms on a superspace $V = V_0 + V_1$ of the appropriate dimension and these are determined up to isomorphism by the isometry class of the real quadratic form on $V_0$. If dim $V_0 = n$, there are $n + 1$ such forms.

To determine the forms of the remaining infinite families, we need the real associative division superalgebras with superinvolution. We have already listed the real associative division superalgebras in the previous example. We now determine, up to isomorphism, the superinvolution on these.

(i) The identity map is an involution of $R$ and $C$. We also have the standard involution $\bar{\,}$ on $C$ and $H$. This last involution is the symplectic involution. $H$ also supports orthogonal involutions. Up to isomorphism, there is only one orthogonal involution, for example, $a1 + bi + cj + dk \mapsto a1 - bi + cj + dk$.

(ii) Since $\ast | D_0$ is of the second kind, $D_0 = C$, $\ast | D_0 = \bar{\,}$ and $u^2$ is a real multiple of $i$. Replacing $u$ by an appropriate real multiple of $u$, we may choose $u$ such that $u^2 = \pm i$.

(iii) In both cases $D \cong C + Cv$, with $v^2 = d \in C$. If $D$ has a superinvolution $\ast$ then $d^\ast = -d$. Since $v^2 \in R1$, this is impossible and such a division superalgebra over the reals does not admit a superinvolution.

We now consider the simple real superalgebras of the form $\mathcal{H}(A, \ast)$. 


(v) Real forms of $\mathcal{M}_{n,m}(C)$ are isomorphic to $\mathcal{M}_{n,m}(R), \mathcal{M}_{n/2,m/2}(H)$ if $n$ and $m$ are even, or to $\mathcal{H}(A, *)$, where $A \cong M_{n+m}(C)$ with $*$ of the second kind. Consider the hermitian superforms on $V = V_0 + V_1$ over $C$. Since the grading can be interchanged without changing the isomorphism class of $M_{n+m}(C)$, we may assume that $\dim V_0 \geq \dim V_1$. Since a hermitian superform can be multiplied by a non-zero real without changing the induced superinvolution, we have $\left\lfloor \frac{n}{2} \right\rfloor + 1$ non-isomorphic $(A, *)$.

(vi) Real forms of $osp_{n,2m}(C)$ are isomorphic to $\mathcal{H}(A, *)$, where $A \cong M_{p+q}(D)$ with $D = R$ or $H$. If $A = M_{n+2m}(R)$, since the skew-symmetric form on $V_1$ is unique, the superinvolution $*$ is determined by a symmetric bilinear form on $V_0$ and there are $\left\lfloor \frac{n}{2} \right\rfloor + 1$ such forms. If $n$ is even, superinvolutions of $M_{n/2+m}(H)$ are induced by a skew hermitian form on $V_0$ and a hermitian form on $V_1$. Since the first is essentially unique, arguing as above, there are $\left\lfloor \frac{m}{2} \right\rfloor + 1$ such $(A, *)$.

(vii) Real forms of $P_n(C)$ are isomorphic to $\mathcal{H}(A, *)$, where $A \cong M_{2k}(D)$ with $D = R$ or $H$ with $*$ induced by an involution $\tilde{\tau}$ of the first kind on $M_k(D)$, $(A_0, * | A_0)$ simple. On $M_n(R)$, there are $\left\lfloor \frac{n}{2} \right\rfloor + 1$ orthogonal involutions. If $n$ is even there is also the symplectic involution. It does not give anything new since the superalgebra of symmetric elements is isomorphic to those under the superinvolution by the transpose involution, interchanging symmetric and skew elements. On $M_{n/2}(H)$, we have $\left\lfloor \frac{n}{4} \right\rfloor + 1$ involutions which induce the standard involution on $H$ and one which induces a non-standard involution on $H$.

(viii) Real forms of $Q_n(C)$ are isomorphic to $A^+$ or $\mathcal{H}(A, *)$, where $A \cong M_k(D)$ with $D$ as in (ii).

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References