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ON INTERSECTIONS OF INTERVAL GRAPHS

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If one can associate with each vertex of a graph an interval of a line, so that two intervals intersect just when the corresponding vertices are joined by an edge, then one speaks of an *interval graph*.

It is shown that any graph on v vertices is the intersection ("product") of at most $[\frac{1}{2}v]$ interval graphs on the same vertex set.

For $v = 2k$, k factors are necessary for, and only for, the complete k-partite graph $K_{2,2,\ldots,2}$.

Some results for the hypergraph generalization of this question are also obtained.

1. Introduction

Given several undirected simple graphs on the same vertex set V, their *edge product* or "intersection" is the graph in which two vertices in V are joined by an edge just when they are so joined in all of the given ("factor") graphs.

An *interval graph* is a graph to each vertex of which can be associated a compact interval of $\mathbf{R}$ so that two vertices are joined by an edge just when the corresponding intervals intersect.

With these definitions, one can state

Theorem 1. (a) *Every graph on v vertices is the edge product of $[\frac{1}{2}v]$ or fewer interval graphs.*

(b) *For $v = 2k$, k factors are necessary for, and only for, the complete k-partite graph $K_{2,2,\ldots,2}$.*

This subject was suggested by Chvátal's investigations [1] of finite families of finite sets with the Helly property. Helly's theorem on convex sets [2, p. 117] states that if a finite family of more than d convex sets in $\mathbf{R}^d$ has the property that any $d+1$ of the sets have a common point, then the whole family has a common point. For $d = 1$, this states an obvious property of intervals. This property extends at once to "bricks" in $\mathbf{R}^d$, that is, to cartesian products of intervals of the d coordinate axes: for arbitrary d, if any 2 bricks of a finite family intersect then the whole family intersects.

A finite family of finite sets has the "Helly-1 property" when any subfamily whose members intersect pairwise has nonvoid intersection. The relation between these notions follows from Theorem 1:

Corollary 1. *A finite family of v finite sets has the Helly-1 property if and only if there is a d and for each set a brick in $\mathbf{R}^d$ such that any subfamily intersects just when the corresponding bricks do.*

Postponing the proof of Theorem 1, we prove the corollary: If such a brick representation exists then the Helly-1 property follows. Conversely, the pairwise intersections of sets in the family determine a graph. By Theorem 1, this graph is the edge product of d interval graphs.[1] The cartesian product in $\mathbf{R}^d$ of the interval representations of the factor graphs is by construction a correct brick representation of pairwise intersections. The Helly-1 property uniquely determines all higher intersections as a function of the pairwise intersections. Bricks always have this property and the given family has it by assumption. □

There is a natural extension of this question to hypergraphs, i.e. finite families of $(k+1)$-tuples from a finite set V. Call such a hypergraph *convex* if one can associate to each element of V a convex set in $\mathbf{R}^k$ such that a $(k+1)$-tuple is in the hypergraph just when the corresponding convex sets intersect.

Theorem 2. *Every hypergraph of $(k+1)$-tuples on v vertices is the intersection of no more than $\frac{1}{2}\binom{v}{k}$ convex hypergraphs on the same vertex set.*

This is only an upper bound, of order v^k, on the number of factors required. For $k=2$, (systems of triples) we obtain a lower bound of the same order.

2. The graph case

Lemma 1. *If vertices a and b of graph G are not joined by an edge and the graph G', obtained from G by deleting a, b and the edges incident on them, is a product of d interval graphs, then G can be obtained as a product of $d+1$ interval graphs.*

Proof. By assumption there is a brick representation of G' in $\mathbf{R}^d$ with vertices v_i mapped to bricks B_i'. Let B_0' be a brick in $\mathbf{R}^d$ meeting all the B_i'. Add one coordinate axis and represent G as follows: For $v_i \in G'$ not joined to either a or b in G, take $B_i = B_i' \times [-1, 1]$. For $v_i \in G$ joined to a but not to b in G, take $B_i = B_i' \times [-1, 3]$. For $v_i \in G'$ joined to b but not to a in G, take $B_i = B_i' \times [-3, 1]$. For $v_i \in G'$ joined to both a and b, take $B_i = B_i' \times [-3, 3]$. For a take $B_0 \times [2, 3]$ and for b take $B_0 \times [-3, -2]$. This realizes G by bricks in $\mathbf{R}^{d+1}$ thus as product of $d+1$ interval graphs. □

[1] Only the fact that a finite d exists is needed. This can be seen directly from the observation that a complete graph with one edge deleted is an interval graph. Thus Theorem 1 is not essential here.

This gives part (a) of Theorem 1 by induction. For $v \leq 3$ all graphs are interval graphs by inspection. Assume Theorem 1(ā) true for up to v-2 vertices and consider G on v vertices. Either $G = K_v$, the complete graph, which is an interval graph, or else there is an unjoined pair, a, b of vertices. Their deletion gives a graph G' realizable with $[\frac{1}{2}(v-2)] = [\frac{1}{2}v] - 1$ or fewer factors, so that G is a product of no more than $[\frac{1}{2}v]$ interval graphs by Lemma 1. □

Part (b) of Theorem 1 is true for $k = 2$ since one can easily check that the only graph with $v \leq 4$ that is not an interval graph is the square (4-cycle)$K_{2,2}$.

Lemma 2. *The complete k-partite graph $K_{2,2,\ldots,2}$ on 2k vertices cannot be obtained as a product of fewer than k interval graphs.*

Proof. Suppose it could be. Let a_i, b_i $(i = 1, \ldots, k)$ be the unjoined vertex pairs. For each such pair the edge a_ib_i must be absent in at least one factor. By the pigeonhole principle some factor has two such edges missing say a_1b_1 and a_2b_2. The four other edges joining these four vertices are in this factor as they are in the product.

As the factor must be an interval graph, this gives an interval representation of the square, which is impossible. □

It remains to show that if a graph G on $2k$ vertices requires k factors, it is isomorphic to $K_{2,2,\ldots,2}$. As this holds for $k = 2$, assume $k > 2$ and the assertion proved up to $k-1$. If G requires k factors it is certainly not the complete graph (which is an interval graph). So there are unjoined vertices, say a, b. Let G' be the graph left when a, b are deleted. If G' were the product of fewer than $k-1$ factors, Lemma 1 would contradict the assumption on G. Thus by the induction hypothesis G' is a $K_{2,2,\ldots,2}$ on $2k-2$ vertices. If all possible edges between $\{a, b\}$ and G' are in G then G is a $K_{2,2,\ldots,2}$. If not, let (a_i, b_i), $i = 1, \ldots, k-1$ be the unjoined pairs of G' and suppose w.l.o.g. that aa_1 is not in G. As $k > 2$ we could have removed from G (a_2, b_2) instead of (a, b) and this removal also must leave at $K_{2,2,\ldots,2}$ which requires (a, a_1) to be present. This contradiction completes the proof of Theorem 1. □

3. The hypergraph case

For a hypergraph of $(k+1)$-tuples, i.e. a (regular)[2] $(k+1)$-hypergraph, the property of having a representation by convex sets in $\mathbf{R}^k$ leaves some freedom. For indeed, knowing which families of $k+1$ of the sets have nonvoid intersection determines intersections of all larger families (by Helly's theorem) while the intersections of families of 2 to k of the sets are not fully determined. Therefore, we define a *tight convex representation* of a $(k+1)$-hypergraph as the association,

[2]"Regular" refers to the fact that all hyperedges have the same cardinality. As all the hypergraphs considered here are such, the qualification will be dropped in the sequel.

to each vertex, of a compact convex body[3] of $\mathbf{R}^k$ such that

(i) any $k+1$ of the bodies intersect iff the corresponding $(k+1)$-tuple is a hyperedge of the $(k+1)$-hypergraph, and

(ii) any k of the bodies have a common point.

We prove the following strengthening of Theorem 2.

Theorem 2'. *Every $(k+1)$-hypergraph on v vertices is the intersection of no more than $\frac{1}{2}\binom{v}{k}$ $(k+1)$-hypergraphs with tight convex representations.*

Proof. For $v > k > 0$, denote by $d_k(v)$ the maximum number of factors required for $(k+1)$-hypergraphs on v vertices (it will turn out to be finite). From Theorem 1. we have $d_1(v) = [\frac{1}{2}v] \le \frac{1}{2}\binom{v}{1}$. Also, $d_k(k+1) = 1 \le \frac{1}{2}\binom{k+1}{k}$ for all $k > 0$: if the only $k+1$ tuple intersects, one can take all bodies the same, if not, take the bodies to be the facets of a k-dimensional simplex. Either way, it is a tight convex representation. Theorem 2 will thus follow by induction if one can show that

$$d_k(v+1) \le d_k(v) + d_{k-1}(v).$$

Consider a $(k+1)$-hypergraph $G = (V, E)$ with vertex set $V = \{a_0, \dots, a_v\}$ and set E of $(k+1)$-tuples from V. Let $V' = V - \{a_0\}$, $E' = E|V'$ the $(k+1)$-tuples from E that are in V'. Then $G' = (V', E')$ is a $(k+1)$-hypergraph on v vertices and can be represented by a product of $p \le d_k(v)$ tight convex $(k+1)$-graphs G_i', $i = 1, \dots, p$. Let C_{ij} be the convex body of R^k representing a_j in G_i', $j = 1, \dots, v$; $i = 1, \dots, p$. For each i, let C_{i0} be a convex body of $\mathbf{R}^k$ covering all the C_{ij} $(j = 1, \dots, v)$, say their convex hull.

By definition, for any k-tuple t of V' and any factor $i = 1, \dots, p$; the sets C_{ij} with $j \in t$ have a common point. For $(k+1)$-tuples this is the case for *all* i iff the $(k+1)$-tuple is in E'.

Let E'' be the set of k-tuples in V' such that, with the addition of a_0, they belong to E. Then $G'' = (V', E'')$ is a k-hypergraph, and as such can be represented as an intersection of $q \le d_{k-1}(v)$ tight convex k-hypergraphs G_i'' $(i = 1, \dots, q)$.

For each $j = 1, \dots, v$ there is a convex body D_{ij} in $\mathbf{R}^{k-1}$ such that for every $(k-1)$-tuple t of j's and fixed i, the $\{D_{ij}\}_{j \in t}$ have a common point, while for a k-tuple this is true for *all* i iff that k-tuple is in E''.

In each of the q factor representations add a coordinate axis and let P_i be a point ($\ne$ origin) on this axis. Let $C_{p+i,j}$ $(i = 1, \dots, q;\ j = 1, \dots, v)$ be the convex hull in $\mathbf{R}^{k+1}$ of $D_{i,j}$ and P_i. Let $C_{p+i,0}$ be the convex hull of the union of the $D_{i,j}$, $(j = 1, \dots, v)$, this is a convex set of $\mathbf{R}^{k+1}$ contained in the original $\mathbf{R}^k$.

Then the C_{ij} $(i = 1, \dots, p+q;\ j = 0, \dots, v)$ form a representation of G as a product of $p+q$ tight convex k-graphs on V. For indeed, it is shown below that

(a) in each factor, all k-tuples intersect,

[3] Here a convex body is a nonvoid compact convex set. An interior is not required.

(b) all $(k+1)$-tuples in E intersect in each factor, and

(c) all other $(k+1)$-tuples fail to intersect in some factor.

Proof of (a). If a_0 is not in the k-tuple then the representing sets intersect in factors $1, \ldots, p$ by construction and in the points P_i of the other factors. If a_0 is in the k-tuple, the k-1 other sets intersect in factors $i = 1, \ldots, p$ and are covered in each by C_{i0}, while in the q other factors the $k-1$ corresponding $D_{i,j}$ already intersect by construction and are covered by $C_{p+i,0}$.

Proof of (b). For a $(k+1)$-tuple not containing a_0 the representing sets intersect by construction in the first p factors and in the points P_i in the other q factors. If a_0 is involved, then in the first p factors the k other sets intersect by construction and are covered by C_{i0}. In the q other factors the k sets D_{ij} already meet by construction and are covered by $C_{p+i,0}$.

Proof of (c). If a_0 is not involved the sets representing the $(k+1)$-tuple have a void intersection in one of the first p factors.

If a_0 belongs to the $(k+1)$-tuple then, by construction, the intersection of the corresponding D_{ij} is void in one of the last q factors, as $C_{p+i,0}$ covers only the D_{ij} (and none of the rest of $C_{p+i,j}$) the intersection of the $k+1$ representing sets $C_{p+i,j}$ (j in $(k+1)$-tuple) is void for that factor.

This completes the induction. □

Turning now to the question of lower bounds, one first needs a generalization of the fact that that $K_{2,2}$ is not an interval graph.

Consider two finite sets A, B such that $|A| = |B| = k+1$, $|A \cap B| < k$. If $V = A \cup B$, then $v = |V|$ satisfies $k+3 \leq v \leq 2(k+1)$.

Lemma 3. *For A, B, V as above, consider the $(k+1)$-hypergraph $G = (V, E)$ in which E consists of all $(k+1)$-tuples on V except A and B. Then G has no tight convex representation.*

Proof. For $k = 1$ this reduces to the fact that $K_{2,2}$ is not an interval graph. If the lemma were false there would be a counterexample G, with minimum k, having a representation by convex bodies $C_i (i = 1, \ldots, v)$ in $\mathbf{R}^k$. If $v < 2(k+1)$ there is an i, say $i = v$, common to A and B, then let $S = C_v$ and let $V' = \{1, \ldots, v-1\}$. Otherwise A and B are disjoint, take one element from each, say $v-1$ from A and v from B then let $S = C_{v-1} \cap C_v$ and let $V' = \{1, \ldots, v-2\}$. In either case $v' = |V'|$ and $k' = k-1$ will satisfy $k'+3 \leq v' \leq 2(k'+1)$. By the tight representation property one has

(i) S is a nonvoid compact convex set

(ii) For all $(k+1)$-tuples from V' the corresponding $k+1$ sets C_i have a common point.

(iii) The intersection of S and k of the $C_i (i \in V')$ is nonvoid *unless* all the indices of the C_i belong to the same one of the sets A or B; there are exactly two such k-tuples A' and B', with $V' = A' \cup B'$.

By Helly's theorem there is a point $P \notin S$ common to all the $C_i (i \in V')$. Let Π be a hyperplane strictly separating P from S.

For $i \in V'$ let $T_i = S \cap C_i$ and let R_i be the intersection of Π with the convex hull of $T_i \cup \{P\}$. When some of the T_i have a common point the construction produces a common point of the corresponding R_i. Conversely, when some R_i's have a common point X, then on the ray from P through X there are points from each of the corresponding T_i and the one nearest to P belongs to all of them.

Thus the R_i $(i \in V')$ form a tight convex representation of the k-hypergraph $G' = (V', E')$ in which E' consist of all k-tuples from V' except A' and B'. This contradicts the minimality of k and completes the proof. □

Let $G(V, E)$ be a 3-hypergraph in which E consists of all triples from V *except* for a Steiner system. Such G thus exist for $v = |V|$ congruent to 1 or 3 modulo 6.

Theorem 3. *For the above 3-hypergraphs, a representation by tight convex 3-hypergraphs requires at least* $\frac{1}{3}\binom{v}{2}$ *factors.*

Proof. A Steiner triple system on V consists of $\frac{1}{3}\binom{v}{2}$ triples. Each of these triples must be missing in at least one factor. If the theorem were false the pigeonhole principle implies that in some factor at least two of the Steiner triples, say A and B, would be missing. The Steiner condition implies that $A \cup B$ contains no other triple of the system. Then the representation of this factor would provide, in particular, a representation of the 3-hypergraph on $A \cup B$ in which only triples A and B are missing. This, however, contradicts the $k = 2$ case of Lemma 3. □

As the maximum required number of factors $d_k(v)$ is clearly monotone non-decreasing in v, one has

$$\frac{1}{3}\binom{v^*}{2} \leq d_2(v) \leq \frac{1}{2}\binom{v}{2}$$

where v^* is the largest integer, not exceeding v, which is congruent to 1 or 3 modulo 6. Thus $d_2(v)$ is of order v^2.

Remark. The use of *compact* convex sets for the representations is only a convenience. It can easily be shown that the classes of intersection patterns realizable by convex sets in $\mathbf{R}^d$ that are (i) compact, (ii) compact with interior, (iii) open, and (iv) general, are exactly the same.

References

[1] V. Chvátal, Private Communication, 1978.
[2] J. Stoer, and C. Witzgall, Convexity and Optimization in Finite Dimensions (Springer-Verlag, Berlin, 1970).