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INTRODUCTION

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1. Introduction

If one were to browse through the table of contents of several well-known books on general graph theory, for example:

"Theorie der Endlichen and Unendlichen Graphen" by D. König (Leipzig, 1936 and Chelsea, New York, 1954),
"Theory of Graphs and its Applications" by C. Berge (Dunod, Paris, 1958 and Methuen, London, 1962),
"Theory of Graphs" by O. Ore (American Mathematical Society Colloquium Publications 38, Providence, RI, 1962),
"Graph Theory" by F. Harary (Addison-Wesley, Reading, MA, 1968),
"Introduction to the Theory of Graphs" by M. Behzad and G. Chartrand (Allyn and Bacon, Boston, 1971),
"Graphs and Hypergraphs" by C. Berge (North-Holland, Amsterdam, 1975),
"Graph Theory with Applications" by J. A. Bondy and U.S.R. Murty (North-Holland, Amsterdam, 1976),
"Graph Theory, An Introductory Course" by B. Bollobás (Springer, Berlin, 1976),
"Graphs and Digraphs" by M. Behzad, G. Chartrand and L. Lesniak-Foster (Woodsworth and Brooks/Cole, 1979),
"Graph Theory" by W.T. Tutte (Encyclopedia of Mathematics and its Applications, Vol. 21, Addison-Wesley, Reading, MA, 1984),

one would find a variety of areas of study, on many of which entire books have been written, for example:

"Connectivity in Graphs" by W.T. Tutte (Toronto Univ. Press, Toronto, 1967),
"Graphical Enumeration" by F. Harary and E.M. Palmer (Academic Press, New York, 1973),
"Extremal Graph Theory" by B. Bollobás (Academic Press, London, 1978),
"Topics on Perfect Graphs", C. Berge and V. Chvátal, eds. (Annals of Discrete

Mathematics, Vol. 21, North-Holland, Amsterdam, 1983),
"Cycles in Graphs", B.R. Alspach and C.D. Godsil, eds. (Annals of Discrete Mathematics, Vol. 27, North-Holland, Amsterdam, 1985),
"Matching Theory" by L. Lovasz and M.D. Plummer (Annals of Discrete Mathematics, Vol. 29, North-Holland, Amsterdam, 1986),
"Planar Graphs: Theory and Algorithms", T. Nishizeki and N. Chiba, eds. (Annals of Discrete Mathematics, Vol. 32, North-Holland, 1988).

With the publication of this volume, we can add Domination in Graphs to this list.

The most common definition given of a *dominating set* is that it is a set of vertices $D \subseteq V$ in a graph $G = (V, E)$ having the property that every vertex $v \in V - D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ is the cardinality of a smallest dominating set of G.

2. History

The earliest ideas of dominating sets, it would seem, date back to the origin of the game of chess in India over 400 years ago, in which one studies sets of chess pieces which cover or dominate various opposing pieces or various squares of the chessboard.

In more recent times the Eight Queens and Five Queens Problems rekindled interest in dominating concepts, e.g., in the books of Ahrens* in 1901 and König* in 1936. Finally with the publications of the books by Berge* in 1958 and Ore* in 1962 the topic of domination was given formal mathematical definition. But by 1972 relatively little work had been done on this topic until Cockayne and Hedetniemi began to study it and ultimately published a 1975 survey of the results that had been obtained by that time. This seems to have brought the subject sufficiently into focus to set research on a much wider scale into motion. In the thirteen years since then over 300 papers have been published on the subject, and in a very real sense domination theory has arrived. Hence this volume.

We divide the contributions in this volume into three sections, entitled 'theoretical', 'new models' and 'algorithmic'. The nine theoretical papers retain a primary focus on properties of the standard domination number $\gamma(G)$. The four papers which we classify as 'new models' are concerned primarily with new variations on the domination theme. The eight algorithmic papers are primarily concerned with finding classes of graphs for which the domination number, and several other domination-related parameters, can be computed in polynomial time.

* All bibliographic citations in this introduction can be found in the Bibliography on Domination in Graphs at the end of this volume, by first author and date of publication.

3. Theoretical

For a variety of reasons we lead off this volume with the paper "Chessboard domination problems" by Cockayne, because of the historical roots of domination in the game of chess, and because Cockayne has done the most definitive work in this area. The follow up paper "On the queen domination problem" by Grinstead, Hahne and Vanstone contains what we believe to be the best approximation to the old problem of placing a minimum number of queens on an arbitrary $n \times n$ chessboard so that all squares are 'covered' by at least one queen.

We are pleased to be able to present next a reprint of a paper by Berge and Duchet entitled "Recent problems and results about kernels in directed graphs", which surveys work that has been done for dominating sets in directed graphs, called kernels. Claude Berge has done more than anyone, we think, to establish the mathematical foundations not only of graph theory in general, but in particular of domination theory. To Claude we extend both our appreciation for his willingness to contribute to this volume and our apologies for not adopting his terminology of "Coefficient of external stability" and choosing instead "The domination number".

David Sumner was one of the early researchers in domination theory and was perhaps the first one to consider the question of domination critical graphs. In this paper "Critical concepts in domination" he considers the problem of characterizing graphs for which adding any edge e decreases the domination number, i.e. $\gamma(G+e) < \gamma(G)$. He also considers the problem of characterizing graphs having minimum dominating sets D which are independent, i.e. no two vertices in D are adjacent.

A related notion, by Fink, Jacobson, Kinch and Roberts in "The bondage number of a graph", is that of finding a set of edges F of smallest order (called the bondage number), whose removal increases the domination number, i.e. $\gamma(G-F) > \gamma(G)$.

It was in a 1978 paper by Cockayne, Hedetniemi and Miller that the following chain of inequalities appeared:

$$\mathrm{ir} \leqslant \gamma \leqslant i \leqslant \beta_0 \leqslant \Gamma \leqslant \mathrm{IR},$$

where ir and IR are the irredundance and upper irredundance numbers, respectively, γ and Γ are the domination and upper domination numbers and i and β_0 are the independent domination and vertex independence numbers. This sequence has been the focus of a large number of papers since then. Among these was a paper by Bollobás and Cockayne in which they proved that for bipartite graphs, $\beta_0 = \Gamma = \mathrm{IR}$. The present paper "Chordal graphs and upper irredundance, upper domination and independence" by Jacobson and Peters expands considerably upon this by presenting several other classes of graphs for which

$\beta_0 = \Gamma = \mathrm{IR}$. We note that this problem has since been made considerably richer by some, as yet unpublished, new results of Cheston and Fricke.

In their original survey paper on domination Cockayne and Hedetniemi introduced the domatic number of a graph, denoted $d(G)$, which equals the maximum order of a partition $\{V_1, V_2, \ldots, V_R\}$ of $V(G)$ such that every set V_i is a dominating set. Today Zelinka has become the world's foremost authority on the domatic number and a variety of related partition numbers. He has published nearly two dozen papers on this topic. We are pleased to have a contribution from Zelinka entitled "Regular totally domatically full graphs" and another from Rall, entitled "Domatically critical and domatically full graphs", on the domatic number of a graph.

We complete the theoretical section of this volume with what we consider to be a particularly noteworthy and significant contribution to domination theory by Bollobás, Cockayne and Mynhardt, who challenge our understanding of the fundamental notion of a minimal dominating set, by introducing the new and challenging notion of a k-minimal dominating set in their article "On generalized minimal domination parameters for paths".

4. New models

The concepts of domination, covering and centrality are so interrelated and so general that it is not at all surprising that so many different types of domination exist; e.g. in a 1985 paper, Hedetniemi, Hedetniemi and Laskar list 30 different types of domination. As of now we know of twice as many types of domination problems which have been studied. Definitions of many of these can be found in the introduction to the Bibliography on Domination at the end of this volume. In this section entitled New Models we present a small sample of some of the newer domination problems currently being studied.

The paper "Dominating cliques in graphs" by Cozzens and Kelleher, studies the existence of families of graphs which contain a complete subgraph whose vertices form a dominating set. They present several forbidden subgraph conditions which are sufficient to imply the existence of dominating cliques and they present a polynomial algorithm for finding a dominating clique for a certain class of graphs.

The paper "Covering all cliques of a graph" by Tuza considers a different kind of domination, in which one seeks a minimum set of vertices which dominates all cliques (i.e. maximal complete subgraphs) of a graph.

The paper by Brigham and Dutton entitled "Factor domination in graphs" considers, among other things, the general problem of finding a minimum set of vertices which is a dominating set of every subgraph in a set of edge-disjoint subgraphs, say $G_1, G_2, \ldots, G_t$, whose union is a given graph G.

The final paper in this section "The least point covering and domination numbers of a graph" by Sampathkumar is one of many papers in which one imposes additional conditions on a dominating set, e.g. the dominating set must induce a connected subgraph (connected domination), a complete subgraph (dominating clique), or a totally-disconnected graph (independent domination). In Sampathkumar's paper the domination number of the subgraph induced by the dominating set must be minimized.

5. Algorithmic

The algorithmic study of domination has exploded onto the scene even more suddenly than the theoretical study of domination. Nearly 100 papers containing domination algorithms or complexity results have been published in the last 10 years; we add another eight papers in this section.

Perhaps the first domination algorithm was an attempt by Daykin and Ng in 1966 to compute the domination number of an arbitrary tree; we say 'attempt' because their algorithm seems to have an error that cannot be easily corrected. Cockayne, Goodman and Hedetniemi apparently constructed the first domination algorithm for trees in 1975 and, at about the same time, David Johnson constructed the first [unpublished] proof that the Domination problem for arbitrary graphs is NP-complete. Since then domination algorithms, of ever increasing sophistication, have been published at a steady rate. We are pleased to present an excellent collection of algorithmic papers on domination in this section.

The first paper by Corneil and Stewart entitled "Dominating sets in perfect graphs" presents both a brief survey of algorithmic results on domination and a discussion of the dynamic-programming-style technique that is commonly used in designing domination algorithms, especially as they are applied to the family of perfect graphs.

The paper "Unit disk graphs" by Clark, Colbourn and Johnson discusses the algorithmic complexity of such problems as domination, independent domination and connected domination, and several other problems, on the intersection graphs of equal size circles in the plane. We think this paper is especially significant since it contains the result that the Domination problem for grid graphs, a subclass of unit disk graphs, is NP-complete. The family of grid graphs includes arbitrary subgraphs of grids as well. As far as we know, however, the complexity of the Domination problem on arbitrary $m \times n$ *complete* grids is still not known.

The paper "Permutation graphs: Connected domination and Steiner trees" by Colbourn and Stewart considers a third class of graphs. To-date a nice variety of NP-complete problems have been shown to have polynomial solutions when restricted to permutation graphs. To this collection of problems we can now add connected domination.

Almost as if by coincidence, it seems, we received the paper on "The discipline number of a graph" by Chvátal and Cook, soon after we had received the paper on "The bondage number of a graph" by Fink, Jacobson, Kinch and Roberts (in the theoretical section of this volume). In their paper, Chvátal and Cook address the question of the computational complexity of the bondage number and show, among other things that it can be formulated as an integer linear program. This paper also provides an example of the relatively recent study of fractional (i.e. real-valued) parameters of graphs. These are the values obtained by real relaxations of the integer linear programs corresponding to various graphical parameters like domination, matching, covering and independence.

Our next paper, "Best location of service centers in a tree-like network under budget constraints" by McHugh and Perl, provides both a nice applications perspective on domination and an illustration of the many papers that have been published on the topic of centrality in graphs. It also provides an example of a pseudo-polynomial domination algorithm and another example of the dynamic programming technique applied to domination problems.

The paper "Dominating cycles in Halin graphs" by Skowrońska and Sysło, discusses both a fourth class of graphs on which polynomial time domination algorithms can be constructed, and the notion of a dominating cycle, i.e. a cycle C in a graph such that every vertex not in C lies at distance at most one from some vertex in C.

The following paper, "Finding dominating cliques efficiently, in strongly chordal graphs and undirected path graphs" by Kratsch is an algorithmic mate of the paper by Cozzens and Kelleher on dominating cliques (in the new models section of this volume). It discusses two more classes of graphs that permit polynomial domination algorithms, in this case, for finding dominating cliques of minimum size.

We conclude the algorithmic section of this volume with a paper "On minimum dominating sets with minimum intersection" by Grinstead and Slater, which is a good representative of the fast developing area of polynomial, and even linear, algorithms on partial k-trees. Grinstead and Slater introduce a difficult, new type of problem, prove that it is in general NP-complete, and give a linear time solution when restricted to trees. This solution also uses a dynamic programming-style approach and a methodology created by Wimer in his 1987 Ph.D. Thesis.

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There can be little doubt that this special issue really has many editors; many people have contributed their time and energy to produce this collection of papers and have in the process left their mark on what is contained herein. We

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Barkauskas, A.; Bascó, B.; Berge, C.; Boland, J.; Bollobás, B.; Brigham, R.C.; Chandrasekharan, N.; Cheston, G.; Chvátal, V.; Clark, B.N.; Cockayne, E.J.; Colbourn, C.J.; Cook, W.J.; Corneil, D.G.; Cozzens, M.B.; Duchet, P.; Dutton, R.; Fink, J.F.; Grinstead, C.M.; Grinstead, D.L.; Hahne, B.; Harary, F.; Hare, E.O.; Hare, W.R.; Jacobson, M.S.; Johnson, D.S.; Kelleher, L.L.; Kinch, L.F.; Kratsch, D.; Livingston, M.; Majumdar, A.; McHugh, J.; Mynhardt, C.M.; Peters, K.; Proskurowski, A.; Rall, D.; Ringeisen, R.D.; Roberts, J.; Sampathkumar, E.; Skowrońska, M.; Slater, P.J.; Stewart, L.K.; Suffel, C.L.; Summer, D.P.; Sysło, M.; Tuza, Z.; Vanstone, D.; Wimer, T.; Zelinka, B.

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