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Skew-adjacency matrices of graphs

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ABSTRACT

The spectra of the skew-adjacency matrices of a graph are considered as a possible way to distinguish adjacency cospectral graphs. This leads to the following topics: graphs whose skew-adjacency matrices are all cospectral; relations between the matchings polynomial of a graph and the characteristic polynomials of its adjacency and skew-adjacency matrices; skew-spectral radii and an analogue of the Perron–Frobenius theorem; and the number of skew-adjacency matrices of a graph with distinct spectra.

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1. Introduction

Given two simple graphs[5] whose adjacency matrices have the same spectrum, what additional information is sufficient to distinguish the graphs?

For example, if $ab = cd$ and $a + b = m + c + d$, where a, b, c, d, m are positive integers then the complete bipartite graph $K_{a,b}$ is (adjacency) cospectral with $K_{c,d} + \overline{K}_m$, the complete bipartite graph $K_{c,d}$ together with m isolated vertices. These graphs may be distinguished by the spectra of their complements, $\overline{K_{a,b}} = K_a + K_b$ and $\overline{K_{c,d} + \overline{K}_m} = (K_c + K_d) \vee K_m$. For, -1 is an eigenvalue of the former complement with multiplicity $a+b-2$ and of the latter with multiplicity $a+b-3$. However, there are many examples of cospectral strongly regular graphs (see, e.g. [3]) and these cannot be distinguished by the spectra of their complements because cospectral regular graphs have cospectral complements.

As an additional test to distinguish a graph, consider the spectra of its set of skew-adjacency matrices; that is, of the set of skew-symmetric $\{0, 1, -1\}$-matrices derived from its adjacency matrix $A = [a_{i,j}]$ by negating one of $a_{i,j}$, $a_{j,i}$ for each unordered pair ij.

Fig. 1 (from [3]) shows all pairs of adjacency cospectral graphs on six vertices. Each graph in the first row is adjacency cospectral with the graph below it. The skew-adjacency matrices of a graph G all have the same spectrum if and only if G has no cycles of even length (Theorem 4.2). We call such a graph an *odd-cycle graph*. All but the second pair of graphs have skew-adjacency matrices with different spectra because one of the graphs is an odd-cycle graph and the other is not.

It is known (and shown in Lemma 5.3) that the coefficients of the characteristic polynomial of the skew-adjacency matrices of a graph are the absolute values of those of its adjacency matrix if and only if the graph is a forest. Thus, two forests are adjacency cospectral if and only if some (or all) of their skew-adjacency matrices are cospectral. In particular, the second pair of graphs in the figure have the same adjacency spectra and the same (unique) skew-adjacency spectra.

It is not clear how often it would be practical or effective to distinguish graphs by the spectra of their derived sets of skew-adjacency matrices, but, as we have just seen, addressing that question leads to interesting results.

Section 2 reviews relations between coefficients of a characteristic polynomial and collections of vertex disjoint directed cycles in a weighted digraph. The relations are specialized to the case of adjacency matrices in Section 3. These relations and those for other matrices of graphs may be found in [8].

In Section 4, the skew co-spectral characterization of odd-cycle graphs is proved (Theorem 4.2). Eq. (8) is the key to that result and most of the other results in this section. It expresses the coefficient s_k of x^{n-k} in the characteristic polynomial $p_S(x)$ of a skew-adjacency matrix S in terms of vertex disjoint collections of edges and even cycles of G that cover k vertices. In particular, if G is an odd-cycle graph, it implies that s_k is the number of matchings in G that cover k vertices.

Section 5 explores relations between the characteristic polynomials of adjacency matrices and skew-adjacency matrices. It is observed there that G is an odd-cycle graph if and only if the coefficients of the characteristic polynomials of *all* of its skew-adjacency matrices are the absolute values of the coefficients of its matchings polynomial. It is not known if this equivalence is still true if the coefficient condition holds for *some* skew-adjacency matrix of G (Problem 1).

Section 6 contains groundwork for an investigation of $\rho_S(G)$, the maximum value of the spectral radii of the skew-adjacency matrices of a graph G. It is not known that G must be an odd-cycle graph if all of its skew-adjacency matrices have the same spectral radius (Problem 2). Also, we conjecture that if G is an odd-cycle graph on n vertices whose skew-adjacency matrices have the greatest spectral radius, then G has a vertex joined to all others (Conjecture 6.1 and following comment). Together with Remark 6.1, Lemma 6.3 may be regarded as an analogue of the Perron–Frobenius theorem, one with nonnegative matrices replaced by those skew-signings of a symmetric nonnegative matrix with zero trace for which the spectral radius is maximum.

Section 7 contains bounds on the number of skew-adjacency matrices of a graph that have distinct spectra.

[5] Terminology in the introduction that is not defined later may be found, e.g., in [22].

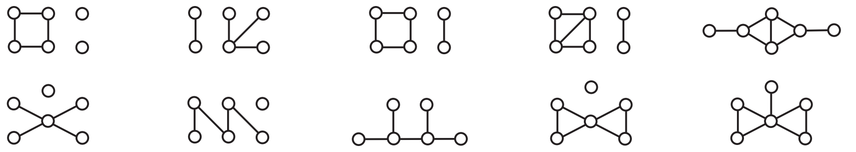


Fig. 1. The adjacency cospectral graphs on six vertices.

2. Characteristic polynomials from weighted digraphs

Given an $n \times n$ matrix $A = [a_{i,j}]$, let $\overrightarrow{G}(A)$ be the arc-weighted digraph on the vertex set $V = [n] = \{1, 2, \ldots, n\}$ with arc set $E(\overrightarrow{G}) = \{(i, j) : a_{i,j} \neq 0\}$ and weight $a_{i,j}$ assigned to arc (i, j). An example is given in Fig. 2.

When $t > 1$, a *dicycle* of length t is a digraph with a vertex set $\{i_1, i_2, \ldots i_t\}$ and arcs (i_k, i_{k+1}), $1 \leq k < t$ and (i_t, i_1). A dicycle of length $t = 1$ is a *loop* (i_1, i_1). For example, in the arc-weighted digraph $\overrightarrow{G}(A)$ in Fig. 2, there is a dicycle of length 1 (or *loop*) at vertex 1, dicycles of length 2 (or *digons*) on each of the vertex sets $\{1, 2\}$, $\{2, 3\}$, $\{2, 4\}$ and a dicycle of length 3 on the vertex set $\{2, 3, 4\}$.

Let $\overrightarrow{\mathcal{U}_k}$ denote the set of all collections $\overrightarrow{U}$ of vertex disjoint dicycles in $\overrightarrow{G}(A)$ (including loops and digons) that cover precisely k vertices of $\overrightarrow{G}(A)$. For $\overrightarrow{U} \in \overrightarrow{\mathcal{U}_k}$, let $e(\overrightarrow{U})$ denote the number of dicycles in $\overrightarrow{U}$ of even length (including digons) and let $\Pi_{\overrightarrow{U}}(A) = \Pi_{(i,j)\in E(\overrightarrow{U})} a_{i,j}$.

Let the characteristic polynomial of A be denoted by

$$p_A(x) = \det(xI - A) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n. \tag{1}$$

Then $(-1)^k a_k$ is equal to the sum of the $k \times k$ principal minors of A. Because dicycles of even length are associated with permutations with negative sign (see, e.g. [2, p. 45]), it follows that

$$a_k = (-1)^k \sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}_k}} (-1)^{e(\overrightarrow{U})} \Pi_{\overrightarrow{U}}(A) = \sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}_k}} (-1)^{|\overrightarrow{U}|} \Pi_{\overrightarrow{U}}(A), \tag{2}$$

where $|\overrightarrow{U}|$ denotes the number of dicycles in $\overrightarrow{U}$. In particular, A has determinant

$$\det A = (-1)^n a_n = (-1)^n \sum_{\overrightarrow{U} \in \overrightarrow{\mathcal{U}_n}} (-1)^{|\overrightarrow{U}|} \Pi_{\overrightarrow{U}}(A). \tag{3}$$

For example, applying (2) and (3) to the arc-weighted digraph $\overrightarrow{G}(A)$ for the 4×4 matrix A above, we see that $\det A = adfh$ and

$$p_A(x) = x^4 - ax^3 + (-bc - de - gh)x^2 + (ade + agh - dfh)x + adfh.$$

3. Characteristic polynomials of adjacency matrices

If G is a simple graph with vertex set $V = [n] = \{1, 2, \ldots, n\}$ and edge set $E(G)$, the *adjacency matrix* of G is the $n \times n$ symmetric $\{0, 1\}$-matrix $A = A(G)$ with $a_{i,j} = 1$ if $ij \in E(G)$ and $a_{i,j} = 0$ if $ij \notin E(G)$. In particular, each diagonal entry of A is 0.

A *routing* $\overrightarrow{U}$ of a vertex disjoint collection U of cycles and (isolated) edges in a simple graph G is obtained by replacing each of the cycles in U by a dicycle and each edge in U by a digon. Thus, if $c(U)$ denotes the number of cycles in U, then U has $2^{c(U)}$ routings.

If A is a symmetric $\{0, 1\}$-matrix with zero diagonal, then A is the adjacency matrix of an (undirected) simple graph $G = G(A)$. The digraph $\overrightarrow{G}(A)$ defined earlier is the doubly-directed graph obtained from $G(A)$ by replacing each edge by a digon and giving each arc a weight of 1. Thus, the summands in (2)

$$A = \begin{bmatrix} a & b & 0 & 0 \\ c & 0 & d & g \\ 0 & e & 0 & f \\ 0 & h & 0 & 0 \end{bmatrix}$$

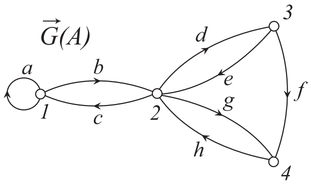


Fig. 2. A square matrix and its associated arc-weighted digraph.

may be grouped according to the members U of the set $\mathcal{U}_k$ of all collections of (undirected) vertex disjoint edges and cycles in G (of length 3 or more) that cover k vertices. Here dicycles of length 3 or more in $\overrightarrow{G}(A)$ are associated with undirected cycles in $G(A)$, digons in $\overrightarrow{G}(A)$ are associated with edges in $G(A)$ and there are no loops in $\overrightarrow{G}(A)$ since A has zero diagonal. Each U in $\mathcal{U}_k$ accounts for $2^{c(U)}$ summands in (2), one for each routing $\overrightarrow{U}$ of U. Thus, if A is the adjacency matrix of a simple graph G, then the characteristic polynomial (1) of A has coefficients

$$a_k = \sum_{U \in \mathcal{U}_k} (-1)^{k+e(U)+m(U)} 2^{c(U)} = \sum_{U \in \mathcal{U}_k} (-1)^{|U|} 2^{c(U)}, \tag{4}$$

where $e(U)$ is the number of even cycles in U, $m(U)$ is the number of disjoint edges in U, $c(U)$ is the number of cycles in U, and $|U|$ is the number of components of U. (See also [8, p. 32,10, p. 20,2, p. 45].)

A *matching* in G on k vertices is a set $M = \{i_1 i_2,\ i_3 i_4, \ldots, i_{k-1} i_k\}$ of vertex disjoint edges in G. A matching M in G is *perfect* if each vertex in G is in some edge of M. If the edge set of $U \in \mathcal{U}_k$ is a matching on k vertices, then k is even, $|U| = k/2$, $c(U) = 0$, and the summand $(-1)^{|U|}2^{c(U)}$ in (4) simplifies to $(-1)^{k/2}$. Thus, if $m_k(G)$ denotes the number of matchings in G that cover k vertices, then $m_k(G) = 0$ if k is odd and the coefficient formula (4) may be rewritten as

$$a_k = (-1)^{k/2} m_k(G) + \sum_{\substack{U \in \mathcal{U}_k, \\ c(U) > 0}} (-1)^{|U|} 2^{c(U)}, \tag{5}$$

where $(-1)^{\frac{k}{2}} m_k(G) = 0$ if k is odd. In particular,

$$(-1)^n \det A = a_n = (-1)^{n/2} m_n(G) + \sum_{\substack{U \in \mathcal{U}_n, \\ c(U) > 0}} (-1)^{|U|} 2^{c(U)}. \tag{6}$$

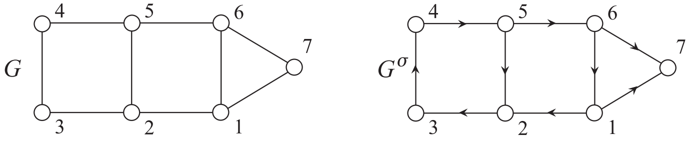
For example, if A is the adjacency matrix of C_n, the cycle on n vertices, then $\det A = 2$ if n is odd and $\det A = 2\left((-1)^{n/2} - 1\right)$ if n is even.

4. Characteristic polynomials of skew-adjacency matrices

An *orientation* of a simple (undirected) graph G is a sign-valued function σ on the set of ordered pairs $\{(i,j), (j,i) \mid ij \in E(G)\}$ that specifies an orientation (or direction) to each edge ij of G. If $ij \in E(G)$, we take $\sigma(i,j) = 1$ when $i \to j$ and $\sigma(i,j) = -1$ when $j \to i$. The resulting oriented graph is denoted by G^σ. Both σ and G^σ are called orientations of G.

The *skew-adjacency* matrix $S^\sigma = S(G^\sigma)$ of G^σ is the $\{0, 1, -1\}$-matrix with (i,j)-entry equal to $\sigma(i,j)$ if $ij \in E(G)$ and 0 otherwise. If there is no confusion, we simply write $S = [s_{i,j}]$ for S^σ. Thus $s_{i,j} = 1$ if $(i,j) \in E(G^\sigma)$, -1 if $(j,i) \in E(G^\sigma)$, and 0 otherwise. An example is shown in Fig. 3.

To obtain the characteristic polynomial of S, we require the arc-weighted digraph $\overrightarrow{G}(S)$. Because $S^\top = -S$, $\overrightarrow{G}(S)$ will be doubly-directed and each digon will be skew-signed: one arc will be weighted 1, and one arc weighted -1. For the example of G^σ and S above, $\overrightarrow{G}(S)$ is shown in Fig. 4.



$$S=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -1 & 1\\ -1 & 0 & 1 & 0 & -1 & 0 & 0\\ 0 & -1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & -1 & 0 & 1\\ -1 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Fig. 3. The skew-adjacency matrix S of an orientation σ of a simple graph G.

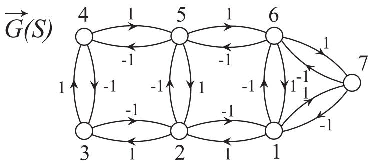


Fig. 4. The $\{-1, 1\}$-arc-weighted doubly-directed digraph of a skew-adjacency matrix.

Recall that $\mathcal{U}_k$ denotes the set of all collections U of (undirected) vertex disjoint edges and cycles (of length 3 or more) in G that cover k vertices, and that a routing $\overrightarrow{U}$ of $U \in \mathcal{U}_k$ is obtained by replacing each edge in U by a digon and each cycle in U by a dicycle.

If σ is an orientation of a simple graph G and $\overrightarrow{U}$ is a routing of $U \in \mathcal{U}_k$, let $\sigma(\overrightarrow{U}) = \Pi_{(i,j)\in E(\overrightarrow{U})}\sigma(i,j)$. We say that $\overrightarrow{U}$ is *positively oriented* (resp. *negatively oriented*) relative to σ if $\sigma(\overrightarrow{U})$ equals 1 (resp. -1), or, equivalently, if an even (resp. odd) number of arcs in $\overrightarrow{U}$ have an orientation that is opposite to that in G^σ. For example, if U is a single edge, then $\overrightarrow{U}$ is a digon and $\sigma(\overrightarrow{U}) = -1$ since one arc of a digon always disagrees with one arc of G^σ. However, if $\overrightarrow{U}$ is a routing of a single cycle U and $\overleftarrow{U}$ is its reversal, then $\sigma(\overleftarrow{U}) = \sigma(\overrightarrow{U})$ if U has even length, while $\sigma(\overleftarrow{U}) = -\sigma(\overrightarrow{U})$ if U has odd length.

If $S = S(G^\sigma)$ is the skew-adjacency matrix of G^σ, then in (2), $\Pi_{\overrightarrow{U}}(S) = \Pi_{(i,j)\in\overrightarrow{U}} s_{i,j} = \Pi_{(i,j)\in\overrightarrow{U}} \sigma(i,j) = \sigma(\overrightarrow{U})$. Also, if the dicycle components (including digons) of $\overrightarrow{U}$ are $\overrightarrow{U_i}$, $i \in [k]$, then $\sigma(\overrightarrow{U}) = \Pi_{i=1}^{k}\sigma(\overrightarrow{U_i})$. Thus, if $S = S(G^\sigma)$ is the skew-adjacency matrix of G^σ and $\overrightarrow{G}(S)$ is the doubly-directed arc-weighted digraph of S, then the summands in (2) over all routings $\overrightarrow{U}$ of a particular U in $\mathcal{U}_k$ will cancel if U contains an odd cycle and will all be equal if U consists only of edges and even cycles.

Let $\mathcal{U}_k^e$ be the set of all members of $\mathcal{U}_k$ with no odd cycles. If σ is an orientation of G and $U \in \mathcal{U}_k^e$, let $c^+(U)$ (resp. $c^-(U)$) denote the number of cycles in U that are positively (resp. negatively) oriented relative to σ when U is given a routing $\overrightarrow{U}$. (Because dicycles in $\overrightarrow{U}$ all have even length, $c^+(U)$ and $c^-(U)$ do not depend on the routing chosen.) Then $c(U) = c^+(U) + c^-(U)$ is the total number of cycles in U and, as before, if $m(U)$ is the number of single edge components of U, then $|U| = c(U) + m(U)$ is the number of components of U. Let $\sigma(U)$ denote the common value of $\sigma(\overrightarrow{U})$ for the routings $\overrightarrow{U}$ of $U \in \mathcal{U}_k^e$. Because each digon associated with an edge in U is negatively oriented, $\sigma(U) = (-1)^{m(U)+c^-(U)} = (-1)^{|U|+c^+(U)}$. It follows from (2) that if the characteristic polynomial of S is

$$p_S(x) = \det(xI - S) = x^n + s_1x^{n-1} + \cdots + s_{n-1}x + s_n,$$

then $s_k = 0$ if k is odd and

$$s_k = \sum_{U \in \mathcal{U}_k^e} (-1)^{|U|} 2^{c(U)} \sigma(U) = \sum_{U \in \mathcal{U}_k^e} (-1)^{c^+(U)} 2^{c(U)} \quad \text{if } k \text{ is even.} \tag{7}$$

If $c(U) = 0$ (i.e., if U is a matching) then $\sigma(U) = (-1)^{|U|}$. Thus, $s_k = 0$ if k is odd and

$$s_k \ = \ m_k(G) + \sum_{\substack{U \in \mathcal{U}_k^e, \\ c(U)>0}} (-1)^{c^+(U)} 2^{c(U)} \quad \text{if } k \text{ is even,} \tag{8}$$

where the sum is taken over all those $U \in \mathcal{U}_k^e$ that have at least one cycle. In particular, $\det S = -s_n = 0$ if n is odd and

$$\det S = s_n \ = \ m_n(G) + \sum_{\substack{U \in \mathcal{U}_n^e, \\ c(U)>0}} (-1)^{c^+(U)} 2^{c(U)}, \quad \text{if } n \text{ is even.} \tag{9}$$

Thus, if the number $m_n(G)$ of perfect matchings in G is odd, then $\det S \neq 0$. The converse statement fails. For example, if S is a skew-adjacency matrix of a negatively oriented even cycle C_n, then $\det S = 4$, but $m_n(C_n) = 2$.

It follows from (8) that if k is even, then

$$s_k \leq \ m_k(G) + \sum_{\substack{U \in \mathcal{U}_k^e, \\ c(U)>0}} 2^{c(U)} \tag{10}$$

with equality if and only if each even cycle in G of length $l \leq k$ that is disjoint from a matching on $k - l$ vertices is negatively oriented relative to σ.

More can be said when $k = n$. Because the union of two distinct perfect matchings of G is a member U of $\mathcal{U}_n^e$ and each $U \in \mathcal{U}_n^e$ with $c(U) > 0$ is determined by $2^{c(U)}$ ordered pairs of perfect matchings, it follows that $m_n(G)(m_n(G) - 1) = \sum_{\substack{U \in \mathcal{U}_n^e, \\ c(U)>0}} 2^{c(U)}$. Thus, when n is even,

$$s_n \leq \ m_n(G) + \sum_{\substack{U \in \mathcal{U}_n^e, \\ c(U)>0}} 2^{c(U)} = m_n(G)^2. \tag{11}$$

A subgraph H of G is termed *nice* [19, p. 125] if $G - V(H)$ has a perfect matching. Note that if $U \in \mathcal{U}_n^e$ and C is a cycle in U, then C must be nice because each of the remaining cycles in U may be replaced by matchings. It follows that when n is even, equality holds in (11) if and only if each nice even cycle in G is negatively oriented relative to σ.

Because S is skew-symmetric, iS is Hermitian and so has real eigenvalues [16, p. 171]. (When not used as an index, i denotes the principal square root of -1.) Thus, S has pure imaginary eigenvalues and, since S has real entries, the eigenvalues occur in complex conjugate pairs. It follows that if S has rank t, then $p_S(x) \ = \ x^{n-t}\Pi_{k=1}^{t/2}(x^2 \ + \ b_k^2)$ for some nonzero scalars b_k. Thus $s_k \ \geq \ 0$ for each k. In particular, $\det S \geq 0$. In fact, $\det S$ is the square of an integer. This follows from a result on the Pfaffian of S (see Eq. (13) and the definition below).

If G is a simple graph with vertex set $V = [n] = \{1, 2, \ldots, n\}$ and edge set $E(G)$, the *generic* skew-adjacency matrix of G is the $n \times n$ skew-symmetric matrix $X(G) = X = [x_{i,j}]$ where the entries $x_{i,j}$ with $i < j$ and $ij \in E(G)$ are independent indeterminates over a field and where $x_{i,j} = 0$ if $ij \notin E(G)$.

If X is a generic skew-adjacency matrix of G, then the *Pfaffian* of X, pf X, is defined by the rule

$$\operatorname{pf} X = \sum_{M \in \mathcal{M}(G)} \operatorname{wt}(X_M), \tag{12}$$

where $\mathcal{M}(G)$ denotes the set of all perfect matchings

$$M = \{i_1 i_2, \ i_3 i_4, \ldots, i_{n-1} i_n\}$$

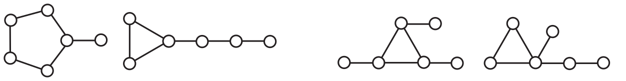


Fig. 5. The only pairs of skew-adjacency cospectral odd-cycle graphs on six or fewer vertices.

in G and where $\mathrm{wt}(X_M)$ is equal to the product $\Pi_{\{i_j,i_{j+1}\}\in M} x_{i_j,i_{j+1}}$ multiplied by the sign of the permutation that takes $(1, 2, \ldots, n)$ to $(i_1, i_2, \ldots, i_n)$. Because X is skew-symmetric, $\mathrm{wt}(X_M)$ is not affected by the order of the edges in M or the order chosen for the vertices of each edge. If n is odd, or if n is even and $\mathcal{M}(G)$ is empty, we take $\mathrm{pf}\, X = 0$.

It is well-known (see, e.g. [6, p. 318]) that

$$\det X = (\mathrm{pf}\, X)^2. \tag{13}$$

Because the entries of X are independent indeterminates, $\det X = (\mathrm{pf}\, X)^2 \neq 0$ if and only if G has a perfect matching (see also [6, pp. 317–323]). Thus, G has a perfect matching if and only if $\mathrm{pf}\, X$ is not identically zero.

In particular, if S is a skew-adjacency matrix of G, then $\det S$ is the square of an integer, and $\det S \geq 0$. Also, if $\det S > 0$ then G must have a perfect matching. However, if G has a perfect matching, it is possible that $\det S = 0$ because of cancellation in $\mathrm{pf}\, S$. But, if the total number of perfect matchings in G is odd (in particular, if G has a unique perfect matching), then $\det S > 0$ for all skew-adjacency matrices S of G.

The *girth* $g(G)$ (resp. *even girth* $g_e(G)$) of a graph G is the length of a shortest cycle (resp. shortest even cycle) in G, if one exists. If G has no cycles (resp. no even cycles) then $g(G)$ (resp. $g_e(G)$) is infinite. Recall that $m_k(G)$ denotes the number of matchings in G that cover precisely k vertices. Thus, $m_k(G) = 0$ if the number of vertices in G is odd. The next lemma follows immediately from formula (8) for s_k.

Lemma 4.1. *In (8), if $1 \leq k < g_e(G)$, then $s_k = m_k(G)$ for all skew-adjacency matrices of G. In particular, if G has no even cycles, then $s_k = m_k(G)$ for all $k \in [n]$, and the skew-adjacency matrices of G all have the same spectrum.*

We have been referring to graphs with no even cycles as *odd-cycle* graphs. A *cactus* is a connected graph each of whose blocks (2-connected subgraphs) is an edge or a cycle. A connected odd-cycle graph is a cactus each of whose blocks is an edge or an odd cycle [4, Ex. 3.2.3]. By comparison, the graphs with no odd cycles (the *even-cycle* graphs) are the bipartite graphs. Graphs with no cycles (the forests) are both even-cycle and odd-cycle graphs.

Example 4.1. Each of the four graphs in Fig. 5 is a connected odd-cycle graph. The first pair of graphs has the same number of matchings on k vertices for each $k = 2, 4, 6$, and the second pair does as well. It follows from Lemma 4.1 that the skew-adjacency matrices of each of the first two graphs all have characteristic polynomial $x^6 + 6x^4 + 8x^2 + 1$ while the skew-adjacency matrices of each of the last two graphs have characteristic polynomial $x^6 + 6x^4 + 6x^2 + 1$. An exhaustive check shows that no other pairs of connected odd-cycle graphs on six or fewer vertices have skew-adjacency matrices with the same characteristic polynomial.

The following lemma shows that the odd-cycle graphs are the only graphs whose skew-adjacency matrices all have the same spectrum.

Theorem 4.2. *The skew-adjacency matrices of a graph G are all cospectral if and only if G has no even cycles.*

Proof. The sufficiency has already been observed in Lemma 4.1.

For the necessity, suppose that G has finite even girth l. Then each collection U in $\mathcal{U}_l^e$ consists either of a single l-cycle in G or a matching in G covering l vertices. By (8), the first l coefficients of the characteristic polynomial of a skew-adjacency matrix $S = S(G^\sigma)$ are

$$s_k = m_k(G) \text{ when } k < l \quad \text{and} \quad s_l = m_l(G) - 2 \sum_{l(C)=l} \sigma(C), \tag{14}$$

where $m_k(G)$ is the number of matchings in G covering k vertices and the sum is taken over all cycles C in G of (smallest even) length l. Thus, s_l is the first coefficient that could possibly be used to distinguish the characteristic polynomials of two skew-adjacency matrices of G.

For an edge e, let $n_+(e)$ be the number of l-cycles C in G that contain e and have $\sigma(C) = 1$, and let $n_-(e)$ be defined analogously. Suppose that $n_+(e) \neq n_-(e)$. If the direction of the arc on e is reversed, then in (14) the contribution from the matchings will be unaffected as will that from the l-cycles not containing e. But the contribution from the l-cycles that contain e equals $-2\,(n_+(C) - n_-(C))$ and will be negated. Consequently, s_l will change. Thus G will have a skew-adjacency matrix whose spectrum differs from that of S and the necessity will have been proved.

Suppose then that $n_+(e) = n_-(e)$ for all edges e in G and all orientations G^σ of G. We shall see that this leads to a contradiction.

For $t \in \{1, \ldots, l\}$, let $n_+(e_1, \ldots, e_t)$ be the number of l-cycles C in G that have $\sigma(C) = 1$ and contain all of $e_1, \ldots, e_t$. Define $n_-(e_1, \ldots, e_t)$ analogously.

We claim that for each $t \in \{1, \ldots, l\}$, $n_+(e_1, \ldots, e_t) = n_-(e_1, \ldots, e_t)$ for all orientations G^σ and all edges $e_1, \ldots, e_t$. We proceed by induction on t.

The case $t = 1$ is assumed. Suppose that the claim holds for some $t < l$ and let G^σ be an orientation of G. For edges $e_1, e_2, \ldots e_t, e_{t+1}$ in G, let $n_+(e_1, \ldots, e_t, \overline{e_{t+1}})$ denote the number of l-cycles C that have $\sigma(C) = 1$ and contain edges $e_1, \ldots, e_t$, but not edge e_{t+1}. Define $n_-(e_1, \ldots, e_t, \overline{e_{t+1}})$ analogously. Then

$$\begin{aligned} n_+(e_1, \ldots, e_t) &= n_+(e_1, \ldots, e_t, e_{t+1}) + n_+(e_1, \ldots, e_t, \overline{e_{t+1}}), \\ n_-(e_1, \ldots, e_t) &= n_-(e_1, \ldots, e_t, e_{t+1}) + n_-(e_1, \ldots, e_t, \overline{e_{t+1}}), \end{aligned}$$

and $n_+(e_1, \ldots, e_t) = n_-(e_1, \ldots, e_t)$ by assumption. Next, consider the orientation $\widetilde{G}$ obtained from G^σ by reversing the orientation of e_{t+1}. Then

$$\begin{aligned} \tilde{n}_+(e_1, \ldots, e_t) &= n_-(e_1, \ldots, e_t, e_{t+1}) + n_+(e_1, \ldots, e_t, \overline{e_{t+1}}), \\ \tilde{n}_-(e_1, \ldots, e_t) &= n_+(e_1, \ldots, e_t, e_{t+1}) + n_-(e_1, \ldots, e_t, \overline{e_{t+1}}), \end{aligned}$$

and $\tilde{n}_+(e_1, \ldots, e_t) = \tilde{n}_-(e_1, \ldots, e_t)$ by assumption. Consequently,

$$\begin{aligned} & n_+(e_1, \ldots, e_t, \overline{e_{t+1}}) - n_-(e_1, \ldots, e_t, \overline{e_{t+1}}) \\ & \quad = n_-(e_1, \ldots, e_t, e_{t+1}) - n_+(e_1, \ldots, e_t, e_{t+1}) \\ & \quad = n_+(e_1, \ldots, e_t, e_{t+1}) - n_-(e_1, \ldots, e_t, e_{t+1}). \end{aligned}$$

Lines 1 and 3 above are equal and sum to zero. Thus $n_+(e_1, \ldots, e_t, e_{t+1}) = n_-(e_1, \ldots, e_t, e_{t+1})$, as desired. This completes the proof of the induction step, and the claim.

In particular, for any orientation G^σ, and edges $e_1, \ldots, e_l$ of an l-cycle, we have $n_+(e_1, \ldots, e_l) = n_-(e_1, \ldots, e_l)$. This is a contradiction, since one member of the equality is 0, while the other is 1. $\square$

In the proof of Theorem 4.2, it was shown that if G is a graph with finite even girth l, then $n_+(e) \neq n_-(e)$ for some orientation G^σ of G and some edge e in G. It was necessary to prove this because it need not hold for all orientations G^σ. For example, for the orientation G^σ of the 4×4 square lattice on a torus with 16 vertices and 16 squares shown in Fig. 6, $l = 4$ and $n_+(e) = n_-(e) = 1$ for all edges e.

If A is an $n \times n$ matrix and R is a sequence with distinct entries from $[n]$, then $A[R]$ is the matrix obtained from A by selecting rows with indices in R and columns with indices in R, taken in the order

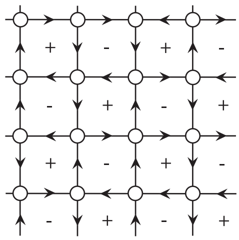


Fig. 6. A square lattice on a torus oriented so that $n_+(e) = n_-(e) = 1$ for all edges e.

that they appear in R. Thus, if R is a strictly increasing sequence, then $A[R]$ is a principal submatrix of A. Also, let $A(R)$ be the submatrix of A obtained by deleting rows and columns with indices in R. (Here, the order of the entries of R is not important).

Note that if S is a skew-adjacency matrix of a graph G of order n and $R \subsetneq [n]$, then $S[R]$ is a skew-adjacency matrix of $G[R] = G - \overline{R}$, the *induced* subgraph of G obtained by deleting the vertices in the complement $\overline{R}$ of the proper subset R. We now have the following theorem.

Theorem 4.3. *Let G be a simple graph with vertex set* $[n]$. *Then G is an odd-cycle graph if and only if any one of the following conditions holds:*

1. *G has no even cycles.*
2. *Each induced subgraph of G has at most one perfect matching.*
3. *For each nonempty subset* $R \subseteq [n]$, *either* $\det S[R] = 1$ *for every skew-adjacency matrix S of G, or* $\det S[R] = 0$ *for every skew-adjacency matrix S of G.*
4. *For each skew-adjacency matrix S of G and each nonempty subset* $R \subseteq [n]$, $\det S[R] = 0$ *or* 1.
5. *For every skew-adjacency matrix S of G and each* $k \in [n]$, *the coefficient* s_k *of the characteristic polynomial of S is equal to* $m_k(G)$, *the number of matchings in G that cover k vertices.*
6. *The skew-adjacency matrices of G all have the same spectrum.*

Proof. Condition 1 is the definition of an odd-cycle graph.

$1 \Rightarrow 2$. If G has no even cycles, no induced subgraph could have two perfect matchings because their symmetric difference would contain an even cycle.

$2 \Rightarrow 3$. Because $\det S[R] = (\mathrm{pf}\, S[R])^2$, it follows that $\det S[R] = 1$ if $G[R]$ has one perfect matching and $\det S[R] = 0$ if $G[R]$ has no perfect matching.

$3 \Rightarrow 4$. This implication is immediate.

$4 \Rightarrow 1$. We argue by contradiction. Suppose that G contains an even cycle and R is the vertex set of a cycle in G of shortest even length. Then the edges of the induced subgraph $G[R]$ consist of the edges of the cycle and perhaps some chords which do not lie on shorter even cycles in $G[R]$. It follows that either $G[R] = K_4$ or that $G[R]$ has at most one chord. If S is a skew-adjacency matrix for G, then $S[R]$ is a skew-adjacency matrix for $G[R]$. If $G[R]$ has no chords then by (9), $G[R]$ (hence G) may be oriented so that $\det S[R] = 4$. If $G[R]$ has one chord, then it may be deleted since neither of the two odd cycles it creates will affect $\det S[R]$. If $G[R] = K_4$ then by the comment following (11), $G[R]$ has a skew-adjacency matrix with determinant $m_4(K_4)^2 = 9$.

$1 \Rightarrow 5$. This is proved in Lemma 4.1.

$5 \Rightarrow 6$. The skew-adjacency matrices of G all have the same characteristic polynomial, and so the same spectrum.

$6 \Rightarrow 1$. This is the result of Theorem 4.2. □

5. Some polynomial comparisons

As before, let

$$p_A(x) = \det(xI - S) = x^n + a_1 x^{n-1} + \cdots + a_n$$

be the characteristic polynomial of the adjacency matrix A of a graph G, and let

$$p_S(x) = \det(xI - S) = x^n + s_1x^{n-1} + \cdots + s_n$$

be the characteristic polynomial of a skew-adjacency matrix S associated with an orientation G^σ of G. Recall that $m_k(G)$ denotes the number of matchings in G on k vertices. Thus $m_k = 0$ if k is odd.

Lemma 5.1. *Let A be the adjacency matrix of a simple graph G with vertex set $[n]$ and let S be the skew-adjacency matrix of G associated with an orientation σ of G. Then the polynomial coefficients a_k and s_k have the following properties:*

1. $s_k \equiv a_k \equiv m_k(G) \pmod 2$ *for all* $k \in [n]$, $s_k = 0$ *for all odd* $k \in [n]$, *and* $s_k = m_k(G)$ *for all even* k *with* $1 < k < g_e(G)$.
2. a_k *is even for all odd* $k \in [n]$ *and* $a_k = 0$ *for all odd* $k < g(G)$.
3. $a_k = (-1)^{k/2}m_k(G) = (-1)^{k/2}s_k$ *for all* $k < g(G)$.
4. *If* $g(G)$ *is odd,* $a_k = 0$ *for all even* $k \in [n]$ *with* $g(G) < k < 2g(G)$.
5. *If* $g(G)$ *is even,* $a_k = 0$ *for all odd* $k \in [n]$ *with* $g(G) < k < 2g(G)$.
6. *If* a_n *is odd, then* G *has a perfect matching.*
7. *If* a_n *is odd, then* n *is even and* $a_n \equiv n + 1 \pmod 4$.
8. $s_n = \det S = (\operatorname{pf} S)^2 \le m_n(G)^2$ *with equality if and only if either* n *is odd (so* $s_n = m_n(G) = 0$*) or* n *is even and each nice even cycle in* G *is negatively oriented relative to* σ.

Proof. Properties 1–6 follow immediately from (5) and (8). Property 7 follows from property 1 and [1, Thm. 1]. Property 8 follows from the definition and properties of the Pfaffian and the comment after inequality (11). □

A graph G of even order is said to be *Pfaffian* if it has an orientation σ such that $|\operatorname{pf} S^\sigma| = m_n(G)$, that is, if the condition for equality in Lemma 5.1(8) holds. For example, an examination of the constant coefficient for each of the characteristic polynomials in Example 7.1 shows that K_4 is Pfaffian but $K_{3,3}$ is not. Clearly, every cactus of even order has an orientation that satisfies the equality condition in Lemma 5.1(8) and so is Pfaffian. In fact, a construction of Kasteleyn [19, p. 322] shows that every planar graph of even order is Pfaffian.

Recall that if G is an odd-cycle graph, then $s_n = m_n(G)$ for all orientations σ of G. Also, the condition for equality in statement 8 of Lemma 5.1 is satisfied vacuously. Thus $m_n(G) = m_n(G)^2$ so $m_n(G) = 0$ or 1. That is, each odd-cycle graph has at most one perfect matching. Of course, this must be the case because the components of the symmetric difference of the edge sets of two distinct perfect matchings are even cycles.

We now examine the polynomials p_A and p_S for two special types of graph: those with no odd cycles (the bipartite graphs), and those with no even cycles (the odd-cycle graphs).

If G has no odd cycles, that is, if G is bipartite, then Lemma 5.1 implies that $a_k = s_k = 0$ for all odd k and *all* skew-adjacency matrices S of G. Lemmas 5.2 and 5.3 imply that more can be said for *some* skew-adjacency matrix of a bipartite graph G. (The equivalence of conditions 1 and 2 in both of the Lemmas 5.2 and 5.3 was proved by Shader and So in [21].)

Lemma 5.2. *Let G be a graph of order n with adjacency matrix A. Then the following conditions are equivalent:*

1. *G is bipartite.*
2. $\operatorname{Spec} S = i \operatorname{Spec} A$ *for some skew-adjacency matrix S of G.*
3. $p_S(x) = (-i)^n p_A(ix)$, *for some skew-adjacency matrix S of G.*
4. *For some skew-adjacency matrix S of G,* $a_k = (-1)^{k/2}s_k$ *for all even* $k \in [n]$ *and* $a_k = s_k = 0$ *for all odd* $k \in [n]$.

Proof. 1 $\Rightarrow$ 2. If G is bipartite, let B be the biadjacency matrix of G as shown in (15). Let G^σ be the orientation of G obtained by taking $\sigma(k,l) = 1$ when $kl \in E(G)$ and $k < l$. Then the skew-adjacency matrix associated with G^σ is the matrix S in (15). Then $iS = P^{-1}AP$ where

$$A = \begin{bmatrix} O & B \\ B^\top & O \end{bmatrix}, \quad S = \begin{bmatrix} O & B \\ -B^\top & O \end{bmatrix}, \quad \text{and} \quad P = \begin{bmatrix} I & O \\ O & iI \end{bmatrix}. \tag{15}$$

Thus, A is similar to iS and so $\operatorname{Spec} A = i \operatorname{Spec} S$. But $\operatorname{Spec} S = \operatorname{Spec} S^\top = \operatorname{Spec}(-S) = -\operatorname{Spec} S$, so $\operatorname{Spec} S = i \operatorname{Spec} A$.

2 $\Rightarrow$ 3. If the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and (2) holds, then the eigenvalues of S are $i\lambda_1, \ldots, i\lambda_n$. Thus, $p_S(x) = \Pi_{k=1}^n (x - i\lambda_k) = \Pi_{k=1}^n (-i)(ix + \lambda_k) = (-i)^n p_A(ix)$, since condition 2 implies that $\operatorname{Spec} A = -\operatorname{Spec} A$.

3 $\Rightarrow$ 4. If condition 3 holds then $s_k = (-i)^n i^{n-k} a_k = i^k a_k$. Since s_k and a_k are real numbers, $a_k = s_k = 0$ if k is odd and $a_k = (-1)^{k/2} s_k$ if k is even.

4 $\Rightarrow$ 1. If condition 4 holds, then $p_A(\lambda) = 0$ if and only if $p_A(-\lambda) = 0$. A standard result [8, p.87] now implies that G is bipartite. $\square$

As a special case of Lemma 5.2, we next consider graphs G that have no cycles at all, either odd or even (that is, forests).

Lemma 5.3. *Let G be a graph of order n with adjacency matrix A. Then the following conditions are equivalent:*

1. *G is a forest.*
2. $\operatorname{Spec} S = i \operatorname{Spec} A$ *for all skew-adjacency matrices S of G.*
3. $p_S(x) = (-i)^n p_A(ix)$, *for all skew-adjacency matrices S of G.*
4. *For all skew-adjacency matrices S of G, $(-1)^{k/2} a_k = m_k(G) = s_k$ for all even $k \in [n]$ and $a_k = s_k = 0$ for all odd $k \in [n]$.*

Proof. If condition 1 holds, then 4 holds by Lemma 5.1(3). If condition 4 holds, then the skew-adjacency matrices of G are all cospectral so G has no even cycles by Theorem 4.2. Also, G has no odd cycles by Lemma 5.2. Thus G is a forest, so 1 holds. The remaining equivalences follow easily. $\square$

In Lemma 5.3(4), when G is bipartite but not a forest, it is possible that $s_n \neq m_n(G)$ for all skew-adjacency matrices of G. For example, if G is the 4-cycle, then $m_4(G) = 2$ but $s_4(G) = \det S$ must be a perfect square.

Since graphs with no even cycles (the odd-cycle graphs) are in a sense the opposite of the well-studied class of graphs with no odd cycles (the bipartite graphs), it is natural to seek properties of the odd-cycle graphs. A feasible task would be to obtain more results on the skew spectrum of an odd-cycle graph because Theorem 4.3(5) can be used to relate its unique skew characteristic polynomial to its matchings polynomial (defined below), and because the latter polynomial is well-studied [10,19].

The *matchings polynomial* of a graph G of order n [10, p. 1] is

$$m(G, x) = \sum_{k=0}^{n} (-1)^{k/2} m_k(G) x^{n-k},$$

where $m_0(G) = 1$ and the k'th summand is 0 if k is odd. Here, as before, $m_k(G)$ denotes the number of matchings in G that cover k vertices, while in the literature, $m_k(G)$ usually denotes the number of matchings in G with k edges. For example, for the graph G in Fig. 3, $m_2(G) = 9$, $m_4(G) = 21$, and $m_6(G) = 11$, so $m(G, x) = x^7 - 9x^5 + 21x^3 - 11x$.

The following lemma is an immediate consequence of the preceding results. In part 2 of the lemma, it is well-known that $m(G, x) = p_A(x)$ if G is a forest (see, e.g. [10, Cor. 1.4, p. 21, 19, Thm. 8.5.3]).

Lemma 5.4. *Let G be a graph of order n with adjacency matrix A.*

1. *G is an odd-cycle graph if and only if $p_S(x) = (-i)^n m(G, ix)$ for all skew-adjacency matrices S of G.*
2. *G is a forest if and only if $m(G, x) = p_A(x)$.*

Problem 1. If $p_S(x) = (-i)^n m(G, ix)$ for *some* skew-adjacency matrix S of G, must G be an odd-cycle graph?

6. Spectral properties of skew-adjacency matrices

If M is an invertible matrix of order n with entries from some field and R is a proper nonempty subset of $[n]$ of cardinality $r = |R|$, Jacobi's identity (see, e.g. [6, p. 301]) implies that

$$(\det M)^{r-1} \det M(R) = \det\left((\operatorname{adj} M)[R]\right), \tag{16}$$

where $\operatorname{adj} M = (\operatorname{cof} A)^\top$, the transpose of the matrix of cofactors of M.

If z is a column n-vector with complex entries, the notation $|z|$ will be reserved for the vector with $|z|_k = |z_k|$ for each $k \in [n]$. The vector z is a *unit* vector if $z^*z = 1$, where $z^* = \bar{z}^\top$, the complex conjugate transpose of z.

Lemma 6.1. *Let G be an odd-cycle graph and let $i\alpha$, α real, be a (common) eigenvalue of the skew-adjacency matrices of G. Let σ be an orientation of G with skew-adjacency matrix S^σ, and let z^σ be a unit $i\alpha$-eigenvector of S^σ. If $i\alpha$ is simple*[6] *then $|z^\sigma|$ is the same vector for all orientations σ of G.*

Proof. Let $M = \lambda I - S^\sigma$. Then $M \operatorname{adj} M = (\det M)I = \det(\lambda I - S^\sigma)I$. Thus, if λ is an eigenvalue of S^σ, then each nonzero column of $\operatorname{adj} M$ (if any) is a λ-eigenvector of S^σ. If λ is a simple eigenvalue of S^σ, then $\operatorname{adj} M$ has a nonzero column because M is similar to a diagonal matrix with one diagonal entry 0, and so has rank equal to $n - 1$.

Because $M = \lambda I - S^\sigma$ is invertible over the field of rational functions in λ, we may apply identity (16) to a 2×2 submatrix of $\operatorname{adj} M$ to obtain the polynomial identity

$$\begin{aligned}\det M \det M(k, l) &= \det\left((\operatorname{adj} M)[k, l]\right)\\ &= C_{k,k}(M)C_{l,l}(M) - C_{k,l}(M)C_{l,k}(M),\end{aligned} \tag{17}$$

where $C_{k,l}(M)$ is the (k, l) cofactor of M. But $\det M$, $\det M(k, l)$, $C_{k,k}(M)$ and $C_{l,l}(M)$ are the characteristic polynomials of skew-adjacency matrices of the odd-cycle graphs G, $G - k - l$, $G - k$ and $G - l$, respectively, and so do not depend on σ. Thus $C_{k,l}(M)C_{l,k}(M)$ does not depend on σ. Also,

$$C_{l,k}(M) = C_{k,l}(M^\top) = C_{k,l}(\lambda I + S^\sigma) = (-1)^{n-1}C_{k,l}(-\lambda I - S^\sigma),$$

so, if $\lambda = i\alpha$, then $C_{l,k}(M) = (-1)^{n-1}\overline{C_{k,l}(M)}$. Thus, if $\lambda = i\alpha$, then $|C_{l,k}(M)|$ does not depend on σ. If $\lambda = i\alpha$ is a simple eigenvalue of S^σ then, as observed earlier, we may choose $l \in [n]$ so that column l of $\operatorname{adj} M$ is an $i\alpha$ eigenvector, w^σ say, of S^σ. Then $|w^\sigma|_k = |C_{l,k}(M)| = |C_{l,k}(i\alpha I - S^\sigma)|$ for $k \in [n]$, so $|w^\sigma|$ does not depend on the orientation σ of G. If z^σ is a unit $i\alpha$-eigenvector of S^σ, then z^σ is a scalar multiple of w^σ since $i\alpha$ is simple. Thus, $|z^\sigma|$ does not depend on σ. $\square$

If M is a square matrix, let $\rho(M)$ denote the *spectral radius* of M, that is, $\rho(M) = \max_\lambda |\lambda|$ where the maximum is taken over all eigenvalues of M. If G is a graph with adjacency matrix A, let $\rho(G) = \rho(A)$ and let $\rho_s(G) = \max_S \rho(S)$ where the maximum is taken over all of the skew-adjacency matrices S of G. We refer to $\rho(G)$ as the *spectral radius* of G and $\rho_s(G)$ as the *maximum skew-spectral radius* of G.

[6] For example, by Lemma 6.3, $i\rho(S^\sigma)$ is simple if G is connected.

Lemma 6.2. *If G is a simple graph, then $\rho_s(G) \le \rho(G)$. Moreover,*

1. *If G is an odd-cycle graph, then $\rho_s(G) = \rho(S)$ for all skew-adjacency matrices S of G, and $\rho_s(G)$ is the largest root of $m(G, x)$.*
2. *If G is bipartite, then $\rho_s(G) = \rho(G)$. If G is connected and not bipartite, then $\rho_s(G) < \rho(G)$.*
3. *If G is connected and bipartite and $A = \begin{bmatrix} O & B \\ B^\top & O \end{bmatrix}$, $\widetilde{S} = \begin{bmatrix} O & \widetilde{B} \\ -\widetilde{B}^\top & O \end{bmatrix}$, are an adjacency and a skew-adjacency matrix of G, then $\rho(A) = \rho(\widetilde{S})$ if and only if $\widetilde{B} = D_1 B D_2$ for some $\{-1, 1\}$-diagonal matrices D_1, D_2.*

Proof. If S is a skew-adjacency matrix of a graph G with adjacency matrix A, then $A = |S|$, where $|S|$ is the matrix with entries $|S|_{k,l} = |s_{k,l}|$ for all k, l. By the Perron–Frobenius theorem [16, p. 509], $\rho(S) \le \rho(A) = \rho(G)$.

1. This follows from Lemmas 4.1 and 5.4(1).
2. By Lemma 5.2, $\rho_s(G) = \rho(G)$.
   Suppose G is connected and $\rho_s(G) = \rho(G)$. Then $\rho(S) = \rho(A)$ for some skew-adjacency matrix S of G. Since $i\rho(S)$ is an eigenvalue of S, the Perron–Frobenius theorem implies $S = iDAD^{-1}$ for some diagonal matrix D with complex diagonal entries $d_1, \ldots, d_n$ of modulus 1. Thus, $id_k\bar{d}_l \in \{-1, 1\}$ when $kl \in E(G)$. We may take $d_1 = 1$, so this implies that the vertices of the connected graph G may be alternately labelled by the two symbols ± 1, $\pm i$ so that adjacent vertices are assigned different labels. Thus G is bipartite.
3. If $\rho(\widetilde{S}) = \rho(A)$ then, because $\rho(S) = \rho(A)$ for S as in (15), it follows easily from the Perron–Frobenius theorem that $\widetilde{S} = DSD^{-1}$ where D may be chosen to be a $\{-1, 1\}$-diagonal matrix since S and $\widetilde{S}$ have real entries. Then $\widetilde{B} = D_1 B D_2$ where $D_1 \oplus D_2$ is a partition of D compatible with that of S.

The converse implication in statement 3 follows easily. □

Problem 2. If G is a connected graph and $\rho(S)$ is the same for all skew-adjacency matrices S of G, must G be an odd-cycle graph?

Example 6.1 (*The extremal skew-spectral radii of trees on n vertices*)**.** Let T be a tree on n vertices. Because T is bipartite, $\rho_s(T) = \rho(T)$. Lovász and Pelikán [18] show that $\rho_s(T) = \rho(T) \le \rho(K_{1,n-1}) = \sqrt{n-1}$, and a result of Hong [15, Thm. 1] implies that equality holds only if $T = K_{1,n-1}$, the star on n vertices. Also, a result of Collatz and Sinogowitz [7] implies that, $\rho_s(T) = \rho(T) \ge \rho(P_n) = \rho_s(P_n)$, with equality only if $T = P_n$, the path on n vertices (see also [18]).

If S is a skew-symmetric real matrix of order n and z is a column n-vector with complex entries, then z^*Sz is pure imaginary:

$$z^*Sz = \sum_{k \ne l} s_{k,l}\bar{z}_k z_l = \sum_{k<l} s_{k,l}(\bar{z}_k z_l - \bar{z}_l z_k) = 2i \sum_{k<l} s_{k,l}\, \mathrm{Im}(\bar{z}_k z_l).$$

Let $w_{kl} = 2s_{k,l}\,\mathrm{Im}(\bar{z}_k z_l) = 2s_{l,k}\,\mathrm{Im}(\bar{z}_l z_k)$. Because $-iS$ is Hermitian, $z^*(-iS)z$ is real, so if S is a skew-adjacency matrix of a graph G, then

$$\mathrm{Im}(z^*Sz) = z^*(-iS)z = \sum_{kl \in E(G)} w_{kl}.$$

Also, $\rho(-iS) = \max_{z^*z=1} z^*(-iS)z$, and an examination of the proof of this fact (in [16, p. 176], say) shows that equality is attained if and only if the unit vector z is an eigenvector of $-iS$ for the eigenvalue $\rho(-iS) = \rho(S)$. Thus

$$\rho(S) = \max_{z^*z=1} \operatorname{Im}(z^*Sz) = \max_{z^*z=1} \sum_{kl\in E(G)} w_{kl} \text{ where } w_{kl} = 2s_{k,l}\operatorname{Im}(\bar{z}_k z_l), \tag{18}$$

and equality is attained if and only if z is a unit $\rho(S)$-eigenvector of $-iS$, or, equivalently, a unit $i\rho(S)$-eigenvector of S.

Lemma 6.3. *Let G be connected and let S be a skew-adjacency matrix of G for which $\rho(S) = \rho_s(G)$. If z is an eigenvector of S for the eigenvalue $i\rho(S)$, then $z_k \neq 0$ for all $k \in [n]$, $i\rho(S)$ is simple, and $w_{kl} > 0$ for all $kl \in E(G)$. Moreover, there is a $\{-1, 1\}$-signed permutation matrix P such that $(PSP^\top)_{k,l} \geq 0$ when $k < l$.*

Proof. By scaling z, we may assume that $z^*z = 1$. Then, by (18), $\rho(S) = \operatorname{Im}(z^*Sz)$. Suppose that $kl \in E(G)$. Let $\widehat{S}$ be the skew-adjacency matrix of G such that $\hat{s}_{k,l} = -s_{k,l}$ and $\hat{s}_{i,j} = s_{i,j}$ when $ij \neq kl$. Then $\rho(S) \geq \rho(\widehat{S})$ and, by (18), $\rho(S) - \rho(\widehat{S}) \leq \operatorname{Im}(z^*Sz) - \operatorname{Im}(z^*\widehat{S}z) = 2w_{kl}$. Thus, $w_{kl} \geq 0$.

Suppose that $w_{kl} = 0$. Then $\rho(S) = \rho(\widehat{S}) = \rho$ say, and $\rho(\widehat{S}) = \operatorname{Im}(z^*\widehat{S}z)$ so z is also an eigenvector of $\widehat{S}$ for the common eigenvalue $i\rho$. Thus $Sz = i\rho z = \widehat{S}z$, so $0 = (Sz)_k - (\widehat{S}z)_k = 2s_{k,l}z_l$. Thus, $z_l = 0$ and so $w_{jl} = 0$ for all vertices j adjacent to l. Since G is connected, by repeating this argument, it follows that if $w_{kl} = 0$, then $z_j = 0$ for all vertices j in G. Since z is not a zero vector, this is a contradiction. Thus, $w_{kl} > 0$ for each edge kl in G. But then $w_k \neq 0$ for all $k \in [n]$, so $i\rho(S)$ is simple.

Let D be the diagonal matrix with k'th diagonal entry equal to 1 if $\arg z_k \in [0, \pi)$ and -1 if $\arg z_k \in [\pi, 2\pi)$. Then $\arg(Dz)_k \in [0, \pi)$ for all $k \in [n]$. Choose a permutation matrix Q so that $\tilde{z} = QDz$ is such that $\arg \tilde{z}_k \leq \arg \tilde{z}_l$ if $k < l$ and let $P = QD$. Then P is a $\{-1, 1\}$-signed permutation matrix and $\tilde{z}$ is an $i\rho(S)$-eigenvector of $\widetilde{S} = PSP^\top$. Also, $\rho(S) = \rho(\widetilde{S})$ since $\widetilde{S}$ is similar to S, and $\operatorname{Im} \bar{\tilde{z}}_k\tilde{z}_l \geq 0$ if $k < l$ since $\arg \bar{\tilde{z}}_k\tilde{z}_l = \arg \tilde{z}_l - \arg \tilde{z}_k \in [0, \pi)$. By the first part of the lemma, $\tilde{w}_{kl} = 2\tilde{s}_{k,l} \operatorname{Im} \bar{\tilde{z}}_k\tilde{z}_l > 0$ for all $kl \in E(G)$, so $\tilde{s}_{k,l} = 1$ when $kl \in E(G)$ and $k < l$. $\square$

Lemma 6.3 may fail if $\rho(S) < \rho_s(G)$. For example, by (8), the characteristic polynomial of the skew-adjacency matrix of a positive orientation (resp. negative orientation) of the 4-cycle C_4 is $z^4 + 4z^2$ (resp. $z^4 + 4z^2 + 4$). Thus $\rho_s(C_4) = 2$, and if S is the skew-adjacency matrix associated with a negative orientation, then $i\rho(S) = \sqrt{2}i$ with multiplicity 2.

If G is a graph with vertex set $[n]$, let $G - kl$ denote the graph obtained by deleting an edge kl of G (but not the vertices k or l), and let $G - k$ and $G - k - l$ be the induced subgraphs obtained by deleting vertex k and vertices k and l, respectively.

Lemma 6.4. *If kl is an edge of G then $\rho(G) \geq \rho(G - kl)$, $\rho_s(G) \geq \rho_s(G - kl)$, $\rho(G) \geq \rho(G - k)$ and $\rho_s(G) \geq \rho_s(G - k)$, with all inequalities strict when G is connected.*

Proof. The statements for $\rho(G)$ follow from the Perron–Frobenius theorem.

Let $\widehat{S}$ be a skew-adjacency matrix of $G - kl$ for which $\rho(\widehat{S}) = \rho_s(G - kl)$ and let z be a unit eigenvector of $\widehat{S}$ for the eigenvalue $i\rho(\widehat{S})$. Let S be the skew-adjacency matrix for G with $s_{i,j} = \hat{s}_{i,j}$ if $ij \neq kl$ and with $s_{k,l} = 1$ or -1 chosen so that $w_{kl} = 2s_{k,l}\operatorname{Im}(\bar{z}_k z_l) \geq 0$. Then by (18), $\rho_s(G) - \rho_s(G - kl) \geq \rho(S) - \rho(\widehat{S}) \geq \operatorname{Im}(z^*Sz) - \operatorname{Im}(z^*\widehat{S}z) = w_{kl} \geq 0$. Thus, $\rho_s(G) \geq \rho_s(G - kl)$. Moreover, if $\rho_s(G) = \rho_s(G - kl)$ then $\rho_s(G) = \rho(S)$, $w_{kl} = 0$ and Lemma 6.3 implies that G is not connected. Thus, $\rho_s(G) > \rho_s(G - kl)$ if G is connected and kl is an edge of G.

By removing edges of G incident to k, we also have $\rho_s(G) \geq \rho_s(G - k)$ with strict inequality when G is connected. $\square$

Example 6.2 (*The complete graph*)**.** If G is a graph on n vertices and K_n is the complete graph of order n, it follows from Lemmas 6.3 and 6.4 that $\rho(G) \leq \rho(K_n) = \rho(A) = n - 1$ and $\rho_s(G) \leq \rho_s(K_n) = \rho(S) = \cot \frac{\pi}{2n}$ where A is the adjacency matrix of K_n and S is the skew-adjacency matrix of K_n which has all entries above the diagonal equal to 1. The second inequality is a special case of Pick's inequality [11,20].

Remark 6.1 (*Generalizations to real skew-symmetric matrices*)**.** Many of the preceding observations hold for skew-adjacency matrices of positive edge-weighted graphs; equivalently, for skew-signings of symmetric matrices with zero diagonal and nonnegative real entries. Suppose that G is an edge-weighted graph with positive edge weights $a_{i,j} = a_{j,i}$ when $ij \in E(G)$ and $a_{i,j} = 0$ when $ij \notin E(G)$. If σ is an orientation of G, we may define an associated skew-weighted matrix S^σ by $S^\sigma_{i,j} = a_{ij} = -S^\sigma_{j,i}$ if $i \to j$ in G^σ. Then, Lemmas 6.1, 6.3 and 6.4 all hold for positive edge-weighted graphs. In particular, if G is a positive edge-weighted odd-cycle graph, the characteristic polynomial of S^σ does not depend on σ, so $\rho(S^\sigma)$ is the same for all σ. Also, Lemma 6.3 may be regarded as an analogue (for those skew-adjacency matrices of weighted connected graphs that have maximum spectral radius) of the Perron–Frobenius theorem.

Example 6.3 (*Minimum skew-spectral radii of connected odd-cycle graphs*)**.** By Lemma 6.4, if G is a connected odd-cycle graph on n vertices with minimum skew-spectral radius, then G must be a tree. From Example 6.1 it follows that among the connected odd-cycle graphs on n vertices, the path P_n has the minimum skew-spectral radius.

Let H_n be the odd-cycle graph formed from the star $K_{1,n-1}$ by adding $\lfloor (n-1)/2 \rfloor$ independent edges between pairs of pendant vertices.

Lemma 6.5.
1. $|E(H_n)| = \lfloor 3(n-1)/2 \rfloor$.
2. $\rho(H_n)$ *equals* $\frac{1}{2} + \sqrt{n - \frac{3}{4}}$ *when n is odd and the largest root of* $x^3 - x^2 - (n-1)x + 1$ *when n is even.*
3. $\rho_s(H_n)$ *equals* $\sqrt{n}$ *when n is odd and* $\sqrt{n + \sqrt{n^2-4}}/\sqrt{2}$ *when n is even.*

Proof.
1. By the definition of H_n, $|E(H_n)| = n - 1 + \lfloor (n-1)/2 \rfloor = \lfloor 3(n-1)/2 \rfloor$.
2. Let A be the adjacency matrix of H_n and let $\rho = \rho(H_n) = \rho(A)$. Because H_n is connected, ρ is a simple eigenvalue of A and $Ax = \rho x$ for some eigenvector x with positive entries. If $\hat{x}$ is a vector obtained by permuting the entries of x by an automorphism of H_n, then $\hat{x}$ is also a ρ-eigenvector of A. Because ρ is simple, each such vector $\hat{x}$ is a multiple of x. It follows that $x_i = x_j$ whenever i and j are vertices of H_n of degree 2. Solving the system $Ax = \rho x$ with this restriction on x gives the values in statement 2.
3. If n is odd, delete the unique vertex of degree $n-1$ in H_n and use the standard identities for the matchings polynomial [10, p. 2] to get

$$\begin{aligned} m(H_n, x) &= x\, m(M_{n-1}, x) - (n-1)m(M_{n-3}, x) \\ &= x(x^2-1)^{\frac{n-1}{2}} - x(n-1)(x^2-1)^{\frac{n-3}{2}}, \end{aligned}$$

where M_{n-1} is a matching on $n-1$ vertices and M_{n-2} is a matching on $n-3$ vertices together with an isolated vertex. Then $m(H_n, x) = x(x^2-1)^{\frac{n-3}{2}}(x^2-n)$ and $\rho_s(H_n) = \sqrt{n}$ by Lemma 6.2(1).

If n is even, delete the unique vertex of degree 1 in H_n to get $m(H_n, x) = x\, m(H_{n-1}, x) - m(M_{n-2}, x)$ and substitute the previous formula with n replaced by $n-1$ to get $m(H_n, x) = (x^2-1)^{\frac{n-4}{2}}(x^4 - nx^2 + 1)$. Then the largest root of $x^4 - nx^2 + 1$ gives the stated value for $\rho_s(H_n)$. □

Recall that a cactus is a connected graph each of whose blocks is either a cycle or an edge. The next lemma asserts that the graph H_n has the greatest size and the greatest spectral radius of the cacti of order n. Part 2 of the lemma is proved in [5].

Lemma 6.6. *If G is a cactus of order n, then*

1. $|E(G)| \le |E(H_n)|$ *and equality holds if and only if at most one block of G is a single edge and all other blocks of G are 3-cycles.*
2. $\rho(G) \le \rho(H_n)$ *and equality holds if and only if* $G \cong H_n$.

Proof. 1. Since a cactus G is planar and each edge of G is on at most one finite face, the number of finite faces is at most $|E(G)|/3$. It follows from Euler's formula for connected planar graphs [4, p. 143] that $|E(G)| \leq \lfloor 3(n-1)/2 \rfloor$. Thus, by Lemma 6.5(1), $|E(G)| \leq |E(H_n)|$ and equality is attained by the graph H_n (and every connected odd-cycle graph whose cycles are all triangles and with at most one edge not in some triangle).

2. See [5, Thm. 3.1]. □

We conjecture that H_n also has the greatest skew-spectral radius of the odd-cycle graphs G of order n.

Conjecture 6.1. If G is an odd-cycle graph of order n, then $\rho_s(G) \leq \rho_s(H_n)$ and equality holds if and only if $G \cong H_n$.

Of the odd cycle graphs with n vertices, if G has the greatest skew-spectral radius, G must be edge maximal by Lemma 6.4. Thus, by Lemma 6.6(1), to prove Conjecture 6.1, it would be sufficient to prove that G must contain a vertex of degree $n-1$.

There are many papers containing techniques for examining the maximum spectral radii of the adjacency matrices and Laplacian matrices of graphs with few cycles (e.g. [9,13,14,23]). Corresponding techniques for the skew-adjacency matrices of odd-cycle graphs may be helpful. One of the standard techniques used to compare spectral radii of adjacency matrices is that of *edge-switching* [23]. For skew-adjacency matrices, the edge-switching technique takes the following form.

Lemma 6.7. *Let S be a skew-adjacency matrix of a simple graph G of order n and let z be a unit eigenvector of S for the eigenvalue $i\rho(S)$. Let u, v be two vertices of G and suppose that $u_1u, \ldots, u_tu$ are edges of G but $u_1v, \ldots, u_tv$ are not. Let $\widehat{G}$ be the graph obtained from G by deleting the edges u_ku and adding the edges u_kv, $1 \leq k \leq t$. If $\sum_{k=1}^{t}\left(|\operatorname{Im}(\bar{z}_{u_k}z_v|) - s_{u_k,u}\operatorname{Im}(\bar{z}_{u_k}z_u)\right) \geq 0$, then $\rho_s(\widehat{G}) \geq \rho(S)$.*

Proof. Let $\widehat{S}$ be the skew-adjacency matrix of $\widehat{G}$ with $\hat{s}_{i,j} = s_{i,j}$ whenever (i,j) is none of (u_k, v) or (v, u_k) for $1 \leq k \leq s$, and let $\hat{s}_{u_k,v} = -\hat{s}_{v,u_k}$ have the same sign as $\operatorname{Im}(\bar{z}_{u_k}z_v)$. Then $\rho(\widehat{S}) - \rho(S) \geq \operatorname{Im}(z^*\widehat{S}z) - \operatorname{Im}(z^*Sz) = \sum_{k=1}^{t}\left(|\operatorname{Im}(\bar{z}_{u_k}z_v)| - s_{u_k,u}\operatorname{Im}(\bar{z}_{u_k}z_u)\right) \geq 0$. □

7. Skew-adjacency matrices of a graph with different spectra

A key notion in estimating the number of skew-adjacency matrices of a graph with distinct spectra is that of *sign similarity*. Two $n \times n$ matrices A and $\widetilde{A}$ are *sign similar* if $\widetilde{A} = DAD$ for some diagonal matrix D with diagonal entries $d_i \in \{-1, 1\}$ for $i \in [n]$. In particular, two skew-adjacency matrices $S, \widetilde{S}$ of a graph G of order n with edge set $E(G)$ are sign similar if and only if there are n scalars $d_i \in \{1, -1\}$ such that $\tilde{s}_{i,j} = d_i d_j s_{i,j}$ whenever $ij \in E(G)$. Sign similar skew-adjacency matrices of a graph must be cospectral but, as the following lemma shows, the converse need not hold.

Lemma 7.1. *Let S be a skew-adjacency matrix of a graph G. Then $S^\top$ is sign similar to S if and only if G is bipartite.*

Proof. If S is a skew-adjacency matrix, then $S^\top = -S$ is sign similar to S if and only there are $d_i \in \{1, -1\}$ such that $d_i d_j = -1$ whenever $ij \in E(G)$; that is, if and only if G is bipartite. □

The following lemma shows that, in determining skew-adjacency matrices S of a graph G that have distinct spectra, it is sufficient to consider those for which $s_{i,j} = 1$ when either $i < j$ and ij is an edge of a prespecified spanning forest of G or $i < j$ and ij is on no even cycle in G.

Lemma 7.2. *Let F be a forest in a graph G and let S be a skew-adjacency matrix of G. Then there is a skew-adjacency matrix $\widetilde{S}$ sign-similar to S with $\tilde{s}_{i,j} = 1$ when either (a) $i < j$ and ij is an edge of F or (b) $i < j$ and ij is an edge of G on no even cycle in G.*

Proof. To prove part (a), we apply induction on the number m of edges of F to show that there is a skew-adjacency matrix $\widetilde{S}$ sign similar to S with $\tilde{s}_{i,j} = 1$ whenever $i < j$ and ij is an edge of F.

If $m = 1$, F has a single edge ij. If $s_{i,j} = 1$ when $i < j$, take $\widetilde{S} = S$. If $s_{i,j} = -1$, let $\widetilde{S} = DSD$ where $d_j = -1$ and $d_k = 1$ for $k \neq j$. Then $\tilde{s}_{i,j} = -s_{i,j} = 1$.

If F has m edges, let r be a leaf of F and let t be its neighbor in F. By induction, there is a diagonal matrix D for which $\widetilde{S} = DSD$ has $\tilde{s}_{i,j} = 1$ when $i < j$ and ij is an edge of $F\backslash\{r\}$. If $r < t$ and $\tilde{s}_{r,t} = 1$ or $t < r$ and $\tilde{s}_{t,r} = 1$, we are done. If not, let $\widehat{D}$ be the diagonal matrix obtained from D by replacing d_r by $-d_r$. Because r is adjacent only to t in F, the product $\widehat{S} = \widehat{D}S\widehat{D}$ will still equal 1 on (i, j) entries for which $i < j$ and ij is an edge of $F\backslash\{r\}$, but the signs of the (r, t) and (t, r) entries will be reversed.

To see part (b), note that if ij is an edge of G in no even cycle in G, then s_k is unchanged in (8) if the direction of on ij is reversed. Thus s_k does not depend on the sign of $s_{i,j}$. $\square$

We note that the previous lemma gives an alternate proof of the fact that the skew-adjacency matrices of an odd-cycle graph all have the same spectra.

If G is a connected graph, to obtain an upper bound on the number of possible skew-adjacency matrices of G with distinct spectra, it would be appropriate to first choose a spanning tree T of G that contains as many edges as possible that are in even cycles of G. Then assign $s_{i,j} = 1$ if $i < j$ and ij is an edge of T or if $i < j$ and ij is on no even cycle of G. If m edges of G that are on even cycles remain unassigned, it follows that G will have at most 2^m skew-adjacency matrices with distinct spectra. The following example shows that although this upper bound can be attained, it is sometimes very poor.

Example 7.1 (*Characteristic polynomials of all skew-adjacency matrices of some graphs*)**.** In the (unoriented) graph G in Fig. 3, the path $1 - 2 - 3 - 4 - 5 - 6 - 7$ is a spanning tree, and the edge 17 is on no even cycle in G. As shown in G^σ, the 7 edges ij on the outer 7-cycle may be oriented so that $i \to j$ when $i < j$, and the corresponding 7 entries of S above the diagonal will equal 1. There are four possible ways that the remaining edges 25 and 16 may be oriented (only the orientation with $5 \to 2$ and $6 \to 1$ is shown). The characteristic polynomials of the skew-adjacency matrices for the four orientations are: $x^7 + 9x^5 + 25x^3 + 21x$, $x^7 + 9x^5 + 21x^3 + 13x$, $x^7 + 9x^5 + 17x^3 + 5x$ and $x^7 + 9x^5 + 21x^3 + 5x$.

On the other hand, if G is the complete graph K_4, then G has 6 edges, 3 of which are in a spanning tree. Thus at most $2^{6-3} = 8$ distinct characteristic polynomials can be obtained from skew-adjacency matrices. But it turns out that there are only two: $x^4 + 6x^2 + 1$ and $x^4 + 6x^2 + 9$.

Also, if G is the complete bipartite graph $K_{3,3}$, then G has 9 edges, all cycles in G are even and a spanning tree has 5 edges. Thus at most $2^{9-5} = 16$ distinct characteristic polynomials are obtained from the skew-adjacency matrices of G. It turns out that there are only three: $x^6 + 9x^4$, $x^6 + 9x^4 + 16x^2$ and $x^6 + 9x^4 + 24x^2 + 16$.

It would be interesting to obtain good estimates on the numbers of skew-adjacency matrices with distinct spectra for all K_n and $K_{n,n}$.

7.1. Recent related work

The independent papers [12,17] (submitted shortly after our original submission in December 2010), overlap ours in places. In particular, both contain expressions for s_k. In this revised submission, formula (7) for s_k has been modified to resemble that in [12].

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