Deriving the Effective Ultrasound Equations for Soft Tissue Interrogation

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Abstract—We are interested in the ultrasound interrogation of soft tissue. As soft tissue is a complicated material, we derive the effective acoustic equations using an averaging method known as homogenization to arrive at suitable constitutive equations. Soft tissue is composed of actin, elastin, collagen, and interstitial fluid. Collagen is packaged in parallel-fibered bundles, or fascicles, in the case of tendon and ligament and a meshwork of fibrils in the case of skin. The interstitial fluid is a hydrophillic gel. The purpose of this paper is to show that one may derive a general form for the constitutive equations for the aggregate soft tissue structure. Subsequent research will be directed on the estimation of the various coefficients and kernels appearing in the effective law. © 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

For living tissue, there is a history-related relationship between stress and strain formulated by Fung [1]. This theory has been adopted for modeling the viscoelastic behavior of soft tissue. Collagen is a basic constituent of soft tissue [2]. The remaining material consists of elastin, reticulum, and a hydrophillic gel. It is known that there is a hysteresis loop in cyclic loading and unloading of connective tissue. However, from curves in Figure 1, it is clear that the mechanical properties of soft tissue are more dependent on their structure rather than the relative amounts of their constituents.

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Elastin is almost perfectly elastic and hysteresis loop is very small. Collagen is a viscoelastic tissue and hysteresis loop is moderate. These loops are for strains that are much larger than usually encountered by the material in normal usage.

The hydrophilic gel varies with the type of tissue but it contains mucopolysaccharides and tissue fluid, i.e., ground material. The collagen fibrils are oriented roughly parallel to one another in the case of of tendon and form a meshwork in the case of skin. Similarly to tendons, muscle is composed of fascicles containing bundles of fibers as shown in Figure 2.

The collagen fibrils are assembled into bundles to form fibers which have diameters ranging from 0.2 to 12 μm. In tendons, collagen fibers are packaged into parallel-fibered bundles or fascicles. The above discussion suggests that we might model tendons and ligaments as a material with microstructure consisting of collagen fibers interspersed with a hydrophilic gel. The structure of collagen fibers in the skin is more like a three-dimensional network of fibrils [4].
As we are interested in modeling the ultrasound interrogation of tissue, we shall employ the methods of homogenization to derive effective acoustic equations for the composite tissue.

2. DESCRIPTION OF PROBLEM

The theory of homogenization can be used to find the effective equations for the acoustic interrogation of complex materials. Although soft tissue does not have a periodic structure, there is a microstructure, hence, in this work we propose to make use of the general microstructure. Suppose that there is a small parameter $\epsilon$, a microscopic length-scale, which characterizes some physical measurement, say, a collagen fibril. Then, we may define a fast variable $y = x/\epsilon$, which allows us to record fast changes from one microcomponent to another. For example, if $K(x/\epsilon)$ were to describe a component appearing in the constitutive relations, then we might suppose it depends on the fast variable as a constant in one component of the material and as a different constant in another component. To be more precise, we need to describe the dependency of the terms in the constitutive relations in terms of the slow variable and the fast variable. To this end, let the tissue sample be contained in $U$, a bounded domain in $\mathbb{R}^n$. For simplicity, we also assume that $U$ has a sufficiently smooth boundary. It is convenient that we let the fast variable $y$ live in $Y = [0, 1]^n$ a the unit cell.

We want to model the acoustic interrogation of soft tissue, hence, let $u(t, x)$ represent the displacement of the acoustic field. The assumption of a small parameter, $\epsilon$, implies that the acoustic pressure $u$ must depend on two spatial variables that are on widely separated scales, namely, $x$, the “macroscopic” variable, and $y = x/\epsilon$, the “microscopic” variable. It is a postulate of the homogenization method that the solution of the differential system describing the displacement field $u$ can be expressed as an asymptotic expansion,

$$u(t, x) = u^e(t, x) = u_0(t, x, y) + \epsilon u_1(t, x, y) + \epsilon^2 u_2(x, y) + \cdots.$$  

In this expansion, we can consider $u_0$ as the macroscopic displacement, while $u_1, u_2, \ldots$, are the microscopic displacements. The physical meaning of the expansion of $u$ is that the actual acoustic displacement $u$ is rapidly oscillating around $u_0$ due to the inhomogeneity from the microscopic point of view. $u_1, u_2, \ldots$, are the perturbing displacements according to the microstructure.

An important technique used in homogenization theory is two-scale convergence method [6,7]. The key goal for a two-scale continuum is to obtain a macrodescription (which is a function of $x$) from a microdescription (which is a function of both the microvariable $y$ and the macrovariable $x$) by averaging with respect to the microvariable $y$. We will make this concept precise in Section 4.

We treat the composite soft tissue as an elastic collagen with interstitial viscoelastic gel. We propose that this may be done using viscoelastic constitutive equations which are history-dependent [2, p. 46; 4, Section 7],

$$\sigma_{ij} = a_{ijkl}\left(x, \frac{x}{\epsilon}\right)e_{kl}(u(t)) + \int_0^t b_{ijkl}\left(x, \frac{x}{\epsilon}, t - s\right)e_{kl}(u(s))\, ds,$$  

where

$$\epsilon(u)_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$  

is the strain tensor. We denote the $\epsilon$ dependency of the coefficients in the constitutive equations by

$$a^e(x) = a_{ijkl}\left(x, \frac{x}{\epsilon}\right), \quad b^e(x, t) = b_{ijkl}\left(x, \frac{x}{\epsilon}, t\right).$$

Recall that the fast variable describes the variation due to the microstructure. The $a_{ijkl}(x, x/\epsilon)$ terms represents an instantaneous elastic effect, which we assume is dominant in the collagen.
Moreover, we assume, as is usual for elastic materials, that these coefficients are bounded smooth periodic in \( y \) and satisfy the symmetry and the ellipticity conditions,

\[
\begin{align*}
  a_{ijkl} &= a_{ijlk} = a_{jikl} = a_{klji}, \\
  a_{ijkl} h_{ij} h_{kl} &\geq \alpha h_{ij} h_{ij}, \quad \alpha > 0
\end{align*}
\] (2.2) (2.3)

Recall that we consider only small acoustic displacements, hence, in the collagen region the response should be primarily elastic with just small viscoelastic components, whereas, in the hydrophilic gel the material is primarily viscoelastic with memory. Other models are mathematically possible. Nevertheless, the averaging method also leads in these cases to effective acoustic equations. The coefficients \( b_{ijkl}(x,y,t) \), which are dominant in the hydrophilic gel, are assumed to be bounded, smooth, periodic in \( y \) and satisfy

\[
b_{ijkl} = b_{jikl},
\] (2.4)

and the regularity assumption, \( \partial_t b_{ijkl} \) are bounded. A change \( O(1) \) in \( x \) results in a change \( O(1/\varepsilon) \) in \( y \), so that \( a \) and \( b' \) fluctuate greatly over macroscopic shifts in position. The huge fluctuation in \( a \) and \( b' \), though, does not necessarily imply correspondingly huge fluctuations in \( u' \). The equation of vibratory motion describing the acoustic displacement is given by

\[
\ddot{u} = \text{div} \sigma' + f, \quad \text{in} \ U.
\] (2.5)

On the boundary \( \partial U \), we assume a homogeneous Dirichlet condition,

\[
u' \big|_{\partial U} = 0,
\] (2.6)

and for simplicity the initial conditions are

\[
\partial_t u' (x,0) = 0, \quad u' (x,0) = 0.
\] (2.7)

This problem is best stated in variational form. In order to formulate this variational problem, it is essential to introduce the function spaces,

\[
H = [H^0_0(U)]^n, \quad V = [L^2(U)]^n.
\] (2.8)

Then, the problem can be formulated in the following way.

**Problem 2.1.** Find

\[
u' \in L^\infty([0,T],H), \quad \partial_t u \in L^\infty([0,T],V) \cap L^2([0,T],H), \quad \partial_{tt} u \in L^\infty([0,T],H'),
\]

so that \( u'|_{t=0} = 0, \partial_t u'|_{t=0} = 0 \), and \( u \) satisfies

\[
\int_{U} \partial_{tt} u' \cdot \phi \, dx + \int_{U} a_{ijkl} e_{kl} (u') e_{ij} (\phi) \, dx \\
+ \int_{0}^{T} \int_{U} b_{ijkl} (t-s) e_{kl} (u'(s)) e_{ij} (\phi(t)) \, dx \, ds = \int_{U} f \cdot \phi, \quad \text{for any} \ \phi \in H,
\] (2.9)

where we assume \( f \in L^2([0,T],[L^2(U)]^n) \).
3. EXISTENCE AND UNIQUENESS OF A SOLUTION TO PROBLEM

For simplicity of notation, we suppress $\epsilon$ dependence in this section. First, we show the uniqueness theorem.

**Theorem 3.1.** A weak solution to Problem 2.1 is unique.

**Uniqueness Proof.** The idea is similar to [3, pp. 186-187]. We adopt the notation used in [8] and write

$$
\left( \int_0^t B(t-s)u(s)\,ds, \phi \right) := \int_0^t \int_U b_{ijkl} \left( x, \frac{x}{\epsilon}, t-s \right) e_k(u(s)) e_i(\phi(t)) \,dx \,ds
$$

and

$$
a(u, \phi) := \int_U a_{ijkl} e_k(u) e_i(\phi) \,dx.
$$

Let $u$ and $u_\ast$ be two possible solutions. If we choose $\phi = \partial_t u - \partial_t u_\ast$, equation (2.9) becomes

$$
\int_U \partial_{tt} u \cdot (\partial_t u - \partial_t u_\ast) \,dx + a(u, \partial_t u - \partial_t u_\ast)
$$

$$
+ \left( \int_0^t B(t-s)u(s) \,ds, \partial_t u - \partial_t u_\ast \right) = \int_U f \cdot (\partial_t u - \partial_t u_\ast)
$$

and

$$
\int_U \partial_{tt} u_\ast \cdot (\partial_t u - \partial_t u_\ast) \,dx + a(u_\ast, \partial_t u - \partial_t u_\ast)
$$

$$
+ \left( \int_0^t B(t-s)u_\ast(s) \,ds, \partial_t u - \partial_t u_\ast \right) = \int_U f \cdot (\partial_t u - \partial_t u_\ast).
$$

It follows, subtracting and setting $v = u - u_\ast$, that

$$
(\partial_{tt} v, \partial_t v) + a(v, \partial_t v) + \left( \int_0^t B(t-s)v(s) \,ds, \partial_t v \right) = 0.
$$

The uniqueness follows directly from the proof in [8, pp. 186-187].

3.1. Galerkin Approximations

By analogy with the approach taken in [8,9], we will construct the weak solution of the hyperbolic integrodifferential problem by Galerkin's method. More precisely, select smooth functions $w_k = w_k(x)$ ($k = 1, 2, \ldots$), such that

$$
\{w_k\}_{k=1}^\infty \text{ is an orthogonal basis of } H
$$

and

$$
\{w_k\}_{k=1}^\infty \text{ is an orthonormal basis of } V.
$$

Such bases always exist for a Hilbert space. Fix now a positive integer $m$. We will look for a function $u_m : [0, T] \rightarrow H$ of the form,

$$
u_m(t) := \sum_{k=1}^m d_k^m(t) w_k,
$$

where we intend to select the coefficient $d_k^m(t)$ ($0 \leq t \leq T$, $k = 1, \ldots, m$) to satisfy

$$
d_k^m(0) = \frac{d_k^m}{d_k^m'}(0) = 0 \quad (k = 1, \ldots, m)
$$
and
\[
(\partial_{tt} u_m, w_j) + a(u_m, w_j) + \left( \int_0^t B(t-s) w_m(s) \, ds, w_j \right)
\]
\[
= d_\ell'' + \sum_{k=1}^m d_m^k \int_U a c(w_k) e(w_j) \, dx + \sum_{k=1}^m \int_0^t \int_U b(t-s) e(w_k) e(w_j) d_m^k(s) \, dx \, ds
\]
\[
= \int_U f \cdot w_j, \quad j = 1, 2, \ldots, m.
\]

This is a linear system of ordinary integro-differential equations subject to homogeneous initial conditions. According to standard theory for ordinary differential equations, there exists a unique $C^1$ function $d_m(t) = (d_1^m(t), \ldots, d_m^m(t))$, satisfying homogeneous initial conditions and solving (3.2) for $0 \leq t \leq T$. We also know $d_m(t) \in L^2([0,T])$. Hereafter, our plan is to send $m \to \infty$ and to show a subsequence of $u_m$ converges to the weak solution. For this, we will need some uniform estimates.

**Theorem 3.2.** There exists a constant $C$, depending only on $U$, $T$, coefficients of $a$ and $b$, such that
\[
\max_{0 \leq t \leq T} \left( \|u_m\|_H + \|\partial_t u_m\|_V \right) + \|\partial_{tt} u_m\|_{L^2(0,T;H^{-1})} \leq C \|f\|_{L^2(0,T;V)}
\]
for $m = 1, 2, \ldots$.

**Energy Estimates Proof.** Multiply equation (3.2) by $d_m^j$, sum $j = 1, \ldots, m$, and recall the construction of $\tilde{u}_m$ to discover
\[
(\partial_{tt} u_m, \partial_t u_m) + a(u_m, \partial_t u_m) = (f, \partial_t u_m) - \left( \int_0^t B(t-s) u_m(s) \, ds, \partial_t u_m \right),
\]
for a.e., $0 \leq t \leq T$. It follows
\[
\frac{d}{dt} \left( \|\partial_t u_m\|_{L^2(U)} + a(u_m, u_m) \right)
\]
\[
\leq \left( \|f\|_{L^2(U)}^2 + \|\partial_t u_m\|_{L^2(U)}^2 \right) - 2 \int_0^t B(t-s) u_m(s) \, ds, \partial_t u_m(t)
\]
and therefore,
\[
\|\partial_t u_m\|_{L^2(U)} + a(u_m, u_m)
\]
\[
\leq \int_0^t \left( \|f\|_{L^2(U)}^2 + \|\partial_t u_m\|_{L^2(U)}^2 \right) \, ds - 2 \int_0^t \left( \int_0^{s_1} B(s_1-s) u_m(s) \, ds, \partial_s u_m(s_1) \right) \, ds_1.
\]
We note that, $\forall \lambda > 0$, there exists $\alpha > 0$, such that
\[
a(u_m, u_m) + \lambda(u_m, u_m) \geq \alpha \|u_m\|_H
\]
and
\[
\int_0^t ds_1 \left( \int_0^{s_1} B(s_1-s) u_m(s) \, ds, \partial_s u_m(s_1) \right) \leq C \left( \int_0^t \|u_m\|^2 \, ds + \|u_m\| \int_0^t \|u_m\| \, ds \right).
\]
It implies
\[
\|\partial_t u_m\|_{L^2(U)}^2 + \|u_m\|_H^2 \leq C \int_0^t \left( \|\partial_t u_m\|_{L^2(U)} + \|u_m\|_H^2 \right) \, ds + \int_0^T \|f\|_{L^2(U)}^2 \, ds.
Thus, Gronwall’s inequality (integral form) yields the estimate,
\[ \| \partial_t u_m \|^2_{L^2(U)} + \| u_m \|^2_H \leq C \int_0^T \| f \|^2_{L^2(U)} \, ds. \]

Finally, fix any \( v \in H \) with \( \| v \|_H \leq 1 \), and write \( v = v^1 + v^2 \), where \( v^1 \in \text{span} \{ w_k \}_{k=1}^m \) and \( (v^2, w_k) = 0 \) (\( k = 1, \ldots, m \)). Then,
\[
(\partial_t u_m, v) = (\partial_t u_m, v^1) = (f, v^1) - a(u_m, v^1) - \left( \int_0^t B(t - s) u_m(s) \, ds, v^1 \right).
\]

Thus,
\[
| (\partial_t u_m, v) | \leq C \left( \| f \|_{L^2(U)} + \| u_m \|_H \right),
\]
since \( \| v^1 \|_H \leq 1 \). Consequently,
\[
\int_0^T \| \partial_t u_m \|^2_{H^{-1}} \, dt \leq C \int_0^T \| f \|^2_{L^2(U)} + \| u_m \|_H \, dt \leq C \| f \|^2_{L^2(0,T;L^2(U))}.
\]

Now, we pass to limits in our Galerkin approximations.

**Theorem 3.3.** There exists a weak solution to Problem 2.1.

**Existence Proof.** According to the energy estimates, there exists a subsequence \( \{ u_{m_k} \}_{k=1}^\infty \) in \( \{ u_m \} \) and \( u \in L^2(0,T;H^1_0(U)) \) with \( \partial_t u \in L^2(0,T;L^2(U)) \), \( \partial_{tt} u \in L^2(0,T;H^{-1}(U)) \), such that

\[
\begin{align*}
 u_{m_k} & \rightharpoonup u \text{ weakly in } L^2(0,T;H^1_0(U)), \\
 u_{m_k}' & \rightharpoonup u' \text{ weakly in } L^2(0,T;L^2(U)), \\
 u_{m_k}'' & \rightharpoonup u'' \text{ weakly in } L^2(0,T;H^{-1}(U)).
\end{align*}
\]

Next, fix an integer \( N \) and choose a function \( v \in C([0,T];H^1_0(U)) \) of the form,
\[
v = \sum_{k=1}^N d^k(t) w_k,
\]
where \( \{ d^k \} \) are smooth functions. We select \( m \geq N \), multiply equation (3.2) by \( d^k \), sum \( k = 1, \ldots, m \), and then integrate with respect to \( t \), to discover
\[
\int_0^T (\partial_{tt} u_m, v) + a(u_m, v) \, dt = \int_0^T (f, v) \, dt - \int_0^T dt \left( \int_0^t B(t - s) u_m(s) \, ds, v(t) \right). \tag{3.7}
\]

We set \( m = m_h \) and recall the weak convergence, to find in the limit that
\[
\int_0^T (\partial_{tt} u, v) + a(u, v) \, dt = \int_0^T (f, v) \, dt - \int_0^T dt \left( \int_0^t B(t - s) u(s) \, ds, v(t) \right). \tag{3.8}
\]

Then, this equality holds for all functions \( v \in L^2(0,T;H^1_0(U)) \), since functions of the form (3.6) are dense in this space. The existence theorem is proved.

Passing to limits in Theorem 3.2, we deduce the following.

**Theorem 3.4.** Regularity.
\[
\sup_{0 \leq t \leq T} (\| u \|_H + \| \partial_t u \|_V) \leq C \| f \|_{L^2(0,T;V)}.
\]
4. TWO-SCALE CONVERGENCE

We are going to use the notion of two-scale convergence which was introduced by Nguetseng [6] and Allaire [7]. Let $C^\infty_\mathbb{P}(Y)$ denote the space of those $C^\infty$ functions periodic on $Y$.

**DEFINITION 4.1.** \(\{u^\varepsilon(x, t)\} \subset [L^2([0, T] \times U)]^n\) two-scale converges to \(u(t, x, y) \in [L^2([0, T] \times U \times Y)]^n\) iff for any \(\phi(t, x, y) \in [C^\infty([0, T] \times U, C^\infty_\mathbb{P}(Y))]^n\), one has

\[
\lim_{\varepsilon \to 0} \int_0^T \int_U u^\varepsilon(t, x) \cdot \phi(t, x, \frac{x}{\varepsilon}) \, dx \, dt = \int_0^T \int_Y u(t, x, y) \cdot \phi(t, x, y) \, dt \, dy. \tag{4.1}
\]

One can establish the following theorem with the obvious modifications of the argument in [6] and [7].

**THEOREM 4.1.**

(i) If \(\{u^\varepsilon(x, t)\}\) is a bounded sequence in \([L^2([0, T], L^2(U))]^n\), then there exists \(u_0(t, x, y) \in [L^2([0, T] \times U, L^2_\mathbb{P}(Y))]^n\), such that a subsequence of \(\{u^\varepsilon(x, t)\}\) two-scale converges to \(u_0(t, x, y)\) in the sense of Definition 4.1.

(ii) If \(\{u^\varepsilon(x, t)\}\) is a bounded sequence in \([L^2([0, T], H^1(U))]^n\), then there exist \(u_0(t, x, y) \in [L^2([0, T] \times U, H^1_\mathbb{P}(Y))]^n\) and \(u_1(t, x, y) \in [L^2([0, T] \times U, H^2_\mathbb{P}(Y))]^n\), such that a subsequence of \(\{u^\varepsilon(x, t)\}\) two-scale converges to \(u_0(t, x, y)\) and a subsequence of \(\nabla_x u^\varepsilon\) two-scale converges to \(\nabla_x u_0 + \nabla_y u_1\) in the sense of Definition 4.1.

(iii) If \(\{u^\varepsilon(x, t)\}\) and \(\{\nabla_x u^\varepsilon(x, t)\}\) are bounded sequence in \([L^2([0, T], L^2(U))]^n\), then there exists \(u_0(t, x, y) \in [L^2([0, T] \times U, L^2_\mathbb{P}(Y))]^n\), such that a subsequence of \(\{u^\varepsilon(x, t)\}\) and \(\{\nabla_x u^\varepsilon(x, t)\}\) two-scale converges to \(u_0(t, x, y)\) and \(\nabla_y u_0\) in the sense of Definition 4.1.

The regularity Theorem 3.4 suggests that two-scale convergence makes sense for Problem 2.1. Let's consider a scalar function \(\psi(t) \in C^2[0, T]\) satisfying \(\psi(T) = \psi'(T) = 0\). Multiplying (2.9) by \(\psi(t)\) satisfying and integrating in time from 0 to T, we have

\[
\int_0^T \psi''(t) \, dt \int_U u^\varepsilon \cdot \phi + \int_0^T \psi(t) \, dt \int_U [A^\varepsilon e(u^\varepsilon)] : e(\phi) + \int_0^T \psi(t) \, dt \int_U f \, \cdot \phi. \tag{4.2}
\]

If we replace the test function by \(\phi(x) + \varepsilon \phi_1(x, x/\varepsilon)\), then, applying Theorem 4.1, we obtain

\[
\int_0^T \psi''(t) \, dt \int_U u^\varepsilon \cdot \left[\phi(x) + \varepsilon \phi_1 \left(x, \frac{x}{\varepsilon}\right)\right] = \int_0^T \psi''(t) \, dt \int_U u_0 \cdot \phi, \tag{4.3}
\]

\[
\int_0^T \psi(t) \, dt \int_U [A^\varepsilon e_x(u^\varepsilon)] : e_x \left[\phi(x) + \varepsilon \phi_1 \left(x, \frac{x}{\varepsilon}\right)\right] \to \int_0^T \psi(t) \, dt \int_U A [e_x(u_0) + e_y(u_1)] : (e_x(\phi) + e_y(\phi_1)), \tag{4.4}
\]

\[
\int_0^T \psi(t) \, dt \int_U A [e_x(u_0) + e_y(u_1)] : (e_x(\phi) + e_y(\phi_1)), \tag{4.5}
\]

\[
\int_0^T \psi(t) \, dt \int_U f \cdot \left[\phi(x) + \varepsilon \phi_1 \left(x, \frac{x}{\varepsilon}\right)\right] \to \int_0^T \psi(t) \, dt \int_U f \cdot \phi. \tag{4.6}
\]
LEMMA 4.1. Let \( \{u_0(t, x), u_1(t, x, y)\} \) be as in Theorem 4.1, then \( \{u_0(t, x), u_1(t, x, y)\} \) satisfy the following equations in distributional sense.

\[
\begin{align*}
\frac{d^2}{dt^2} \int_U u_0 \phi + \int_U \int_Y A [e_x(u_0) + e_y(u_1)] : e_x(\phi) \\
+ \int_0^t ds \int_U \int_Y B(t-s) [e_x(u_0(s)) + e_y(u_1(s))] : e_x(\phi) = \int_U f \cdot \phi,
\end{align*}
\]

\[\text{(4.7)}\]

\[
\begin{align*}
\int_U \int_Y A [e_x(u_0) + e_y(u_1)] : e_y(\phi_1) \\
+ \int_0^t ds \int_U \int_Y B(t-s) [e_x(u_0(s)) + e_y(u_1(s))] : e_y(\phi_1) = 0,
\end{align*}
\]

\[\text{(4.8)}\]

\[
\begin{align*}
u_0(0) = \partial_t u_0(0) = 0, \quad u_1(0) = 0.
\end{align*}
\]

\[\text{(4.9)}\]

PROOF. Let \( \epsilon \to 0 \) in (4.2) and use (4.3)-(4.6) and integration by parts in \( t \),

\[
\begin{align*}
\int_0^T \psi''(t) \int_U u_0 \phi + \int_0^T \psi(t) \int_U \int_Y A [e_x(u_0) + e_y(u_1)] : e_x(\phi) & \\
+ \int_0^T \psi(t) \int_U \int_Y B(t-s) [e_x(u_0(s)) + e_y(u_1(s))] : e_x(\phi) & \\
+ \int_0^T \psi(t) \int_U \int_Y A [e_x(u_0) + e_y(u_1)] : e_y(\phi_1) & \\
+ \int_0^T \psi(t) \int_U \int_Y B(t-s) [e_x(u_0(s)) + e_y(u_1(s))] : e_y(\phi_1) = \int_U f \cdot \phi.
\end{align*}
\]

\[\text{(4.10)}\]

We are now able to establish the lemma by setting \( \phi_1 = 0 \) and \( \phi = 0 \), respectively.

LEMMA 4.2. The system (4.7)-(4.9) has a unique solution.

PROOF. It is sufficient to prove that for \( f = 0, g = 0 \), we have only the trivial solution \( u_0 = u_1 = 0 \). Set \( \phi = \partial_t u_0, \phi_1 = \partial_t u_1 \) as test function in (4.7) and (4.8), respectively,

\[
\begin{align*}
\int_U \partial_t u_0 \partial_t u_0 + \int_U \int_Y A (e_x(u_0) + e_y(u_1)) : e_x(\partial_t u_0) & \\
+ \int_0^t ds \int_U \int_Y B(t-s) (e_x(u_0(s)) + e_y(u_1(s))) : e_x(\partial_t u_0(t)) & = 0,
\end{align*}
\]

\[\text{(4.11)}\]

\[
\begin{align*}
\int_U \int_Y A (e_x(u_0) + e_y(u_1)) : e_y(\partial_t u_1) & \\
+ \int_0^t ds \int_U \int_Y B(t-s) (e_x(u_0(s)) + e_y(u_1(s))) : e_y(\partial_t u_1(t)) & = 0.
\end{align*}
\]

\[\text{(4.12)}\]

Adding (4.11) and (4.12) and integrating in time,

\[
\begin{align*}
\int_U (\partial_t u_0)^2 + \int_U \int_Y A (e_x(u_0) + e_y(u_1)) : (e_x(u_0) + e_y(u_1)) & \\
= -2 \int_0^t ds \int_0^s ds_1 \int_U \int_Y B(s-s_1) (e_x(u_0(s_1)) + e_y(u_1(s_1))) : (e_x(\partial_t u_0) + e_y(\partial_t u_1)).
\end{align*}
\]

\[\text{(4.13)}\]

Note that

\[
\begin{align*}
\left| \int_0^t ds \int_0^s ds_1 \int_U \int_Y B(s-s_1) e(\varphi(s_1)) : e(\partial_t \varphi(s)) \right| \\
\leq C \left[ \int_0^t \| e(s) \|^2 ds + \| \varphi(t) \| \int_0^t \| \varphi(s) \| ds \right].
\end{align*}
\]

Gronwall's inequality, initial conditions \( u_0 \equiv 0 \) and \( u_1 \equiv 0 \), imply \( u_0 = 0, u_1 = 0 \).
5. DERIVATION OF THE EFFECTIVE EQUATION OF $u_0$

Since it is known that the system (4.7)-(4.9) has a unique solution, we seek this solution $u_1(t,x,y)$ in the form,

$$u_1(t,x,y) = \sum_{ij} \int_0^t K^{ij}(y, t-s) (e_x (\partial_x u_0))_{ij}(x, s) \, ds. \tag{5.1}$$

Plugging (5.1) into (4.8),

$$\int_U \int_Y A \left\{ e_x(u_0) + \sum_{ij} \int_0^t e_y(K^{ij})(y, t-s) (\partial_x e_x(u_0))_{ij}(x, s) \, ds \right\} : e_y(\phi_1)$$

$$+ \int_0^t ds_1 \int_U \int_Y B(t-s_1) \left\{ e_x(u_0(s_1)) + \sum_{ij} e_y(K^{ij})(y, s_1-s) (\partial_x e_x(u_0))_{ij}(x, s) \, ds \right\} : e_y(\phi_1(t)) = 0. \tag{5.2}$$

Interchanging order of integration and integrating by parts with respect to $s$, we have

$$\int_0^t ds_1 \int_U \int_Y B(t-s_1) \int_0^{s_1} e_y(K^{ij})(y, s_1-s) (\partial_x e_x(u_0))_{ij}(x, s) \, ds$$

$$= \int_0^t (\partial_x e_x(u_0))_{ij}(x, s) \, ds \int_U \int_Y B(t-s_1) e_y(K^{ij})(y, s_1-s) \, ds_1$$

$$= \int_0^t (e_x(u_0))_{ij}(x, s) \int_U \int_Y \left\{ B(t-s) e_y(K^{ij})(y, 0) + \right.$$

$$\left. + \int_t^s B(t-s_1) e_y(\partial_2 K^{ij})(y, s_1-s) \, ds_1 \right\} \, ds.$$

Here, we use $\partial_2(\cdot)$ to represent the partial derivative with respect to the second argument. Applying integration by parts,

$$\int_0^t e_y(K^{ij})(y, t-s) (\partial_x e_x(u_0))_{ij}(x, s) \, ds = e_y(K^{ij})(y, 0) (e_x(u_0))_{ij}(x, t)$$

$$+ \int_0^t e_y(\partial_2 K^{ij})(y, t-s) (e_x(u_0))_{ij}(x, s) \, ds.$$

Therefore, (5.2) becomes

$$\int_U \int_Y A \left\{ e_x(u_0) + \sum_{ij} \left[ e_y(K^{ij})(y, 0) (e_x(u_0))_{ij}(x, t) \right. \right.$$
Collecting $e_x(u_0)$ terms, and choosing $K^{ij}(0)$ so that
\begin{equation}
\int_U \int_Y A \left\{ e_x(u_0) + \sum_{ij} e_y(K^{ij})(y,0)(e_x(u_0))_{ij}(x,t) \right\} : e_y(\phi_1) = 0,
\end{equation}
we have
\begin{equation}
\int_Y A \{ \delta_{ij} + e_y(K^{ij})(y,0) \} : e_y(\phi_1) = 0, \quad \text{for any } \phi_1 \in [H^1_0(Y)]^n,
\end{equation}
where $\delta_{ij}$ are the discrete version of the delta functions. Consequently, $K^{ij}(0)$ is determined by the following linear elastic system,
\begin{equation}
\text{div}_Y \left\{ A(\delta_{ij} + e_y(K^{ij})(y,0)) \right\} = 0, \quad K^{ij}(y,0) \in [H^1_0(Y)]^n.
\end{equation}
Collecting time integral terms in (5.3) and choosing $K^{ij}(t,y)$ so that
\begin{equation}
\int_0^t ds \int_Y B(t-s) e_x(u_0(s)) + \sum_{ij} e_x(u_0)_{ij}(x,s) \left\{ A e_y(\partial_2 K^{ij})(y,t-s) + B(t-s) e_y(K^{ij})(y,0) + \int_s^t B(t-s_1) e_y(\partial_2 K^{ij})(y,s_1-s) \, ds_1 \right\} : e_y(\phi_1) = 0,
\end{equation}
which implies
\begin{equation}
\int_Y \left\{ B(t-s) \delta_{ij} + A e_y(\partial_2 K^{ij})(y,t-s) + B(t-s) e_y(K^{ij})(y,0) + \int_s^t B(t-s_1) e_y(\partial_2 K^{ij})(y,s_1-s) \, ds_1 \right\} : e_y(\phi_1) = 0,
\end{equation}
i.e., $K^{ij}(\tau,y)$ is determined by the following system,
\begin{equation}
\text{div}_Y \left\{ B(\tau) \delta_{ij} + A e_y(\partial_2 K^{ij})(y,\tau) + B(\tau) e_y(K^{ij})(y,0) + \int_0^\tau B(s_2) e_y(\partial_2 K^{ij})(y,s_2+\tau) \, ds_2 \right\} = 0,
\end{equation}
\begin{equation}
K^{ij}(t,y) \in L^2([0,T],[H^1_0(Y)]^n), \quad K^{ij}(0,y) \text{ is given by (5.6).}
\end{equation}
We proceed to get effective equation. Now, plugging (5.1) into (4.7)
\begin{equation}
\frac{d^2}{dt^2} \int_U u_0 \phi + \int_U \int_Y A \left\{ e_x(u_0) + \sum_{ij} \int_0^t e_x(K^{ij})(y,t-s)(\partial_2 e_x(u_0))_{ij}(x,s) \, ds \right\} : e_x(\phi)
+ \int_0^t ds \int_U \int_Y B(t-s) e_x(u_0)
+ \sum_{ij} \int_0^t e_y(K^{ij})(y,s-s_1)(e_x(\partial_2 u_0))_{ij}(x,s_1) \, ds_1 : e_x(\phi) = \int_U f \cdot \phi.
\end{equation}
Performing integration by parts and interchanging order of integration, we have
\[
\frac{d^2}{dt^2} \int_U u_0 \phi + \int_U \int_Y A \left\{ e_x (u_0) + \sum_{ij} e_y (K^{ij}) (y, 0) (e_x (u_0))_{ij} (x, t) \right\} : e_x (\phi)
\]
\[
+ \int_0^t ds \int_U \int_Y B (t - s) e_x (u_0 (s))
\]
\[
+ \sum_{ij} e_x (u_0)_{ij} (x, s) \left\{ B (t - s) e_y (K^{ij}) (y, 0) + A e_y (\partial_2 K^{ij}) (y, t - s)
\right\}
\]
\[
+ \int_s^t B (t - s_1) e_y (\partial_2 K^{ij}) (y, s_1 - s) \, ds_1 \right\} : e_x (\phi) = \int_U \int_0^t f \cdot \phi.
\]
(5.11)

To simplify the above equation it is useful to introduce the symmetric tensors \(A\) and \(B\) as
\[
A_{ijkl} = \int_Y A_{ijkl} + \int_Y \sum_{mn} \frac{1}{2} A_{kmn} \left( \frac{\partial K^{ij}}{\partial y_m} (0) + \frac{\partial K^{ij}}{\partial y_n} (0) \right),
\]
(5.12)
\[
B_{ijkl} (\tau) = \int_Y B_{ijkl} + \int_Y \sum_{mn} \frac{1}{2} A_{kmn} \left( \frac{\partial (\partial_2 K^{ij})}{\partial y_m} (0) + \frac{\partial (\partial_2 K^{ij})}{\partial y_n} (0) \right)
\]
\[
+ \frac{1}{2} B_{kmn} \left( \frac{\partial K^{ij}}{\partial y_m} (0) + \frac{\partial K^{ij}}{\partial y_n} (0) \right) + \int_\tau^{s_2} B (s_2) e_y (\partial_2 K^{ij}) (y, s_2 + \tau) \, ds_2 \right\}.
\]
(5.13)

The effective equation for \(u_0\) then may be seen to take on the form,
\[
\partial_t u_0 - \text{div} A e (u_0) - \text{div} \int_0^t B (t - s) e (u_0) \, ds = f.
\]
(5.14)

We may summarize this approach by stating our result as a theorem.

**Theorem 5.1.** Let \(u_0 (t, x)\) be as in Theorem 4.1, then \(u_0 (t, x)\) satisfies equation (5.14) and (4.9).

From effective equation (5.14), we get the effective constitutive relation between stress and strain,
\[
\sigma_0 = A e (u_0) + \int_0^t B (t - s) e (u_0) \, ds.
\]
(5.15)

**REFERENCES**