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J. Math. Anal. Appl. 327 (2007) 1130–1142

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Global attractors for p-Laplacian equation ☆

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Received 23 March 2005

Available online 6 June 2006

Submitted by J.A. Ball

Abstract

The existence of a $(L^2(\Omega), W_0^{1,p}(\Omega)\cap L^q(\Omega))$-global attractor is proved for the p-Laplacian equation $u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(u) = g$ on a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ with Dirichlet boundary condition, where $p \geqslant 2$. The nonlinear term f is supposed to satisfy the polynomial growth condition of arbitrary order $c_1|u|^q - k \leqslant f(u)u \leqslant c_2|u|^q + k$ and $f'(u) \geqslant -l$, where $q \geqslant 2$ is arbitrary. There is no other restriction on p and q. The asymptotic compactness of the corresponding semigroup is proved by using a new a priori estimate method, called asymptotic a priori estimate.

Keywords: Attractors; p-Laplacian equation; Asymptotic a priori estimate

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the existence of global attractors in $W_0^{1,p}(\Omega)$ and $L^q(\Omega)$ for the following p-Laplacian equation:

$$u_t - \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + f(u) = g \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

with the Dirichlet boundary condition

$$u|_{\partial\Omega} = 0, \tag{1.2}$$

☆ Supported in part by the NSFC Grant (10471056) and Trans-Century Training Program Foundation for the Talents by the Ministry of Education.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.04.085

and initial condition

$$u(x,0) = u_0(x), \tag{1.3}$$

where $p \geqslant 2$; $f : \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions

$$f'(u) \geqslant -l \quad \text{for some } l \geqslant 0 \tag{1.4}$$

and

$$c_1|u|^q - k \leqslant f(u)u \leqslant c_2|u|^q + k, \quad q \geqslant 2; \tag{1.5}$$

and $g \in L^s(\Omega)$, here s satisfies

$$s \geqslant \min\left\{2, \frac{n(q+p-2)}{p(n+q-1)-n}\right\}. \tag{1.6}$$

This problem has been studied extensively in many monographs and lectures; see, e.g., [1–5,10] and references therein. In [1], Babin and Vishik provided detailed discussion about this problem, however, even though they got that the corresponding semigroup $\{S(t)\}_{t\geqslant 0}$ has a $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-bounded absorbing set, they only established the existence of $(L^2(\Omega), (W_0^{1,p}(\Omega) \cap L^q(\Omega))_w)$-global attractor (see [1, p. 127, Theorems 3.1 and 3.2]), and the existence of $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-global attractor remained unknown; in Temam [10], only the special case of $f = ku$ was discussed; in [2], Carvalho, Cholewa and Dlotko considered the existence of global attractors for problems with monotone operators, and as an application, they got the existence of $(L^2(\Omega), L^2(\Omega))$-global attractor for p-Laplacian equation, in which the nonlinear term satisfies a condition similar to (1.5), see also Cholewa and Dlotko [5].

Recently, Carvalho and Gentile in [4], combining with their comparison results developed in [3], obtained that the corresponding semigroup has a $(L^2(\Omega), W_0^{1,p}(\Omega))$-global attractor. However, they need some additional assumptions, i.e., either assume that $p > \frac{n}{2}$ and $f = f_1 + f_2$, where f_1 satisfying $(f_1, u) \geqslant 0$ and f_2 being a global $(L^2(\Omega), L^2(\Omega))$-Lipschitz mapping, or assume that f satisfies some growth condition such that it can be dominated by the p-Laplacian operator.

It is well known that if we want to prove the existence of the $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-global attractor, we need to verify that the semigroup associated with Eq. (1.1) has some kind of compactness, which is necessary, in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$. As we know, under assumptions (1.4)–(1.6), the solutions of Eq. (1.1) are at most in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ and have no higher regularities. Therefore, there is no compact embedding results for these cases. Moreover, since the solutions have no higher regularities and the semigroup associated with the solutions is only $(L^2(\Omega), (W_0^{1,p}(\Omega) \cap L^q(\Omega))_w)$-continuous if we do not impose any restrictions on p and q (see Babin and Vishik [1]), it seems to be difficult that we can directly verify the asymptotic compactness in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$. Hence, in order to obtain the strong attracting property of global attractors, by the usual methods, one must impose some conditions on p, q or directly on the nonlinearity, which guarantee that the nonlinear term can be dominated by the principal part.

In this paper, motivated by the ideas in [8,9,11], we give a new method for verifying that the semigroup is $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-asymptotically compact, and then establish the existence of the $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-global attractor for problem (1.1)–(1.3) under assumptions (1.4)–(1.6).

Our main result is the following:

Theorem 1.1. *Assume that Ω is a bounded smooth domain in $\mathbb{R}^n$ $(n \geqslant 3)$, f satisfies* (1.4)–(1.5) *and $g \in L^s(\Omega)$, where s satisfies* (1.6)*. Then the semigroup $\{S(t)\}_{t\geqslant 0}$ generated by* (1.1)–(1.3) *with initial data $u_0 \in L^2(\Omega)$ has a $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-global attractor $\mathcal{A}$, that is, $\mathcal{A}$ is compact, invariant in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $W_0^{1,p}(\Omega) \cap L^q(\Omega)$.*

In some sense, this conclusion improves the previous relevant results.

For convenience, hereafter let $\|\cdot\|_q$ be the norm of $L^q(\Omega)$ $(q \geqslant 1)$, $|u|$ the modular (or absolute value) of u, $m(e)$ (sometimes we also write it as $|e|$) the Lebesgue measure of $e \subset \Omega$, $\Omega(u \geqslant M) = \{x \in \Omega\colon u(x) \geqslant M\}$ and $\Omega(u \leqslant -M) = \{x \in \Omega\colon u(x) \leqslant -M\}$, and C an arbitrary positive constant, which may be different from line to line and even in the same line. Since $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, we take the equivalent norm in $W_0^{1,p}(\Omega)$ to be

$$\|\nabla u\|_p = \left(\sum_{i=1}^{n} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^p\right)^{1/p} \quad \text{for any } u \in W_0^{1,p}(\Omega).$$

2. Abstract results

2.1. Preliminaries

In this subsection, we recall some basic concepts about the bi-spaces global attractor, see [1,6,8,11] for more details.

Definition 2.1. Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on Banach space X and Z be a metric space. A set $\mathcal{A} \subset X \cap Z$, which is invariant, closed in X, compact in Z and attracts bounded subsets of X in the topology of Z is called an (X, Z)-global attractor.

Definition 2.2. Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on Banach space X. A bounded subset B_0 of Z satisfies that for any bounded subset $B \subset X$, there is $T = T(B)$, such that $S(t)B \subset B_0$ for any $t \geqslant T$ is called an (X, Z)-bounded absorbing set.

Definition 2.3. Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on Banach space X. $\{S(t)\}_{t\geqslant 0}$ is called (X, Z)-asymptotically compact, if for any bounded(in X) sequence $\{x_n\}_{n=1}^{\infty} \subset X$ and $t_n \geqslant 0$, $t_n \to \infty$ as $n \to \infty$, $\{S(t_n)x_n\}_{n=1}^{\infty}$ has a convergent subsequence with respect to the topology of Z.

Proposition 2.4. *For any $\varepsilon > 0$, the bounded subset B of $L^q(\Omega)$ $(q > 0)$ has a finite ε-net in $L^q(\Omega)$ if there exists a positive constant $M = M(\varepsilon)$ which depends on ε, such that*:

(i) *B has a finite $(3M)^{(p-q)/p}(\frac{\varepsilon}{2})^{q/p}$-net in $L^p(\Omega)$ for some p, $p > 0$*; *and*

(ii) $$\left(\int_{\Omega(|u|\geqslant M)} |u|^q\right)^{1/q} < 2^{-(2q+2)/q}\varepsilon \quad \textit{for all } u \in B.$$

Proposition 2.5. *Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on $L^p(\Omega)$ $(p \geqslant 1)$ and suppose that $\{S(t)\}_{t\geqslant 0}$ has a bounded absorbing set in $L^p(\Omega)$. Then for any $\varepsilon > 0$ and any bounded subset $B \subset L^p(\Omega)$, there exist positive constants $T = T(B)$ and $M = M(\varepsilon)$ such that*

$$m\big(\Omega\big(\big|S(t)u_0\big| \geqslant M\big)\big) \leqslant \varepsilon \quad \text{for all } u_0 \in B \text{ and } t \geqslant T.$$

2.2. Abstract results

Generalizing the idea of [8,11], we give the following results, which are very useful in our later discussion for the existence of the bi-spaces global attractor:

Proposition 2.6. *Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on $L^p(\Omega)$ and have a $(L^p(\Omega), L^q(\Omega))$-global attractor, where $p \leqslant q < \infty$. Then for any $\varepsilon > 0$ and any bounded subset B of $L^p(\Omega)$, there exist $M = M(\varepsilon)$ and $T = T(\varepsilon, B)$, such that*

$$\int\limits_{\Omega(|S(t)u_0|\geqslant M)} |S(t)u_0|^q \leqslant 2^{q+1}\varepsilon \quad \text{for all } t \geqslant T \text{ and } u_0 \in B. \tag{2.1}$$

Proof. Let $\mathcal{A}$ be the $(L^p(\Omega), L^q(\Omega))$-global attractor, then there is a $T = T(\varepsilon, B)$, such that

$$S(t)B \subset \mathcal{N}_q\left(\mathcal{A}, \frac{\varepsilon^{1/q}}{2}\right) \quad \text{for all } t \geqslant T,$$

where $\mathcal{N}_q(\mathcal{A}, \frac{\varepsilon^{1/q}}{2})$ means the $\frac{\varepsilon^{1/q}}{2}$ neighborhood of $\mathcal{A}$ with respect to the L^q-norm.

Hence, by the standard Lebesgue integral theory, combining with the compactness of $\mathcal{A}$ in $L^q(\Omega)$, (2.1) follows immediately. □

Theorem 2.7. *Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on $L^p(\Omega)$ and have a $(L^p(\Omega), L^p(\Omega))$-global attractor, where $p \leqslant q < \infty$. Then $\{S(t)\}_{t\geqslant 0}$ has a $(L^p(\Omega), L^q(\Omega))$-global attractor provided that the following conditions hold*:

(i) *$\{S(t)\}_{t\geqslant 0}$ has a $(L^p(\Omega), L^q(\Omega))$-bounded absorbing set*;
(ii) *for any $\varepsilon > 0$ and any bounded subset B of $L^p(\Omega)$, there exist positive constants $M(= M(\varepsilon))$ and $T(= T(\varepsilon, B))$, such that*

$$\int\limits_{\Omega(|S(t)u_0|\geqslant M)} \big|S(t)u_0\big|^q < \varepsilon \quad \text{for all } u_0 \in B \text{ and } t \geqslant T. \tag{2.2}$$

Proof. Using Proposition 2.4, from condition (ii) and the assumption that $\{S(t)\}_{t\geqslant 0}$ has a $(L^p(\Omega), L^p(\Omega))$-global attractor, we know that $\{S(t)\}_{t\geqslant 0}$ is $(L^p(\Omega), L^q(\Omega))$-asymptotically compact.

Set

$$\mathcal{A}_q = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} S(t)B_0}^{L^q}, \tag{2.3}$$

where B_0 is a $(L^p(\Omega), L^q(\Omega))$-bounded absorbing set. Obviously, $\mathcal{A}_q$ is nonempty, compact in $L^q(\Omega)$ and attracts every bounded subset B of $L^p(\Omega)$ with respect to the L^q-norm (e.g., see [7,10]).

In what follows, we need only to verify that $\mathcal{A}_q$ is invariant, i.e., $S(t)\mathcal{A}_q = \mathcal{A}_q$ for all $t \geqslant 0$. Denote by $\mathcal{A}_p$ the $(L^p(\Omega), L^p(\Omega))$-global attractor, then, from the definitions of $\mathcal{A}_p$, $\mathcal{A}_q$ and B_0, we can easily show that $\mathcal{A}_q = \mathcal{A}_p$. Therefore, $\mathcal{A}_q$ is invariant since $\mathcal{A}_p$ is invariant. □

Remark 2.8. As in the above theorem, by establishing some kinds of a priori estimates as (2.2), from the $(L^p(\Omega), L^p(\Omega))$-asymptotic compactness, we can get the $(L^p(\Omega), L^q(\Omega))$-asymptotic compactness, where $q \geqslant p$. We call this method, getting a "strong" compactness from a "weak" compactness by establishing some kinds of a priori estimates as (2.2), as *asymptotic a priori estimate method.*

Remark 2.9. For our problems, after establishing the existence of a $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-bounded absorbing set, we can see that the existence of a $(L^2(\Omega), L^2(\Omega))$-global attractor is obvious. Hence, if we want to obtain the existence of a $(L^2(\Omega), L^q(\Omega))$-global attractor, we need only to verify condition (ii).

3. Absorbing set in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$

We start with the following general existence and uniqueness of solutions which can be obtained by the standard Galerkin methods; see also in Babin and Vishik [1].

Theorem 3.1. *Assume that Ω is a bounded smooth domain in $\mathbb{R}^n$ $(n \geqslant 3)$, f satisfies* (1.4)–(1.5) *and $g \in L^s(\Omega)$, where s satisfies* (1.6). *Then for any initial data $u_0 \in L^2(\Omega)$ and any $T > 0$, there exists a unique solution u for Eqs.* (1.1)–(1.3) *which satisfies*

$$u \in L^\infty\big(0, T; L^2(\Omega)\big) \cap L^p\big(0, T; W_0^{1,p}(\Omega)\big) \cap L^q\big(0, T; L^q(\Omega)\big),$$

and the mapping $u_0 \to u(t)$ is both $(L^2(\Omega), L^2(\Omega))$-continuous and $(L^2(\Omega), (W_0^{1,p}(\Omega))_w \cap (L^q(\Omega))_w)$-continuous.

By Theorem 3.1, we can define the operator semigroup $\{S(t)\}_{t \geqslant 0}$ in $L^2(\Omega)$ as

$$S(t)u_0 : L^2(\Omega) \times \mathbb{R}^+ \to L^2(\Omega), \tag{3.1}$$

which is both $(L^2(\Omega), L^2(\Omega))$-continuous and $(L^2(\Omega), (W_0^{1,p}(\Omega))_w \cap (L^q(\Omega))_w)$-continuous.

In what follows, we always assume that $\{S(t)\}_{t \geqslant 0}$ is the semigroup generated by the weak solutions of Eq. (1.1) with initial data $u_0 \in L^2(\Omega)$.

Theorem 3.2. $\{S(t)\}_{t \geqslant 0}$ *has a $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-bounded absorbing set, that is, there is a positive constant ρ, such that for any bounded subset B in $L^2(\Omega)$, there exists a positive constant T which depends only on the L^2-norm of B such that*

$$\int_\Omega \big|\nabla u(t)\big|^p + \int_\Omega \big|u(t)\big|^q \leqslant \rho \quad \textit{for all } t \geqslant T \textit{ and } u_0 \in B.$$

Proof. Multiplying (1.1) by u, after the standard integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_p^p + \int_\Omega f(u)u = \int_\Omega gu,$$

combining with assumption (1.5), which implies that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2+\|\nabla u\|_p^p+C\int_\Omega |u|^q \leqslant \int_\Omega gu + C|\Omega|. \tag{3.2}$$

Thus, from assumption (1.6), we get

$$\frac{d}{dt}\|u\|_2^2+C\left(\|\nabla u\|_p^p+\int_\Omega |u|^q\right)\leqslant C\big(\|g\|_s,|\Omega|\big).$$

Noticing that $p\geqslant 2$, we have

$$\frac{d}{dt}\|u\|_2^2+C\int_\Omega |u|^2\leqslant C\big(\|g\|_s,|\Omega|\big). \tag{3.3}$$

Applying the Gronwall lemma, from (3.3), we know that $\{S(t)\}_{t\geqslant 0}$ has a $(L^2(\Omega),L^2(\Omega))$-bounded absorbing set, i.e., for arbitrary bounded subset B in $L^2(\Omega)$, there exists $T_1(=T_1(B))$ which depends only on the L^2-norm of B such that

$$\big\|S(t)u_0\big\|_2^2\leqslant \rho_0 \quad \text{for all } t\geqslant T_1,\ u_0\in B, \tag{3.4}$$

where the constant ρ_0 is independent of B. Taking $t\geqslant T_1$, integrating (3.2) on $[t,t+1]$ and combining with (3.4), we have

$$\int_t^{t+1}\left(\|\nabla u\|_p^p+\|u\|_q^q-\int_\Omega gu\right)\mathrm{d}s\leqslant C\big(\|g\|_s,|\Omega|,\rho_0\big)\quad \text{for all } t\geqslant T_1. \tag{3.5}$$

Meanwhile, let $F(s)=\int_0^s f(\tau)\,d\tau$; then by (1.5) again, we can deduce that

$$\tilde{C}_1|u|^q-k_1\leqslant F(u)\leqslant \tilde{C}_2|u|^q+k_1, \tag{3.6}$$

and then,

$$\tilde{C}_1\|u\|_q^q-k_1|\Omega|\leqslant \int_\Omega F(u)\leqslant \tilde{C}_2\|u\|_q^q+k_1|\Omega|. \tag{3.7}$$

Hence, from (3.5), we get

$$\int_t^{t+1}\left(\|\nabla u\|_p^p+\int_\Omega F(u)-\int_\Omega gu\right)\mathrm{d}s\leqslant C\big(\|g\|_s,|\Omega|,\rho_0,\tilde{C}_2,k_1\big). \tag{3.8}$$

On the other hand, multiplying (1.1) by u_t, we obtain

$$\|u_t\|_2^2+\frac{d}{dt}\left(\frac{1}{p}\int_\Omega |\nabla u|^p+\int_\Omega F(u)-\int_\Omega gu\right)=0. \tag{3.9}$$

Therefore, from (3.8) and (3.9), by virtue of the uniform Gronwall lemma, we get

$$\int_\Omega |\nabla u|^p+\int_\Omega F(u)-\int_\Omega gu\leqslant C\big(\|g\|_s,|\Omega|,\rho_0,\tilde{C}_2,k_1\big). \tag{3.10}$$

Thanks to (1.6) and (3.7), (3.10) implies that for all $t \geqslant T_1 + 1$,

$$\int_\Omega |\nabla u|^p + \int_\Omega |u|^q \leqslant C\big(\|g\|_s, |\Omega|, \rho_0, \tilde{C}_2, k_1\big) + C\big(\|g\|_s, |\Omega|\big). \tag{3.11}$$

Now, take $\rho = C(\|g\|_s, |\Omega|, \rho_0, \tilde{C}_1, k_1) + C(\|g\|_s, |\Omega|)$ and $T = T_1 + 1$, and we complete the proof of Theorem 3.2. □

From Theorem 3.2 and the compactness of Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, we have the following result immediately:

Corollary 3.3. *Assume that Ω is a bounded smooth domain in $\mathbb{R}^n$ $(n \geqslant 3)$, f satisfies* (1.4)–(1.5) *and $g \in L^s(\Omega)$, where s satisfies* (1.6). *Then the semigroup $\{S(t)\}_{t\geqslant 0}$ generated by* (1.1)–(1.3) *with initial data $u_0 \in L^2(\Omega)$ has a $(L^2(\Omega), L^2(\Omega))$-global attractor $\mathcal{A}_2$, that is, $\mathcal{A}_2$ is compact, invariant in $L^2(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $L^2(\Omega)$.*

4. Asymptotic a priori estimate and the $(L^2(\Omega), L^q(\Omega))$-global attractor

The main purpose of this section is to give an asymptotic a priori estimate for the unbounded part of the modular $|u|$ for the solution u of Eq. (1.1) in L^q-norm. Namely, we have the following theorem, which plays a crucial role in our paper:

Theorem 4.1. *For any $\varepsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exist two positive constants $T = T(B, \varepsilon)$ and $M = M(\varepsilon)$ such that*

$$\int_{\Omega(|u|\geqslant M)} |u|^q \leqslant C\varepsilon \quad \textit{for all } t \geqslant T \textit{ and } u_0 \in B,$$

where the constant C is independent of ε and B.

Proof. For any fixed $\varepsilon > 0$, there exists $\delta > 0$ such that for any $e \subset \Omega$ with $m(e) \leqslant \delta$, we have

$$\int_e |g|^s < \varepsilon. \tag{4.1}$$

Moreover, from Propositions 2.5, 2.6 and Corollary 3.3, we know that there exist $T_1 = T_1(B, \varepsilon)$ and $M_1 = M_1(\varepsilon)$, such that for any $u_0 \in B$ and $t \geqslant T_1$,

$$m\big(\Omega\big(\big|u(t)\big| \geqslant M_1\big)\big) \leqslant \min\{\delta, \varepsilon\} \tag{4.2}$$

and

$$\int_{\Omega(|u(t)|\geqslant M_1)} \big|u(t)\big|^2 \leqslant \varepsilon. \tag{4.3}$$

In addition, it follows from (1.5) that $f(s) \geqslant 0$ provided that $s > (k/C_1)^{1/q}$. Let $M_2 = \max\{M_1, (k/C_1)^{1/q}\}$ and $t \geqslant T_1$.

Multiplying (1.1) by $(u - M_2)_+$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \big|(u - M_2)_+\big|^2 + \int_\Omega \big|\nabla(u - M_2)_+\big|^p + \int_\Omega f(u)(u - M_2)_+ = \int_\Omega g(u - M_2)_+,$$

where $(u-M)_+$ denotes the positive part of $u-M$, that is,

$$(u-M)_+ = \begin{cases} u-M, & u \geqslant M, \\ 0, & u \leqslant M. \end{cases}$$

Integrating the above inequality on $[t, t+1]$, and combining with (1.5), (1.6), (4.1), (4.3) and (4.2), we have

$$\int_t^{t+1} \int_{\Omega(u \geqslant 2M_2)} f(u)u \, dx \, ds \leqslant C\varepsilon. \tag{4.4}$$

On the other hand, let $\Omega_1 = \Omega(u \geqslant 2M_2)$, we multiply (1.1) by $(u-M_2)_+^{q-1}$ and integrating over Ω, we get that

$$\begin{aligned} &\int_{\Omega(u \geqslant M_2)} (u-M_2)^{q-1} u_t + (q-1) \int_{\Omega(u \geqslant M_2)} (u-M_2)^{q-2} |\nabla u|^p \\ &\quad + \int_{\Omega(u \geqslant M_2)} f(u)(u-M_2)^{q-1} \leqslant \int_{\Omega(u \geqslant M_2)} g(u-M_2)^{q-1}, \end{aligned} \tag{4.5}$$

where

$$\frac{1}{2^{q-1}} \int_{\Omega(u \geqslant 2M_2)} f(u) u^{q-1} \leqslant \int_{\Omega(u \geqslant M_2)} f(u)(u-M_2)^{q-1}. \tag{4.6}$$

Note that, from (1.6) we have

$$\left| \int_{\Omega(u \geqslant M_2)} g(u-M_2)^{q-1} \right| \leqslant \left(\int_{\Omega(u \geqslant M_2)} |g|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega(u \geqslant M_2)} |u|^{2(q-1)} \right)^{\frac{1}{2}} \tag{4.7}$$

or

$$\int_{\Omega(u \geqslant M_2)} g(u-M_2)^{q-1} \leqslant \left(\int_{\Omega(u \geqslant M_2)} |g|^s \right)^{\frac{1}{s}} \left(\int_{\Omega(u \geqslant M_2)} |u|^{\frac{(q-1)s}{s-1}} \right)^{\frac{s-1}{s}}, \tag{4.8}$$

where

$$\frac{(q-1)s}{s-1} \leqslant \frac{(q-2+p)n}{n-p} = \left(\frac{q-2}{p} + 1 \right) \frac{np}{n-p}.$$

Hence, from inequalities (4.5)–(4.8), we have

$$\frac{d}{dt} \int_{\Omega(u \geqslant 2M_2)} |u|^q \leqslant C\varepsilon.$$

Using the uniform Gronwall lemma, we obtain

$$\int_{\Omega(u \geqslant 2M_2)} |u|^q \leqslant C\varepsilon, \tag{4.9}$$

where the constant C is independent of ε and M_2.

Repeating the same steps above, just taking $(u+M_2)_-$ and $(u+2M_2)_-^{q-1}$ instead of $(u-M_2)_+$ and $(u-2M_2)_+^{q-1}$, respectively, we deduce that

$$\int_{\Omega(u\leqslant -2M_2)} |u|^q \leqslant C\varepsilon. \tag{4.10}$$

Taking $M=2M_2$ and combining (4.9) with (4.10), we immediately obtain

$$\int_{\Omega(|u|\geqslant M)} |u|^q \leqslant C\varepsilon,$$

where the constant C is independent of ε and M. $\square$

Theorem 4.2. *Assume that Ω is a bounded smooth domain in $\mathbb{R}^n$ $(n\geqslant 3)$, f satisfies* (1.4)–(1.5) *and $g\in L^s(\Omega)$, where s satisfies* (1.6). *Then the semigroup $\{S(t)\}_{t\geqslant 0}$ generated by* (1.1)–(1.3) *with initial data $u_0\in L^2(\Omega)$ has a $(L^2(\Omega), L^q(\Omega))$-global attractor $\mathcal{A}_q$, that is, $\mathcal{A}_q$ is compact, invariant in $L^q(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $L^q(\Omega)$.*

Proof. From Theorems 3.1, 3.2 and 4.1 and Corollary 3.3, we know the hypotheses of Theorem 2.7 are all satisfied. $\square$

Remark 4.3. In fact, if we are only concerned with the existence of the $(L^2(\Omega), L^q(\Omega))$-global attractor for Eq. (1.1), then assumption (1.4) can be replaced by the following weaker assumption:

$$\big(f(s_1)-f(s_2)\big)(s_1-s_2)\geqslant -C|s_1-s_2|^2 \quad \text{for any } s_1,s_2\in\mathbb{R},$$

which guarantees the existence and uniqueness of weak solution of Eq. (1.1) and the continuity of the semigroup in $L^2(\Omega)$.

5. $(L^2(\Omega), W_0^{1,p}(\Omega)\cap L^q(\Omega))$-global attractor

In this section, we prove the existence of a $(L^2(\Omega), W_0^{1,p}(\Omega)\cap L^q(\Omega))$-global attractor. For this purpose, at first, we will give some a priori estimates about u_t endowed with L^2-norm.

Lemma 5.1. *Assume that Ω is a bounded smooth domain in $\mathbb{R}^n$ $(n\geqslant 3)$, f satisfies* (1.4)–(1.5) *and $g\in L^s(\Omega)$, where s satisfies* (1.6). *Then for any bounded subset B in $L^2(\Omega)$, there exists a positive constant $T=T(B)$ such that*

$$\big\|u_t(s)\big\|_2^2\leqslant \rho_1 \quad \text{for all } u_0\in B \text{ and } s\geqslant T,$$

where $u_t(s)=\frac{d}{dt}(S(t)u_0)|_{t=s}$ and ρ_1 is a positive constant which is independent of B.

Proof. *Step 1.* At first, we will give some formal calculations. By differentiating (1.1) in time and denoting $v=u_t$, we get

$$v_t-\operatorname{div}\big(|\nabla u|^{p-2}\nabla v\big)-(p-2)\operatorname{div}\big(|\nabla u|^{p-4}(\nabla u\cdot\nabla v)\nabla u\big)+f'(u)v=0, \tag{5.1}$$

where "$\cdot$" denotes the dot product in $\mathbb{R}^n$. Multiplying the above equality by v, integrating over Ω and using (1.4), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \int_\Omega |\nabla u|^{p-2}|\nabla v|^2 + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u \cdot \nabla v)^2 \leqslant l\|v\|_2^2, \tag{5.2}$$

hence,

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 \leqslant l\|v\|_2^2. \tag{5.3}$$

On the other hand, integrating (3.9) from t to $t+1$ and using (3.10), we get

$$\int_t^{t+1} \|u_t\|_2^2 \leqslant C\big(\rho, \|g\|_s, |\Omega|\big) \tag{5.4}$$

as t large enough.

Combining (5.3) with (5.4), and using the uniform Gronwall lemma, we have

$$\int_\Omega |u_t|^2 \leqslant C\big(\rho, \|g\|_s, |\Omega|\big)$$

as t large enough.

Step 2. Now, we will give a rigorous proof by use of Galerkin approximations. Let ω_i, $i = 1, 2, \ldots$, be the eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$. Then by the standard elliptic theory, we know that $\omega_i \in C^\infty(\Omega)$ and $\{\omega_i\}_{i=1}^\infty$ consists of an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $W_0^{1,p}(\Omega)$ and $L^q(\Omega)$ for any $1 < p, q < \infty$. Let P_m be the orthogonal projector

$$P_m : L^2 \to H_m = \operatorname{span}\{\omega_1, \omega_2, \ldots, \omega_m\}.$$

Set $u_m(t) = \sum_{j=1}^m a_{j,m}(t)\omega_j$, and consider the following ODEs:

$$u_{mt} - P_m\big(\operatorname{div}\big(|\nabla u_m|^{p-2}\nabla u_m\big)\big) + P_m f(u_m) = P_m g, \tag{5.5}$$

with initial condition

$$u_m|_{t=0} = P_m u_0. \tag{5.6}$$

Existence and uniqueness results of ODEs imply that there is a unique solution for (5.5)–(5.6) (at least on some short time internal $[0, T_m)$). Moreover, since f is a C^1-function, we know $a_{j,m}(t) \in C^2(0, T_m)$ and as a consequence, we can differentiate (5.5) with respect to t:

$$\begin{aligned} &v_{mt} - P_m \operatorname{div}\big(|\nabla u_m|^{p-2}\nabla v_m\big) - (p-2)P_m \operatorname{div}\big(|\nabla u_m|^{p-4}(\nabla u_m \cdot \nabla v_m)\nabla u_m\big) \\ &\quad + P_m f'(u_m)v_m = 0, \end{aligned} \tag{5.7}$$

where $v_m(t) = \frac{d}{dt}u_m(t) = \sum_{j=1}^m \frac{d}{dt}a_{j,m}(t)\omega_j$, and (5.7) holds in the classical sense. Therefore, the formal calculations in Step 1 are well defined for (5.7) with $v_m(t)$, and we can obtain that

there exist positive constants ρ_1 (which depends on ρ, $|\Omega|$ and $\|g\|_s$, but independent of m) and T (which depends on the L^2-norm of u_0, but independent of m), such that

$$\int_\Omega \big|u_{mt}(s)\big|^2 \leqslant \rho_1 \quad \text{for all } s \geqslant T \text{ and } m \in \mathbb{N}. \tag{5.8}$$

Thus, we have

$$u_{mt}(s) \rightharpoonup h(s) \quad \text{in } L^2(\Omega) \text{ for any } s \geqslant T \tag{5.9}$$

and

$$\int_\Omega \left|h(s)\right|^2 \leqslant \rho_1 \quad \text{for all } s \geqslant T. \tag{5.10}$$

At the same time, from the a priori estimates, e.g., see Babin and Vishik [1], we know

$$u_{mt}(s) \rightharpoonup u_t(s) \quad \star\text{-weakly in } L^\infty\big(0,\infty; W^{-1,p'}(\Omega) + L^{q'}(\Omega)\big), \tag{5.11}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. By the uniqueness of weak solution (Theorem 3.1) and the uniqueness of limits, we have $h(s) = u_t(s)(s > T)$. Thus,

$$\int_\Omega \left|u_t(s)\right|^2 \leqslant \rho_1 \quad \text{for all } s \geqslant T. \tag{5.12}$$

Since T only depends on the L^2-norm of u_0, Lemma 5.1 follows from (5.12) immediately. □

Now, we prove that the semigroup $\{S(t)\}_{t\geqslant 0}$ is $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-asymptotically compact.

Theorem 5.2. *Assume that Ω is a bounded smooth domain in $\mathbb{R}^n$ $(n \geqslant 3)$, f satisfies* (1.4)–(1.5) *and $g \in L^s(\Omega)$, where s satisfies* (1.6). *Then the semigroup $\{S(t)\}_{t\geqslant 0}$ generated by* (1.1)–(1.3) *with initial data $u_0 \in L^2(\Omega)$ is $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-asymptotically compact.*

Proof. Let B_0 be a $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$-bounded absorbing set obtained in Theorem 3.2, then we need only to show that

$$\text{for any } \{u_{0n}\} \subset B_0 \text{ and } t_n \to \infty,\ \{u_n(t_n)\}_{n=1}^\infty \text{ is precompact in } W_0^{1,p}(\Omega) \cap L^q(\Omega),$$

where $u_n(t_n) = S(t_n)u_{0n}$.

Thanks to Theorem 4.2, it is sufficient to verify that

$$\text{for any } \{u_{0n}\} \subset B_0 \text{ and } t_n \to \infty, \{u_n(t_n)\}_{n=1}^\infty \text{ is precompact in } W_0^{1,p}(\Omega). \tag{5.13}$$

In fact, from Corollary 3.3 and Theorem 4.2, we know that $\{u_n(t_n)\}_{n=1}^\infty$ is precompact in $L^2(\Omega)$ and $L^q(\Omega)$. Without loss of generality, we assume that $\{u_{n_k}(t_{n_k})\}$ is a Cauchy sequence in $L^q(\Omega)$.

In the following, we prove that $\{u_{n_k}(t_{n_k})\}$ is a Cauchy sequence in $W_0^{1,p}(\Omega)$.

For this purpose, we need only to apply a simple property of p-Laplacian for $p \geqslant 2$: there exists a positive constant δ, such that for all $u_1, u_2 \in W_0^{1,p}(\Omega)$,

$$\langle Au_1 - Au_2, u_1 - u_2\rangle \geqslant \delta \|u_1 - u_2\|^p_{W_0^{1,p}(\Omega)},$$

where $Au = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(p \geqslant 2)$ and $\langle \cdot,\cdot\rangle$ is the L^2-inner product.

Then, we get

$$\begin{aligned}
&\delta\left\|u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\right\|^p_{W_0^{1,p}(\Omega)}\\
&\quad\leqslant\left\langle Au_{n_k}(t_{n_k})-Au_{n_j}(t_{n_j}),u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\right\rangle\\
&\quad=\left\langle-\frac{d}{dt}u_{n_k}(t_{n_k})-f\big(u_{n_k}(t_{n_k})\big)+\frac{d}{dt}u_{n_j}(t_{n_j})+f\big(u_{n_j}(t_{n_j})\big),u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\right\rangle\\
&\quad\leqslant\int_\Omega\left|\frac{d}{dt}u_{n_k}(t_{n_k})-\frac{d}{dt}u_{n_j}(t_{n_j})\right|\left|u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\right|\\
&\qquad+\int_\Omega\left|f\big(u_{n_k}(t_{n_k})\big)-f\big(u_{n_j}(t_{n_j})\big)\right|\left|u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\right|\\
&\quad\leqslant\left\|\frac{d}{dt}u_{n_k}(t_{n_k})-\frac{d}{dt}u_{n_j}(t_{n_j})\right\|_2\left\|u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\right\|_2\\
&\qquad+C\big(1+\left\|u_{n_k}(t_{n_k})\right\|_q^q+\left\|u_{n_j}(t_{n_j})\right\|_q^q\big)\left\|u_{n_k}(t_{n_k})-u_{n_j}(t_{n_j})\right\|_q,
\end{aligned}$$

which, combining with Lemma 5.1, yields (5.13) immediately. □

For our problem, since we do not impose any restriction on q, in general, we do not get the $(L^2(\Omega), W_0^{1,p}(\Omega)\cap L^q(\Omega))$-continuity of the corresponding semigroup. Hence, we will use the idea from Zhong, Yang and Sun [11], that is, we use the norm-to-weak continuity instead of the norm-to-norm (or weak-to-weak) continuity of the semigroup in the usual criterions for the existence of global attractors.

Proof of Theorem 1.1. Let B_0 be a $(L^2(\Omega), W_0^{1,p}(\Omega)\cap L^q(\Omega))$-bounded absorbing set obtained in Theorem 3.2. Set

$$\mathcal{A}=\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}S(t)B_0}^{W_0^{1,p}(\Omega)\cap L^q(\Omega)}.$$

Then, from Theorems 3.2 and 5.2, we know that $\mathcal{A}$ is nonempty, compact in $W_0^{1,p}(\Omega)\cap L^q(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $W_0^{1,p}(\Omega)\cap L^q(\Omega)$.

In what follows, we will verify the invariance of $\mathcal{A}$, i.e., $S(t)\mathcal{A}=\mathcal{A}$ for all $t\geqslant 0$.

By the equivalent characterization of ω-limit set, for any $x_0\in\mathcal{A}$ we know that there exist $\{x_n\}\subset B_0$ and t_n with $t_n\to\infty$ as $n\to\infty$ such that $S(t_n)x_n\to x_0$ in $W_0^{1,p}(\Omega)\cap L^q(\Omega)$.

At first, using the $(L^2(\Omega), (W_0^{1,p}(\Omega))_w\cap(L^q(\Omega))_w)$-continuity of $\{S(t)\}_{t\geqslant 0}$, we have

$$S(t+t_n)x_n=S(t)S(t_n)x_n\rightharpoonup S(t)x_0\quad\text{in } W_0^{1,p}(\Omega)\cap L^q(\Omega).$$

On the other hand, from Theorem 5.2, we know that $\{S(t+t_n)x_n\}_{n=1}^\infty$ has a subsequence which is convergent in $W_0^{1,p}(\Omega)\cap L^q(\Omega)$. Without loss of generality, we assume that $S(t+t_{n_k})x_{n_k}\to x$ in $W_0^{1,p}(\Omega)\cap L^q(\Omega)$. Hence, by the uniqueness of limits, we have $S(t)x_0=x$, and from the definition of x we know that $x\in\mathcal{A}$. Therefore, combining with the arbitrariness of x_0 and t, we get

$$S(t)\mathcal{A}\subset\mathcal{A}\quad\text{for all } t\geqslant 0. \tag{5.14}$$

Now, we prove the converse inclusion. Since $t_n\to\infty$ as well as t_n-t, without loss of generality, we assume that $t_n-t\geqslant 0$ for each $n\in\mathbb{N}$. Therefore, by Theorem 5.2, $\{S(t_n-t)x_n\}_{n=1}^\infty$

has a convergent subsequence in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$, without loss of generality, we assume $S(t_{n_m} - t)x_{n_m} \to y_0 \in \mathcal{A}$ in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$, and then by use of the $(L^2(\Omega), (W_0^{1,p}(\Omega))_w \cap (L^q(\Omega))_w)$-continuity again, we have

$$x_0 \leftarrow S(t_{n_m})x_{n_m} = S(t)S(t_{n_m} - t)x_{n_m} \rightharpoonup S(t)y_0 \quad \text{in } W_0^{1,p}(\Omega) \cap L^q(\Omega).$$

Hence, notice the uniqueness of limits again, we have $x_0 = S(t)y_0$, and from the definition of y_0 we know that $y_0 \in \mathcal{A}$, which implies

$$\mathcal{A} \subset S(t)\mathcal{A} \quad \text{for all } t \geqslant 0. \tag{5.15}$$

The invariance of $\mathcal{A}$ follows from (5.14) and (5.15) immediately. □

Remark 5.3. Although the semigroup $\{S(t)\}_{t\geqslant 0}$ is only $(L^2(\Omega), (W_0^{1,p}(\Omega))_w \cap (L^q(\Omega))_w)$-continuous, we can also obtain that $\mathcal{A}_2$, $\mathcal{A}_q$ and $\mathcal{A}$ coincide with each other and are connected in $W_0^{1,p}(\Omega) \cap L^q(\Omega)$.

Acknowledgment

The authors would like to express their sincere thanks to the referee for his/her valuable comments and suggestions, especially about Lemma 5.1.

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