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Almost periodic linear differential equations with non-separated solutions

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Abstract

A celebrated result by Favard states that, for certain almost periodic linear differential systems, the existence of a bounded solution implies the existence of an almost periodic solution. A key assumption in this result is the separation among bounded solutions. Here we prove a theorem of anti-Favard type: if there are bounded solutions which are non-separated (in a strong sense) sometimes almost periodic solutions do not exist. Strongly non-separated solutions appear when the associated homogeneous system has homoclinic solutions. This point of view unifies two fascinating examples by Zhikov–Levitan and Johnson for the scalar case. Our construction uses the ideas of Zhikov–Levitan together with the theory of characters in topological groups.

Résumé

Un résultat célébré de Favard affirme que, pour certains systèmes différentiels linéaires presque périodiques, l'existence d'une solution bornée implique l'existence d'une solution presque périodique. Une supposition essentielle dans ce résultat est la séparation des solutions bornées. Ici nous prouvons un théorème de type anti-Favard : s'il y a des solutions bornées qui sont non-séparées (dans un sens fort), parfois il n'existent pas des solutions presque périodiques. Des solutions fortement non-séparées apparaissent quand le système homogène associé a des solutions homoclines. Ce point de vue unifie deux exemples fascinants de Zhikov–Levitan et Johnson pour le cas scalaire. Notre construction utilise des idées de Zhikov–Levitan ainsi que la théorie de caractères dans groupes topologiques.

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1. Introduction

This paper is concerned with the theory of linear differential equations with almost periodic coefficients. Sometimes this field is called Favard's theory, due to the contributions made by this author in [5]. In that paper, published in 1927, Favard discussed the existence of almost periodic solutions of an equation in $\mathbb{R}^N$ of the type:

$$\dot{x} = A(t)x + b(t), \tag{1}$$

where the matrix $A(t)$ and the vector $b(t)$ both are almost periodic (in the classical sense of Bohr).

An important tool in the Favard's work was the characterization of almost periodicity which had been recently obtained by Bochner, leading to the concept of *hull* of an almost periodic function. For the matrix-valued function the hull $\mathcal{H}_A$ is composed by those functions $A_*(t)$ which can be obtained as uniform limits on the real line of the type:

$$A_*(t) = \lim_{n\to+\infty} A(t + h_n),$$

where $\{h_n\}$ is some sequence of real numbers. This notion was employed by Favard and its main assumption was concerned with the bounded solutions of the so-called *homogeneous hull* of Eq. (1), namely of the family of equations:

$$\dot{z} = A_*(t)z, \quad A_* \in \mathcal{H}_A. \tag{2}$$

The assumption was:

(H) For every $A_* \in \mathcal{H}_A$, any nontrivial bounded solution to (2) is separated from zero.

For a solution $\varphi(t)$, to be separated from zero means that

$$\inf_{t\in\mathbb{R}} \left|\varphi(t)\right| > 0.$$

It must be noticed that condition (H) is automatically satisfied if the equations in (2) have no bounded solutions excepting $z \equiv 0$. It also holds when the matrix $A(t)$ is periodic.

In general, if (H) holds, the bounded solutions of the inhomogeneous hull of (1) must be separated. This means that, if $\varphi_1(t)$ and $\varphi_2(t)$ are two different bounded solutions of the same equation, arbitrarily chosen in the family:

$$\dot{x} = A_*(t)x + b_*(t), \quad (A_*, b_*) \in \mathcal{H}_{(A,b)},$$

then $\inf_{t\in\mathbb{R}} |\varphi_1(t) - \varphi_2(t)| > 0$. This separation property is crucial in Favard's proof of the following statement ("*mod*" denotes the module of an almost periodic function).

Theorem 1. (Favard [5]) *Assume that* (H) *holds and* (1) *has a bounded solution. Then there exists an almost periodic solution ϕ with $mod(\phi) \subset mod(A, b)$.*

The main question we address in the present paper is what happens when (H) fails. This question was already discussed by Zhikov and Levitan in [17] (see also [10]) and by Johnson in [9]. In both cases the authors constructed fascinating examples of scalar linear equations for which all the solutions are bounded, but none of them is almost periodic.

The example by Zhikov and Levitan dealt with quasi-periodic functions with two frequencies. The function $A(t)$ was taken from an example due to Bohr. It satisfied certain symmetry assumptions, and the primitive $\mathcal{A}(t) = \int_0^t A(s)\,ds$ was such that

$$\lim_{|t|\to+\infty} \frac{\mathcal{A}(t)}{|t|^{1/2}} = -\infty.$$

In Johnson's example A and b where uniform limits of periodic functions. The function A was taken from an example by Conley and Miller in [3] and, among other properties, its primitive satisfies:

$$\mathcal{A}(t) \to -\infty \quad \text{as } |t| \to +\infty.$$

In both examples $\varphi(t) = \exp(\mathcal{A}(t))$ is a solution of the homogeneous equation which takes arbitrary small values, but, in fact, we have more: $\varphi(t)$ is homoclinic to zero in the sense that $\varphi(t) \to 0$ as $|t| \to +\infty$. The aim of this paper is to show that Favard's result does not hold when such a decay takes place, unifying the two situations in [17] and [9]. Our main assumption is:

(H_0) For some $A_* \in \mathcal{H}_A$, Eq. (2) has nontrivial bounded solutions, and all of them are homoclinic to zero.

The result is summed up in the following statement.

Theorem 2. *Assume that* (H_0) *holds. Then there exists an almost periodic vector $b(t)$ such that $mod(b) \subset mod(A)$ and* (1) *has bounded solutions, but none of them is almost periodic.*

We do not know if (H_0) could be replaced in the previous theorem by the negation of (H), namely:

(wH_0) For some $A_* \in \mathcal{H}_A$, Eq. (2) has a bounded solution φ with

$$\inf_{t\in\mathbb{R}} \left|\varphi(t)\right| = 0, \qquad \varphi \not\equiv 0.$$

We observe that the main result in [11] implies that a scalar equation with $A(t)$ quasi-periodic, and represented by a sufficiently smooth function on the appropriate torus, cannot satisfy (H_0). On the other hand, such an equation will satisfy (wH_0) as soon as $\mathcal{A}(t)$ is unbounded and $A(t)$

has zero mean value, disregarding its smoothness. Favard [5] proved indeed that, in this case, one may always choose $A_* \in \mathcal{H}(A)$ such that

$$\int_0^t A_*(s)\,ds \leqslant 0 \quad \forall t \in \mathbb{R},$$

and the claim follows trivially from the unboundedness of this primitive.

An additional criticism about the previous theorem could be the lack of regularity of the coefficients A and b. A function is in the class AP^∞ if it belongs to C^∞ and it is almost periodic together with all its derivatives. The next statement says that smooth counterexamples may be constructed.

Corollary 3. *Assume that A satisfies the hypothesis of Theorem* 2 *and all its coefficients belong to AP^∞. Then in the conclusions we may take b with coefficients in AP^∞.*

At the end of the paper, in Appendix A, we give an example of an equation with coefficients in AP^∞ and satisfying (H_0).

To conclude, let us notice that Theorem 2 states the nonexistence of almost periodic solutions to (1) whatever their module is, either contained or not contained in $mod(A)$.

In fact, in Section 2 we will prove that, under the same assumptions of Theorem 2, an almost periodic solution to (1) must automatically satisfy the module containment property

$$mod(x) \subset mod(A).$$

This result was suggested to the authors by the proof of Theorem 2.2 in [9]. Then, in Section 6 we will show that these solutions may be seen as the *continuous* solutions of a suitable abstract partial differential equation on $\mathcal{H}_A$:

$$D\mathfrak{x} = \mathfrak{A}(\omega)\mathfrak{x} + \mathfrak{b}(\omega), \quad \omega \in \mathcal{H}_A. \tag{3}$$

Here the coefficients $\mathfrak{A}$ and $\mathfrak{b}$ are the "extensions by continuity" of the almost periodic functions $A(t)$ and $b(t)$ to $\mathcal{H}_A$ (see Section 5). The operator D is a sort of directional derivative which can be defined because $\mathcal{H}_A$ has a well-known structure of topological group. This is a standard approach when $A(t)$ is a quasi-periodic function. In this case $\mathcal{H}_A$ is isomorphic to a torus and (3) can be identified with a first order partial differential equation on the torus.

In the same Section 6, we will show that the existence of solutions to (3), which are *discontinuous* in some unavoidable way, prevents the existence of continuous ones. Hence, the proof of Theorem 2 reduces to the construction of an almost periodic vector $b(t)$, for which Eq. (3) admits discontinuous solutions. This will be done in Section 7, by adapting to $\mathcal{H}_A$ an argument by Zhikov and Levitan on $\mathbb{T}^2$ (see [17, Section 8]). This is a main step in our approach, and the needed abstraction process justifies a strategical choice we made all through the paper: instead of dealing with the concrete group $\mathcal{H}_A$, we deal with an abstract compact, commutative topological group. In Sections 3–5, we discuss some generalities of the theory of topological groups and characters and its relationship with almost periodic functions. The reader who is familiar with the theory of continuous groups will find that the results in these sections are either well known or easy exercises. However, they are a needed preparation for Section 8 and they could be of help to the reader who is an expert in differential equations but is not familiar with groups.

Finally, in Section 8, we prove Theorem 2 and Corollary 3.

To finish this introduction we mention some other works related to Favard's theory [2,4,12, 14,16].

Notations. Given a function $f(t)$, by $T_\tau f(t)$ we denote the translated function $f(t+\tau)$. Moreover, the hull of $f(t)$ is the uniform closure (on all the real line) of the set $\{T_\tau f\colon \tau \in \mathbb{R}\}$. It is denoted by $\mathcal{H}_f$ and, when a topology is considered on it, this is the one associated with the uniform convergence.

2. Module containment property

The module $mod(f)$ of an almost periodic function $f(t)$ is the least additive subgroup of the real numbers containing the Fourier exponents of $f(t)$ (see [1,10]). If $g(t)$ is another almost periodic function then the module containment property

$$mod(f) \subset mod(g) \tag{4}$$

can be characterized in several ways (see [7]). For periodic functions this inclusion just means that the minimal period of $g(t)$ is a multiple of the minimal period of $f(t)$.

We are interested in finding linear equations of the type (1) for which all almost periodic solutions satisfy

$$mod(x) \subset mod(A, b). \tag{5}$$

In the periodic case (A and b have the same period) this happens if the Floquet multipliers of the homogeneous equation do not lie on $\mathbb{S}^1$. The next proposition generalizes this fact. As previously mentioned, it is inspired by [9].

Proposition 4. *Assume that the trivial one is the unique almost periodic solution of the homogeneous equation associated to* (1). *Then every almost periodic solution $x(t)$ to* (1) *has the module containment property* (5).

Proof. Assume by contradiction that $x(t)$ is an almost periodic solution to (1) which does not satisfy (5). Because of the characterizations of (4) previously mentioned, this implies the existence of a sequence (τ_n) such that

$$T_{\tau_n} A \to A, \qquad T_{\tau_n} b \to b \quad \text{but} \quad T_{\tau_n} x \nrightarrow x.$$

Here, the involved limits are uniform on the real line. On the other hand, up to subsequences $T_{\tau_n} x \to y$, almost periodic solution to (1) with $y \neq x$. Thus $z = y - x$ should be a nontrivial almost periodic solution to the homogeneous equation associated to (1). $\square$

Next we discuss how condition (H_0) is related to the module containment property. We recall that this condition refers to the equations in the homogeneous hull of (1), namely:

$$\dot{z} = A_*(t) z, \tag{6}$$

where $A_* \in \mathcal{H}_A$. It will be sufficient to impose a condition that is less restrictive than (H_0), namely:

(H_0') For some $A_* \in \mathcal{H}_A$, all the bounded solutions to (6) are homoclinic to zero.

Lemma 5. *Assume* (H_0') *holds true. Then, for every* $A_* \in \mathcal{H}_A$*, the only almost periodic solution to* (6) *is the trivial one.*

Proof. By a contradiction argument assume that (6) has a nontrivial almost periodic solution for some A_* in $\mathcal{H}_A$. Then the same happens to every equation in the family (6). This is a well-known fact in the theory of linear equations (see for instance [6]). Since almost periodic functions cannot decay to zero, we find a situation that is not compatible with (H_0'). □

3. Topological groups and characters

The Greek letter Ω will always denote a topological group which is commutative, metrizable, compact and connected. A generic element will be denoted by $\omega \in \Omega$ and the notation will be additive, so that the operation will be '+' and the neutral element '0.' The category of these groups will be denoted by $\mathcal{G}$, its morphisms being the continuous homomorphisms of groups.

An important topological group is the unit circle

$$\mathbb{S}^1 = \{z \in \mathbb{C}\colon\ |z| = 1\}.$$

This is a multiplicative group, but it is in the category $\mathcal{G}$.

Given an topological group Ω, a character is a morphism $\Omega \to \mathbb{S}^1$. A classical result says that nontrivial characters always exist as soon as Ω is a nontrivial compact group [15, p. 241]. The set of all characters of Ω is usually called its dual group and denoted by Ω^*. It is itself a group. For instance

$$(\mathbb{S}^1)^* = \{z \mapsto z^n\colon\ n \in \mathbb{Z}\}.$$

Another important element of $\mathcal{G}$ is the d-torus $\mathbb{T}^d$. Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is an additive group isomorphic to $\mathbb{S}^1$, and a generic element will be denoted by $\bar{\theta}$, where $\theta \in \mathbb{R}$. From the knowledge of the characters of $\mathbb{S}^1$ it is easy to compute those of $\mathbb{T}^d$, obtaining

$$(\mathbb{T}^d)^* = \{(\overline{\theta_1}, \dots, \overline{\theta_d}) \mapsto e^{2\pi i(k_1\theta_1 + \dots + k_d\theta_d)}\colon\ k_j \in \mathbb{Z}\ \forall j\}.$$

As a last example, we will present the dual group of a noncompact topological group, namely the additive group $\mathbb{R}$ of the real numbers:

$$\mathbb{R}^* = \{t \mapsto e^{i\alpha t}\colon\ \alpha \in \mathbb{R}\}.$$

Given a nontrivial character $\sigma \in \Omega^*$, we are interested in its kernel

$$\ker(\sigma) = \{\omega \in \Omega\colon\ \sigma(\omega) = 1\}.$$

It is a (closed and then) compact subgroup of Ω, and then is either discrete (and hence finite) or perfect. For instance, when $\Omega = \mathbb{S}^1$, it has to coincide with the n-roots of the unity for some positive integer n. The next result says this is the only case we can have a finite set.

Proposition 6. *Let be $\Omega \in \mathcal{G}$ and let σ be a nontrivial character. If Ω is not isomorphic to $\mathbb{S}^1$ ($\Omega \ncong \mathbb{S}^1$) then* $\ker(\sigma)$ *is perfect.*

Proof. Since the image of σ is a compact and connected subgroup of $\mathbb{S}^1$, we deduce it has to coincide with $\mathbb{S}^1$ [15, Section 36]. From the isomorphy theorems [15, Section 20] we then deduce that the quotient group $\Omega/\ker(\sigma)$ is isomorphic to $\mathbb{S}^1$.

Assume now that $\ker(\sigma)$ is discrete, and let U be an open subset of Ω such that $(U-U)\cap\ker(\sigma)=\{0\}$, where $U-U=\{\omega_1-\omega_2\colon\ \omega_1,\omega_2\in U\}$. The projection $\pi:\Omega\to\Omega/\ker(\sigma)$ is open and so $V=\pi(U)$ is open in the quotient group. It is now easy to prove that $\pi:U\to V$ is a homeomorphism. Thus Ω and $\Omega/\ker(\sigma)$ are locally homeomorphic and, in particular, Ω is locally arcwise connected. This is important in order to apply the theory of covering maps. Since

$$\pi^{-1}(V)=\bigcup_{\omega\in\ker(\sigma)}(\omega+U),$$

and ω_1+U and ω_2+U are disjoint if $\omega_1\neq\omega_2$, we conclude that π is a covering map. Every covering group of $\mathbb{S}^1$ must be isomorphic to $\mathbb{R}$ or $\mathbb{S}^1$ (see [15, Chapter 9]). Since Ω is compact we conclude that Ω is isomorphic to $\mathbb{S}^1$. $\square$

We shall be interested in topological groups having a one-parameter dense subgroup. More precisely, we shall consider pairs (Ω,Ψ) where $\Omega\in\mathcal{G}$ and $\Psi:\mathbb{R}\to\Omega$ is a continuous homomorphism whose image is dense in Ω. These pairs will be the objects of a category denoted by $\mathcal{P}$.

The trivial group belongs to $\mathcal{P}$, endowed by the trivial homomorphism $t\in\mathbb{R}\mapsto 0$. The simplest nontrivial element of $\mathcal{P}$ is given by $\mathbb{S}^1$, together with any nontrivial element of $\mathbb{R}^*$. The next lemma characterizes the elements of $\mathcal{P}$ having groups which are isomorphic to $\mathbb{S}^1$.

Lemma 7. *Assume $(\Omega,\Psi)\in\mathcal{P}$. Then $\Omega\cong\mathbb{S}^1$ if and only if Ψ is periodic and nonconstant.*

Proof. Assume first $\Omega\cong\mathbb{S}^1$ in $\mathcal{G}$, and let Φ be the isomorphism. We deduce that $\Phi\circ\Psi$ is a nontrivial element of $\mathbb{R}^*$, and then periodic and nonconstant function. The same happens to Ψ.

Assume now Ψ is periodic and nonconstant, and let $T>0$ be its minimal period. The kernel of Ψ is composed by the multiples of T, $\ker(\Psi)=T\mathbb{Z}$, while the image $\Psi(\mathbb{R})=\Psi[0,T]$ is at the same time compact and dense in Ω. Thus Ψ is an epimorphism and the theorem of isomorphy applies to conclude that $\Omega\cong\mathbb{R}/T\mathbb{Z}\cong\mathbb{S}^1$. $\square$

Another important element of $\mathcal{P}$ is the d-dimensional torus $\mathbb{T}^d$ together with an homomorphism of the type

$$t\in\mathbb{R}\mapsto(\overline{\nu_1 t},\ldots,\overline{\nu_d t})\in\mathbb{T}^d.$$

Here the frequency vector $(\nu_1,\ldots,\nu_d)$ is a vector of $\mathbb{R}^d$, and the previous homomorphism may be visualized as a winding on the torus. In order to satisfy the density assumption, its components $\nu_1,\ldots,\nu_d$ must be taken linearly independent over $\mathbb{Z}$. These vectors are usually called *nonresonant*.

We end the section by introducing the notion of morphism $(\Omega_1,\Psi_1)\to(\Omega_2,\Psi_2)$ between two elements of $\mathcal{P}$. This is a morphism $\Phi:\Omega_1\to\Omega_2$ in the category $\mathcal{G}$ which preserves the dense subgroups, namely such that

$$\Phi \circ \Psi_1 = \Psi_2.$$

Notice that every morphism of $\mathcal{P}$ has some extra properties, which are hidden in the definition and which can be easily checked by means of standard density and compactness arguments: between any two elements of $\mathcal{P}$ there is at most one morphism, and it is in fact an epimorphism.

In some sense, the class of the morphisms of $\mathcal{P}$ induces a partial order structure on $\mathcal{P}$ itself, up to isomorphisms. As we will see in Section 5, this order structure plays a key role when handling almost periodic functions. The next lemma is helpful in deciding when morphisms do exist.

Lemma 8. *There exists a morphism $(\Omega_1, \Psi_1) \to (\Omega_2, \Psi_2)$ if and only if, for every sequence (τ_n), $\Psi_1(\tau_n) \to 0$ implies $\Psi_2(\tau_n) \to 0$.*

Proof. The necessity of the condition follows from the continuity of any morphism Φ. Concerning the sufficiency, first define Φ on $\operatorname{Im}\Psi_1$ by setting $\Phi(\Psi_1(t)) = \Psi_2(t)$. That this is well done, and, in fact, defines a uniformly continuous map on $\operatorname{Im}\Psi_1$, is a consequence of the following property:

$$\Psi_1(t_n) - \Psi_1(s_n) \to 0 \quad \text{implies} \quad \Psi_2(t_n) - \Psi_2(s_n) \to 0,$$

which holds for every sequences (t_n) and (s_n). Thus Φ can be uniquely extended to a continuous map on Ω_1. Finally, an easy computation shows that Φ is a group homomorphism. □

4. Minimal flows and sections

Let us fix, from now on in this section, a pair $(\Omega, \Psi) \in \mathcal{P}$. A flow may be defined on Ω by means of

$$\omega \cdot t := \omega + \Psi(t).$$

Since Ψ has dense image, the flow is minimal, i.e. it has no closed invariant subsets but the empty set and Ω itself. Indeed, the equality

$$\overline{\omega \cdot \mathbb{R}} = \overline{\omega + \Psi(\mathbb{R})} = \omega + \overline{\Psi(\mathbb{R})} = \Omega$$

holds for every $\omega \in \Omega$.

It is worthwhile to notice that the flow cannot have equilibria or periodic orbits unless $\Omega = 0$ or $\Omega \cong \mathbb{S}^1$. This is a consequence of Lemma 7.

The problem we address in this section is how to construct a global section for the above considered flow.

As an example, consider first the case $\Omega = \mathbb{T}^d$, with Ψ the winding map associated to the nonresonant vector $(\nu_1, \dots, \nu_d) \in \mathbb{R}^d$. This flow is induced by the differential equation

$$\dot{\theta}_1 = \nu_1, \quad \dots, \quad \dot{\theta}_d = \nu_d,$$

and, due to the nonresonance condition, the frequencies must be nonzero. Choose for instance $\nu_1 \neq 0$. Then any flow line crosses infinitely many times the subset of $\mathbb{T}^d$ described by the equation $\theta_1 = 0 \pmod 1$, and consecutive intersections are separated by a time interval of length $1/\nu_1$. Hence this set is a global section for the flow.

The question is how to do the same in the general case. To this aim notice that, in the previous example, the section was in fact the kernel of the nontrivial character $(\overline{\theta_1}, \dots, \overline{\theta_d}) \mapsto e^{i2\pi\theta_1}$. Moreover, the kernel of any nontrivial character of $\mathbb{T}^d$ plays exactly the same role. A kernel is defined by the equation

$$k_1\theta_1 + \cdots + k_d\theta_d = 0 \pmod 1$$

for some suitable integers $k_1, \dots, k_d$. They cannot be all zero, because the character is nontrivial, and from the nonresonance of the frequency vector we deduce that

$$k_1\nu_1 + \cdots + k_d\nu_d \neq 0.$$

Denoted by $1/T$ the above number, it not difficult to see that any flow line crosses the kernel at time intervals of length T.

Next proposition says this is the typical situation. To simplify the notations, from now on we will assume that $\sigma \in \Omega^*$ is a given *nontrivial character* (which, of course, implies $\Omega \neq 0$), and we will define:

$$\Sigma = \ker(\sigma) = \{\omega \in \Omega \colon \sigma(\omega) = 1\}.$$

Proposition 9. *There exists a $T > 0$ having the following property*: *for every $\omega \in \Omega$ there exists a unique $\tau(\omega) \in [0, T)$ such that $\omega \cdot t \in \Sigma$ if and only if $t \in \tau(\omega) + T\mathbb{Z}$.*

Proof. By the construction of the flow, the equality

$$\sigma(\omega \cdot t) = \sigma(\omega)\sigma\big(\Psi(t)\big)$$

holds for every ω and t. Since $\sigma \circ \Psi \in \mathbb{R}^*$, we deduce the existence of a (unique) real α such that

$$\sigma(\omega \cdot t) = \sigma(\omega)e^{i\alpha t} \quad \forall \omega, t.$$

Since the flow is minimal and σ is nontrivial, the number α cannot be zero. Take $T = 2\pi/|\alpha|$. The thesis follows from the existence of a unique real number τ satisfying

$$\sigma(\omega) = e^{-i\alpha\tau}$$

and, moreover, such that $0 \leqslant |\alpha|\tau < 2\pi$. $\square$

The flow induces the map

$$\Sigma \times \mathbb{R} \to \Omega, \quad (\omega, t) \mapsto \omega \cdot t.$$

The previous proposition implies that it is onto. The next lemma states an additional property.

Lemma 10. *The map defined above is open.*

Proof. Fix $\hat{\omega}_0 \in \Sigma$, $\tau_0 \in \mathbb{R}$ and define $\omega_0 = \hat{\omega}_0 \cdot \tau_0$. It is enough to prove that any point ω in Ω which is close to ω_0 can be expressed in the form $\omega = \hat{\omega} \cdot \tau$ with $\hat{\omega} \in \Sigma$ close to $\hat{\omega}_0$ and $\tau \in \mathbb{R}$ close to τ_0. In the notations of the proof of Proposition 9, $\sigma(\omega_0) = e^{i\alpha\tau_0}$. For ω close to ω_0 we can find τ close to τ_0 so that $\sigma(\omega) = e^{i\alpha\tau}$. The proof is finished if we define $\hat{\omega} = \omega \cdot (-\tau)$. $\square$

To finish the section, let us show how to use the global section Σ to parametrize long and thin strips of Ω along the flow. Take for instance the usual nonresonant flow on the torus. Let us consider a single flow line: since it has not self-intersections, any finite portion may be enlarged a little bit without overlapping. As we already noticed at the beginning of the section, the flow has no self-intersections (equilibria or closed orbits) if $\Omega \not\cong S^1$. The next statement says that, in the general case, enlargement without overlapping is possible.

Lemma 11. *Assume $\Omega \not\cong \mathbb{S}^1$. For every $\omega_0 \in \Sigma$ and every bounded interval $I \subset \mathbb{R}$, there exists a U, neighborhood in Σ of ω_0, such that the induced flow is injective on $U \times I$.*

Proof. Arguing by contradiction, assume the conclusion is false. There should exist pairs $(x_n, t_n) \neq (y_n, s_n)$ in $\Sigma \times I$ with $x_n \to \omega_0$, $y_n \to \omega_0$ and such that

$$x_n \cdot t_n = y_n \cdot s_n \quad \forall n.$$

Rewriting the above equality in the form

$$x_n \cdot (t_n - s_n) = y_n \tag{7}$$

we deduce that $t_n \neq s_n$ for every n. After extracting a subsequence we can assume $t_n - s_n \to \tau$ for some $\tau \in \mathbb{R}$ and, passing to the limit in (7), we obtain

$$\omega_0 \cdot \tau = \omega_0.$$

This implies that $\tau = 0$ because the flow has no periodic orbits. However, $\tau = 0$ contradicts the fact that Σ is a section for the flow. Indeed, in this case Eq. (7) says that, starting at $x_n \in \Sigma$, we should be back at Σ after time intervals which become smaller and smaller with n. This contradicts the conclusions of Proposition 9. $\square$

5. Almost periodic functions and groups

Given a pair $(\Omega, \Psi) \in \mathcal{P}$ and $\mathfrak{f} \in \mathcal{C}(\Omega)$, the function

$$f(t) = \mathfrak{f}\big(\Psi(t)\big) \tag{8}$$

is almost periodic. This can be easily checked because the group Ω is compact. We will say that the function $f(t)$ is *representable* over (Ω, Ψ). By density arguments, it is clear that $\mathfrak{f} \in \mathcal{C}(\Omega)$ is unique. As an example consider the case $\Omega \cong \mathbb{S}^1$, then the formula (8) gives rise to periodic functions. This is a consequence of Lemma 7.

An important and well-known point, is that any given almost periodic function $f(t)$ may be obtained in this way via the notion of hull. The hull $\mathcal{H}_f$ is a compact metric space (uniform convergence) which becomes an element of $\mathcal{G}$ with the operation obtained as the extension by continuity of the rule

$$T_\tau f + T_s f = T_{\tau+s} f \quad \forall \tau, s.$$

The neutral element is f. If we define

$$\Psi_f(\tau) = T_\tau f,$$

the pair $(\mathcal{H}_f, \Psi_f)$ is in $\mathcal{P}$. The representation formula (8) holds with

$$\mathfrak{f}(f_*) = f_*(0) \quad \forall f_* \in \mathcal{H}_f. \tag{9}$$

This function $\mathfrak{f}$ is sometimes called the "extension by continuity" of the almost periodic function $f(t)$ to its hull $\mathcal{H}_f$. All those are standard facts in the theory of almost periodic functions: for a more detailed discussion see [13].

The next lemma proves that the canonical representation $(\mathcal{H}_f, \Psi_f)$ is minimal.

Lemma 12. *The almost periodic function $f(t)$ is representable over $(\Omega, \Psi) \in \mathcal{P}$ if and only if there exists a morphism $(\Omega, \Psi) \to (\mathcal{H}_f, \Psi_f)$.*

Proof. If the morphism exists, the representation is trivially obtained by composition with the canonical representation. Assume now $f(t)$ is representable over the given pair, and (8) holds for a suitable $\mathfrak{f} \in \mathcal{C}(\Omega)$. By means of Lemma 8, to conclude the proof we have to show that $\Psi(\tau_n) \to 0$ implies $T_{\tau_n} f \to f$ uniformly. This is a consequence of the identity

$$T_{\tau_n} f(t) = f(t + \tau_n) = \mathfrak{f}\big(\Psi(t + \tau_n)\big) = \mathfrak{f}\big(\Psi(t) + \Psi(\tau_n)\big)$$

together with the uniform continuity of $\mathfrak{f}(\omega)$. $\square$

It is worthwhile to notice what the representability is, in the case Ω is the hull of another almost periodic function, say $g(t)$. By definition, $\Psi_g(\tau_n) \to 0$ if and only if $T_{\tau_n} g \to g$ uniformly, and the same holds for $f(t)$. Hence $f(t)$ is representable over (or extends by continuity to) $(\mathcal{H}_g, \Psi_g)$ if and only if

$$mod(f) \subset mod(g).$$

This can be easily checked using the previously mentioned characterizations of this inclusion.

To finish the section, let us consider the derivatives of almost periodic functions and their representations. Given a function $\mathfrak{f}: \Omega \to \mathbb{R}$ let us first introduce a notion of derivative along the flow, by means of

$$D_\Psi \mathfrak{f}(\omega) = \lim_{\tau \to 0} \frac{1}{\tau} \big\{ \mathfrak{f}(\omega \cdot \tau) - \mathfrak{f}(\omega) \big\}.$$

Notice that, if $\Omega = \mathbb{T}^d$ and the flow is associated to the nonresonant vector $\nu \in \mathbb{R}^d$, then the previous derivative is in fact a directional derivative, namely:

$$D_\Psi = \frac{\partial}{\partial \nu} = \sum_{i=1}^{d} \nu_i \frac{\partial}{\partial \theta_i}.$$

Assume now that an almost periodic function $f(t)$ is representable over (Ω, Ψ) and has a derivative $f'(t)$. If (8) holds then $D_\Psi \mathfrak{f}$ exists on the points of the image of Ψ and we have

$$f'(t) = D_\Psi \mathfrak{f}\big(\Psi(t)\big) \quad \forall t. \tag{10}$$

The next result explores when this formula induces a representation for $f'(t)$.

Lemma 13. *In the previous conditions for $f(t)$, the derivative $f'(t)$ is almost periodic if and only if $D_\Psi \mathfrak{f}$ exists everywhere and belongs to $\mathcal{C}(\Omega)$.*

Proof. If $D_\Psi \mathfrak{f}$ is continuous one can apply (10) to deduce that $f'(t)$ is almost periodic. Let us now assume that $f'(t)$ is almost periodic and prove the converse. Since $mod(f') = mod(f)$, Lemma 12 applies to show that $f'(t)$ is representable over (Ω, Ψ). Assume

$$f'(t) = \mathfrak{g}\big(\Psi(t)\big)$$

for a suitable $\mathfrak{g} \in \mathcal{C}(\Omega)$. Because of (10), the equality $D_\Psi \mathfrak{f} = \mathfrak{g}$ holds on the image of Ψ. To show that it holds on an arbitrary point ω of Ω we pick a sequence $\omega_n = \Psi(\tau_n) \to \omega$ and notice that the following limits:

$$f_{\omega_n}(t) := \mathfrak{f}\big(\Psi(\tau_n) + \Psi(t)\big) \to f_\omega(t) := \mathfrak{f}\big(\omega + \Psi(t)\big),$$
$$f'_{\omega_n}(t) = f'(t + \tau_n) = \mathfrak{g}\big(\Psi(\tau_n) + \Psi(t)\big) \to \mathfrak{g}\big(\omega + \Psi(t)\big)$$

are uniform on the real line. This implies that f_ω has derivative and $f'_\omega(t) = \mathfrak{g}(\omega \cdot t)$. In consequence $D_\Psi \mathfrak{f}$ exists on the orbit passing through ω and one has:

$$D_\Psi \mathfrak{f}(\omega \cdot t) = \mathfrak{g}(\omega \cdot t) \quad \forall t.$$

For arbitrary ω and $t = 0$ one deduces that $D_\Psi \mathfrak{f} = \mathfrak{g}$ is continuous. $\square$

6. An abstract partial differential equation

Assume that $\mathfrak{A}$ is a square matrix of dimension N with coefficients in $\mathcal{C}(\Omega)$ and $\mathfrak{b} \in \mathcal{C}(\Omega, \mathbb{R}^N)$, and consider the equation

$$D_\Psi \mathfrak{x} = \mathfrak{A}(\omega)\mathfrak{x} + \mathfrak{b}(\omega). \tag{11}$$

A *solution* of (11) is a function $\mathfrak{x} : \Omega \to \mathbb{R}^N$ such that $D_\Psi \mathfrak{x}$ exists on Ω and the previous identity is satisfied for all $\omega \in \Omega$.

Equation (11) is related to the family of ordinary differential equations

$$\dot{x} = \mathfrak{A}(\omega \cdot t)x + \mathfrak{b}(\omega \cdot t), \tag{12}$$

where $\omega \in \Omega$ acts as a parameter. For instance, if $\mathfrak{x}$ is a solution of (11) then, for every $\omega \in \Omega$, the function

$$x_\omega(t) = \mathfrak{x}(\omega \cdot t)$$

is a solution of (12). This follows from the definition of D_Ψ. Moreover, if $\mathfrak{x}$ is continuous, then x_ω is almost periodic for every $\omega \in \Omega$.

Concerning the converse, let us first consider the case where $\Omega = \mathbb{T}^d$ and $\Psi(t) = (\nu_1 t, \dots, \nu_d t)$, where $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{R}^d$ is a nonresonant vector. Equation (11) becomes

$$\frac{\partial}{\partial \nu}\mathfrak{x} = \mathfrak{A}(\omega)\mathfrak{x} + \mathfrak{b}(\omega),$$

while (12) is a quasi-periodic equation. If this last ordinary equation admits a quasi-periodic solution with the same frequencies as the coefficients, namely such that

$$x(t) = \mathfrak{x}(\omega \cdot t), \quad \mathfrak{x} \in \mathcal{C}\big(\mathbb{T}^d, \mathbb{R}^N\big),$$

then $\mathfrak{x}$ is a solution of the partial differential equation. This is a standard fact in the theory of the quasi-periodic equations: the next lemma extends it to the general case.

Lemma 14. *Assume that, for a given $\omega_* \in \Omega$, Eq.* (12) *has a solution $x_*(t) = \mathfrak{x}(\omega_* \cdot t)$ with $\mathfrak{x} \in \mathcal{C}(\Omega, \mathbb{R}^N)$. Then $\mathfrak{x}$ is a solution to* (11).

Proof. The proof is the same as for the quasi-periodic case; we give it hereafter just for the convenience of the reader. It is clear that, for every $\tau \in \mathbb{R}$, the translated function $T_\tau x_*$ is an almost periodic solution to (12) with respect to $\omega_* \cdot \tau$. Fix now an $\omega \in \Omega$ and choose τ_n such that $\omega_* \cdot \tau_n \to \omega$. By the uniform continuity of $\mathfrak{x}$, we know that $\mathfrak{x}(\omega_* \cdot \tau_n \cdot t) \to \mathfrak{x}(\omega \cdot t)$ uniformly on t. On the other hand, standard arguments apply to show that the function $x(t) = \mathfrak{x}(\omega \cdot t)$ solves the limiting equation (12). Since

$$\dot{x}(t) = D_\Psi \mathfrak{x}(\omega \cdot t)$$

holds true, the thesis follows by taking $t = 0$ in (12). $\square$

By summarizing the previous discussion, we showed that those almost periodic solutions to (12) which are representable over (Ω, Ψ), are in one-to-one correspondence with the continuous solutions of (11). The next proposition is concerned with the *discontinuous* solutions to the same equation.

Before stating the result, let us notice that the set of the continuity points of any solution of (11) cannot be arbitrary: because of the uniqueness of the initial value problem associated to (12), it has to be a subset of Ω which is invariant for the flow induced by Ψ. Moreover, let us point out that we are interested in discontinuities which are in some sense unavoidable: if

$\mathfrak{y}$ is a function on Ω, and $C \subset \Omega$ is the set where $\mathfrak{y}$ is continuous, we will say that $\mathfrak{y}$ has *non-removable discontinuities* when it cannot be modified outside C, obtaining a continuous function. For instance, this implies that $C \neq \emptyset$.

The next result will show that, under certain circumstances, continuous and discontinuous solutions to (11) cannot coexist. To this aim, consider the class of homogeneous ordinary differential equations

$$\dot{z} = \mathfrak{A}(\omega \cdot t)z \tag{13}$$

with $\omega \in \Omega$, and rewrite condition (H_0') as

(K_0') For some $\omega \in \Omega$, all the bounded solutions to (13) are homoclinic to zero.

Proposition 15. *Assume that* (K_0') *holds. If* (11) *admits a bounded solution with non-removable discontinuities, then it has no continuous solutions.*

Proof. Let $\mathfrak{y}$ be a bounded solution to (11) with non-removable discontinuities, and assume by contradiction that $\mathfrak{x}$ is a continuous solution of the same equation. Due to the compactness of Ω, $\mathfrak{x}$ is also bounded. If $\omega_* \in \Omega$ denotes a point where (K_0') is satisfied, we must have

$$\mathfrak{y}(\omega_* \cdot t) - \mathfrak{x}(\omega_* \cdot t) \to 0 \quad \text{as } |t| \to +\infty. \tag{14}$$

Let now $\omega \in \Omega$ be a continuity point of $\mathfrak{y}$. Since the flow on Ω is minimal, the ω-limit of ω_* coincides with Ω. Hence there is a sequence $t_k \to +\infty$ such that $\omega_* \cdot t_k \to \omega$. As a consequence of (14) we have

$$\mathfrak{y}(\omega) = \lim_k \mathfrak{y}(\omega_* \cdot t_k) = \lim_k \mathfrak{x}(\omega_* \cdot t_k) = \mathfrak{x}(\omega).$$

This is impossible for $\mathfrak{y}$. $\square$

7. Construction of discontinuous solutions

Assume that $\mathfrak{A}$ is an $N \times N$ matrix with coefficients in $\mathcal{C}(\Omega)$, and consider the following condition:

(K_0'') For some $\omega \in \Omega$, Eq. (13) admits a nontrivial solution homoclinic to zero.

The aim of this section is to prove the proposition.

Proposition 16. *If* (K_0'') *holds, then there exists a term* $\mathfrak{b} \in \mathcal{C}(\Omega, \mathbb{R}^N)$*, for which Eq.* (11) *has a bounded solution with non-removable discontinuities.*

From now on, we will assume that condition (K_0'') is satisfied. In the remaining part of the section, we will prove the previous proposition.

The next lemma is a first step in this direction.

Lemma 17. *Assume that condition* (K_0'') *holds. Then* Ω *is nontrivial and, moreover, we have* $\Omega \not\cong \mathbb{S}^1$.

Proof. Due to Lemma 7, we have to prove that Ψ is aperiodic. By contradiction assume it is periodic. Then Eq. (13) is periodic for every $\omega \in \Omega$. Then, as a consequence of the Floquet theory, its bounded solutions must be almost periodic. In particular, they must be separated from zero (see for instance [7, Theorem 5.7]). This is impossible for the ω's which condition (K_0'') refers to. □

From now on, let $\varphi(t)$ be a nontrivial solution to

$$\dot{z} = \mathfrak{A}(\omega_* \cdot t)z \tag{15}$$

satisfying

$$\lim_{|t|\to+\infty} \varphi(t) = 0. \tag{16}$$

Then, define on Ω the following function:

$$\mathfrak{z}(\omega) = \begin{cases} \varphi(t) & \text{if } \omega = \omega_* \cdot t,\ t \in \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the flow on Ω is aperiodic, $\mathfrak{z}(\omega)$ is well defined for every $\omega \in \Omega$. Moreover, by construction, it is a bounded solution to the homogeneous equation

$$D_\Psi \mathfrak{z} = \mathfrak{A}(\omega)\mathfrak{z}. \tag{17}$$

Concerning its continuity properties: because $\varphi(t) \neq 0$ for every t, it is discontinuous at any point of $\omega_* \cdot \mathbb{R}$ while, because of (16), it is continuous on $\Omega \setminus \omega_* \cdot \mathbb{R}$.

Summing up, $\mathfrak{z}$ is a bounded and discontinuous solution to (17). Unfortunately, however, the discontinuities are removable: indeed, it may be extended to the trivial solution, which on $\Omega \setminus \omega_* \cdot \mathbb{R}$ coincides with $\mathfrak{z}$.

To overcome the problem, we will modify the function $\mathfrak{z}$ on a suitable enlargement of the flow line $\omega_* \cdot \mathbb{R}$, in such a way that $\varphi(t)$ will be the jump one has by crossing that line. This was done by Zhikov and Levitan in [17, Section 8, p. 159]), in the particular case where $\Omega = \mathbb{T}^2$, $N = 1$ and $\varphi(t)$ is integrable on $\mathbb{R}$.

To start with, let us rewrite $\varphi(t)$ in a more convenient way, as a sum

$$\varphi(t) = \sum_{n\in\mathbb{Z}} \varphi_n(t) \tag{18}$$

of compactly supported functions. To this aim, chose a sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ of strictly positive real numbers satisfying

$$\alpha_0 \geqslant \|\varphi\|_\infty, \qquad \alpha_{-n} = \alpha_n \quad \text{and} \quad \sum_{n\neq 0} \alpha_n < +\infty. \tag{19}$$

Then cover $\mathbb{R}$ with a sequence of open and bounded intervals $\{I_n\}_{n\in\mathbb{Z}}$, in such a way that each one has length

$$|I_n| > 4 \quad \forall n, \tag{20}$$

the $(n+1)$th interval stays on the right of I_n, but overlaps it by a fixed length

$$|I_{n+1} \cap I_n| = 2 \quad \forall n, \tag{21}$$

and finally the following estimate holds:

$$\left|\varphi(t)\right| \leqslant \alpha_n \quad \forall t \in I_n.$$

Notice that, because of conditions (20) and (21), the covering is locally finite: indeed, each point has a neighborhood which intersect at most two consecutive intervals.

To construct the covering, one can for instance choose $I_0 = (a_0, b_0)$ such that $b_0 - a_0 > 4$ and $a_0 < 0 < b_0$, and so long that $|\varphi(t)| \leqslant \alpha_1$ holds outside $[a_0+2, b_0-2]$. Then define $b_{-1} = a_0+2$ and $a_1 = b_0 - 2$, and choose $I_{-1} = (a_{-1}, b_{-1})$ and $I_1 = (a_1, b_1)$ so long that $|\varphi(t)| \leqslant \alpha_2$ outside $[a_{-1}+2, b_1-2]$. By repeating the previous process, we obtain the searched intervals. This construction is possible because of (16), and because each I_n can be taken arbitrarily large.

Once the covering $\{I_n\}_{n\in\mathbb{Z}}$ is given, let us consider a $\mathcal{C}^1$ partition of unity subordinated to it. This means to consider a sequence of $\mathcal{C}^1$ maps $\chi_n : \mathbb{R} \to [0, 1]$ which satisfy

$$\operatorname{supp}(\chi_n) \subset I_n \quad \forall n$$

and, moreover:

$$\sum_{n\in\mathbb{Z}} \chi_n(t) = 1 \quad \forall t.$$

Notice that, if $t \notin I_n$, then $\chi_n(t) \equiv 0$ in a neighborhood of t. Hence, if we define

$$\mathbb{Z}_t = \{n \in \mathbb{Z}:\ t \in I_n\}$$

then we have

$$\sum_{n\in\mathbb{Z}_t} \chi_n(t) = 1 \quad \text{and} \quad \sum_{n\in\mathbb{Z}_t} \dot{\chi}_n(t) = 0. \tag{22}$$

The set $\mathbb{Z}_t$ will contain one or two elements. Moreover, again due to (20) and (21), the partition can be chosen so that the derivative of χ_n satisfies

$$\beta_1 := \sup_n \|\dot{\chi}_n\|_\infty < +\infty.$$

Finally, we are ready to define

$$\varphi_n(t) = \chi_n(t)\varphi(t) \quad \forall n, t.$$

The identity (18) holds by construction. Moreover, $\mathrm{supp}(\varphi_n) \subset I_n$, and the estimates

$$\|\varphi_n\|_\infty \leqslant \alpha_n, \qquad \|\dot{\varphi}_n\|_\infty \leqslant \big(\beta_1 + \|\mathfrak{A}\|_\infty\big)\alpha_n \tag{23}$$

trivially hold for every n. The second one depends on the identity

$$\dot{\varphi}_n(t) = \dot{\chi}_n(t)\varphi(t) + \mathfrak{A}(\omega_* \cdot t)\varphi_n(t).$$

The idea is now the following: for every given n, to interpret φ_n as a function defined on the flow line segment $\omega_* \cdot I_n$, and to extend it to a function $\mathfrak{u}_n : \Omega \to \mathbb{R}^N$ which has its support in a thin strip around that segment.

To this aim, from now on let us fix a *nontrivial* character $\sigma \in \Omega^*$, and denote by

$$\Sigma = \big\{\omega \in \Omega\colon\ \sigma(\omega) = 1\big\}$$

its kernel. The condition (K_0'') is invariant by the flow and we prove in Section 4 that Σ is a global section for the flow on Ω, so it is not restrictive to assume that

$$\omega_* \in \Sigma.$$

Next we construct a sequence $\{U_n\}_{n\in\mathbb{Z}}$ of open neighborhoods of ω_* in Σ satisfying that for each $n \geqslant 0$ (respectively $n \leqslant 0$): $U_{n+1} \subset U_n$ [respectively $U_{n-1} \subset U_n$] and the flow is injective on $U_n \times (I_n \cup I_{n+1})$ (respectively $U_n \times (I_n \cup I_{n-1})$). The existence of these neighborhoods is guaranteed by Lemmas 11 and 17. Let Ω_n be the image by the flow of $U_n \times I_n$, that is

$$\Omega_n = \{\eta \cdot t\colon\ \eta \in U_n,\ t \in I_n\}.$$

We know by Lemma 10 that Ω_n is open in Ω and the flow $(\eta, t) \mapsto \eta \cdot t$ describes a homeomorphism between $U_n \times I_n$ and Ω_n for every $n \in \mathbb{Z}$. This fact will be implicitly used several times. Another property that can be easily checked and will be useful in the proof of Lemma 19 is the following: for every $n \in \mathbb{Z}$, the flow also describes an homeomorphism between $[U_n \cap U_{n+1}] \times [I_n \cap I_{n+1}]$ and $\Omega_n \cap \Omega_{n+1}$.

Define:

$$\mathfrak{u}_n(\omega) = \begin{cases} s_n(\eta)\varphi_n(t) & \text{if } \omega = \eta \cdot t,\ \eta \in U_n,\ t \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\{s_n\}_{n\in\mathbb{Z}}$ is a sequence of functions $\Sigma \to [0, 1]$, whose definition will be responsible for the continuity properties of the $\mathfrak{u}_n$'s. To construct them, let us start by fixing two sequences $\{x_k\}$ and $\{y_k\}$ such that

$$x_k, y_k \in \Sigma \setminus (\omega_* \cdot \mathbb{R}), \qquad x_k, y_k \to \omega_*, \qquad x_k \neq y_h \quad \forall h, k.$$

To justify the existence of these sequences notice that Σ is compact and perfect (Lemma 17 and Proposition 6) while $\Gamma = \Sigma \cap (\omega_* \cdot \mathbb{R})$ is countable (Proposition 9). Thus, Baire theorem implies that $\Sigma \setminus \Gamma$ is dense in Σ. Once we have these sequences we fix a function $s : \Sigma \to [0, 1]$, such that

$$s \in \mathcal{C}\big(\Sigma \setminus \{\omega_*\}\big), \tag{24}$$

and

$$s(x_k) \to 0, \qquad s(y_k) \to 1.$$

See Appendix A for an explicit construction. Let us point out that the value $s(\omega_*)$ will be immaterial for the arguments below. Then, localize this function inside each U_n, by defining

$$s_n(\eta) = \gamma_n(\eta)s(\eta),$$

where:

(i) γ_n is continuous on Σ;
(ii) the support of γ_n is contained in U_n;
(iii) $\gamma_n \equiv 1$ on a neighborhood of ω_*.

Again, see Appendix A for a concrete construction of these functions.

Let us now look at the consequences on the maps $\mathfrak{u}_n$'s. Because of condition (ii), the inclusion

$$\operatorname{supp}(\mathfrak{u}_n) \subset \Omega_n \tag{25}$$

holds true. In particular, $\mathfrak{u}_n$ is continuous on every point $\omega \in \Omega \setminus \Omega_n$. Inside Ω_n, one can use condition (i) and (24) to show that $\mathfrak{u}_n$ is continuous at points in $\Omega_n \setminus \omega_* \cdot I_n$.

Summing up, $\mathfrak{u}_n$ is continuous outside the flow line segment $\omega_* \cdot I_n$, while we expect some discontinuity here, because of condition (24).

Again distinguishing between the inside and the outside of Ω_n, one can easily check that the derivative of $\mathfrak{u}_n$ along the flow exists everywhere in Ω, and equals

$$D_\Psi \mathfrak{u}_n(\omega) = \begin{cases} s_n(\eta)\dot{\varphi}_n(t) & \text{if } \omega = \eta \cdot t,\ \eta \in U_n,\ t \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by the same arguments used for $\mathfrak{u}_n$, also $D_\Psi \mathfrak{u}_n$ is continuous at the points outside $\omega_* \cdot I_n$.

Finally notice that, due to (23), each $\mathfrak{u}_n$ is bounded on Ω together with its derivative, namely:

$$\|\mathfrak{u}_n\|_\infty \leqslant \alpha_n, \qquad \|D_\Psi \mathfrak{u}_n\|_\infty \leqslant \big(\beta_1 + \|\mathfrak{A}\|_\infty\big)\alpha_n. \tag{26}$$

We are now ready to define:

$$\mathfrak{u}(\omega) = \sum_{n\in\mathbb{Z}} \mathfrak{u}_n(\omega) \quad \forall \omega \in \Omega. \tag{27}$$

Because of (26) and (19), the previous sum is uniformly convergent on Ω, and the same happens to the sum of the derivatives. Hence, elementary arguments apply to show that $D_\Psi \mathfrak{u}$ exists everywhere, and equals

$$D_\Psi \mathfrak{u}(\omega) = \sum_{n\in\mathbb{Z}} D_\Psi \mathfrak{u}_n(\omega) \quad \forall \omega \in \Omega.$$

If we define

$$\mathfrak{b}(\omega) = D_\Psi \mathfrak{u}(\omega) - \mathfrak{A}(\omega)\mathfrak{u}(\omega), \quad \omega \in \Omega, \tag{28}$$

then clearly $\mathfrak{u}$ is a bounded solution to the partial differential equation

$$D_\Psi \mathfrak{x} = \mathfrak{A}(\omega)\mathfrak{x} + \mathfrak{b}(\omega).$$

To prove Proposition 16, we have to show that $\mathfrak{b}$ is continuous on Ω, while $\mathfrak{u}$ has non-removable discontinuities. This is the content of the next two lemmas.

Lemma 18. *The function* $\mathfrak{u}$*, as defined by* (27)*, has non-removable discontinuities.*

Proof. Since each $\mathfrak{u}_n$ is continuous outside $\omega_* \cdot \mathbb{R}$, the same happens to $\mathfrak{u}$. Next we will prove that $\mathfrak{u}$ has a non-removable discontinuity at any point of the flow line $\omega_* \cdot \mathbb{R}$.

Let us fix a $t \in \mathbb{R}$, and study the behaviour of $\mathfrak{u}$ around $\omega_* \cdot t$. Consider the set of indexes:

$$\mathbb{Z}_t = \{n \in \mathbb{Z}\colon\ t \in I_n\}.$$

If $n \notin \mathbb{Z}_t$ then $\omega_* \cdot t \notin \omega_* \cdot I_n$, for otherwise the flow should be periodic. Hence $\mathfrak{u}_n$ is continuous at $\omega_* \cdot t$, and the same happens to the sum $\sum_{n \notin \mathbb{Z}_t} \mathfrak{u}_n$. It remains to show that the function

$$\hat{\mathfrak{u}} = \sum_{n \in \mathbb{Z}_t} \mathfrak{u}_n$$

has a non-removable discontinuity at the point $\omega_* \cdot t$. To this aim, consider the sequences x_k and y_k that we have fixed in the construction. We claim that

$$\hat{\mathfrak{u}}(y_k \cdot t) - \hat{\mathfrak{u}}(x_k \cdot t) \to \varphi(t).$$

Notice that, if this is true, then the same jump occurs for $\mathfrak{u}$ and the proof is complete. Indeed, we know that $\varphi(t) \neq 0$ for every t (by the uniqueness of the initial value problem), and all the points $x_k \cdot t$ and $y_k \cdot t$ live in the region where $\mathfrak{u}$ is continuous.

Next we compute the limit of $\hat{\mathfrak{u}}(y_k \cdot t)$. Take $n \in \mathbb{Z}_t$, so that $t \in I_n$. Since $y_k \to \omega_*$, then also $y_k \in U_n$ eventually holds as $k \to +\infty$. Hence eventually we may compute:

$$\mathfrak{u}_n(y_k \cdot t) = \varphi_n(t)\gamma_n(y_k)s(y_k) = \varphi_n(t)s(y_k),$$

where condition (iii) has been used. Making the sum over $n \in \mathbb{Z}_t$, and using (22), we then obtain

$$\hat{\mathfrak{u}}(y_k \cdot t) = s(y_k) \sum_{n \in \mathbb{Z}_t} \varphi_n(t) = s(y_k)\varphi(t).$$

Until now, we only used that $y_k \to \omega_*$. Considering the full information about y_k we may conclude that

$$\hat{\mathfrak{u}}(y_k \cdot t) \to \varphi(t).$$

On the other hand, if the sequence $\hat{\mathfrak{u}}(x_k \cdot t)$ is now considered, then the same arguments apply to show that

$$\mathfrak{u}_n(x_k \cdot t) = \varphi_n(t)\gamma_n(x_k)s(x_k) \to 0$$

holds for every $n \in \mathbb{Z}_t$. Hence $\hat{\mathfrak{u}}(x_k \cdot t) \to 0$, so proving the claim and the lemma. $\square$

Lemma 19. *The function* $\mathfrak{b}$*, as defined by* (28)*, is continuous on* Ω.

Proof. We follow the line of the proof of Lemma 18, using the same notations.

First of all notice that $\mathfrak{b}$ may be written as a uniform convergent sum:

$$\mathfrak{b}(\omega) = \sum_{n \in \mathbb{Z}} \mathfrak{b}_n(\omega), \quad \omega \in \Omega,$$

where we defined

$$\mathfrak{b}_n(\omega) = \begin{cases} s_n(\eta)\{\dot{\varphi}_n(t) - \mathfrak{A}(\eta \cdot t)\varphi_n(t)\} & \text{if } \omega = \eta \cdot t,\ \eta \in U_n,\ t \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

The same arguments as in the proof of Lemma 18 apply to show that $\mathfrak{b}$ is continuous outside $\omega_* \cdot \mathbb{R}$ and that, if we fix $t \in \mathbb{R}$, the sum $\sum_{n \notin \mathbb{Z}_t} \mathfrak{b}_n$ is continuous at $\omega_* \cdot t$.

The only delicate point is the continuity at $\omega_* \cdot t$ of

$$\hat{\mathfrak{b}} = \sum_{n \in \mathbb{Z}_t} \mathfrak{b}_n.$$

Notice that, as a consequence of (22) we have

$$\sum_{n \in \mathbb{Z}_t} \dot{\varphi}_n(t) = \left[\sum_{n \in \mathbb{Z}_t} \dot{\chi}_n(t)\right]\varphi(t) + \left[\sum_{n \in \mathbb{Z}_t} \chi_n(t)\right]\dot{\varphi}(t) = \dot{\varphi}(t).$$

Hence we may compute:

$$\begin{aligned} \hat{\mathfrak{b}}(\omega_* \cdot t) &= s(\omega_*) \sum_{n \in \mathbb{Z}_t} \bigl\{\dot{\varphi}_n(t) - \mathfrak{A}(\omega_* \cdot t)\varphi_n(t)\bigr\} \\ &= s(\omega_*)\bigl\{\dot{\varphi}(t) - \mathfrak{A}(\omega_* \cdot t)\varphi(t)\bigr\} = 0. \end{aligned}$$

Take now a sequence $\omega_k \to \omega_* \cdot t$ in Ω, and let us show that $\hat{\mathfrak{b}}(\omega_k) \to 0$ as $k \to +\infty$, possibly up to subsequences. This is enough to show the continuity at $\omega_* \cdot t$. To compute $\hat{\mathfrak{b}}(\omega_k)$ explicitly, notice that

$$\omega_k \in \bigcap_{n \in \mathbb{Z}_t} \Omega_n$$

eventually holds as $k \to +\infty$, for this set is an open neighborhood of $\omega_* \cdot t$. After the construction of the neighborhoods U_n we mentioned a couple of properties of the sets Ω_n. They imply that, whatever $\mathbb{Z}_t$ is, the flow describes an homeomorphism

$$\Omega_n \cap \Omega_{n+1} \cong \Bigg[\bigcap_{n\in\mathbb{Z}_t} U_n\Bigg] \times \Bigg[\bigcap_{n\in\mathbb{Z}_t} I_n\Bigg].$$

Write $\omega_k = \eta_k \cdot t_k$ where $\eta_k \in \bigcap_{n\in\mathbb{Z}_t} U_n$ and $t_k \in \bigcap_{n\in\mathbb{Z}_t} I_n$. Since $\omega_k \to \omega_* \cdot t \in \bigcap_{n\in\mathbb{Z}_t} \Omega_n$, then we must have $\eta_k \to \omega_*$ and $t_k \to t$. As a consequence, for large k's we may compute:

$$\begin{aligned}\hat{\mathfrak{b}}(\omega_k) &= \sum_{n\in\mathbb{Z}_t} \gamma_n(\eta_k)s(\eta_k)\big\{\dot\varphi_n(t_k) - \mathfrak{A}(\eta_k \cdot t_k)\varphi_n(t_k)\big\}\\ &= s(\eta_k)\sum_{n\in\mathbb{Z}_t}\big\{\dot\varphi_n(t_k) - \mathfrak{A}(\eta_k \cdot t_k)\varphi_n(t_k)\big\}.\end{aligned}$$

Now, since the sequence $s(\eta_k)$ is bounded, we may assume that

$$s(\eta_k) \to \theta$$

holds true for some $\theta \in [0, 1]$, possibly up to subsequences. Hence, we obtain again

$$\begin{aligned}\hat{\mathfrak{b}}(\omega_k) &\to \theta\sum_{n\in\mathbb{Z}_t}\big\{\dot\varphi_n(t) - \mathfrak{A}(\omega_* \cdot t)\varphi_n(t)\big\}\\ &= \theta\big\{\dot\varphi(t) - \mathfrak{A}(\omega_* \cdot t)\varphi(t)\big\} = 0\end{aligned}$$

which concludes the proof. □

The next proposition is concerned with the regularity of the term $\mathfrak{b}$, namely with the existence and continuity of its derivatives along the flow. To this aim, we need some further regularity of the partition of unity we considered. From now on, it will be assumed that we deal with a $\mathcal{C}^\infty$ partition of unity and that

$$\beta_r := \sup_n \big\|\chi_n^{(r)}\big\|_\infty < +\infty \quad \forall r.$$

This is again possible because of conditions (20) and (21).

Proposition 20. *Assume that, for some $k \geqslant 1$, the first k derivatives along the flow of $\mathfrak{A}$ ($D_\Psi^r\mathfrak{A}$, $r \leqslant k$) exist and are continuous on Ω. Then the same happens to $\mathfrak{b}$, as defined by* (28).

Proof. We will first show the existence of the derivatives of $\mathfrak{b}$ and then, following the proof of Lemma 19, their continuity. The notations will be the same of that proof.

Since $D^r\mathfrak{A}$ exists for $r \leqslant k$, then the map $t \mapsto \mathfrak{A}(\omega_* \cdot t)$ is of class $\mathcal{C}^k$, and hence $\varphi \in \mathcal{C}^{k+1}$. As a consequence, each $\mathfrak{b}_n$ is k-times derivable along the flow, and for every $j \leqslant k$ we have

$$D_\Psi^j\mathfrak{b}_n(\omega) = \begin{cases} s_n(\eta)\frac{d^j}{dt^j}\{\dot\varphi_n - A(\eta\cdot t)\varphi_n\}(t) & \text{if } \omega = \eta\cdot t,\ \eta\in U_n,\ t\in I_n,\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, using the chain rule it is not difficult to check that

$$\left\| D_\Psi^j \mathfrak{b}_n \right\|_\infty \leqslant c_j \alpha_n$$

holds for every $j \leqslant k$, where c_j depends only on the constants $\beta_1, \dots, \beta_{j+1}$ and $\|\mathfrak{A}\|_\infty, \dots, \|D_\Psi^j \mathfrak{A}\|_\infty$. Standard arguments of uniform convergence then apply to show that $D_\Psi^j \mathfrak{b}$ exists for $j \leqslant k$ and equals:

$$D_\Psi^j \mathfrak{b} = \sum_{n \in \mathbb{Z}} D_\Psi^j \mathfrak{b}_n.$$

To complete the proof, we have to show they are in fact continuous. Using that $D_\Psi^j \mathfrak{A} \in \mathcal{C}(\Omega)$ for every $j \leqslant k$, everything works fine outside $\omega_* \cdot \mathbb{R}$. Moreover, at a given point $\omega_* \cdot t$, the only delicate part is to show the continuity of the sum

$$\widehat{D_\Psi^j \mathfrak{b}} = \sum_{n \in \mathbb{Z}_t} D_\Psi^j \mathfrak{b}_n.$$

This can be done exactly as in the proof of Lemma 19. The only difference is that here we have to use the identity

$$\sum_{n \in \mathbb{Z}_t} \frac{d^j}{dt^j} \left\{ \dot{\varphi}_n(t) - \mathfrak{A}(\omega_* \cdot t)\varphi_n(t) \right\} = 0. \tag{29}$$

This can be obtained by differentiating, with respect to s, the identity

$$\sum_{n \in \mathbb{Z}_t} \left\{ \dot{\varphi}_n(s) - \mathfrak{A}(\omega_* \cdot s)\varphi_n(s) \right\} = 0,$$

which holds for s close to t. □

8. Conclusions

In this section we put together what we obtained in the last sections, in order to prove the results stated in the introduction.

To this aim, consider the homogeneous almost periodic differential equation

$$\dot{z} = A(t)z,$$

where $A(t)$ satisfies condition (H_0).

If we take:

$$(\Omega, \Psi) = (\mathcal{H}_A, \Psi_A)$$

then both the abstract conditions (K_0') and (K_0'') are satisfied, in correspondence to the same point $\omega = A_*$. Indeed, if $\mathfrak{A} \in \mathcal{C}(\mathcal{H}_A)$ is the canonical representation of $A(t)$ (see (9)), then

$$\mathfrak{A}(\omega \cdot t) = \mathfrak{A}(A_* + T_t A) = T_t A_*(0) = A_*(t).$$

This depends on the definition of the addition in $\mathcal{H}_A$ (see Section 5).

As a consequence of Proposition 16, there exists a $\mathfrak{b} \in \mathcal{C}(\mathcal{H}_A)$ such that Eq. (11) admits a bounded solutions with non-removable discontinuities. Hence, Proposition 15 applies to show that the same equations cannot have continuous solutions. If we define:

$$b(t) = \mathfrak{b}(T_t A),$$

this means that the inhomogeneous equation

$$\dot{x} = A(t)x + b(t)$$

has bounded solutions and cannot have almost periodic solutions satisfying $mod(x) \subset mod(A)$. This is a consequence of the discussions after Lemmas 12 and 14. On the other hand, Proposition 4 applies (via Lemma 5) to show they are the unique admissible almost periodic solutions. This proves Theorem 2.

Concerning Corollary 3, assume now that the coefficients of A are in AP^∞, and prove that also those of b are in AP^∞.

From Lemma 13, the derivatives along the flow $D^k_\Psi \mathfrak{A}$ exists everywhere in Ω and the coefficients of $D^k_\Psi \mathfrak{A}$ belong to $\mathcal{C}(\mathcal{H}_A)$ for every k. Thus, Proposition 20 guarantees that $D^k_\Psi \mathfrak{b} \in \mathcal{C}(\mathcal{H}_A, \mathbb{R}^N)$ for every k, and we may conclude that the coefficients of b are in AP^∞.

Appendix A

A.1. Discontinuous functions on metric spaces

Let X be a metric space, and denote by d its metric. Moreover, assume that $p \in X$ is an accumulation point for X itself.

First we will show the existence of a function $s : X \to [0, 1]$ such that the following conditions are satisfied:

$$s \in \mathcal{C}\big(X \setminus \{p\}\big), \qquad \liminf_{x \to p} s(x) = 0, \qquad \limsup_{x \to p} s(x) = 1. \tag{A.1}$$

To this aim, use that p is an accumulation point to construct two sequences $\{y_k\}_{k\in\mathbb{N}}$ and $\{z_k\}_{k\in\mathbb{N}}$ in X such that:

$$y_k, z_k \neq p \quad \forall k,$$
$$y_k, z_k \to p \quad \text{as } k \to +\infty,$$
$$y_h \neq z_k \quad \forall h, k.$$

Then define $s(p)$ arbitrarily in $[0, 1]$, and

$$s(x) = \frac{d(x, Y)}{d(x, Y) + d(x, Z)} \quad \forall p \neq x,$$

where $Y = \{y_k\colon\ k \in \mathbb{N}\}$ and $Z = \{z_k\colon\ k \in \mathbb{N}\}$. Since $\bar{Y} \cap \bar{Z} = \{p\}$, the denominator vanishes only at p. Hence the function is well defined and continuous outside p. On the other hand,

$d(y_k, Y) = d(z_k, Z) = 0$. Thus

$$s(y_k) = 0, \qquad s(x_k) = 1$$

hold true for every k, so proving (A.1).

To conclude the discussion, let us show how to localize the previous function around p. If we choose two open neighborhoods U and V of p such that

$$\bar{V} \subset U$$

this can be done by considering the product map $x \in X \mapsto \gamma(x)s(x)$, where $\gamma : X \to [0, 1]$ satisfies the following three conditions:

$$\gamma \in \mathcal{C}(X), \qquad \operatorname{supp}(\gamma) \subset \bar{U}, \qquad \gamma \equiv 1 \quad \text{on } V. \tag{A.2}$$

An explicit realization is $\gamma \equiv 1$, when $U = X$, and

$$\gamma(x) = \frac{d(x, U^C)}{d(x, V) + d(x, U^C)}$$

in all the other cases. Notice that, since $\bar{V} \cap U^C = \emptyset$, the function γ is indeed well defined and continuous on X. The remaining two conditions in (A.2) can be checked easily.

A.2. An example referred to Corollary 3

We construct a function $A(t)$ in AP^∞ such that (H_0) holds with $A = A_*$. We deal with scalar equations ($N = 1$) and so it is enough to prove that the primitive $\mathcal{A}(t) = \int_0^t A(s)\,ds$ satisfies

$$\mathcal{A}(t) \to -\infty \quad \text{as } |t| \to \infty.$$

Define $A = -f$ with

$$f(t) = \sum_{n=0}^{\infty} a^n \sin\bigl(b^n t\bigr), \quad 0 < b < a < 1.$$

It is easy to check that f is in AP^∞ and the primitive $F(t)$ is given by

$$F(t) = 2\sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n \sin^2\left(\frac{b^n t}{2}\right).$$

Define $N(t) = \min\{n\colon\ b^n|t| \leqslant \pi\}$. The inequality

$$|\sin\theta| \geqslant \frac{2}{\pi}|\theta| \quad \text{if } \theta \in [-\pi/2, \pi/2]$$

implies that

$$F(t) \geqslant 2\left(\frac{t}{\pi}\right)^2 \sum_{n \geqslant N(t)} (ab)^n = 2\left(\frac{t}{\pi}\right)^2 \frac{(ab)^{N(t)}}{1-ab}.$$

Some computations, valid for large $|t|$, show that

$$N(t) \leqslant \frac{\log(|t|/\pi)}{|\log b|} + 1 \quad \text{and}$$

$$F(t) \geqslant (constant)|t|^{1-|\log a/\log b|} \to +\infty \quad \text{as } |t| \to \infty.$$

This fact was already observed by Haraux in [8, p. 110].

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