European Journal of Combinatorics 32 (2011) 1317–1328

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

On prime inductive classes of graphs

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ARTICLE INFO

Article history:
Received 7 January 2009
Accepted 13 May 2011
Available online 12 August 2011

ABSTRACT

Let $H[G_1, \ldots, G_n]$ denote a graph formed from unlabelled graphs $G_1, \ldots, G_n$ and a labelled graph $H = (\{v_1, \ldots, v_n\}, E)$ replacing every vertex v_i of H by the graph G_i and joining the vertices of G_i with all the vertices of those of G_j whenever $\{v_i, v_j\} \in E(H)$. For unlabelled graphs $G_1, \ldots, G_n, H$, let $\varphi_H(G_1, \ldots, G_n)$ stand for the class of all graphs $H[G_1, \ldots, G_n]$ taken over all possible orderings of $V(H)$.

A prime inductive class of graphs, $I(\mathcal{B}, \mathcal{C})$, is said to be a set of all graphs, which can be produced by recursive applying of $\varphi_H(G_1, \ldots, G_{|V(H)|})$ where H is a graph from a fixed set $\mathcal{C}$ of prime graphs and $G_1, \ldots, G_{|V(H)|}$ are either graphs from the set $\mathcal{B}$ of prime graphs or graphs obtained in the previous steps. Similar inductive definitions for cographs, k-trees, series–parallel graphs, Halin graphs, bipartite cubic graphs or forbidden structures of some graph classes were considered in the literature (Batagelj (1994) [1] Drgas-Burchardt et al. (2010) [6] and Hajós (1961) [10]).

This paper initiates a study of prime inductive classes of graphs giving a result, which characterizes, in their language, the substitution closed induced hereditary graph classes. Moreover, for an arbitrary induced hereditary graph class $\mathcal{P}$ it presents a method for the construction of maximal induced hereditary graph classes contained in $\mathcal{P}$ and substitution closed.

The main contribution of this paper is to give a minimal forbidden graph characterization of induced hereditary prime inductive classes of graphs. As a consequence, the minimal forbidden graph characterization for some special induced hereditary prime inductive graph classes is given

There is also offered an algebraic view on the class of all prime inductive classes of graphs of the type $I(\{K_1\}, \mathcal{C})$.

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doi:10.1016/j.ejc.2011.05.001

1. Introduction

There are many different possibilities for defining a class of graphs: by forbidden structures, features, which each member of such a family has to have, construction and so on. Among others, inductive definitions of graph classes were done in the literature [1,2,6,10]. The last option is useful mainly because of an algorithmic aspect. An efficient algorithm for a problem restricted to an inductively (recursively) constructed graph class typically uses the following approach: solve the problem on the base graphs defined for the given class, and then combine the solutions for subgraphs into a solution for a larger graph that is formed by the specific composition rules that govern the construction of members in the class [2].

In this paper, we are focusing on a wide class of special inductive graph classes, defined as prime. The minimal forbidden graph characterization of those of them which are induced hereditary is given (Theorem 3) and in particular all induced hereditary graph classes with prime forbidden subgraphs are described in the language of prime inductive classes (Theorem 6). Moreover, maximal prime inductive graph classes, which are substitution closed and contained in a given induced hereditary graph class are found in Theorem 7. Finally, an algebraic description of prime inductive graph classes in Theorems 8 and 9 is given. The examples of the presented results application can be found in the paper.

2. Preliminaries

Throughout this paper all graphs are undirected, simple and finite. Let G denote a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a given $v \in V(G)$ let $N_G(v)$, $deg_G(v)$ stand for the open neighbourhood of v and the degree of v in G, respectively. The symbols $\overline{G}$ and $G[V']$ for $V' \subseteq V(G)$ denote the complement of the graph G and the graph induced in G by the set of vertices V'. If G' is an induced subgraph of G we write $G' \leq G$. The notation K_n, C_n, P_n is used for a complete graph, a cycle and a path of order n, respectively.

A set $W \subseteq V(G)$ is *a module* in a graph G if for any two vertices $x, y \in W$, the equality $N_G(x) \setminus W = N_G(y) \setminus W$ is satisfied. The *trivial modules* in G are $V(G)$, $\emptyset$ and the singletons. A graph having only trivial modules is called *prime*. A module M of a graph G is *strong* if for each other module M' of G either M and M' are disjoint sets or one of them is a proper subset of the second one.

For a given graph G its strong modules can be organized in a tree, T_G, to represent their inclusion order, that is, a strong module M is an ancestor of another strong module M' in T_G if and only if $M' \subseteq M$. T_G is called the modular decomposition tree of G. Its root is $V(G)$ and its leaves are the singletons $\{v\}$, for $v \in V(G)$.

To explain interesting interdependence between the structure of a graph and its prime induced subgraphs we cite the modular decomposition theorem, due to Gallai.

By *a maximal prime induced subgraph* of G we mean a prime induced subgraph of G, which is contained as an induced subgraph in no other prime induced subgraph of G.

Theorem 1 ([7]). *Let G be any graph with at least two vertices. Then exactly one of the following conditions holds:*

1. *G is disconnected and can be uniquely decomposed into its connected components,*
2. *$\overline{G}$ is disconnected and G can be uniquely decomposed into complements of connected components of $\overline{G}$,*
3. *G and $\overline{G}$ are connected and there is some $U \subseteq V(G)$ and a unique partition Π of $V(G)$ such that*
   (a) *$|U| \geq 4$, and*
   (b) *every part S of the partition Π is a maximal strong module in G with $|S \cap U| = 1$, and*
   (c) *the subgraph of G induced by U is the maximal prime induced subgraph of G, whose form does not depend on the choice of U.*

Because for a given graph G, its unique decomposition parts described in Theorem 1 correspond to its maximal strong modules, one can see that T_G has exactly three types of internal vertices. Let M be a strong module in G and v_M be the internal vertex of T_G corresponding to M. If $G[M]$ is disconnected, then v_M is called series, and its children in T_G are sets of vertices of the connected components of

$G[M]$. If $\overline{G}[M]$ is disconnected, then v_M is called parallel, and its children in T_G are sets of vertices of the connected components of $\overline{G}[M]$. If both $G[M]$ and $\overline{G}[M]$ are connected, then v_M is called prime, and its children are labelled by the maximal strong modules included in M and different from M. These modules define the partition of M denoted as Π in Theorem 1 used for the graph $G[M]$.

Let T_G be a modular decomposition tree of a graph G. We assign to each internal vertex v of T_G a graph H_v so that if v is parallel then $H_v = K_2$, if v is series then $H_v = \overline{K_2}$, otherwise $H_v = G[U_v]$, where $U_v \subseteq V(G)$ is a set having exactly one common vertex with each maximal strong module which corresponds to the child of v in T_G ($|V(H_v)| = deg_{T_G}(v)$).

Theorem 1 and the definitions mentioned above imply that G is uniquely determined by T_G with assignment $\{H_v : v \text{ is an internal vertex of } T_G\}$, where each H_v is viewed as the graph labelled by children of v. In the rest of the paper the couple $(T_G, \{H_v : v \text{ is an internal vertex of } T_G\})$ will be called the modular pair of G.

Let us denote by $\mathcal{I}$ the class of all graphs having at least one vertex, and let **PRIME** be its subclass of prime graphs on at least two vertices. Note that both $\mathcal{I}$ and **PRIME** contain unlabelled graphs. From now on $Z(G) = \{H \in \mathcal{I} : \text{there exists an internal vertex } v \text{ of } T_G \text{ such that } H \text{ is isomorphic to } H_v\}$.

Let $Z^*(G)$ stand for a superset of $Z(G)$ containing all induced subgraphs of graphs from $Z(G)$ being in **PRIME**.

Observe that for each $G \in$ **PRIME** the modular decomposition tree T_G is a star on $|V(G)|+1$ vertices having only one internal vertex v with $H_v = G$. Hence, in such a case $Z(G) = \{G\}$.

Lemma 1. *Let $G, G' \in \mathcal{I}$ and $G' \in$* **PRIME**. *Then $G' \leq G$ if and only if $G' \in Z^*(G)$.*

Proof. First we assume that $G' \in Z^*(G)$. It means that there exists $H \in Z(G)$ satisfying $G' \leq H$. By Theorem 1 we know that each $H \in Z(G)$ is an induced subgraph of G and by the transitivity of $\leq$-relation one can observe that $G' \leq G$. To prove the converse implication, assume that **PRIME** $\ni G' \leq G$ and there is no $H \in Z(G)$ such that $G' \leq H$. It means that there are disjoint strong modules $V_1, \ldots, V_n$, $n \geq 2$, satisfying $G' \leq G[V_1 \cup \cdots \cup V_n]$, $V(G') \cap V_i \neq \emptyset$ for each $i \in [n]$ and there exists $k \in [n]$ for which $|V(G') \cap V_k| \geq 2$. Trivially, if $G' \leq G$ and M is a module of G, then $V(G') \cap M$ is a module of G'. Hence, $V_k \cap V(G')$ is a non-trivial module in G'. This contradicts the fact that $G' \in$ **PRIME**. □

Below we give an immediate observation which follows from Lemma 1.

Remark 1. Let $G_1, G_2 \in \mathcal{I}$ and $G_1 \leq G_2$. Then $Z^*(G_1) \subseteq Z^*(G_2)$.

For given graphs $G_1, \ldots, G_n \in \mathcal{I}$ and a labelled graph $H = (\{v_1, \ldots, v_n\}, E)$ we will use the symbol $H[G_1, \ldots, G_n]$ to denote the graph whose vertex set is the union of $V(G_1), V(G_2), \ldots, V(G_n)$ and whose edge set consists of the union of $E(G_1), E(G_2), \ldots, E(G_n)$ with the additional edge set $\{\{x, y\} : x \in V(G_i),\ y \in V(G_j),\ \{v_i, v_j\} \in E\}$. A description of the symbol $H[G_1, \ldots, G_n]$ implies that $V(H)$ is ordered, which will be sometimes omitted in assumptions. It is worth mentioning that the presented above graph operation was defined as a generalized lexicographic product of graphs and analysed in the literature in different aspects (see [11]).

For $C \in$ **PRIME** and unlabelled graphs $G_1, \ldots, G_{|V(C)|}$, which can be isomorphic, by $\varphi_C(G_1, \ldots, G_{|V(C)|})$ we denote the set of all graphs $C[G_1, \ldots, G_{|V(C)|}]$ taken over all possible labellings of $V(C)$. For instance, $\varphi_{K_2}(G_1, G_2)$ ($\varphi_{\overline{K_2}}(G_1, G_2)$) produces a one-element set consisting of the join (union) of G_1 and G_2.

3. Prime inductive graph classes

Let $\mathcal{B}, \mathcal{C}$ be classes of prime graphs such that $K_1 \in \mathcal{B} \setminus \mathcal{C}$. We define a *prime inductive class* $I(\mathcal{B}, \mathcal{C})$ as a class of all graphs which can be obtained starting from graphs of $\mathcal{B}$ and using recursively φ_C with $C \in \mathcal{C}$. It means that if $(T_G, \{H_v : v \in V\})$ is the modular pair of $G \in I(\mathcal{B}, \mathcal{C})$, then for each $v \in V$ we have $H_v \in \mathcal{B} \cup \mathcal{C}$ and if $H_v \in \mathcal{B} \setminus \mathcal{C}$, then either v is the unique vertex of V and T_G is the star or v is adjacent to $|V(H_v)|$ leaves and one ancestor labelled by a graph from $\mathcal{C} \setminus \mathcal{B}$. It follows that if G belongs to the prime inductive class $I(\mathcal{B}, \mathcal{C})$, T_G is of height at least two and v is its root, then $H_v \in \mathcal{C}$. Moreover, for an arbitrary $\mathcal{C} \subseteq$ **PRIME**, the equality $I(\{K_1\}, \mathcal{C}) = \{G \in \mathcal{I} : Z(G) \subseteq \mathcal{C}\}$ is satisfied.

A class $\mathcal{A}$ of graphs is called *prime induced hereditary* if $G \in \mathcal{A}$ implies $Z^*(G) \subseteq \mathcal{A}$.

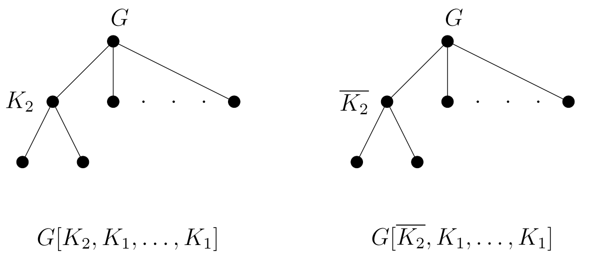


Fig. 1. Modular pairs.

Lemma 2. *Let $\mathcal{B}, \mathcal{C}$ be classes of prime graphs which are prime induced hereditary. Let $K_1 \in \mathcal{B} \setminus \mathcal{C}$ and $\{K_2, \overline{K_2}\} \subseteq \mathcal{C}$. Then $F \in$ **PRIME** $\cap I(\mathcal{B}, \mathcal{C})$ if and only if $F \in \mathcal{B} \cup \mathcal{C}$.*

Proof. Obviously, if $F \in \mathcal{B} \cup \mathcal{C}$ then $F \in$ **PRIME** and because $K_1 \in \mathcal{B}$ we have $F \in I(\mathcal{B}, \mathcal{C})$. If $F \in I(\mathcal{B}, \mathcal{C}) \setminus (\mathcal{B} \cup \mathcal{C})$, then the modular decomposition tree T_F has at least two internal vertices. It is so because we must build F from elements of $\mathcal{B}$ at least once using some $\varphi_{\mathcal{C}}$ and if we only once use only one $\varphi_{\mathcal{C}}$ then at least one out of the used elements from $\mathcal{B}$ has at least two vertices. Thus such a graph $F \notin$ **PRIME** which follows from the previously observed (before Lemma 1) fact that for each prime graph its modular decomposition tree has exactly one internal vertex. □

A graph class P is any isomorphism closed non-empty subclass of $\mathcal{I}$. A graph class P is *induced hereditary* if it is closed under taking induced subgraphs. Among others $\mathcal{O} = \{G \in \mathcal{I} : G \text{ is edgeless}\}$ and $\mathcal{K} = \{G \in \mathcal{I} : G \text{ is a complete graph}\}$ are induced hereditary graph classes. By $\mathbf{L}_{\leq}$ we denote the class of all induced hereditary graph classes (for more details, see [3]).

Let $\mathcal{B}, \mathcal{C}$ be classes of prime graphs such that $K_1 \in \mathcal{B} \setminus \mathcal{C}$. It has to be mentioned that not all prime inductive classes $I(\mathcal{B}, \mathcal{C})$ are in $\mathbf{L}_{\leq}$. For instance $C_6 \in I(\{C_6, K_1\}, \{\overline{K_2}\})$, $P_4 \notin I(\{C_6, K_1\}, \{\overline{K_2}\})$ and $P_4 \leq C_6$. It follows that $I(\{C_6, K_1\}, \{\overline{K_2}\}) \notin \mathbf{L}_{\leq}$. Using Lemma 1, it can be checked at once that any prime inductive class of graphs $I(\{K_1\}, \mathcal{C})$, for which $\mathcal{C}$ is prime induced hereditary, is in $\mathbf{L}_{\leq}$. On the other hand, the assumption that $\mathcal{B}, \mathcal{C}$ are prime induced hereditary is not sufficient for the condition $I(\mathcal{B}, \mathcal{C}) \in \mathbf{L}_{\leq}$. Let us consider the example $\mathcal{B} =$ **PRIME** $\cup \{K_1\}$ and $\mathcal{C} = \{K_2, \overline{K_2}\}$. Evidently, $\mathcal{B}, \mathcal{C}$ are prime induced hereditary and $K_1 \in \mathcal{B} \setminus \mathcal{C}$. Moreover, the graph Q_{20} presented in Fig. 2 belongs to $I(\mathcal{B}, \mathcal{C})$ because $Q_{20} \in \mathcal{B} =$ **PRIME** $\cup \{K_1\}$ but its induced subgraph Q_{16}, presented in Fig. 2 too, does not belong to $I(\mathcal{B}, \mathcal{C})$. It is so because $T_{Q_{16}}$ cannot be produced inductively as an element of $I(\mathcal{B}, \mathcal{C})$ ($T_{Q_{16}}$ is illustrated in Fig. 1 for $G = P_4$). In the next theorem some sufficient conditions for the assertion $I(\mathcal{B}, \mathcal{C}) \in \mathbf{L}_{\leq}$ will be given.

Let $\mathcal{B}, \mathcal{C}$ be graph classes which are prime induced hereditary. By $\mathcal{D}_{\mathcal{B},\mathcal{C}}$ we denote the class of graphs $\bigcup_G(\varphi_G(K_2, K_1, \ldots, K_1) \cup \varphi_G(\overline{K_2}, K_1, \ldots, K_1))$, where the union is taken over all graphs G being minimal with respect to $\leq$-relation in $(\mathcal{B} \setminus (\mathcal{C} \cup \{K_1\}))$.

Theorem 2. *Let $\mathcal{B}, \mathcal{C}$ be classes of prime graphs which are prime induced hereditary, $K_1 \in \mathcal{B} \setminus \mathcal{C}$ and $\{K_2, \overline{K_2}\} \subseteq \mathcal{C}$. If the conditions $H' \in \mathcal{D}_{\mathcal{B},\mathcal{C}}$ and $H \in \mathcal{B}$ imply $H' \not\leq H$, then $I(\mathcal{B}, \mathcal{C}) \in \mathbf{L}_{\leq}$.*

Proof. Suppose that $G \in I(\mathcal{B}, \mathcal{C})$ and $G_1 \leq G$. We shall prove that $G_1 \in I(\mathcal{B}, \mathcal{C})$.

By the definition of $I(\mathcal{B}, \mathcal{C})$ we have the inclusion $Z(G) \subseteq \mathcal{B} \cup \mathcal{C}$. Because $\mathcal{B}, \mathcal{C}$ are prime induced hereditary $Z^*(G) \subseteq \mathcal{B} \cup \mathcal{C}$ holds. It follows from Remark 1 that $Z^*(G_1) \subseteq Z^*(G)$ and because $Z(G_1) \subseteq Z^*(G_1)$ we have the observation $Z(G_1) \subseteq \mathcal{B} \cup \mathcal{C}$.

To obtain the contradiction, suppose that $G_1 \notin I(\mathcal{B}, \mathcal{C})$. Our earlier considerations lead to the conclusion that if $(T_{G_1}, \{H_v : v \in V\})$ is the modular pair of G_1, then there exists a vertex $w \in V$ satisfying that $H_w \in \mathcal{B} \setminus (\mathcal{C} \cup \{K_1\})$ and there exists a son x of w such that $x \in V$ and $H_x \in \mathcal{C}$. It follows that there exists at least one graph $G^* \in \varphi_{H_w}(K_2, K_1, \ldots, K_1) \cup \varphi_{H_w}(\overline{K_2}, K_1, \ldots, K_1)$ satisfying $G^* \leq H_w[H_x, K_1, \ldots, K_1]$. Note that H_x has at least two vertices and H_w is an induced supergraph of at least one graph from $\min_{\leq}(\mathcal{B} \setminus (\mathcal{C} \cup \{K_1\}))$. Thus G^* contains as an induced subgraph at least one graph H' from $\mathcal{D}_{\mathcal{B},\mathcal{C}}$. Next, combining $H_w \leq G \in I(\mathcal{B}, \mathcal{C})$ with $H_w \in \mathcal{B} \setminus (\mathcal{C} \cup \{K_1\})$ we get the existence of $H \in \mathcal{B}$ satisfying $H_w[H_x, K_1, \ldots, K_1] \leq H$. Evidently, $H' \leq H$, giving a contradiction. □

Now we refer to a minimal forbidden graph set $\mathbf{F}(\mathcal{P})$, which uniquely determines the graph class $\mathcal{P} \in \mathbf{L}_{\le}$ and is defined as follows:

$\mathbf{F}(\mathcal{P}) = \{G \in \mathcal{I} : G \notin \mathcal{P}$ but for each proper induced subgraph H of $G, H \in \mathcal{P}\}$.

Theorem 3. *Let $\mathcal{B}, \mathcal{C}$ be classes of prime graphs which are prime induced hereditary. Let $K_1 \in \mathcal{B} \setminus \mathcal{C}$ and $\{K_2, \overline{K_2}\} \subseteq \mathcal{C}$. If the conditions $H' \in \mathcal{D}_{\mathcal{B},\mathcal{C}}$ and $H \in \mathcal{B}$ imply $H' \not\le H$, then $\mathbf{F}(I(\mathcal{B}, \mathcal{C})) = \mathcal{D}_{\mathcal{B},\mathcal{C}} \cup \min_{\le}(\mathcal{D}_{\mathcal{B},\mathcal{C}} \cup (\mathbf{PRIME} \setminus (\mathcal{B} \cup \mathcal{C})))$.*

Proof. From Theorem 2, it immediately follows that $I(\mathcal{B}, \mathcal{C}) \in \mathbf{L}_{\le}$. We start by showing that if $G \in \min_{\le}(\mathcal{B} \setminus (\mathcal{C} \cup \{K_1\}))$, then each graph G^* being either of the form $G[K_2, K_1, \ldots, K_1]$ or $G[\overline{K_2}, K_1, \ldots, K_1]$ is forbidden for $I(\mathcal{B}, \mathcal{C})$. Clearly, G^* cannot be an element of $I(\mathcal{B}, \mathcal{C})$ because it has the modular pair (see Fig. 1), which does not correspond to any of the graphs from $I(\mathcal{B}, \mathcal{C})$.

Assume that v is an arbitrary vertex of G^*. We claim that $G^* \setminus \{v\} \in I(\mathcal{B}, \mathcal{C})$. If v is one of the vertices of $K_2(\overline{K_2})$ then $G^* \setminus \{v\} \in \mathcal{B}$, which implies $G^* \setminus \{v\} \in I(\mathcal{B}, \mathcal{C})$. Otherwise ($v$ is neither a vertex of K_2 nor $\overline{K_2}$) we obtain that $Z(G^* \setminus \{v\}) \subseteq \mathcal{C}$. Hence $G^* \setminus \{v\} \in I(\{K_1\}, \mathcal{C}) \subseteq I(\mathcal{B}, \mathcal{C})$.

Now, we assume that $F \in \min_{\le}(\mathcal{D}_{\mathcal{B},\mathcal{C}} \cup (\mathbf{PRIME} \setminus (\mathcal{B} \cup \mathcal{C})))$ and F is prime. Evidently, by Lemma 2 the statement $F \notin I(\mathcal{B}, \mathcal{C})$ is true. Let v be an arbitrary vertex of F. If $F \setminus \{v\}$ is prime then by choice of F we know that $F \setminus \{v\} \in \mathcal{B} \cup \mathcal{C}$ and Lemma 2 guarantees that $F \setminus \{v\} \in I(\mathcal{B}, \mathcal{C})$.

Assume that $F \setminus \{v\}$ is not prime. Then again by choice and minimality arguments we can observe that $Z(F \setminus \{v\}) \subseteq \mathcal{B} \cup \mathcal{C}$. Let us assume that $F \setminus \{v\} \notin I(\mathcal{B}, \mathcal{C})$. Hence in the modular decomposition tree $T_{F\setminus\{v\}}$ there exists a vertex x corresponding to the operation φ_C with $C = H_x$ and $C \in \mathcal{B} \setminus (\mathcal{C} \cup \{K_1\})$ and x has a son x' corresponding to the operation $\varphi_{C'}$ with $C' = H_{x'}$, where $C' \in \mathcal{C}$. Because either $K_2 \le C'$ or $\overline{K_2} \le C'$ we have that $G_1 = C[K_2, K_1, \ldots, K_1]$ or $G_2 = C[\overline{K_2}, K_1, \ldots, K_1]$ is an induced subgraph of $F \setminus \{v\}$. Of course both G_1, G_2 contain induced subgraphs from $\mathcal{D}_{\mathcal{B},\mathcal{C}}$, contrary to the choice of F.

Actually we have shown that $\mathcal{D}_{\mathcal{B},\mathcal{C}} \cup \min_{\le}(\mathcal{D}_{\mathcal{B},\mathcal{C}} \cup (\mathbf{PRIME} \setminus (\mathcal{B} \cup \mathcal{C}))) \subseteq \mathbf{F}(I(\mathcal{B}, \mathcal{C}))$. Finally, assume that $F \in \mathbf{F}(I(\mathcal{B}, \mathcal{C}))$ and $F \notin \mathcal{D}_{\mathcal{B},\mathcal{C}} \cup \min_{\le}(\mathcal{D}_{\mathcal{B},\mathcal{C}} \cup (\mathbf{PRIME} \setminus (\mathcal{B} \cup \mathcal{C})))$. The last part of the proof is divided into two cases.

1. F is not prime. Thus the minimality argument guarantees that $Z(F) \subseteq \mathcal{B} \cup \mathcal{C}$. Since $F \notin I(\mathcal{B}, \mathcal{C})$, like the previous part of the proof (for $F \setminus \{v\}$), the existence of a graph from $\mathcal{D}_{\mathcal{B},\mathcal{C}}$ being an induced subgraph of F can be shown, which is impossible.
2. F is prime. Then from Lemma 2 once again $F \in \mathbf{PRIME} \setminus (\mathcal{B} \cup \mathcal{C})$ and F does not contain any induced subgraph from $\mathcal{D}_{\mathcal{B},\mathcal{C}}$. It is easy to see that there is no F satisfying all these requirements. □

Below we present the examples of Theorem 3 application giving the minimal forbidden graph characterizations for some prime inductive classes of graphs. The following lemma will be a useful tool in our consideration.

Lemma 3 ([15]). *Let H be a prime supergraph of the path on four vertices, which is different from this path. Then H contains as an induced subgraph at least one of the graphs $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ depicted in Fig. 2.*

Theorem 4. *The class $\mathbf{F}(I(\{K_1, K_2, \overline{K_2}, P_4, P_5\}, \{K_2, \overline{K_2}, P_4\}))$ consists of the graphs Q_i, $i \in \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 20, 21, 22\}$ depicted in Fig. 2.*

Proof. Clearly, we can use Theorem 3 with $\mathcal{B} = \{K_1, K_2, \overline{K_2}, P_4, P_5\}$ and $\mathcal{C} = \{K_2, \overline{K_2}, P_4\}$. Because P_5 is the unique graph from $\mathcal{B} \setminus (\mathcal{C} \cup \{K_1\})$ then $\mathcal{D}_{\mathcal{B},\mathcal{C}} = \{Q_8, \ldots, Q_{13}\}$. Moreover, Lemma 3 implies that graphs Q_1, Q_2, Q_3, Q_5, Q_6 are minimal elements in $\mathbf{PRIME} \setminus (\mathcal{B} \cup \mathcal{C})$ in the sense of $\le$-relation, so that $A = \{\mathcal{Q}_i : i \in \{1, 2, 3, 5, 6, 8, 9, 10, 11, 12, 13\}\}$ is a set of incomparable elements in $(\mathcal{I}, \le)$. Next, we can observe that $S = A \cup \{Q_7, Q_{20}, Q_{21}, Q_{22}\}$ is an antichain in $(\mathcal{I}, \le)$ and graphs from $S \setminus A$ are at the same time from $\mathbf{PRIME} \setminus (\mathcal{B} \cup \mathcal{C})$. We shall show that each other graph G^* from $\mathbf{PRIME} \setminus (\mathcal{B} \cup \mathcal{C})$ contains as an induced subgraph at least one element of S. Using Lemma 3 once again we can assume that G^* has at least six vertices and contains P_5 as an induced subgraph. Suppose that P_5 is induced by $V' = \{v_1, v_2, v_3, v_4, v_5\} \subseteq V(G^*)$ and let $v_6 \in V(G^*) \setminus V'$. If $|N(v_6) \cap V'| = 1$, then Q_7, Q_8 or Q_{20} is induced subgraph of G^*. If $|N(v_6) \cap V'| = 2$, then $V' \cup \{v_6\}$ induces in G^* one of the graphs: Q_2, Q_3, Q_9, Q_{10}, Q_{12}, Q_{21} as a subgraph. If $|N(v_6) \cap V'| = 3$, then we can find induced Q_1, Q_2, Q_3, Q_{11}, Q_{13} or Q_{22} in G^*. If $|N(v_6) \cap V'| = 4$, then G^* contains Q_1 or Q_3 as an

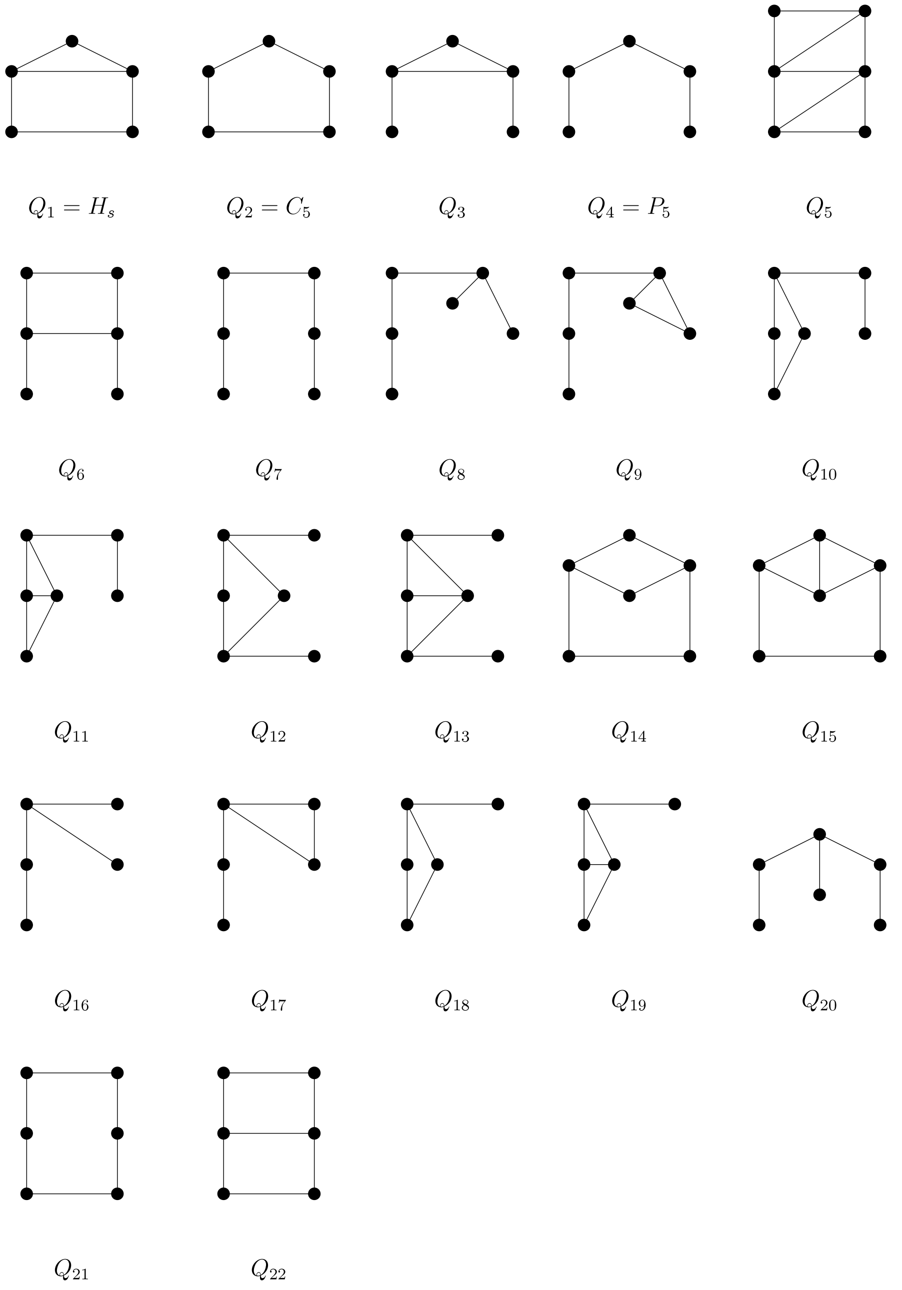


Fig. 2. Special graphs.

induced subgraph. If $|N(v_6) \cap V'| \in \{0, 5\}$ then either there exists other vertex of G^* satisfying one of the properties analysed previously or V' is a module in G^* contrary to the assumption about its primality. $\square$

In the same fashion one can derive many other sets of forbidden graphs for prime inductive graph classes, which are induced hereditary. We present some of these results in Table 1 omitting their proofs which imitate the proof of Theorem 4.

Table 1

$\mathcal{B}$	$\mathcal{C}$	$\mathbf{F}(I(\mathcal{B},\mathcal{C})) = \{Q_i : i \in J\}$
$\{K_1, K_2, \overline{K_2}, P_4\}$	$\{K_2, \overline{K_2}\}$	$J = \{1, 2, 3, 4, 16, 17, 18, 19\}$
$\{K_1, K_2, \overline{K_2}, P_4, P_5\}$	$\{K_2, \overline{K_2}\}$	$J = \{1, 2, 3, 7, 16, 17, 18, 19, 21\}$
$\{K_1, K_2, \overline{K_2}, P_4, C_5\}$	$\{K_2, \overline{K_2}\}$	$J = \{1, 3, 4, 16, 17, 18, 19\}$
$\{K_1, K_2, \overline{K_2}, P_4, C_5\}$	$\{K_2, \overline{K_2}, P_4\}$	$J = \{1, 3, 4, 5, 6, 14, 15\}$
$\{K_1, K_2, \overline{K_2}, P_4, P_5\}$	$\{K_2, \overline{K_2}, P_4\}$	$J = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 20, 21, 22\}$

It is worth mentioning that the last row of Table 1 coincides with Theorem 4 and the first one concerns the subclass of P_4-sparse graphs and can be deduced from [12].

Other consequences of Theorem 3 can be formulated as the following facts.

Theorem 5. *Let $G^* \in$ **PRIME** be a graph on at least four vertices, $M = \varphi_{G^*}(K_2, K_1, \ldots, K_1) \cup \varphi_{G^*}(\overline{K_2}, K_1, \ldots, K_1)$ and $\mathcal{P} \in \mathbf{L}_{\leq}$. Then $\mathbf{F}(\mathcal{P}) = M$ if and only if $\mathcal{P} = I(\mathcal{B}, \mathcal{C})$, where $\mathcal{B} = \{G \in \mathbf{PRIME}$: for each $G' \in M$ holds $G' \not\leq G\} \cup \{K_1\}$ and $\mathcal{C} = \{G \in \mathbf{PRIME} : G^* \not\leq G\}$.*

Proof. If $G^* \in$ **PRIME** and $\mathcal{P} = I(\mathcal{B}, \mathcal{C})$, where $\mathcal{B} = \{G \in \mathbf{PRIME}$: for each $G' \in M$ holds $G' \not\leq G\} \cup \{K_1\}$ and $\mathcal{C} = \{G \in \mathbf{PRIME} : G^* \not\leq G\}$, then Theorem 3 gives that $\mathbf{F}(\mathcal{P}) = \varphi_{G^*}(K_2, K_1, \ldots, K_1) \cup \varphi_{G^*}(\overline{K_2}, K_1, \ldots, K_1)$. The uniqueness of description of $\mathcal{P} \in \mathbf{L}_{\leq}$ by $\mathbf{F}(\mathcal{P})$ leads to the assertion. □

For instance it follows that if Q_{14}, Q_{15} are graphs presented in Fig. 2, then $\mathbf{F}(\mathcal{P}) = \{Q_{14}, Q_{15}\}$ characterizes $\mathcal{P} = I(\{G \in \mathbf{PRIME} : Q_{14} \not\leq G$ and $Q_{15} \not\leq G\} \cup \{K_1\}, \{G \in \mathbf{PRIME} : C_5 \not\leq G\})$. Fortunately, Theorem 5 can be generalized to the case $\mathbf{F}(\mathcal{P}) = \bigcup_{\{G: G\in \mathcal{G}\}}(\varphi_G(K_2, K_1, \ldots, K_1) \cup \varphi_G(\overline{K_2}, K_1, \ldots, K_1))$, where $\mathcal{G}$ is an arbitrary class of at least 4-vertex prime graphs whose elements are incomparable in $(\mathcal{I}, \leq)$. Such $\mathbf{F}(\mathcal{P})$ describes the class $I(\mathcal{B}, \mathcal{C})$ with $\mathcal{B} = \{G \in \mathbf{PRIME}$: G does not contain any element of $\bigcup_{\{G: G\in \mathcal{G}\}}(\varphi_G(K_2, K_1, \ldots, K_1) \cup \varphi_G(\overline{K_2}, K_1, \ldots, K_1))$ and $\mathcal{C} = \{G \in \mathbf{PRIME}$: for each $G^* \in \mathcal{G},\ G^* \not\leq G\}$.

Theorem 6. *Let $\mathcal{P} \in \mathbf{L}_{\leq}$. $\mathbf{F}(\mathcal{P}) \subseteq$ **PRIME** if and only if there exists a class of graphs $\mathcal{C} \subseteq$ **PRIME**, which is prime induced hereditary such that $\mathcal{P} = I(\{K_1\}, \mathcal{C})$.*

Proof. Let $\mathcal{P} = I(\{K_1\}, \mathcal{C})$ for some $\mathcal{C} \subseteq$ **PRIME**, which is prime induced hereditary. First we assume that $\{K_2, \overline{K_2}\} \subseteq \mathcal{C}$. By Theorem 3, $\mathbf{F}(\mathcal{P}) = \min_{\leq}(\mathbf{PRIME} \setminus \mathcal{C})$, which gives us $\mathbf{F}(\mathcal{P}) \subseteq$ **PRIME**. If the condition $\{K_2, \overline{K_2}\} \subseteq \mathcal{C}$ is not satisfied then because $\mathcal{C}$ is prime induced hereditary three possibilities are allowed:

1. $\overline{K_2} \in \mathcal{C}$ and $K_2 \notin \mathcal{C}$. In this case $\mathcal{P} = \mathcal{O}$ and $\mathbf{F}(\mathcal{O}) = \{K_2\} \subseteq$ **PRIME**,
2. $\overline{K_2} \notin \mathcal{C}$ and $K_2 \in \mathcal{C}$. In this case $\mathcal{P} = \mathcal{K}$ and $\mathbf{F}(\mathcal{K}) = \{\overline{K_2}\} \subseteq$ **PRIME**,
3. $\overline{K_2} \notin \mathcal{C}$ and $K_2 \notin \mathcal{C}$. In this case $\mathcal{P} = \{K_1\}$ and $\mathbf{F}(\mathcal{P}) = \{\overline{K_2}, K_2\} \subseteq$ **PRIME**.

Assume that $\mathbf{F}(\mathcal{P}) \subseteq \mathbf{PRIME} \setminus \{K_2, \overline{K_2}\}$. Put $\mathcal{C} = \{G \in \mathbf{PRIME} : Z^*(G) \cap \mathbf{F}(\mathcal{P}) = \emptyset\}$. Remark 1 implies that defined $\mathcal{C}$ is prime induced hereditary. The assumption $\{K_2, \overline{K_2}\} \cap \mathbf{F}(\mathcal{P}) = \emptyset$ and the definition of $\mathcal{C}$ yield $\{K_2, \overline{K_2}\} \subseteq \mathcal{C}$. Hence, by Theorem 3 we have $\mathbf{F}(I(\{K_1\}, \mathcal{C})) = \min_{\leq}(\mathbf{PRIME} \setminus (\{K_1\} \cup \{G \in \mathbf{PRIME} : Z^*(G) \cap \mathbf{F}(\mathcal{P}) = \emptyset\})) = \min_{\leq}(\{G \in \mathbf{PRIME} : Z^*(G) \cap \mathbf{F}(\mathcal{P}) \neq \emptyset\}) = \mathbf{F}(\mathcal{P})$. But $\mathbf{F}(\mathcal{P})$ uniquely determines $\mathcal{P}$ which completes the proof in that case. Now we assume that $\{K_2, \overline{K_2}\} \cap \mathbf{F}(\mathcal{P}) \neq \emptyset$. Then

1. if $K_2 \in \mathbf{F}(\mathcal{P})$ and $\overline{K_2} \notin \mathbf{F}(\mathcal{P})$ we have $\mathcal{P} = \mathcal{O} = I(\{K_1\}, \mathcal{C})$ for $\mathcal{C} = \{\overline{K_2}\} \subseteq$ **PRIME**,
2. if $\overline{K_2} \in \mathbf{F}(\mathcal{P})$ and $K_2 \notin \mathbf{F}(\mathcal{P})$ we have $\mathcal{P} = \mathcal{K} = I(\{K_1\}, \mathcal{C})$ for $\mathcal{C} = \{K_2\} \subseteq$ **PRIME**,
3. if $\{K_2, \overline{K_2}\} \subseteq \mathbf{F}(\mathcal{P})$ we have $\mathcal{P} = \{K_1\} = I(\{K_1\}, \mathcal{C})$ for $\mathcal{C} = \emptyset \subseteq$ **PRIME**. □

4. Closure under substitution

A *substitution* graph G of two graphs G_1, G_2 is obtained by first removing a vertex $v \in V(G_2)$ and then making every vertex of G_1 adjacent to all the neighbours of v in G_2. We call a class $\mathcal{P}$ of graphs *substitution closed* if for each graphs $G_1, G_2 \in \mathcal{P}$ the substitution graph of G_1, G_2 is in $\mathcal{P}$.

In 1997 Giakoumakis [8] proved that for each $\mathcal{P} \in \mathbf{L}_{\leq}$ its closure under substitution $\mathcal{P}^*$ consisting of all the graphs from $\mathcal{P}$ and all their substitution graphs can be characterized by $\mathbf{F}(\mathcal{P}^*)$, which contains all minimal prime extensions of all the graphs from $\mathbf{F}(\mathcal{P})$. It has to be said that G' is a minimal prime extension of G if it is prime induced supergraph of G and it does not contain as an induced subgraph any other prime graphs containing induced G. Let $\mathbf{L}^*_{\leq}$ stand for a class of all induced hereditary and substitution closed graph classes. Because for each class $\mathcal{P} \in \mathbf{L}^*_{\leq}$ we have $\mathcal{P} = \mathcal{P}^*$ then the above consideration leads to the following conclusion.

Remark 2. Let $\mathcal{P} \in \mathbf{L}_{\leq}$. Then $\mathcal{P} \in \mathbf{L}^*_{\leq}$ if and only if $\mathbf{F}(\mathcal{P}) \subseteq \mathbf{PRIME}$.

In the light of Remark 2 and Theorem 6 we are in a position to formulate the following fact.

Corollary 1. *$\mathcal{P} \in \mathbf{L}^*_{\leq}$ if and only if $\mathcal{P} = I(\{K_1\}, \mathcal{C})$ for some $\mathcal{C} \subseteq$ **PRIME** being a prime hereditary class of graphs.*

There are interesting known graph classes from $\mathbf{L}^*_{\leq}$. Two of the most notable are perfect graphs and cographs. Probably the simple inductive description produce polynomial solvability of many, hard in general, problems for such classes [9]. Corollary 1 gives us a lot of graph classes being inductive and induced hereditary, simultaneously. To define such a class it is enough to forbid a set of prime graphs which is an antichain in $(\mathcal{I}, \leq)$. If a graph class $\mathcal{P} \in \mathbf{L}_{\leq}$ has a non-prime forbidden graph then it does not have a prime inductive description of the type $I(\{K_1\}, \mathcal{C})$. It motivates the searching of set inclusion maximal graph classes contained in $\mathcal{P}$ and having such a description. This question is stated as a main problem of this section.

To explore this topic we have to use the following notions. Let $\mathcal{M}$ be a set of graph classes. A graph class $\mathcal{T}$ is called a *transversal* (respectively, *an antichain transversal*) of $\mathcal{M}$ if for each $\mathcal{P} \in \mathcal{M}$ we have $\mathcal{T} \cap \mathcal{P} \neq \emptyset$ (respectively, $\mathcal{T} \cap \mathcal{P} \neq \emptyset$ and $\mathcal{T}$ is an antichain in $(\mathcal{I}, \leq)$).

Lemma 4 *([3]). Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathbf{L}_{\leq}$. Then $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if and only if for every $F \in \mathbf{F}(\mathcal{P}_2)$ there exists a graph $F' \in \mathbf{F}(\mathcal{P}_1)$ such that $F' \leq F$.*

Lemma 5. *Let $\mathcal{P} \in \mathbf{L}_{\leq}$. A property $\mathcal{Q} \in \mathbf{L}^*_{\leq}$ satisfies $\mathcal{Q} \subseteq \mathcal{P}$ if and only if $\mathbf{F}(\mathcal{Q})$ is an antichain transversal of the family $\{Z^*(F) : F \in \mathbf{F}(\mathcal{P})\}$.*

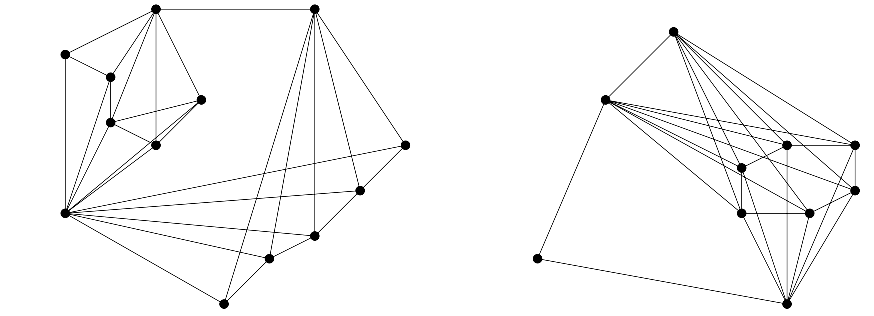
Proof. Assume that $\mathcal{Q} \in \mathbf{L}^*_{\leq}$ and $\mathcal{Q} \subseteq \mathcal{P}$. From Lemma 4 we know that for each $F \in \mathbf{F}(\mathcal{P})$ there exists $F' \in \mathbf{F}(\mathcal{Q})$ such that $F' \leq F$. Moreover, by Remark 2 each $F' \in \mathbf{F}(\mathcal{Q})$ is prime. Hence, Lemma 1 implies that for each $F \in \mathbf{F}(\mathcal{P})$ there exists $F' \in \mathbf{F}(\mathcal{Q})$ satisfying $F' \in Z^*(F)$. It means that $\mathbf{F}(\mathcal{Q})$ is a transversal of the family $\{Z^*(F) : F \in \mathbf{F}(\mathcal{P})\}$. Using the definition of the set $\mathbf{F}(\mathcal{Q})$ we know that it has to be an antichain transversal.

To prove the opposite implication let us assume that T is an antichain transversal of the family $\{Z^*(F) : F \in \mathbf{F}(\mathcal{P})\}$. Because T is an antichain in $(\mathcal{I}, \leq)$, it can be viewed as a family of minimal forbidden graphs for some property $\mathcal{Q} \in \mathbf{L}_{\leq}$. The fact that T contains only prime graphs and Remark 2 imply that $\mathcal{Q} \in \mathbf{L}^*_{\leq}$. We shall see that $\mathcal{Q} \subseteq \mathcal{P}$. Since T is a transversal of $\{Z^*(F) : F \in \mathbf{F}(\mathcal{P})\}$ one can observe that for each $F \in \mathbf{F}(\mathcal{P})$ there is $F' \in T$ such that $F' \leq F$ which, using Lemma 4, implies claimed inclusion. □

Let $\mathcal{K}, \mathcal{M}$ be two antichains in the partially ordered set $(\mathcal{I}, \leq)$. We write that $\mathcal{K} \rho \mathcal{M}$ if and only if for each $F \in \mathcal{M}$ there exists $F' \in \mathcal{K}$ such that $F' \leq F$. It is easy to verify that ρ is a partial order in the class of all antichains in $(\mathcal{I}, \leq)$. In the light of Lemmas 4 and 5 the relation ρ defined in the family of all antichains in $(\mathcal{I}, \leq)$, gives us a new insight into the main problem of this section.

Theorem 7. *Let $\mathcal{P} \in \mathbf{L}_{\leq}$ and let $\mathcal{T}_{\mathcal{P}}$ be the class of all antichain transversals for the family $\{Z^*(F) : F \in \mathbf{F}(\mathcal{P})\}$. $\mathcal{Q} \in \mathbf{L}^*_{\leq}$ is a maximal in the sense of $\subseteq$-relation element which is contained in $\mathcal{P}$ if and only if $\mathbf{F}(\mathcal{P})$ is a maximal element in $(\mathcal{T}_{\mathcal{P}}, \rho)$.*

Let us consider an example. Let $\mathcal{P}$ be a graph class for which $\mathbf{F}(\mathcal{P}) = \{G_1, G_2\}$ (see Fig. 3, where H_s is the house graph denoted by Q_1 in Fig. 2). Note that $\mathcal{P}$ is well defined because $\underline{G_1, G_2}$ are incomparable in $(\mathcal{I}, \leq)$. We can verify that $Z^*(G_1) = \{K_2, \overline{K_2}, P_4, P_5, C_5\}$ and $Z^*(G_2) = \{K_2, \overline{K_2}, H_s, P_4, P_5, C_6\}$. According to Theorem 7, to construct a maximal $\mathcal{Q} \in \mathbf{L}^*_{\leq}$, which is contained in $\mathcal{P}$, we have to find the



$G_1 = C_5[K_1, P_5, K_1, P_4[K_1, K_1, K_1, K_2], K_1]$ $\qquad$ $G_2 = H_s[C_6, K_1, K_1, K_1, K_1]$

Fig. 3. Forbidden graphs.

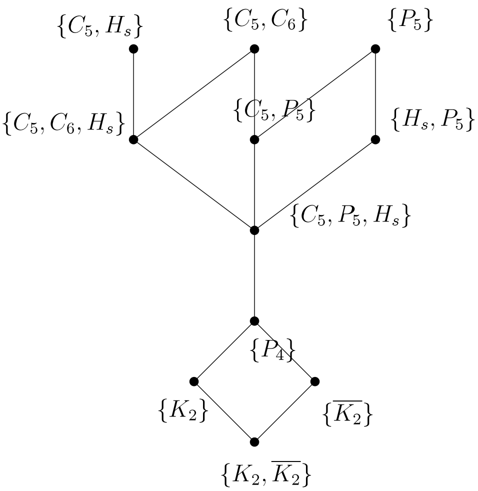


Fig. 4. Hasse diagram.

set of all antichain transversals $\mathcal{T}_{\mathcal{P}}$ of the family $\{Z^*(G_1), Z^*(G_2)\}$ and then the maximal, with respect to ρ-relation elements in this one. The Hasse diagram of the partial ordered set $(\mathcal{T}_{\mathcal{P}}, \rho)$ is illustrated in Fig. 4.

The presented consideration defines three set inclusion maximal properties $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ contained in $\mathcal{P}$ and being in $\mathbf{L}^*_{\leq}$. These properties are described by classes of minimal forbidden graphs $\mathbf{F}(\mathcal{Q}_1) = \{C_5, H_s\}, \mathbf{F}(\mathcal{Q}_2) = \{\overline{C}_5, C_6\}, \mathbf{F}(\mathcal{Q}_3) = \{P_5\}$. One of the resulting classes, $\mathcal{Q}_3$, was characterized in [13] as the class of those graphs for which each of its connected induced subgraphs has a dominating clique or a dominating C_5.

It must be noted that the procedure, which constructs a $\subseteq$-maximal class $\mathcal{Q} \in \mathbf{L}^*_{\leq}$ contained in the given class $\mathcal{P} \in \mathbf{L}_{\leq}$ guarantees the finiteness of $\mathbf{F}(\mathcal{Q})$ provided that $\mathbf{F}(\mathcal{P})$ is finite. In fact $|\mathbf{F}(\mathcal{Q})| \leq |\bigcup_{\{F \in \mathbf{F}(\mathcal{P})\}} Z^*(F)|$. Moreover, as we can observe in the example, $\mathcal{Q}$ is not uniquely determined. Both these facts are different from the corresponding features of the class $\mathcal{P}^* \in \mathbf{L}^*_{\leq}$, which contains $\mathcal{P}$ and in that sense is minimal with respect to $\subseteq$-relation [8].

We now turn our attention to other observations concerning classes $\mathcal{Q}$ from $\mathbf{L}^*_{\leq}$ contained in a given $\mathcal{P} \in \mathbf{L}_{\leq}$. One can easily see that if $\mathcal{P}$ has at least one forbidden cograph, then Theorem 5 implies that it can contain only $\mathcal{Q} = \mathcal{O}$ or $\mathcal{Q} = \mathcal{K}$. This is a case of planar graphs, claw-free graphs (especially line graphs), diamond-free graphs and so on.

Next, we illustrate Theorem 7 for $\mathcal{P}$ described by $\mathbf{F}(\mathcal{P})$ consisting of a finite or an infinite family of wheels (*a wheel of order n* can be defined as $K_2[K_1, C_{n-1}]$ with an arbitrary ordering of $V(K_2)$). Let $\{n_i : i \in \mathcal{J} \subseteq \mathbb{N}\}$ be the set of all orders of wheels from $\mathbf{F}(\mathcal{P})$, where for $i > j$ the condition $n_i > n_j$ holds. If $n_1 \geq 6$, then the unique $\subseteq$-maximal class $\mathcal{Q} \in \mathbf{L}^*_{\leq}$ included in $\mathcal{P}$ is characterized by $\mathbf{F}(\mathcal{P}) = \{C_{n_i-1} : i \in \mathcal{J}\}$. A quite different situation is for $n_1 = 4$. In that case the unique $\mathcal{Q}$ equals $\mathcal{O}$. For $n_1 = 5$, both properties $\mathcal{O}$ and $\mathcal{K}$ can play the role of $\mathcal{Q}$. The last two cases are consequences of the fact that the wheels on four or five vertices are cographs.

5. Comments on clique-with

One of the most interesting graph invariants, which could be considered in the context of presented results is the clique-with of a graph. The notion of *clique-with*, $cwd(G)$, of a graph G is defined in [4] as the minimum number of labels needed to construct G by means of the following four operations:

- creation of a new vertex v with label i,
- disjoint union of two labelled graphs,
- joining by an edge every vertex labelled i to every vertex labelled j for $i \neq j$,
- renaming label i to label j.

In 2000 Courcelle and Olariu [5] published the result stating that for every graph G, $cwd(G) = \max\{cwd(H) : H \leq G \text{ and } H \in \mathbf{PRIME}\}$. Hence by Lemma 1 we deduce that $cwd(G) = \max\{cwd(H) : H \in Z^*(G)\}$. In the light of this fact, if for fixed $k \in \mathbb{N}$ we define $\mathcal{CWD}_k$ as the class of graphs $\{G \in \mathcal{I} : cwd(G) \leq k\}$, then the definition of $I(\{K_1\}, \mathcal{C})$ and Theorem 6 force the following fact.

Remark 3. If $k \in \mathbb{N}$, then

1. $\mathcal{CWD}_k = I(\{K_1\}, \mathcal{C})$, where $\mathcal{C} = \{G \in \mathbf{PRIME} : cwd(G) \leq k\}$, and
2. $\mathbf{F}(CWD_k) \subseteq \mathbf{PRIME}$.

Moreover, if we know a class $\mathcal{P} \in \mathbf{L}_{\leq}$ with clique-with bounded from above by k, then $\mathcal{P}^*$ has the same property. It means that a hereditary graph class having at least one non-prime forbidden graph can be extended to the wider superclass with the same upper bound of clique-with. In addition, because $I(\mathcal{B}, \mathcal{C}) \subseteq I(\{K_1\}, \mathcal{B} \cup \mathcal{C})$ we know that all the classes presented in Table 1 have clique-with parameter bounded from above by three.

6. Algebraic structures

This section is intended for noting some interesting facts concerning prime inductive classes, which can be described in an algebraic language. First we shall show two results (Theorems 8 and 9) investigating the considered graph classes in the context of algebraic structures. Then, using them, the unique representation result shall be proved (Corollary 2) and direction of new research will be given.

Let $\mathcal{C}^* = \{I(\{K_1\}, \mathcal{C}) : \mathcal{C} \subseteq \mathbf{PRIME}\}$.

Theorem 8. *$(\mathcal{C}^*, \subseteq)$ is a Boolean algebra.*

Proof. First we shall show that for two elements of $\mathcal{C}^*$, say $I(\{K_1\}, \mathcal{C}_1)$ and $I(\{K_1\}, \mathcal{C}_2)$, the elements $I(\{K_1\}, \mathcal{C}_1 \cap \mathcal{C}_2)$ and $I(\{K_1\}, \mathcal{C}_1 \cup \mathcal{C}_2)$ are their meet and join in $(\mathcal{C}^*, \subseteq)$, respectively. Obviously, $I(\{K_1\}, \mathcal{C}_1 \cap \mathcal{C}_2) \subseteq I(\{K_1\}, \mathcal{C}_i) \subseteq I(\{K_1\}, \mathcal{C}_1 \cup \mathcal{C}_2)$ for $i \in \{1, 2\}$. Let $I(\{K_1\}, \mathcal{C}) \subseteq I(\{K_1\}, \mathcal{C}_i)$ for $i \in \{1, 2\}$ and let $G \in I(\{K_1\}, \mathcal{C})$. Then, as we have mentioned previously, $Z(G) \subseteq \mathcal{C}_1 \cap \mathcal{C}_2$ and $G \in I(\{K_1\}, \mathcal{C}_1 \cap \mathcal{C}_2)$, which means that $I(\{K_1\}, \mathcal{C}_1 \cap \mathcal{C}_2)$ is the meet for elements $I(\{K_1\}, \mathcal{C}_1)$ and $I(\{K_1\}, \mathcal{C}_2)$ in $(\mathcal{C}^*, \subseteq)$. In the same fashion the join element is verified. The distributivity of $(\mathcal{C}^*, \subseteq)$ follows immediately by the distributivity of union and intersection of each to the other one. Clearly $I(\{K_1\}, \emptyset)$ and $I(\{K_1\}, \mathbf{PRIME})$ are the minimum and maximum elements in $(\mathcal{C}^*, \subseteq)$, respectively. These considerations lead to the fact that $I(\{K_1\}, \mathbf{PRIME} \setminus \mathcal{C}_1)$ is the complement of $I(\{K_1\}, \mathcal{C}_1)$. □

Next, we can give the observation arranging $\mathbf{L}^*_{\leq}$ in $\mathcal{C}^*$.

Theorem 9. *$(\mathbf{L}^*_{\leq}, \subseteq)$ is a distributive sublattice of $(\mathcal{C}^*, \subseteq)$.*

Proof. It is an immediate consequence of the fact that the union and intersection of two prime induced hereditary families $C_1, C_2 \subseteq$ **PRIME** are prime induced hereditary too. □

6.1. Theoretical application

Below we present the examples of applications of two theoretical results on lattices (Theorems 10 and 11) to $(\mathbf{L}^*_{\leq}, \subseteq)$ and $(\mathcal{C}^*, \subseteq)$.

Let $(\mathcal{L}, \preceq)$ be a lattice with a join operation $\vee$ and let $x_i, i \in [m]$ be elements of $\mathcal{L}$. We call an expression $x_1 \vee \ldots \vee x_m$ *irredundant* if no x_i can be omitted without altering the value of a join.

An element $x \in \mathcal{L}$ is said to be $\vee$-*reducible*, or briefly *reducible*, if $x = x_1 \vee x_2$, where x, x_2 are arbitrary elements of $\mathcal{L}$ for which $x_1 \prec x, x_2 \prec x$, otherwise x is said to be *irreducible*.

The unique factorization result presented below can be found in [14].

Theorem 10 *([14]). If $\bigvee_{i=1}^{n} x_i$ and $\bigvee_{j=1}^{m} y_j$ are two irredundant representations of x as a join of irreducible elements of a distributive lattice, then $n = m$ and there exists a bijection $\varphi : [n] \to [n]$ such that $x_i = y_{\varphi(i)}, i \in [n]$.*

Taking into account Theorems 8 and 9 we now obtain an interesting fact on (induced hereditary) graph properties, which are closed under substitution.

Corollary 2. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be irreducible properties in $(\mathbf{L}^*_{\leq}, \subseteq)$ (respectively, in $(\mathcal{C}^*, \subseteq)$). If the expression $\bigvee_{i=1}^{n} \mathcal{P}_i = \mathcal{P}$ is irredundant in $(\mathbf{L}^*_{\leq}, \subseteq)((\mathcal{C}^*, \subseteq))$, then it is the unique irredundant representation of $\mathcal{P}$ as a join of finitely many irreducible properties in $(\mathbf{L}^*_{\leq}, \subseteq)$ (respectively, in $(\mathcal{C}^*, \subseteq)$).*

For example let $i \in \mathbb{N}$ and $\mathcal{C}^i$ denote the class of graphs consisting of the cycle C_i and all the paths on at least two and at most $i - 1$ vertices. For each $i \geq 3$ the class $I(\{K_1\}, \mathcal{C}^i) = \mathcal{R}_i$ is reducible in $(\mathcal{C}^*, \subseteq)$ and irreducible in $(\mathbf{L}^*_{\leq}, \subseteq)$. It is so because $\mathcal{R}_i$ can be expressed in $(\mathcal{C}^*, \subseteq)$ as $I(\{K_1\}, \{P_2\}) \vee \ldots \vee I(\{K_1\}, \{P_{i-1}\}) \vee I(\{K_1\}, \{C_i\})$ and there are no two proper subsets of $\mathcal{C}^i$, which are prime induced hereditary and incomparable in the sense of $\subseteq$-relation. In particular Theorems 9 and 10 show that for $\{i_1, \ldots, i_m\} \subseteq \mathbb{N}$ the join $\mathcal{R}_{i_1} \vee \ldots \vee \mathcal{R}_{i_m} = \mathcal{R}$ is the unique irredundant representation of $\mathcal{R}$ as a join of finitely many irreducible classes in $(\mathbf{L}^*_{\leq}, \subseteq)$ giving the uniqueness of the factorization of $\mathcal{R}$, like the uniqueness of factorization of an integer, which follows from the same theorem.

Of particular interest in connection with distributive lattices is the concept of an *interval* $[y, x]$ of a lattice $(\mathcal{L}, \preceq)$ consisting of all the elements z of $(\mathcal{L}, \preceq)$ such that $y \preceq z \preceq x$. It is easily verified that $[y, x]$ forms a sublattice of $(\mathcal{L}, \preceq)$ [14].

Theorem 11 *([14]). In a distributive lattice all the intervals $[x, x \vee y]$ and $[x \wedge y, y]$ are isomorphic.*

Using Theorems 9 and 11 for $x = \mathcal{R}_t \vee \mathcal{R}_s$ and $y = \mathcal{R}_k$, where $k > s > t > 4$, we can explore the interval $[x \wedge y, y] = [I(\{K_1\}, \{\overline{K}_2, K_2, P_4, \ldots P_{t-1}\}), \mathcal{R}_k]$, which evidently is a chain, instead of the interval $[\mathcal{R}_t \vee \mathcal{R}_s, \mathcal{R}_k \vee \mathcal{R}_s \vee \mathcal{R}_t]$, whose exploring seems to be difficult.

These two theoretical applications show us a new direction in the investigation of prime inductive graph classes.

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