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On k nearest points of a finite set in a normed linear space

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Abstract

Given a finite set $A=\{a_1,a_2,\dots,a_n\}$ in a normed linear space X; for $x\in X$, let $\pi_i(x)$ be a permutation of $\{1,2,\dots,n\}$ such that $\|x-a_{\pi_1(x)}\|\leqslant\|x-a_{\pi_2(x)}\|\leqslant\cdots\leqslant\|x-a_{\pi_n(x)}\|$. We consider the following problem: for $1\leqslant k\leqslant n$, let $\frac{1}{k}\sum_{i=1}^{k}\|x-a_{\pi_i(x)}\|$ be the average distance to the k nearest points from a point x of the space; we are interested in minimizing this average when x describes the space X and in finding optimal solutions. This problem, which has a clear practical meaning, seems to have received little attention. Several properties of the solutions are proved.

Keywords: Median; Fermat point; Location

1. Introduction

Suppose that we are a service supplier (for example, a bank) with n potential users in a geographical area and we want to locate a branch to serve k clients among the existing n. If it is assumed that in a given period of time each user utilizes the service in the same way, we are interested in minimizing the sum of the distances from the facility to the k users; also, we want to know this sum for different values of k, since that will facilitate our choice of an optimal value of k (for example, if serving $k+1$ clients instead of k implies a big additional effort, the game is not worth the candle). We would like to know the place of suitable locations for the branch, and the specific k clients to be served.

Several other examples of similar problems can be given: for example, we could think of the location of a food take away, a video rent, a laundry, or also of situations concerning military or rescue operations.

Of course, the above problems can be generalized in several directions, for example, we may consider clients with different importance, and/or needing different numbers of visits and this implies the use of weighted distances. Generalizations of this type seem to be not very difficult and can be studied in the future.

Here we study the problem indicated above, in a general context: we do not limit ourselves to the plane, but we think of points spread in a normed linear space; measuring distances by norms seems to be a well accepted idea. Our results generalize some of the results known concerning medians (solutions of Fermat problem), to which our study reduces for $k=n$.

Let $(X,\|\cdot\|)$ be a real normed linear space; let we consider a finite set $A=\{a_1,a_2,\dots,a_n\}$ containing n different elements ($a_i\neq a_j$ for $i\neq j$): we shall write $\#A=n$; in general, we shall take $n\geqslant 3$, otherwise our problems become trivial.

For $x\in X$, let $\pi_i(x)$ be a permutation of $\{1,2,\dots,n\}$ such that

$$\|x-a_{\pi_1(x)}\|\leqslant\|x-a_{\pi_2(x)}\|\leqslant\cdots\leqslant\|x-a_{\pi_n(x)}\|, \tag{1.1}$$

this will be called in the following *a permutation determined by x*.

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Set $\|x - a_{\pi_i(x)}\| = d_i(x)$ and

$$\mu_k(A,x) = \frac{1}{k}\sum_{i=1}^{k} d_i(x) \quad (1 \leqslant k \leqslant n). \tag{1.2}$$

Define

$$\mu_k(A) = \inf_{x\in X} \mu_k(A,x). \tag{1.3}$$

Given A and k, $1 \leqslant k \leqslant n = \#A$, we set, for $\varepsilon > 0$:

$$\gamma_k(A,\varepsilon) = \{x \in X;\ \mu_k(A,x) \leqslant \mu_k(A) + \varepsilon\}$$

and

$$\gamma_k(A) = \bigcap_{\varepsilon>0} \gamma_k(A,\varepsilon) = \{x \in X;\ \mu_k(A,x) = \mu_k(A)\}.$$

We are interested in estimating $\mu_k(A)$ and in finding elements in $\gamma_k(A)$ i.e., elements minimizing $\mu_k(A,x)$, (for k and A given).

We shall call k—*medium* of A a point in $\gamma_k(A)$; note that an n—*medium* of A ($\#A = n$) is usually called a *median*, or a *Fermat point* of A; we shall also write $\mu(A)$, instead of $\mu_n(A)$.

For $k = 1$, our problem becomes trivial: for every A, we have $\mu_1(A) = 0$ and $\gamma_1(A) = A$.

We recall that the term k—*median* is used in the literature with respect to a different problem (concerning the location of k different "service centers" to serve all users of a finite set).

The problem considered here, has a kind of counterpart in terms of Weber problem; in fact, the following problem was briefly discussed in [4] for the special case of the plane, with the Euclidean norm:

let A be a finite set; given a desired coverage of A, what is the minimum radius allowing to obtain this coverage, and where should the center be placed?

At first sight, the problem considered here appears to be not completely trivial, since—even in inner product spaces—the function defined by (1.2) is neither convex (or quasi-convex) nor concavė: the next example illustrates this fact. Such apparent difficulty (recall that we are interested in minimizations) can be overcome with respect to some of the problems considered.

Example 1.1. Let X be the Euclidean plane; let $a_1 = (0,1)$, $a_2 = (0,-1)$, $a_3 = (\sqrt{3},0)$. For $A = \{a_1,a_2,a_3\}$ $(n = 3)$ and $k = 2$, it is easy to see that the function $\mu_2(F,x_\alpha)$, with $x_\alpha = (\alpha,0)$, is concave for $0 \leqslant \alpha \leqslant \sqrt{3}/3$ and convex for $\sqrt{3}/3 \leqslant \alpha \leqslant \sqrt{3}$.

We recall that a space X is said to be *strictly convex* (SC), if $\|x+y\| = \|x\| + \|y\|$ implies $y = \lambda x$ for some real $\lambda \geqslant 0$.

2. Preliminary results

We start with a few simple remarks.

Clearly, given A, $\#A = n$, for any $x \in X$ we have

$$\mu_1(A,x) \leqslant \mu_2(A,x) \leqslant \cdots \leqslant \mu_n(A,x) \tag{2.1}$$

which implies

$$\mu_1(A) \leqslant \mu_2(A) \leqslant \cdots \leqslant \mu_n(A). \tag{2.2}$$

Theorem 2.1. *Given A and k, $1 \leqslant k \leqslant \#A$, the function $\mu_k(A,x)$ $(x \in X)$ is* 1*-Lipschitz.*

Proof. Take x, y in X and by using a permutation determined by x, we obtain

$$k\mu_k(A,x) = \sum_{i=1}^{k} \|x - a_{\pi_i(x)}\| \geqslant \sum_{i=1}^{k} (\|y - a_{\pi_i(x)}\| - \|y - x\|)$$
$$\geqslant k\mu_k(A,y) - k\|y - x\|,$$

so $k\ (\mu_k(A,y) - \mu_k(A,x)) \leqslant k\|y - x\|$.

By reversing the role of x and y, we easily obtain $|\mu_k(A,y)-\mu_k(A,x)| \leqslant \|y-x\|$, which is the thesis. □

Proposition 2.2. *Let* $A' \subset A$, $\#A' \geqslant k$ *then we have*

$$\mu_k(A) \leqslant \mu_k(A'), \tag{2.3}$$

moreover, if $\mu_k(A)=\mu_k(A')$, *then*

$$\gamma_k(A') \subset \gamma_k(A). \tag{2.4}$$

Proof. The first statement is a consequence of the inequalities $\mu_k(A,x) \leqslant \mu_k(A',x)$ for every $x \in X$.

Concerning (2.4), note that $x \in \gamma_k(A')$ means $\mu_k(A',x)=\mu_k(A')$, so under the assumption $\mu_k(A)=\mu_k(A')$, there exist in A' k elements $a_1,a_2,\ldots,a_k$, which are also in A, such that

$(1/k)\sum_{i=1}^{k} \|x-a_i\| = \mu_k(A')=\mu_k(A)$, so $x \in \gamma_k(A)$. □

Let us consider the case $k=2$. We have the following result.

Theorem 2.3. *Given* A, *we have*

$$2\mu_2(A) = \min\{\|a_i-a_j\|; a_i,a_j \in A; i \neq j\}, \tag{2.5}$$

$$\gamma_2(A) \neq \emptyset;$$

if X *is* SC, *then*

$$\gamma_2(A) \text{ is the union of segments joining pairs of points } a_i,a_j, \text{ with } \|a_i-a_j\| = 2\mu_2(A). \tag{2.6}$$

Proof. Clearly, $2\mu_2(A) \geqslant \min\{\|a_i-a_j\|; a_i,a_j \in A, i \neq j\}$, but given a pair $\bar{\imath}$, $\bar{\jmath}$ realizing the minimum on the right, all points in the segment joining $a_{\bar{\imath}}$ and $a_{\bar{\jmath}}$, do the same: this proves (2.5), and moreover that $\gamma_2(A)$ contains all points of segments joining pairs with this property.

If $x \in \gamma_2(A)$, then $2\mu_2(A) = \|x-a_{\bar{\imath}}\| + \|x-a_{\bar{\jmath}}\| \geqslant \|a_{\bar{\imath}}-a_{\bar{\jmath}}\|$ for a pair of indexes $\bar{\imath},\bar{\jmath}$; by using (2.5), we obtain equality, so $\|x-a_{\bar{\imath}}\| + \|x-a_{\bar{\jmath}}\| = \|a_{\bar{\imath}}-a_{\bar{\jmath}}\|$; if X is SC, this implies (2.6). □

Remark. Clearly (2.6) characterizes SC: in fact, if X is not SC, there are pairs a_1,a_2 such that $\|x-a_1\|+\|x-a_2\|=\|a_1-a_2\|$ for points x not on the segment joining them; so the set $A=\{a_1,a_2\}$ violates (2.6).

Already for $k=3$, it is known that $\gamma_3(A)$ can be empty: in fact examples of sets of cardinality three without medians are known (see e.g. [7, Remark]).

In case X is SC, medians of sets (if they exist) are unique, unless n is even and A is contained in a line. Trivial examples, also in the Euclidean plane, show that in general k-media, also for $k \geqslant 3$, are not unique if $\#A \geqslant 4$.

3. Main results

In order to give estimates of the size of $\gamma_k(A)$, we recall some definitions.

Set

$$\delta(A) = \sup\{\|a_i-a_j\|; a_i,a_j \in A\} \quad (\textit{diameter} \text{ of } A).$$

Define, for $x \in X$,

$$r(A,x) = \sup_{a_i \in A} \|x-a_i\|$$

then set

$$r(A) = \inf_{x \in X} r(A,x) = \inf_{x \in X} \sup_{a_i \in A} \|x-a_i\| \quad (\textit{radius} \text{ of } A).$$

A point $x \in X$ such that $r(A,x)=r(A)$ is called a *center* of A.

Theorem 3.1. *Let $m \in \gamma_k(A)$, $m' \in \gamma_j(A)$ with $k \leqslant j$, then we have*

$$\|m - m'\| \leqslant \mu_k(A) + \delta(A) + \mu_j(A). \tag{3.1}$$

Proof. Let $\pi_1(m), \dots, \pi_k(m) \dots$ be a permutation determined by m and $\pi_1(m'), \dots, \pi_j(m') \dots$ one determined by m'; for $i = 1, \dots, k$ we have

$$\begin{aligned}\|m - m'\| &\leqslant \|m - a_{\pi_i(m)}\| + \|a_{\pi_i(m)} - m'\| \\ &\leqslant \|m - a_{\pi_i(m)}\| + \|a_{\pi_i(m)} - a_{\pi_i(m')}\| + \|a_{\pi_i(m')} - m'\|\end{aligned}$$

by adding these k inequalities, we obtain:
$k\|m - m'\| \leqslant k\mu_k(A) + k\delta(A) + k\mu_j(A)$, so the conclusion. □

Remark. For $j = k$, we obtain $(m, m' \in \gamma_k(A))$

$$\|m - m'\| \leqslant 2\mu_k(A) + \delta(A); \tag{3.2}$$

also, for $k = 1$, we obtain for every $i = 1, \dots, n$

$$\|a_i - m'\| \leqslant \delta(A) + \mu_j(A). \tag{3.2$'$}$$

Moreover, the upper bound in (3.2) is achieved, for example, in the following case: let X be the plane with the max norm;
$A = \{(0,1); (0,-1); (2,1); (2,-1)\}$; $k = 2$; $m = (3,0)$; $m' = (-1,0)$;
we have $\delta(A) = 2$; $\mu_2(A) = 1$; $\|m - m'\| = 4$.

Given A, say that two points a, b of A form a *minimal pair* of A if $\|a - b\| = 2\mu_2(A)$. Given $A = A_0$, if a_{01}, a_{02} is a minimal pair of A, set $A_1 = A_0 \setminus \{a_{01}, a_{02}\}$; then define, by induction, a (finite) sequence of sets in this way:
$A_k = A_{k-1} \setminus \{a_{k-1,1}, a_{k-1,2}\}$ $(2k \leqslant n)$, where $a_{k-1,1}, a_{k-1,2}$ is a minimal pair of A_{k-1}.
It is not difficult to prove the following result (we shall not give the details of the proof).

Proposition 3.2. *We always have, for $\#A = n \geqslant 2h$ (h integer):*

$$\mu(A) \geqslant \mu_{2h}(A) \geqslant \frac{1}{h}(\mu_2(A_0) + \cdots + \mu_2(A_{h-1})).$$

Proposition 3.3. *Given two disjoint sets A, A' with $\#A \geqslant k$, $\#A' \geqslant k'$, if $h = k + k'$, then we have*

$$\mu_h(A \cup A') \leqslant \frac{1}{2}(\mu_k(A) + \mu_{k'}(A') + \delta(A \cup A')).$$

Proof. We assume that c, c' exist such that $c \in \gamma_k(A)$ and $c' \in \gamma_{k'}(A')$; for the general case the proof is similar by using elements in $\gamma_k(A, \varepsilon)$, $\gamma_k(A', \varepsilon)$, ε being arbitrarily small. We have assumed $A \cap A' = \emptyset$.

Let $m = (c + c')/2$; if $a_1, \dots, a_k$ are the points of A nearest to c, and $a_{k+1}, \dots, a_{k+k'}$ are the points of A' nearest to c', we obtain

$$\begin{aligned}\sum_{i=1}^{k+k'} \|m - a_i\| &= \left\|\frac{c+c'}{2} - a_1\right\| + \cdots + \left\|\frac{c+c'}{2} - a_k\right\| + \left\|\frac{c+c'}{2} - a_{k+1}\right\| + \cdots \\ &+ \left\|\frac{c+c'}{2} - a_{k+k'}\right\| \leqslant \left\|\frac{c-a_1}{2}\right\| + \cdots + \left\|\frac{c-a_k}{2}\right\| + \left\|\frac{c'-a_1}{2}\right\| + \cdots + \left\|\frac{c'-a_k}{2}\right\| \\ &+ \left\|\frac{c-a_{k+1}}{2}\right\| + \cdots + \left\|\frac{c-a_{k+k'}}{2}\right\| + \left\|\frac{c'-a_{k+1}}{2}\right\| + \cdots + \left\|\frac{c'-a_{k+k'}}{2}\right\|\end{aligned}$$

$$\leqslant \frac{k}{2}\mu_k(A) + \frac{k}{2}r(A, c') + \frac{k'}{2}r(A', c) + \frac{k'}{2}\mu_{k'}(A');$$

now observe that if a is an element of A nearest to c, then for any $a' \in A'$ we have $\|c - a'\| \leqslant \|c - a\| + \|a - a'\| \leqslant \mu_k(A, c) + \delta(A \cup A')$, so $r(A', c) \leqslant \mu_k(A) + \delta(A \cup A')$,

similarly
$r(A, c') \leqslant \mu_{k'}(A') + \delta(A \cup A')$.
So we obtain

$$(k+k')\mu_{k+k'}(A \cup A') \leqslant \sum_{i=1}^{k+k'} \|m - a_i\| \leqslant \frac{k\mu_k(A) + k'\mu_{k'}(A')}{2} + \frac{k}{2}(\mu_{k'}(A') + \delta(A \cup A'))$$
$$+\frac{k'}{2}(\mu_k(A) + \delta(A \cup A')),$$

which implies the thesis. □

The following example shows that the inequality given in Proposition 3.3 is sharp.

Example 3.1. Let X be the plane with the Euclidean norm. Let $A = \{(-1, \varepsilon); (-1, 0)\}; A' = \{(1, -\varepsilon); (1, 0)\}, k = k' = 2$. We have $\mu_2(A) = \varepsilon/2 = \mu_2(A')$; $\delta(A \cup A')) = 2\sqrt{1+\varepsilon^2}$; $\mu_4(A \cup A') = \frac{1}{2}(1 + \sqrt{1+\varepsilon^2})$. For $\varepsilon \to 0$, we approach the equality related to Proposition 3.3.

Given a finite set A, for $x \in X$ let, according to [3] ($k \leqslant \#A$) $r_k(A, x) = (1/k)\sum_{i=1}^{k} \|x - a_i\|$, where $a_1, \ldots, a_k$ are k elements of A farthest to x, and

$$r_k(A) = \inf_{x \in X} r_k(A, x).$$

Clearly, $r_1(A) \geqslant \cdots \geqslant r_n(A) = \mu(A)$.
Also, say that $c \in X$ is a *k-centrum* of A if

$$r_k(A, c) = r_k(A).$$

Note that $r_1(A, c) = r(A, c)$; i.e. a 1-*centrum* of A is a *center* of A.

Theorem 3.4. *Given a set A, let k, j, integers between 1 and $n = \#A$; if $m \in \gamma_k(A)$ and c is a j-centrum of A, $j \leqslant k$, then we have*

$$\|m - c\| \leqslant r_j(A) + \mu_k(A), \tag{3.3}$$

moreover

$$(n-k)r_{n-k}(A) + k\mu_k(A) \leqslant n\mu(A), \tag{3.4}$$

or also

$$(n-k)(r_{n-k}(A) - \mu(A)) \leqslant k(\mu(A) - \mu_k(A)). \tag{3.4$'$}$$

Proof. Let $\pi_1(m), \ldots, \pi_k(m) \ldots$ be a permutation determined by m; for $i = 1, \ldots, k$ we have

$$\|m - c\| \leqslant \|m - a_{\pi_i(m)}\| + \|a_{\pi_i(m)} - c\|, \quad i = 1, \ldots, k,$$

by adding on i from 1 to k, we obtain $k\|m - c\| \leqslant k\mu_k(A) + \sum_{i=1}^{k} \|c - a_{\pi_i(m)}\| \leqslant k\mu_k(A) + kr_k(A, c)$, so $\|m - c\| \leqslant \mu_k(A) + r_k(A, c)$, if $k \geqslant j$, then the thesis follows from $r_k(A, c) \leqslant r_j(A, c) = r_j(A)$.
To prove (3.4), note that for every $x \in X$ we have

$$\sum_{i=1}^{k} \|x - a_i\| = (n-k)r_{n-k}(A, x) + k\mu_k(A, x) = n\mu(A, x).$$

By taking the infimum for $x \in X$ we obtain (3.4). □

Remark. Inequality (3.3) is not true in general when $j > k$; for example, if $j = n$ and $k = 1$, if c is a center and m is a median of A, then $\|m - c\|$ can be larger than $r(A)$: see e.g. Section 3 in [1].
In case n is even and we take $k = j = n/2$, then (3.3) gives the following estimate

$$\|m - c\| \leqslant 2r_{n/2}(A), \tag{3.3$'$}$$

similarly, for $k = n - k (= n/2)$, (3.4) gives

$$r_{n/2}(A) + \mu_{n/2}(A) \leqslant 2\mu(A). \tag{3.4$''$}$$

Again, the last inequalities are sharp: this can be seen by using the example given in the Remark to Theorem 3.1.

Corollary 3.5. *Assume that equality holds in* (3.3) *and that X is* SC *then there are at least k points of A on the line joining c and m.*

Proof. The thesis follows from proof of Theorem 3.4: with the notation used there, we have $\|m-a_{\pi_i(m)}\|+\|a_{\pi_i(m)}-c\|=\|m-c\|$ for $i=1,\dots,k$. □

Theorem 3.6. *Let A be a finite set A, $1\leqslant k\leqslant n=\#A$; write, for sake of simplicity, μ_k instead of $\mu_k(A)$ then we have*

$$\mu_n\leqslant\frac{k}{n}\mu_k+\frac{n-k}{n}\mu_{n-k}+\frac{2k(n-k)}{n^2}\,\delta(A).$$

Proof. We assume that $\gamma_k(A)$, $\gamma_{n-k}(A)$ are non-empty (otherwise the proof is similar, by using approximate solutions).

Let $m_1\in\gamma_k(A)$, $m_2\in\gamma_{n-k}(A)$; take $m=(k/n)m_1+[(n-k)/n]m_2$; we have

$$\|m-a_i\|\leqslant\left\|\frac{k}{n}(m_1-a_i)\right\|+\left\|\frac{n-k}{n}(m_2-a_i)\right\|,$$

so, by adding on $i=1,\dots,n$, and using the inequalities $(1\leqslant i\leqslant n)$:

$\|m_j-a_i\|\leqslant\min_{1\leqslant l\leqslant n}\|m_j-a_l\|+\delta(A)$, $j=1,2$, we obtain

$$\sum_{i=1}^{n}\|m-a_i\|\leqslant\frac{k}{n}\,(k\mu_k+(n-k)(\mu_k+\delta(A)))+\frac{n-k}{n}((n-k)\mu_{n-k}$$

$$+k(\mu_{n-k}+\delta(A))=k\mu_k+(n-k)\mu_{n-k}+\frac{2k(n-k)}{n}\,\delta(A).$$

Therefore $\mu_n\leqslant\sum_{i=1}^{n}\|m-a_i\|/n=(k/n)\mu_k+((n-k)/n)\mu_{n-k}+(2k(n-k)/n^2)\delta(A)$, which is the thesis. □

Remark 1. In particular, if $k=n-k=n/2$, then Theorem 3.6 gives $\mu_n\leqslant\mu_{n/2}+\delta(A)/2$.

Also, the last inequality, together with (3.4″), gives

$$r_{n/2}(A)-\mu(A)\leqslant\delta(A)/2. \tag{3.4‴}$$

Remark 2. With an argument similar to that used in proving Theorem 3.6, we can prove the following: assume that A contains at least k points; then given x and y, if $x_\lambda=\lambda x+(1-\lambda)y$, $0\leqslant\lambda\leqslant 1$ then

$$\mu_k(A,x_\lambda)\leqslant\tfrac{1}{2}\,(\mu_k(A,x)+\mu_k(A,y)+\|y-x\|).$$

We discuss now the situation $\mu_k(A)=\mu_{k+1}(A)$.

Theorem 3.7. *Let $\mu_k(A)=\mu_{k+1}(A)$ $(3\leqslant k+1\leqslant n=\#A)$; then*

$$\gamma_{k+1}(A)\subset\gamma_k(A). \tag{3.5}$$

Proof. Let $c\in\gamma_{k+1}(A)$; if $\pi_i(c)$, $i=1,\dots,n$, is a permutation of $\{1,\dots,n\}$ determined by c, $a_{\pi_i(c)}$ being the corresponding elements of A, we obtain

$$\mu_{k+1}(A)=\mu_{k+1}(A,c)=\frac{1}{k+1}\sum_{i=1}^{k+1}\|c-a_{\pi_i(c)}\|\geqslant\frac{1}{k}\sum_{i=1}^{k}\|c-a_{\pi_i(c)}\|$$

$$=\mu_k(A,c)\geqslant\mu_k(A),$$

the assumption $\mu_{k+1}(A)=\mu_k(A)$ then implies $\sum_{i=1}^{k}\|c-a_{\pi_i(c)}\|=k\mu_k(A)$, so $c\in\gamma_k(A)$, and also $\|c-a_{\pi_1(c)}\|=\cdots=\|c-a_{\pi_{k+1}(c)}\|$. □

Remark 1. In general, $\mu_k(A)=\mu_{k+1}(A)$ does not imply $\gamma_k(A)=\gamma_{k+1}(A)$: for example, in the plane with the max norm, consider $A=\{(1,1);(1,-1);(-1,0)\}$, $k=2$: we have $\mu_2(A)=\mu_3(A)=1$; $\gamma_3(A)=\{0\}$; $\gamma_2(A)$ contains all points (x,y) with $0\leqslant x\leqslant 1,|y|\leqslant 1$, as well as all points (x,y) with $-1\leqslant x\leqslant 0$, $-x-1\leqslant y\leqslant x+1$ and $1\leqslant x\leqslant 2$, $x-2\leqslant y\leqslant -x+2$.

Remark 2. As shown in the proof of Theorem 3.7, if $\mu_k(A) = \mu_{k+1}(A)$, then a point $c \in \gamma_{k+1}(A)$ is also a center of a subset A' of A containing (at least) $k+1$ elements of A: in fact, c is a median of $a_{\pi_1(c)}, \ldots, a_{\pi_{k+1}(c)}$ and is at the same distance from these points.

Theorem 3.8. *Let X be* SC *and assume that $\gamma_3(A) \neq \emptyset$ for some A with $\#A \geqslant 3$; then $\mu_2(A) < \mu_3(A)$.*

Proof. Assume that $\mu_2(A)=\mu_3(A)$ for some set A; it is not a restriction, eventually after scaling, to assume $\mu_2(A)=\mu_3(A)=1$.

Take $m \in X$ such that $\mu_3(A,m)=1$ this implies that $\|m-a_i\|+\|m-a_j\|+\|m-a_k\|=3$ for a triplet $a_i,\ a_j,\ a_k$; also, since $\mu_2(A)=1,\ \|m-a_i\|+\|m-a_j\|=\|m-a_j\|+\|m-a_k\|=\|m-a_k\|+\|m-a_i\|=2=\|a_i-a_j\|=\|a_j-a_k\|=\|a_k-a_i\|$.

Therefore, by SC, $a_i\, a_j\, a_k$ and m are all on the same line. But this is a contradiction proving the thesis. □

Given a set A, say that a point $x \in X$ is a *weakly efficient point* of A if there is no point $x' \neq x$ such that

$$\|x'-a\| < \|x-a\| \quad \text{for all } a \in A,$$

and a *strictly efficient point* if there is no point $x' \neq x$ such that

$$\|x'-a\| \leqslant \|x-a\| \quad \text{for all } a \in A.$$

We have the following simple, standard result.

Theorem 3.9. *Given a set A, if $x \in X$ is not a weakly efficient point of A, then $x \notin \gamma_k(A)$ $(1 \leqslant k \leqslant n = \#A)$; if $x \in X$ is not a strictly efficient point of A, then it cannot be the unique element of $\gamma_k(A)$ $(1 \leqslant k \leqslant n = \#A)$.*

Proof. Let $A=\{a_1,\ldots,a_n\}$. If x is not weakly efficient for A, then we can find $x' \neq x$ such that $\|x'-a_i\| < \|x-a_i\|$ for $i=1,\ldots,n=\#A$, so $\mu_k(A,x') < \mu_k(A,x)$ for all k.

If x is not strictly efficient for A, then given $x \in X$ we can find $x' \neq x$ so that $\|x'-a_i\| \leqslant \|x-a_i\|$ for $i=1,\ldots,n$, thus $\mu_k(A,x') \leqslant \mu_k(A,x)$; therefore, if $x \in \gamma_k(A)$, then also $x' \in \gamma_k(A)$. □

4. Perturbations

In this section we make a few remarks concerning the size of "changes" for varying sets.

(1) Given $A=\{a_1,a_2,\ldots,a_n\}$, let $B=\{b_1,b_2,\ldots,b_n\}$ with $\|b_i-a_i\| \leqslant \varepsilon$ for $i=1,\ldots,n$. Then, for $k \leqslant n$, we have $|\mu_k(A,x)-\mu_k(B,x)| \leqslant \varepsilon$ for every $x \in X$, therefore $|\mu_k(A)-\mu_k(B)| \leqslant \varepsilon$.

But $\gamma_k(A)$ and $\gamma_k(B)$ can be very different from each other, also in case they are both singletons.

(2) Given a set A, the sets $\gamma_k(A)$ and $\gamma_k(A,\varepsilon)$ can be very different, also for ε small. Clearly, $\gamma_k(A,\varepsilon)$ is non-empty for every $\varepsilon \neq 0$; in case $\lim_{\varepsilon\to 0}\delta(\gamma_k(A,\varepsilon))=0$, then $\gamma_k(A)$ is a singleton.

(3) Embed A, containing n elements, in a set A' containing $n+m$ elements; then for $k \leqslant n$, we have:

$$k\mu_k(A') \leqslant k\mu_k(A) \leqslant (k+m)\mu_{k+m}(A').$$

In particular, if we add to A only one element, then we obtain

$$\frac{\mu_k(A)}{\mu_{k+1}(A')} \leqslant 1+\frac{1}{k}.$$

5. Numerical aspects

The problem studied in the present paper can be considered as a special case of the Ordered Weber Problem (OWP), proposed, for instance, in [5,6]. In fact, OWP provides a generalization and a common framework for the classical continuous location problems, by defining region-wise the objective function. Better results concerning OWP were indicated when increasing weights are assigned to farther points: in this case the resulting function to be minimized is convex; but this is not the situation we consider in the present paper, which is related to non-negative, decreasing weights. In [5] only theoretical results were indicated, for the convex case. In [6] some geometrical properties are indicated for distances measured by polyhedral gauges; also, an efficient algorithm is given for increasing weights, while two possible algorithms for the general case are indicated. Finally, concerning decreasing weights, or the general case (or non-convex case), results were indicated in [2]: such paper deals with algorithms, but only for networks, or for the continuous case with the rectilinear norm.

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