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Note

Collapsible biclaw-free graphs

Hong-Jian Lai[a,c], Xiangjuan Yao[b,*]

[a] *Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA*
[b] *College of Sciences, China University of Mining and Technology, Jiangsu, Xuzhou 221008, PR China*
[c] *College of Science, Chongqing Technology and Business University, Chongqing 400067, PR China*

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Abstract

A graph is called biclaw-free if it has no biclaw as an induced subgraph. In this note, we prove that if G is a connected bipartite biclaw-free graph with $\delta(G) \geqslant 5$, then G is collapsible, and of course supereulerian. This bound is best possible.

Keywords: Supereulerian graphs; Collapsible graphs; Biclaw-free graphs

1. Introduction

Graphs in this paper are finite and simple. Undefined terms and notations are from [2]. For a graph G, let $O(G)$ denote the set of odd degree vertices of G. A graph G is *eulerian* if G is connected with $O(G) = \emptyset$, and is *supereulerian* if G has a spanning eulerian subgraph. Since a spanning eulerian subgraph H with maximum degree $\Delta(H) = 2$ is a hamiltonian cycle, supereulerian graphs are viewed as a relaxed version of hamiltonian graphs. Boesch et al. in [1] indicated that the problem of characterizing supereulerian graphs might be very difficult. In 1979, Pulleyblank [9] showed that determining if a graph is supereulerian is NP-complete.

Catlin [3] introduced the concept of collapsible graphs. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph Γ_R such that $O(\Gamma_R) = R$. For example, K_1 and cycles of length less than 4 are collapsible, but C_4 is not. Note that when $R = \emptyset$, a spanning connected subgraph Γ_R of G is a spanning eulerian subgraph of G, and so collapsible graphs must be supereulerian. For more in the literature, please see the survey paper of Catlin [4] and its update [5].

A *claw* is a graph isomorphic to the complete bipartite graph $K_{1,3}$. A *bilcaw* is defined as the graph obtained from two vertex disjoint claws by adding an edge between the two vertices of degree 3 in each of the claws (see Fig. 1).

A graph is called *biclaw-free* if it does not have a biclaw as an induced subgraph. In 1992, Li conjectured that high minimum degree may assure a biclaw-free graph to be hamiltonian.

Conjecture 1.1 (*Li, Conjecture 2b.32 of Faudree et al. [6], see also Li [8]*). *There exists a constant c such that every connected bipartite biclaw-free graph G with $\delta(G) \geqslant c$ is hamiltonian.*

* Corresponding author.
E-mail address: yxjcumt@126.com (X. Yao).

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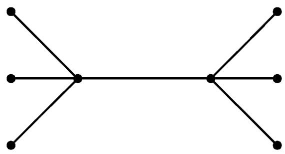

Fig. 1. The biclaw.

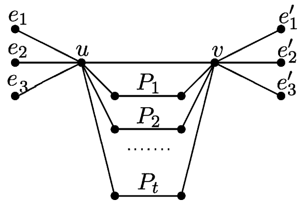

Fig. 2. One section of graph G.

A bipartite graph G with bipartition $\{A, B\}$ is *balanced* if $|A| = |B|$. If a bipartite graph G is hamiltonian, then G must be balanced. For any integer $c > 0$, the complete bipartite graph $K_{c,c+1}$ is clearly biclaw-free, has minimum degree c, but is not hamiltonian. Therefore, Conjecture 1.1 should be rephrased as that there exists a constant c such that every connected balanced bipartite biclaw-free graph G with $\delta(G) \geqslant c$ is hamiltonian. While this conjecture is still open, we in this note will prove the following.

Theorem 1.2. *Every connected bipartite biclaw-free graph G with $\delta(G) \geqslant 5$ is supereulerian.*

The proof of this theorem will be given in the next section. We shall also show that the bound $\delta(G) \geqslant 5$ is best possible.

2. Proof of the main result

We shall prove the following stronger result, which implies Theorem 1.2.

Theorem 2.1. *Every connected bipartite biclaw-free graph G with $\delta(G) \geqslant 5$ is collapsible.*

We start with some lemmas.

Lemma 2.2. *Let G be a bipartite biclaw-free graph with $\delta(G) = \delta \geqslant 4$. Then for any two adjacent vertices u and v in G, there are at least $\delta - 3$ internally disjoint (u, v)-paths of length* 3.

Proof. By contradiction. Suppose that there exist two adjacent vertices u and v, but there are only $t \leqslant \delta - 4$ internally disjoint (u, v)-paths of length 3 (which are denoted by $P_1, P_2, \ldots, P_t$, see Fig. 2).

Then in the graph $G - \bigcup_{i=1}^{t} E(P_i)$, there must be three edges e_1, e_2, e_3 that are incident with u, and other three edges e'_1, e'_2, e'_3 that are incident with v. By bipartiteness and the contradiction assumption, e_i $(i = 1, 2, 3)$ and e'_j $(j = 1, 2, 3)$ cannot be joined by any edge except uv. But then $G[uv, e_1, e_2, e_3, e'_1, e'_2, e'_3]$ will be an induced biclaw of G, contrary to the assumption that G is biclaw-free. $\square$

This lemma has a few corollaries.

Corollary 2.3. *Let G be a bipartite biclaw-free graph with $\delta \geqslant 4$. Then every edge $e \in E(G)$ lies in a* 4*-cycle of G.*

This can be easily deduced from Lemma 2.2.

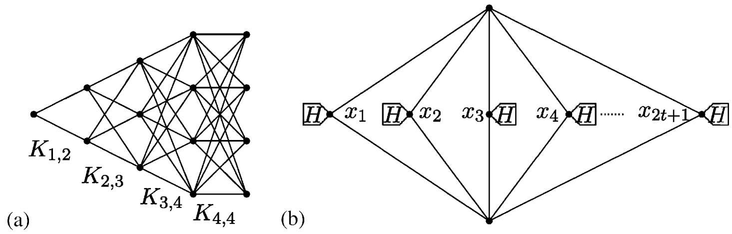

Fig. 3. (a) H and (b) $K_{2,2t+1}(H)$.

Corollary 2.4. *Let G be a bipartite biclaw-free graph with* $\delta(G)=\delta \geqslant 4$, *then* $\kappa'(G)\geqslant \delta-2$, *where* $\kappa'(G)$ *represents edge connectivity.*

Proof. For an arbitrary edge cut X of G, let u and v be two vertices that are adjacent in G but belong to different components in $G-X$. By Lemma 2.2, there are at least $\delta-2$ internally disjoint (u, v)-paths (include the edge uv), so X should include at least $\delta-2$ edges. By the arbitrariness of X, $\kappa'(G)\geqslant\delta-2$. $\square$

Lemma 2.5 (*Theorem 1 of Lai [7]*). *If* $\kappa'(G)\geqslant 2$, $\delta(G)\geqslant 3$, *and if every edge of G lies in a* 4-*cycle, then G is collapsible.*

Corollary 2.6. *If* $\kappa'(G)\geqslant 3$ *and if every edge of G lies in a cycle of length at most* 4, *then G is collapsible.*

Proof. Every block of G satisfies the hypothesis of Lemma 2.5. $\square$

Proof of Theorem 2.1. Let G be a connected bipartite biclaw-free graph with $\delta(G)=\delta\geqslant 5$. By Corollary 2.4, $\kappa'(G)\geqslant\delta-2\geqslant 3$. By Corollary 2.3, every edge of G lies in a cycle of length 4. It follows by Corollary 2.6 that G must be collapsible. $\square$

To see that the bound $\delta(G)\geqslant 5$ is best possible, we consider the following family of graphs. Let $K_{2,2t+1}$ have bipartition (X, Y), where $X=\{x_1, x_2, \ldots, x_{2t+1}\}$ $(t\geqslant 2)$. Let H denote the graph depicted in Fig. 3(a). We call the vertex of degree 2 in H its *peak*. Let $G(t)=K_{2,2t+1}(H)$ be the graph obtained from the disjoint union of a $K_{2,2t+1}$ and $2t+1$ copies of H, by identifying x_i $(i=1, 2, \ldots, 2t+1)$ of $K_{2,2t+1}$ with the peak of one H, see Fig. 3(b).

Since $G(t)=K_{2,2t+1}(H)$ can be contracted to $K_{2,2t+1}$, which is not supereulerian, $G(t)$ is not supereulerian, and so not collapsible also. On the other hand, it is straightforward to verify that $G(t)$ is a connected bipartite biclaw-free graph with $\delta(G(t))=4$. Therefore, the condition $\delta(G)\geqslant 5$ in Theorems 1.2 and 2.1 cannot be improved.

Note that $G(t)$ has a cut vertex. We have the following surmise:

Conjecture 2.7. *Every* 2-*connected bipartite biclaw-free graph G with* $\delta(G)\geqslant 4$ *is collapsible.*

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