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On the Mann Iteration Process in a Hilbert Space

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1. INTRODUCTION

For a self-mapping T of a compact interval of the real line having a unique fixed point, Mann [12] proved that the iteration process $v_{n+1} = (1 - d_n) v_n + d_n T v_n$, with $d_n = 1/(n + 1)$, converges to the fixed point. Franks and Marzec [3] proved that the uniqueness requirement was unnecessary. Outlaw and Groetsch [14] obtained convergence for T a nonexpansive mapping on a convex compact subset of the complex plane. Groetsch [5] generalized the procedure for nonexpansive mappings on uniformly convex Banach spaces. Also, Dotson [2] used the procedure for quasi-nonexpansive mappings on strictly convex Banach spaces. In a Hilbert space we enlarge the class of mappings for which the iteration process converges to a fixed point to demicontractive mappings. Also, we give some examples and look at an unsolved problem. Throughout this paper we let C denote a subset of a Hilbert space H and T a mapping of C into itself.

2. THEOREMS AND EXAMPLES

A self-mapping T on C is said to be *pseudocontractive* [1] if for all $x, y \in C$,

$$\| Tx - Ty \|^2 \leqslant \| x - y \|^2 + \|(I - T)x - (I - T)y \|^2,$$

where I denotes the identity mapping. T is said to be *strictly pseudocontractive* [1] if there exists a constant $k < 1$ such that, for all $x, y \in C$,

$$\| Tx - Ty \|^2 \leqslant \| x - y \|^2 + k \|(I - T)x - (I - T)y \|^2.$$

We say T is *demicontractive* if there exists a constant $k < 1$ such that, for each fixed point p of T and each $x \in C$,

$$\| Tx - p \|^2 \leqslant \| x - p \|^2 + k \| x - Tx \|^2.$$

We call k the *contraction coefficient*. Clearly, a strictly pseudocontractive mapping

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is both pseudocontractive and demicontractive. Also, Kannan [7–11] and then Wong [16] studied mappings that satisfy the condition:

$$\| Tx - Ty \| \leqslant (\| x - Tx \| + \| y - Ty \|)/2. \tag{1}$$

For $y = p = Tp$, this becomes $\| Tx - p \| \leqslant \| x - Tx \|/2$. Hence $\| Tx - p \|^2 \leqslant \| x - Tx \|^2/4 \leqslant \| x - p \|^2 + \| x - Tx \|^2/4$. Thus the class of demicontractive mappings includes the mappings studied by Kannan and Wong.

EXAMPLE 1. We give an example of a demicontractive mapping which is not pseudocontractive, hence not strictly pseudocontractive. Let H be the real line and $C = [-1, 1]$. Define T on C by $Tx = \frac{2}{3}x \sin(1/x)$ if $x \neq 0$ and $T(0) = 0$ Clearly, 0 is the only fixed point of T. Also, for $x \in C$, $| Tx - 0 |^2 = | Tx |^2 = | \frac{2}{3}x \sin(1/x)|^2 \leqslant | 2x/3 |^2 \leqslant | x |^2 \leqslant | x - 0 |^2 + k | Tx - x |^2$ for any $k < 1$. Thus T is demicontractive. We show that T is not pseudocontractive. Let $x = 2/\pi$ and $y = 2/3\pi$. Then $| Tx - Ty |^2 = 256/81\pi^2$. However,

$$| x - y |^2 + |(I - T) x - (I - T) y |^2 = 160/81\pi^2.$$

Let $\{d_n\}$ be a sequence from (0, 1) such that $\sum_{n=1}^{\infty} d_n(1 - d_n)$ diverges. With an initial value v_1, we consider the iteration process defined by

$$v_{n+1} = (1 - d_n) v_n + d_n T v_n , \tag{2}$$

for $n \geqslant 1$. Let $A = (a_{i,j})$ be the infinite matrix defined by $a_{1,1} = 1$, $a_{1,k} = 0$ for $k > 1$; $a_{n+1,n+1} = d_n$ for $n \geqslant 1$; $a_{n+1,j} = a_{j,j} \prod_{k=j}^{n} (1 - d_k)$ for $1 \leqslant j \leqslant n$ and $a_{n+1,j} = 0$ for $j > n + 1$. Dotson [2] has shown that A yields a normal Mann process. A frequent choice for the sequence $\{d_n\}$ is $d_n = 1/(n + 1)$. The matrix A then becomes the Cesaro matrix, originally used by Mann [12]. If $0 < k < 1$, one can choose $d > 0$ and a sequence $\{d_n\}$ from (0, 1) such that $\sum d_n(1 - d_n)$ diverges and $d_n \to d < 1 - k$. Just let $d = (1 - k)/2$ and $d_n = ((1 - k)/2) + (1/(n + 1))$.

THEOREM 1. *Suppose C is a convex subset of H. Suppose T is a demicontractive mapping of C into itself with contraction coefficient k. Suppose the set of fixed points $F(T)$ is nonempty. Suppose $\sum d_n(1 - d_n)$ diverges and $d_n \to d < 1 - k$. Then* $\liminf \| v_n - Tv_n \| = 0$ *for each $v_1 \in C$, where v_{n+1} is defined by* (2).

Proof. Ishikawa [6] has shown that for any x, y, z in a Hilbert space and real number λ, $\| \lambda x + (1 - \lambda) y - z \|^2 = \lambda \| x - z \|^2 + (1 - \lambda) \| y - z \|^2 - \lambda(1 - \lambda) \| x - y \|^2$. Thus for each $p \in F(T)$ and each integer n,

$$\begin{aligned} 0 \leqslant \| v_{n+1} - p \|^2 &= \|(1 - d_n) v_n + d_n T v_n - p \|^2 \\ &= (1 - d_n) \| v_n - p \|^2 + d_n \| Tv_n - p \|^2 - d_n(1 - d_n) \| v_n - Tv_n \|^2 \\ &\leqslant (1 - d_n) \| v_n - p \|^2 + d_n(\| v_n - p \|^2 + k \| v_n - Tv_n \|^2) \\ &\quad - d_n(1 - d_n) \| v_n - Tv_n \|^2 \\ &= \| v_n - p \|^2 - d_n(1 - d_n - k) \| v_n - Tv_n \|^2. \end{aligned} \tag{3}$$

By induction, we obtain

$$0 \leqslant \| v_1 - p \|^2 - \sum_{j=1}^{n} d_j(1 - d_j - k) \| v_j - Tv_j \|^2.$$

Thus

$$\sum_{n=1}^{\infty} d_n(1 - d_n - k) \| v_n - Tv_n \|^2 \leqslant \| v_1 - p \|^2. \tag{4}$$

Since $0 \leqslant d_n < 1$, $d_n > d_n$ $(1 < d_n)$. Also $\sum d_n(1 - d_n)$ diverges. Thus $\sum d_n$ diverges. Let $\eta = 1 - d - k$. Then $\eta > 0$, and there exists an integer N such that $d_n < d + \eta/2$ for $n \geqslant N$. Thus $1 - d_n - k > 1 - k - d - \eta/2 = \eta/2$. Therefore $\sum d_n(1 - d_n - k) \geqslant (\eta/2) \sum d_n$, which diverges. Hence $\sum d_n(1 - d_n - k)$ diverges. Thus, from (4), we obtain $\liminf \| v_n - Tv_n \| = 0$.

Remark 1. In Theorem 1, if $d \neq 0$, the terms of the series $\sum d_n(1 - d_n - k)$ are bounded away from zero. Hence we can conclude that $\lim \| v_n - Tv_n \| = 0$.

Remark 2. Since $1 - d_n - k \to 1 - d - k > 0$, there exists an integer N_0 such that $1 - d_n - k > 0$ for $n \geqslant N_0$. Thus, from Eq. (3), we obtain $\| v_{n+1} - p \| \leqslant \| v_n - p \|$ for $n \geqslant N_0$.

Groetsch [5] proved the following for T nonexpansive and H a uniformly convex Banach space. Our result shows that the theorem holds for a larger class of mappings in a Hilbert space.

COROLLARY 1. *Suppose C is a closed bounded convex subset of H. Suppose T and $\{d_n\}$ satisfy the hypothesis of Theorem* 1. *Suppose $I - T$ maps closed bounded subsets of C into closed subsets of C (in particular, if T is demicompact* [1]). *Then for each $v_1 \in C$, the iteration process defined by* (2) *converges to a fixed point of T.*

Proof. Let A denote the closure of the set of iterates v_n. By Theorem 1, 0 is a cluster point of $(I - T)A$. But $(I - T)A$ is closed so $0 \in (I - T)A$. Thus there exists a subsequence v_{n_j} converging to a fixed point p. Remark 2 and $v_{n_j} \to p$ imply that $v_n \to p$.

COROLLARY 2. *Suppose C is a closed bounded convex subset of H. Suppose T and $\{d_n\}$ satisfy the hypothesis of Theorem* 1 *with $d > 0$. Suppose p is a cluster point of $\{v_n\}$ and T is continuous at p. Then v_n converges to p and $p \in F(T)$.*

Proof. Suppose $v_{n_j} \to p$. By continuity at p, $\| Tv_{n_j} - v_{n_j} \| \to \| Tp - p \|$. Also, by Remark 1, $\| Tv_n - v_n \| \to 0$. Thus $\| Tp - p \| = 0$ and hence $Tp = p$. Again $v_{n_j} \to p$ and Remark 2 implies that $v_n \to p$.

Remark 3. Wong [16] showed that if C is closed, bounded, and convex, then a continuous mapping T that satisfies (1) has a unique fixed point P. If the T in Corollary 1 satisfies (1), then $v_n \to p$. Thus we have a way of finding the fixed point.

LEMMA (Opial [13]). *Suppose H is a Hilbert space and the sequence $\{x_n\}$ is weakly convergent to x_0. Then for any $x \neq x_0$,* $\liminf \| x_n - x_0 \| < \liminf \| x_n - x \|$.

The following theorem generalizes a theorem of Dotson [2, Theorem 8].

THEOREM 2. *Suppose C is a closed convex subset of H. Suppose $T: C \to C$ such that:*

(a) $F(T) \neq \phi$.

(b) *T is demicontractive with contraction coefficient k.*

(c) *If any sequence x_n converges weakly to x and $(I - T)(x_n)$ converges strongly to* 0 *then* $(I - T)(x) = 0$.

Then for $v_1 \in C$ and $d_n \to d$, $0 < d < 1 - k$, the iteration process defined by (2) *converges weakly to a fixed point of T.*

Proof. Suppose $Tp = p$. By Remark 2, there exists an integer N such that $\| v_{n+1} - p \| \leqslant \| v_n - p \|$ for all $n \geqslant N$. If $v_N = p$, then clearly $v_n \to p$. If $v_N \neq p$, $\| v_N - p \| = r > 0$. Let $S_r(p) = \{x: \| x - p \| \leqslant r\}$, and let $D = C \cap S_r(p)$. Then $\{v_n\}_{n=N}^{\infty} \subset D$. Also, D is weakly compact since it is closed, bounded, and convex. Thus there exists a subsequence $\{v_{n_j}\}$ which converges weakly to $y \in D \subset C$. By Remark 1, $(I - T) v_{n_j} \to 0$; hence, by condition (c), $Ty = y$.

Suppose $\{v_n\}$ does not converge weakly to y. Then the sequence $\{v_n\}_{n=N}^{\infty}$ has at least one other weak cluster point $q \neq y$. Suppose $\{v_{m_i}\}$ converges weakly to q. As for y, $Tq = q$. From Remark 2, we see that the sequences $\{\| v_n - y \|\}$ and $\{\| v_n - q \|\}$ are nonincreasing for sufficiently large n. Thus $\lim \| v_n - y \|$ and $\lim \| v_n - q \|$ both exist. Using Opial's lemma, we obtain the following contradiction:

$$\begin{aligned}
\lim_n \| v_n - y \| &= \liminf_j \| v_{n_j} - y \| \\
&< \liminf_j \| v_{n_j} - q \| \\
&= \liminf_i \| v_{m_i} - q \| \\
&< \liminf_i \| v_{m_i} - y \| \\
&= \lim_n \| v_n - y \| .
\end{aligned}$$

Therefore, v_n converges weakly to $y \in F(T)$.

Remark 4. A mapping T is said to be *demiclosed* if weak convergence of any sequence $\{x_n\}$ to x and strong convergence of $\{Tx_n\}$ to y implies $Tx = y$. For C closed and convex, every weakly continuous self-mapping T of C is weakly closed, every weakly closed self-map is demiclosed, and if $I - T$ is demiclosed, then condition (c) holds.

Dotson [2, Theorem 6] proved the following for T quasi-nonexpansive and H a uniformly convex Banach space. In a Hilbert space, we can enlarge the class of mappings and obtain the result as a corollary of Theorem 2.

COROLLARY 3. *Suppose C is a closed convex subset of H with T and $\{v_n\}$ as in Theorem* 1 *and $d > 0$.*

(1) *There exists a subsequence of $\{v_n\}$ which converges weakly to $y \in C$, and if $I - T$ is demiclosed, $y \in F(T)$.*

(2) *If $I - T$ is demiclosed and T has only one fixed point $p \in C$, then $\{v_n\}$ converges weakly to p.*

(3) *If $I - T$ is weakly closed, then each weak cluster point of $\{v_n\}$ is a fixed point of T.*

Proof. $I - T$ is demiclosed implies condition (c); thus we have part (1). We have part (2) without assuming $F(T)$ is a singleton. For (3), note that the last part of the proof of Theorem 2 shows that $\{v_n\}$ can have only one weak cluster point.

EXAMPLE 2. We give an example to show that the iteration process defined by (2) does not necessarily converge for a pseudocontractive mapping.

Let H be the complex plane and $C = \{z: |z| \leqslant 1\}$. Define $T: C \to C$ by

$$T(re^{i\theta}) = 2re^{i(\theta+\pi/3)}, \qquad \text{for } 0 \leqslant r \leqslant \tfrac{1}{2}$$

$$= e^{i(\theta+2\pi/3)}, \qquad \text{for } \tfrac{1}{2} < r \leqslant 1.$$

Clearly zero is the only fixed point of T. With $d_n = 1/(n+1)$, we show that the sequence $\{z_n\}$ defined by (2) does not converge to 0. If $0 < |z_n| \leqslant \frac{1}{2}$, then $|z_{n+1}| > |z_n|$. If $|z_n| > \frac{1}{2}$, then $|z_{n+1}| = (n^2 |z_n|^2 - n|z_n| + 1)^{1/2}/(n+1) > (n^2 - 2n + 4)^{1/2}/2(n+1)$. This last quantity is bounded away from zero. Thus for $z_1 \neq 0$, the sequence $\{z_n\}$ does not converge to the fixed point 0.

We outline the procedure to show that T is pseudo-contractive. For $z_1, z_2 \in C$, $\| Tz_1 - Tz_2 \| \leqslant \| z_1 - z_2 \|^2 + \|(I - T) z_1 - (I - T) z_2 \|^2$ if and only if

$$\operatorname{Re}\{\overline{(z_1 - z_2)}\, [(I - T) z_1 - (I - T) z_2]\} \geqslant 0. \tag{5}$$

Since distance between points is preserved under rotation about the origin, without loss of generality we may assume $z_1 = |z_1|$ and $|z_2| \leqslant |z_1|$. Let $z_1 = x$ and $z_2 = ye^{i\theta}$ where $y \leqslant x$. Then the left side of (5) can be considered as a function of three real variables,

$$f(x, y, \theta) = \operatorname{Re}\{|x - ye^{i\theta}|^2 - (x - ye^{-i\theta})(T(x) - T(ye^{i\theta}))\}.$$

We consider three cases:

(I) $0 \leqslant y \leqslant x \leqslant \frac{1}{2}$, $0 \leqslant \theta \leqslant 2\pi$,

(II) $0 \leqslant y \leqslant \frac{1}{2} < x \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$, and

(III) $\frac{1}{2} < y \leqslant x \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$.

Case I. With $Tx = 2xe^{i\pi/3}$ and $T(ye^{i\theta}) = 2ye^{i(\theta+\pi/3)}$, it is easy to show that $f(x, y, \theta) = 0$.

Case II. With $Tx = e^{i(2\pi/3)}$ and $Tye^{i\theta} = 2ye^{i(\theta+\pi/3)}$,

$$f(x, y, \theta) = x^2 + x/2 - (xy + y/2) \cos \theta - 3^{1/2}(xy - y/2) \sin \theta.$$

We show that $f(x, y, \theta) \geqslant 0$ for arbitrary but fixed x, y. With x, y fixed, f can be considered a function of the single variable θ. Clearly for $y = 0, f(x, y, \theta) > 0$. For $y \neq 0$, elementary calculus yields a minimum at

$$\theta_0 = \arctan[3^{1/2}(x - \tfrac{1}{2})/(x + \tfrac{1}{2})].$$

Since $0 < y < \frac{1}{2} < x \leqslant 1$,

$$\begin{aligned} f(x, y, \theta_0) &= x^2 + x/2 - [y(x + \tfrac{1}{2})^2 + 3y(x - \tfrac{1}{2})^2]/(4x^2 - 2x + 1)^{1/2} \\ &\geqslant x^2 + x/2 - [(x + \tfrac{1}{2})^2 + 3(x - \tfrac{1}{2})^2]/2 \\ &= \tfrac{1}{16} - (x - \tfrac{3}{4})^2 \geqslant 0. \end{aligned}$$

Case III. With $Tx = e^{i(2\pi/3)}$ and $Tye^{i\theta} = e^{i(\theta+2\pi/3)}$,

$$\begin{aligned} &f(x, y, \theta) \\ &\quad = x^2 + y^2 + (x - y)/2 - [2xy + (x + y)/2] \cos \theta - (3^{1/2}/2)(x - y) \sin \theta. \end{aligned}$$

We consider f defined on the compact set $D = \{(x, y, \theta) \mid \frac{1}{2} \leqslant y \leqslant x \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$. Since f is continuous, it assumes its minimum value. We show this minimum is nonnegative on D, hence nonnegative on the subset of D which comprises Case III. Setting the three partial derivatives, f_x, f_y, and f_θ, of f equal to zero, we find no relative extrema in the interior of D. We consider the five faces of D.

(a) For faces $\theta = 0$, $\theta = 2\pi$, or $x = y$, it is easy to show that f is nonnegative.

(b) Consider the face $x = 1$. Setting $f_y(1, y, \theta) = 0$ and $f_\theta(1, y, \theta) = 0$, we find no relative extrema in the interior of this face. Thus the minimum occurs along an edge of the face. The only edge not included in (a) is for $y = \frac{1}{2}$, and it is easy to see that $f(1, \frac{1}{2}, \theta) \geqslant 0$.

(c) Consider the face $y = \frac{1}{2}$. Setting $f_x(x, \frac{1}{2}, \theta) = 0$ and $f_\theta(x, \frac{1}{2}, \theta) = 0$, we again see that the minimum occurs along an edge. However, all edges have been considered in (a) and (b). Thus f is nonnegative for $y = \frac{1}{2}$. Therefore f is nonnegative on D. Consequently, T is a pseudocontractive mapping for which the iteration process defined by (2) does not converge to a fixed point of T.

PROBLEM. Does the iteration process always converge for continuous pseudocontractive mappings or Lipschitz pseudocontractive mappings?

The referee pointed out the interesting paper by Reinermann [17]. Section 3 of his paper contains similar results. In particular, his Theorem 6 has the same conclusion as our Corollary 1. Theorem 6 has a slightly weaker hypothesis and it applies to a smaller class of functions. If $F(t) \neq \phi$, the demicontractive mappings include those of Goëbel *et al.* [4] since they were shown to be nonexpansive in [15].

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