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Weak Purity for Gorenstein Rings

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1. INTRODUCTION

The notion of a "purity of branch locus" theorem dates back at least to Zariski [42], who proved the theorem in a geometric context for algebraic functions. Soon thereafter, the theorem was translated into the language of local algebra and generalized by Nagata [34] (also see [35, Theorem 41.1]) and then Auslander [5], its form being essentially as follows: Let R be a regular local ring and A a normal domain which is a module finite ring extension of R. If the extension $R \hookrightarrow A$ is unramified in codimension one, then it is unramified (and in this setting, flat as well), hence étale.

Work continued in this area: Abhyankar (see [1–4]) investigated ramification theory in algebraic geometry, and other generalizations emerged. For example, Grothendieck [23, Exposé 10, Theorem 3.4] proved that complete intersections of dimension ≥ 3 are "pure." It follows from Cutkosky [10, Theorem 5] that if we ease the restriction on R, allowing R to be a normal complete intersection and requiring $R \hookrightarrow A$ to be unramified in codimension two, then the extension is unramified.[2]

The search for "purity-type" theorems when the hypotheses on the normal domains $R \hookrightarrow A$ and on the extension are adjusted is a focus in Griffith [21]. Moreover, "weaker" purity results are considered: Let $B \hookrightarrow A$ be a module finite extension of normal domains, with unramification in some fixed low codimension. When do "good" properties of B (say, Gorenstein) guarantee that A is also "good"? The adjective "weak" refers to the fact that the result states that A inherits properties from B, but does not assert that the extension itself becomes good (that is, unramified).

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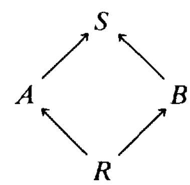
In [21], Griffith, via a construction he attributes to Abhyankar [4], illustrates that module finite extensions $B \hookrightarrow S$ which are unramified in codimension one, with B a Gorenstein ring and S not Cohen–Macaulay, are plentiful. This demonstrates that, in the classical setting, weak purity fails in a convincing fashion when the ring B is merely Gorenstein. In more detail, starting with any complete local normal domain A which is of characteristic zero and which contains an algebraically closed field (it is enough that A contain all roots of unity), Griffith constructs ring extensions described by the following diagram:



Each extension is module finite; R is a complete regular local ring provided by the Cohen structure theory; S is a local normal domain which is a normal extension of R (in the sense of Galois theory); and B is a Gorenstein cyclic extension of R such that $B \hookrightarrow S$ is unramified in codimension one. Since $R \hookrightarrow S$ and $A \hookrightarrow S$ are normal extensions, we have the reduced trace map $\mathrm{tr}_{S/A}\colon S \to A$ so that if the original A is not Cohen–Macaulay, then S cannot be Cohen–Macaulay either [28, Proposition 12].

In the following investigation, we study module finite ring extensions $B \hookrightarrow A$ of normal rings where B, the base ring, is always a local excellent domain. Given unramification in some low codimension and restrictions on B (often B will denote a Gorenstein ring), the purity/weak purity of the extension is examined. The primary theorem of this project asserts that if the Gorenstein ring B is "regular" enough (that is, if B satisfies (R_k)), then A will inherit a certain amount of depth (that is, A will satisfy (S_{k-1})). An example will illustrate that this purity is of the weak kind: the A can be Cohen–Macaulay, yet the extension is not unramified. In addition, we show how the technique can be used to recover a result which is a consequence of Grothendieck's much stronger purity theorem in the case when B is a hypersurface ring.

The method of proof involves maximal Cohen–Macaulay (MCM) modules, so the main result takes place in the equicharacteristic case (that is, all the rings contain a field), where such modules have been shown to exist by Hochster (see [27]). Subsequently, under certain conditions, the mixed characteristic case is considered by reducing to the equicharacteristic case. Another important ingredient is certain module isomorphisms for normal ring extensions which are unramified in codimension one, much in the spirit of those used in Auslander's version of the proof of the purity of branch locus theorem.

After the main results, we apply the ideas to extensions $B \hookrightarrow A$ as described, which are also normal. One such application involves investigating codimension two primes in A which are fixed under the group action of $G = \mathrm{Gal}(L/K)$ (here L and K are the fraction fields of A and B, respectively), or what amounts to the same thing, codimension two primes in A which are non-split. Another application considers depth properties of torsion divisorial ideals in the divisor class group of the normal domain B.

We begin with preliminaries—some basic definitions, necessary results, and a survey of the general setting. Next, we describe how the original problem can be reduced to one which is more tractable via completion and how the normal closure enters into the arguments. Subsequently, we discuss the necessary module isomorphisms and construct two exact B-complexes which are used to deduce the main results. Finally we consider applications of the ideas, including the issue of codimension two primes in normal extensions which are fixed under the action of the Galois group.

2. PRELIMINARIES: DEFINITIONS, RESULTS, SETTING, AND REDUCTIONS

In general, any unexplained terms or notations which appear in this presentation can be found in a standard commutative algebra text, for example, in Bourbaki [9] or Matsumura [31, 32]. What follows is a very brief collection of definitions which are prominent.

DEFINITION 2.1. Let M be a module over the local ring (R, m). We define the depth of M, denoted $\mathrm{depth}_R(M)$, to be $\inf\{i \mid \mathrm{Ext}^i_R(R/m, M) \neq 0\}$.

We remark that, for infinitely generated M, it is possible that $\mathrm{depth}_R(M) = \infty$. However, it is shown in Foxby [17] that $mM \neq M$ implies that $\mathrm{depth}_R(M) \leq \dim(R)$. To see how this definition coincides with the usual M-sequence definition of depth, refer to [12, Chap. 1].

DEFINITION 2.2. Let M be a module over (R, m). M is said to be a Cohen–Macaulay module if $\mathrm{depth}_R(M) = \dim_R(M)$; M is said to be a maximal Cohen–Macaulay module (or MCM-module) if $mM \neq M$ and $\mathrm{depth}_R(M) = \dim(R)$; M is said to be a balanced MCM-module (Sharp [40]) in the event it is a Cohen–Macaulay module for every system of parameters of R.

Every finitely generated MCM R-module is balanced; however, this need not be the case for infinitely generated MCM-modules (which are known to exist over equicharacteristic local rings by Hochster's famous work—see [27]). Additionally, balanced MCM-modules exist: in fact, it is shown in Griffith [18], that over a complete local ring, starting with an MCM-module as provided by Hochster, a countably generated balanced MCM-module with additional properties can be produced. In [40], it is shown that balanced MCM-modules localize properly (that is, to balanced MCM-modules) over complete local rings.

Assume that A is a B-algebra and M and N are B-modules. Consider the B-module $\operatorname{Hom}_B(M, N)$. When M is an A-module, we can define an A-module structure on $\operatorname{Hom}_B(M, N)$ as follows: for $a \in A$ and $\phi \in \operatorname{Hom}_B(M, N)$, define $a\phi$ by $(a\phi)(m) := \phi(am) \in N$ for $m \in M$. We refer to this structure as domain induced. When N is an A-module, we can define the following A-structure on $\operatorname{Hom}_B(M, N)$: for $a \in A$ and $\phi \in \operatorname{Hom}_B(M, N)$, define $a\phi$ by $(a\phi)(m) := a\phi(m) \in N$ for $m \in M$. We refer to this structure as codomain induced.

Our typical setting is that of ring extensions $B \hookrightarrow A$ with additional properties on the rings and on the extension. The notions of ramification and normal extension play a key role.

DEFINITION 2.3. Let B be a ring, A a B-algebra, and $P \in \operatorname{Spec}(A)$. P is said to be unramified over B provided that, setting $p = P \cap B$, we have:

(i) $pA_P = PA_P$ and

(ii) $B_p/pB_p \hookrightarrow A_P/pA_P$ is a finite separable field extension.

A is said to be unramified over B when every $P \in \operatorname{Spec}(A)$ is unramified over B.

We note that, in order for A to be unramified over B, it is enough that every maximal ideal $m \in \operatorname{Spec}(A)$ is unramified over B (see [7, Theorem 2.5] or [35, Corollary 38.8]).

DEFINITION 2.4. With $B \hookrightarrow A$ as above, we say that A is unramified over B in codimension i when every $P \in \operatorname{Spec}(A)$ of codimension $\leq i$ is unramified over B. In the case where $B \subset A$ and A is a domain, to say that A is unramified over B in codimension zero is to say that the corresponding extension of fraction fields is separable.

For equivalent definitions of unramified (and étale) extensions and various properties, we refer the reader to [32, 38].

DEFINITION 2.5. Let $B \hookrightarrow A$ be a module finite extension of normal (that is, integrally closed) domains. The extension is said to be normal if the corresponding extension of fraction fields $K \hookrightarrow L$ is Galois. In the case of a normal extension, the group $G = \mathrm{Gal}(L/K)$ acts on A, and the fixed ring A^G is equal to B.

Results and Other Preliminaries

In the presentation, we refer to various results and constructions. For convenience, we provide a brief list of several of these with appropriate reference, and occasionally with a sketch of a proof or some explanation.

To detect unramification in codimension one, we have the following due to Auslander and Goldman [8, Proposition A.4], which is utilized in Griffith (see [20–22]) in the following modified form.

THEOREM 1 (Auslander–Goldman). *Let $B \hookrightarrow A$ be a module finite extension of local normal domains and assume that the extension of fraction fields is separable. Then, $B \hookrightarrow A$ is unramified in codimension one if and only if the trace map $\mathrm{tr}_{A/B}\colon A \to B$ generates $\mathrm{Hom}_B(A, B)$ as an A-module.*

In the situation where $B \hookrightarrow A$ is a module finite extension of complete local rings, B being Gorenstein, the module $\mathrm{Hom}_B(A, B)$ is called the canonical module for A and is often denoted by Ω_A^0. For a local Cohen–Macaulay ring R, the following statements are equivalent:

(i) the canonical module Ω_R^0 exists and is isomorphic to R

(ii) R is a Gorenstein ring.

As a result of this equivalence, in the case when $B \hookrightarrow A$ is unramified in codimension one, we have the following: A is Cohen–Macaulay if and only if A is Gorenstein. This is immediate, for the Auslander–Goldman result gives that $\Omega_A^0 = \mathrm{Hom}_B(A, B) \simeq A$. For information regarding the canonical module, we refer the reader to Herzog and Kunz [25].

Additionally, the following results will be useful in the sequel:

LEMMA 2.6 (Depth Lemma (see [12, Lemma 1.1])). *Let (R, m) be a local ring, and*

$$0 \to L \to M \to N \to 0$$

an exact sequence of R-modules.

(i) *If* $\mathrm{depth}(N) < \infty$ *and* $\mathrm{depth}(M) > \mathrm{depth}(N)$, *then* $\mathrm{depth}(L) = \mathrm{depth}(N) + 1$,

(ii) $\mathrm{depth}(L) \geq \min\{\mathrm{depth}(M), \mathrm{depth}(N)\}$.

THEOREM 2 (see Serre [39, Chap. IV, Proposition 12]). *Let $B \xrightarrow{\phi} A$ be a homomorphism of Noetherian local rings which makes A a finite B-module. For any finite A-module M, we have that*

$$\mathrm{depth}_B(M) = \mathrm{depth}_A(M) \quad \textit{and} \quad \dim_B(M) = \dim_A(M).$$

THEOREM 3 (see Fossum, Foxby, Griffith and Reiten [15, Corollary 2.6]). *Suppose that $B \hookrightarrow A$ is a module finite extension of complete local rings, where B is Gorenstein, that A satisfies (S_n) where $n > 0$, and that $\Omega_A^0 \simeq A$. Then* $\Omega_A^j = \operatorname{Ext}_B^j(A, B) = 0$ *for* $1 \le j \le n - 2$. *In particular, if it is the case that* $\operatorname{depth}(A) + n - 2 \ge \dim(A)$, *then A is Cohen–Macaulay.*

THEOREM 4 (see Kaplansky [30, Theorem 208]). *Let (R, m) be a local ring; N, M two R-modules with N finitely generated over R and* $\operatorname{pd}_R(N) = 1$. *If* $\operatorname{Ext}_R^1(N, M) = 0$, *then $mM = M$.*

Proof. Let

$$0 \longrightarrow G \xrightarrow{\ \psi\ } F \longrightarrow N \longrightarrow 0$$

be a minimal finite free R-resolution of N. Then the entries of the matrix giving ψ are in m. Since $\operatorname{Ext}_R^1(N, M) = 0$, then

$$\operatorname{Hom}_R(F, M) \xrightarrow{\ \psi^*\ } \operatorname{Hom}_R(G, M) \longrightarrow 0$$

is exact. As ψ^* is given by the same matrix as ψ (note the observation following the proof), we have that $\operatorname{Hom}_R(G, M) = \operatorname{Im}(\psi^*) \subseteq m \operatorname{Hom}_R(G, M)$, that is $\oplus_{i=1}^t M \subseteq m \oplus_{i=1}^t M$ where $t = \operatorname{rank}(G)$. So $M = mM$. ▮

Finally, we note the following basic observation: For a ring S, when the map of S-modules $N \xrightarrow{r} N$ is multiplication by an element $r \in S$, the same is true of the induced maps

$$\operatorname{Ext}_S^i(M, N) \xrightarrow{\operatorname{Ext}_S^i(M, r)} \operatorname{Ext}_S^i(M, N).$$

So, given an S-linear map

$$\coprod S \xrightarrow{\ \psi\ } \coprod S$$

of free modules, the additivity of the functor $\operatorname{Ext}_S^i(M, \cdot\,)$ (here our M is a finite S-module) implies that the induced map

$$\operatorname{Ext}_S^i\Big(M, \coprod S\Big) \xrightarrow{\ \psi_*^i\ } \operatorname{Ext}_S^i\Big(M, \coprod S\Big)$$

can be given by the same matrix as ψ. We think of ψ_*^i as operating on a basis of $\coprod S = F$, and have the commutative diagram where the vertical identifications are the natural ones:

$$\begin{array}{ccc}
\operatorname{Ext}_S^i(M, \coprod S) & \xrightarrow{\psi_*^i} & \operatorname{Ext}_S^i(M, \coprod S) \\
\downarrow \cong & & \downarrow \cong \\
\coprod \operatorname{Ext}_S^i(M, S) & \xrightarrow{\psi_*^i} & \coprod \operatorname{Ext}_S^i(M, S) \\
\downarrow \cong & & \downarrow \cong \\
F \otimes_S \operatorname{Ext}_S^i(M, S) & \xrightarrow{\psi \otimes 1} & F \otimes_S \operatorname{Ext}_S^i(M, S)
\end{array}$$

With appropriate modifications, a similar result holds when considering the functor $\operatorname{Ext}_S^i(\cdot, N)$.

Setting and Reductions

The general setting in which we work—referred to as $(*)$ in this presentation—is the following:

$B \hookrightarrow A$ is a module finite ring extension,
B is an equicharacteristic local normal excellent domain of dimension d,
A is a normal ring.

Notice that the assumptions force A to be excellent semilocal of dimension d.

The main result is in the situation $(*)$, with additional assumptions on B (i.e., Gorenstein and satisfying (R_k)) and on the extension (unramified in codimension one). The aim is to draw a conclusion regarding the depth of A. Two essential considerations in the proof are (i) a reduction to the complete case, and (ii) the introduction of the normal closure.

Completion

The excellence of B and A ensures that these rings are analytically normal. Hence, applying $(\cdot) \otimes_B \hat{B}$ (where $(\hat{\ })$ denotes completion with respect to the m_B-adic topology) gives a module finite extension of normal rings where $\hat{B}$ is a local domain. Excellence guarantees that certain good properties of B pass to $\hat{B}$: if B satisfies (R_k) and is Gorenstein, then the same is true of $\hat{B}$. The semilocal $\hat{A}$ decomposes as $A_1 \times A_2 \times \cdots \times A_t$, each A_i being a complete local normal ring, hence a domain. Since we are interested in the depth properties of A, it is enough to work with $\hat{A}$, hence with each A_i—that is, with the extension $\hat{B} \hookrightarrow A_i$. It is important to observe that unramification in codimension one is preserved under completion: for if the trace map $\mathrm{tr}_{A/B}$ generates the free A-module $\mathrm{Hom}_B(A, B)$, then after applying $(\cdot) \otimes_B \hat{B}$ we see that $\mathrm{Hom}_{\hat{B}}(\hat{A}, \hat{B})$ is generated by the corresponding trace map.

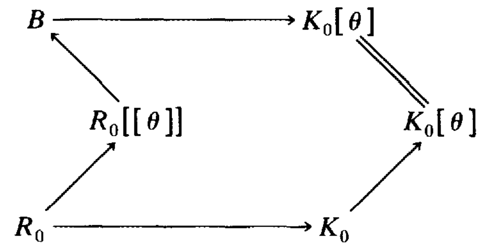
Normal Closure

In our general situation described above, we assume that A is a local domain and that the corresponding extension of fraction fields $K \hookrightarrow L$ is separable. We form the normal closure of the extension $B \hookrightarrow A$ as in [21, Sect. 1]: let E be the Galois closure of $K \hookrightarrow L$ and denote by S the integral closure of A in E, which, because of the excellence of A, is a finite A-module. If the base ring B is complete, then S is a complete local normal domain. The objective in the consideration of the normal closure lies in Section 3—we need to make use of the module isomorphism of Proposition 3.1. However, it is subsequently demonstrated (Proposition 3.3) that the depth of the module $\mathrm{Hom}_B(A, M)$ is the same as that of $\mathrm{Hom}_B(S, M)$ where $B \hookrightarrow A$ is the original extension, S the corresponding normal closure, and M an S-module with appropriate properties (refer to Section 3 for details). It is the depth of $\mathrm{Hom}_B(A, M)$ for suitable M which is critical for the proof of the main results.

It is important to understand how unramification behaves under such a "normal closure" procedure. Auslander [5, p. 117], which in part he attributes to Abhyankar [1], observes that composites of unramified extensions are unramified (also see [7, Proposition A.1]). This notion is translated and used in our context in [22], where the statement is as follows: If $B \hookrightarrow A$ is unramified in codimension i, then the normal closure $B \hookrightarrow S$ is unramified in codimension i (as is the extension $A \hookrightarrow S$).

Primitive Element Assumption

Finally, we remark that for our approach to succeed for a Gorenstein ring B which is not a hypersurface, it is necessary to locate a hypersurface $\overline{R} = R/fR \subset B$ such that the extension is module finite with $\overline{R}$ and B sharing a common fraction field (that is, B is the integral closure of $\overline{R}$ in K). This is always achievable in the case where there is a regular local ring $R_0 \subset B$ so that the extension is module finite and the extension of fraction fields is simple. In this case we may take the primitive element $\theta \in K$ to be in B. When we are in the complete case, the situation appears as follows:



Now $R_0[[X]]/fR_0[[X]] \simeq R_0[[\theta]]$ where $f \in R_0[[X]]$ is the monic polynomial satisfied by θ. Putting $R = R_0[[X]]$, we have our complete local hypersurface $\overline{R} \subset B$.

When B is complete local and the characteristic is zero, the Cohen-structure theory provides the desired R_0; when B is a finitely generated algebra over a perfect field k, one can produce the desired R_0 via a Noether-normalization type argument (using the fact that $K = Q(B)$ is separably generated over k). In the proof of the main theorem for a Gorenstein ring B, we assume the existence of the complete local hypersurface $\overline{R} \subset B$ with $(\overline{R})' = B$ (here, $(\cdot)'$ denotes the integral closure in $K = Q(B)$). We refer to this condition as the *primitive element assumption.*

3. MODULE ISOMORPHISMS AND APPLICATIONS

In his module theoretic version of the proof of the purity of branch locus theorem, Auslander [5] reduced to the case where the extension $B \hookrightarrow S$ (with B a regular local ring) is normal. He then utilized the ring isomorphism $\Delta(S;G) \simeq \mathrm{Hom}_B(S,S)$ where $G = \mathrm{Gal}(L/K)$ (L and K being the fraction fields of S and B, respectively), $\Delta(S;G)$ denotes the twisted group algebra (that is, $\Delta(S;G) = \oplus_{\sigma_i \in G} S\sigma_i$, a free S-module of rank $g = [L:K]$ where multiplication is extended from $a\sigma_i \cdot b\sigma_j = a\sigma_i(b)\sigma_i\sigma_j$ for $a, b \in S$, $\sigma_i, \sigma_j \in G$), and where the S-structure of $\mathrm{Hom}_B(S,S)$ is obtained via the codomain.

In [22], this isomorphism is used to obtain other module isomorphisms for ring extensions which are unramified in codimension one. We proceed to prove a proposition giving one of these isomorphisms.

PROPOSITION 3.1. *Let $B \hookrightarrow S$ be a module finite extension of local normal domains which is normal (with group G of order g) and which is unramified in codimension $i \geq 1$. Let M be an S-module satisfying* $\mathrm{depth}(M_p) \geq 2$ *for all $p \in \mathrm{Spec}(B)$ of codimension $> i$. Then, $M^g \simeq_S \mathrm{Hom}_B(S, M)$ where the codomain S-structure is used on the module* $\mathrm{Hom}_B(S, M)$.

Remarks (see [7, Sects. 3, 4; 35, Theorem 41.5]). (a) For a finite extension of normal domains $B \hookrightarrow S$, unramified in codimension one with B local, the following statements are equivalent:

(i) S is locally free over B in codimension i ($i \geq 1$), that is, S_p is a free B_p-module for all $p \in \operatorname{Spec}(B)$ of codimension $\leq i$,

(ii) $B \hookrightarrow S$ is unramified in codimension i.

Moreover, if B satisfies (R_i), then the unramification in codimension one "lifts" to codimension i via localization and the classical purity of branch locus theorem.

(b) When $i = 1$, then any balanced (S_2) S-module satisfies the conditions of the proposition, yielding the isomorphism $M^g \simeq_S \operatorname{Hom}_B(S, M)$.

COROLLARY 3.2. *For $B \hookrightarrow S$ and M as in the proposition,*

$$\operatorname{depth}(M_p) = \operatorname{depth}\big(\operatorname{Hom}_{B_p}(S_p, M_p)\big)$$

for all $p \in \operatorname{Spec}(B)$.

Proof of Corollary 3.2. This is immediate since $\operatorname{Hom}_{B_p}(S_p, M_p) \simeq (\operatorname{Hom}_B(S, M))_p$ because S is a finite B-module. ∎

Proof of Proposition 3.1. As in [22] we define ϕ: $M \otimes_S \operatorname{Hom}_B(S, S) \to \operatorname{Hom}_B(S, M)$ by ϕ: $m \otimes f \mapsto \phi_{m \otimes f}$ where $\phi_{m \otimes f}(x) = f(x) \cdot m \in M$ for $x \in S$. This is a well-defined S-linear map when the codomain S-structure is used for the homomorphism modules. It is evident that ϕ is an isomorphism for $S = B$, consequently ϕ is an isomorphism when S is a finite free B-module. Thus, the assumption of unramification in codimension i (which, as remarked, is equivalent to local freeness in codimension i when unramification in codimension one is present) gives that ϕ is an isomorphism when localized at any $p \in \operatorname{Spec}(B)$ of codimension $\leq i$. So we have (†): $\ker \phi$ and $\operatorname{coker} \phi$ have their B-support contained in a subset of $\operatorname{Spec}(B)$ consisting of primes of codimension $> i$.

In our setting, we have from [5] that $\operatorname{Hom}_B(S, S) \simeq \Delta(S; G)$ where, as mentioned before, the twisted group algebra $\Delta(S; G)$ is S-free of rank $g = |G| = [L : K]$. So,

$$M \otimes_S \operatorname{Hom}_B(S, S) \simeq_S M \otimes_S S^g \simeq_S M^g.$$

Now, $\ker \phi \subset M \otimes_S \operatorname{Hom}_B(S, S) \simeq_S M^g$ so that $\operatorname{Ass}_B(\ker \phi) \subset \operatorname{Ass}_B(M)$. By assumption, $\operatorname{depth}(M_p) \geq 2$ for $p \in \operatorname{Spec}(B)$ of codimension $> i$, so that M and hence $\ker \phi$ have no B-associated primes of codimension $> i$. But $\operatorname{Ass}_B(\ker \phi) \subset \operatorname{Supp}_B(\ker \phi)$, and since (†) holds, it must be that $\operatorname{Ass}_B(\ker \phi) = \varnothing$ and hence $\ker \phi = 0$.

Next we consider the exact sequence

$$0 \longrightarrow M \otimes_S \operatorname{Hom}_B(S,S) \xrightarrow{\phi} \operatorname{Hom}_B(S,M) \longrightarrow \operatorname{coker}\phi \longrightarrow 0.$$

Let $p \in \operatorname{Ass}_B(\operatorname{coker}\phi)$ and localize the above sequence at p, noting that $\operatorname{depth}(M_p) > 0$ implies that $\operatorname{depth}(\operatorname{Hom}_{B_p}(S_p, M_p)) > 0$. An appeal to the Depth Lemma (see Section 2) gives that

$$\operatorname{depth}\big((M \otimes_S \operatorname{Hom}_B(S,S))_p\big) = \operatorname{depth}(M_p^g) = \operatorname{depth}(M_p) = 1,$$

a contradiction (since $\operatorname{Ass}_B(\operatorname{coker}\phi) \subset \operatorname{Supp}_B(\operatorname{coker}\phi)$ and (†) imply that $\operatorname{codim}(p) > i$). So again, $\operatorname{Ass}_B(\operatorname{coker}\phi) = \varnothing$, hence $\operatorname{coker}\phi = 0$, and the proposition is proved. ▮

Remarks. (a) To prove that $\ker\phi = 0$ in the proposition, it is enough that $\operatorname{depth}(M_p) \geq 1$ for $p \in \operatorname{Spec}(B)$ of codimension $> i$, and

(b) When S is locally free over B on the punctured spectrum $\operatorname{Spec}^\circ(B)$, then the required assumption on M is that $\operatorname{depth}(M) \geq 2$.

We now proceed to a companion proposition to Proposition 3.1 which gives information regarding the depth of $\operatorname{Hom}_B(A, W)$ (for suitable S-modules W) where the extension $B \hookrightarrow A$ has normal closure $B \hookrightarrow S$.

PROPOSITION 3.3. *Let $B \hookrightarrow A$ be a module-finite extension of local normal domains which is unramified in codimension $i \geq 1$, and denote by S the normal closure of $B \hookrightarrow A$. Let W be an S-module which satisfies the depth condition of Proposition* 3.1 *and denote* $\operatorname{Hom}_B(A, W)$ *by M. Then* $\operatorname{depth}_B(W) = \operatorname{depth}_B(M)$.

As with Corollary 3.2, an immediate result is the following.

COROLLARY 3.4. *For $B \hookrightarrow A$ and W as in Proposition* 3.3,

$$\operatorname{depth}(W_p) = \operatorname{depth}\big(\operatorname{Hom}_{B_p}(A_p, W_p)\big)$$

for all $p \in$ $\operatorname{Spec}(B)$.

Proof of Proposition 3.3. First we remark that $M = \operatorname{Hom}_B(A, W)$ is an S-module via the codomain structure on $\operatorname{Hom}_B(A, W)$, and that the module M satisfies the depth condition of Proposition 3.1 since W does. Consider the module-finite extensions $B \hookrightarrow A \hookrightarrow S$ with fraction fields $K \hookrightarrow L \hookrightarrow E$, where $K \hookrightarrow E$ is Galois with group G. Then $L \hookrightarrow E$ is also a Galois extension with group $H < G$, and moreover, $S^H = A$. Noting that the normal extension $A \hookrightarrow S$ is also unramified in codimension i, and that, because of the module-finiteness of the extensions $B \hookrightarrow A \hookrightarrow S$, we may compute depths over A instead of B, we apply Proposition 3.1

to the extension $A \hookrightarrow S$ and module M to obtain the isomorphism $\mathrm{Hom}_A(S, M) \simeq_S M^h$ where $h = |H|$. Using the Adjoint Isomorphism Theorem, we see that

$$\begin{aligned} M^h &\simeq \mathrm{Hom}_A(S, M) \\ &= \mathrm{Hom}_A(S, \mathrm{Hom}_B(A, W)) \simeq_B \mathrm{Hom}_B(S \otimes_A A, W) \simeq \mathrm{Hom}_B(S, W). \end{aligned}$$

But, an application of Proposition 3.1 to $B \hookrightarrow S$ and the S-module W yields the S-isomorphism $\mathrm{Hom}_B(S, W) \simeq W^g$ where $g = |G|$. So, as B-modules, $M^h \simeq W^g$, hence we conclude that $\mathrm{depth}_B(M) = \mathrm{depth}_B(W)$. ∎

To compute depth properties in the sequel, we require the following, which is a direct consequence of the Acyclicity Lemma of Peskine and Szpiro [36, Lemma 1.8], and which applies to modules M for which the above propositions hold.

PROPOSITION 3.5. *Suppose B is local, $B \hookrightarrow A$ is a module finite ring extension, and M is a B-module such that*:

(i) $\mathrm{depth}_B(\mathrm{Hom}_B(A, M)) = \mathrm{depth}_B(M) = t$ $(t \geq 3)$ *and*

(ii) $\mathrm{Supp}_B\{\mathrm{Ext}_B^i(A, M)\} \subset \{m_B\}$ *for* $i = 1, \ldots, t-2$.

Then $\mathrm{Ext}_B^i(A, M) = 0$ *for* $i = 1, \ldots, t-2$.

4. CONSTRUCTION OF THE EXACT B-COMPLEXES

The proofs of the main results found in the next section rely on the exact complexes which we construct here. Our setting is the following: $B \hookrightarrow A$ is a module finite extension of complete local normal domains. We consider two cases:

(i) B is a hypersurface ring. By the Cohen-structure theory, there is a complete regular local ring R_0 such that $R_0 \subset B \subset A$, where the extensions are module finite.

(ii) B is a Gorenstein ring which is not a hypersurface. We remind the reader that in the event $\mathrm{char}(B) \neq 0$, then the primitive element assumption (refer to Section 2) is in force. As above, by the Cohen-structure theory, there is a complete regular local ring R_0 such that $R_0 \subset \overline{R} \subset B \subset A$ where $\overline{R}$ is the complete local hypersurface with $(\overline{R})' = B$ (here $(\cdot)'$ denotes integral closure in K, the field of fractions of B). Recall that given a complete regular local $R_0 \subset B$ so that $R_0 \hookrightarrow B$ is module finite and the extension of fraction fields is simple, we may produce $\overline{R}$ by taking $\overline{R} = R_0[[\theta]]$, $\theta \in B$ a primitive element.

As shown in [18], there is a countably generated MCM A-module C so that C is free as an R_0-module. In addition, such a C is a balanced MCM A-module.

PROPOSITION 4.1. *Let $B \hookrightarrow A$ be a module finite extension of complete local normal domains and C an MCM A-module of the type described above. Then:*

(i) *If B is a hypersurface, then there is an exact B-complex*

$$0 \longrightarrow C \longrightarrow F_B \xrightarrow{\phi_B} F_B \longrightarrow C \longrightarrow 0,$$

where F_B is a countably generated free B-module.

(ii) *If B is Gorenstein ring and not a hypersurface, then there is an exact B-complex*

$$0 \longrightarrow C \longrightarrow F_B \xrightarrow{\phi_B} F_B \longrightarrow T \longrightarrow 0,$$

where F_B is countably generated B-free and T has an A-module structure.

Proof. (i) Since B is a complete local hypersurface, then $B = R/fR$ where R is a complete regular local ring and $f \in m_R$. From [19, Theorem 1.7] we have that $\operatorname{pd}_R(C) = \dim(R) - \dim(B) = 1$. Let

$$0 \longrightarrow G \xrightarrow{\psi} F \longrightarrow C \longrightarrow 0$$

be a free R-resolution of C. Since C is a torsion R-module, then $G \simeq_R F$, so we consider the free resolution

$$0 \longrightarrow F \xrightarrow{\phi} F \longrightarrow C \longrightarrow 0. \tag{4.1}$$

Here, F is a countably generated free R-module and ϕ is represented by a column-finite matrix. We apply $B \otimes_R (\cdot)$ to (4.1) and obtain

$$0 \longrightarrow \operatorname{Tor}_1^R(B,C) \longrightarrow \bar{F} \xrightarrow{\phi \otimes 1_B} \bar{F} \longrightarrow C \longrightarrow 0.$$

Putting $F_B = \bar{F}$, a free B-module and $\phi_B = \phi \otimes 1_B$, we have our desired complex as soon as it is verified that $\operatorname{Tor}_1^R(B,C) \simeq C$. From the exact sequence

$$0 \longrightarrow R \xrightarrow{f} R \longrightarrow R/fR = B \longrightarrow 0,$$

we obtain

$$0 \longrightarrow \operatorname{Tor}_1^R(B,C) \longrightarrow C \xrightarrow{f} C \longrightarrow B \otimes_R C \longrightarrow 0$$

by applying $(\cdot) \otimes_R C$. But $fC = 0$, hence $\operatorname{Tor}_1^R(B,C) \simeq C$, and we have the exact complex

$$0 \longrightarrow C \longrightarrow F_B \xrightarrow{\phi_B} F_B \longrightarrow C \longrightarrow 0.$$

(ii) We have $R_0 \hookrightarrow \bar{R} \hookrightarrow B \hookrightarrow A$, all module finite extensions, where $\bar{R} = R/fR$ is a complete local hypersurface with $(\bar{R})' = B$. From [19, Theorem 1.7], we have $\mathrm{pd}_R(C) = 1$, so that there is an R-free resolution

$$0 \longrightarrow F \xrightarrow{\phi} F \longrightarrow C \longrightarrow 0, \tag{4.2}$$

where F is a countably generated free R-module and ϕ is represented by a column-finite matrix. To produce the desired complex, we apply $\mathrm{Hom}_R(B, \cdot)$ to (4.2) and obtain the exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_R(B,F) \longrightarrow \mathrm{Hom}_R(B,F) \longrightarrow \mathrm{Hom}_R(B,C) \\ &\xrightarrow{\delta} \mathrm{Ext}^1_R(B,F) \longrightarrow \mathrm{Ext}^1_R(B,F) \longrightarrow \mathrm{Ext}^1_R(B,C) \\ &\xrightarrow{\delta} \mathrm{Ext}^2_R(B,F). \end{aligned} \tag{4.3}$$

Regarding (4.3), we make the following observations:

(a) Since B is R-torsion and F is R-torsion-free, then $\mathrm{Hom}_R(B,F) = 0$;

(b) Since $\mathrm{pd}_R(B) = 1$, then $\mathrm{Ext}^2_R(B,F) = 0$ follows from the depth theorem of Auslander and Buchsbaum;

(c) From the facts that $fB = 0$ and f is F-regular, we obtain $\mathrm{Ext}^1_R(B,F) \simeq \mathrm{Hom}_{\bar{R}}(B,\bar{F})$;

(d) $\mathrm{Hom}_{\bar{R}}(B,\bar{F}) = \mathrm{Hom}_{\bar{R}}(B, \coprod \bar{R}) \simeq \coprod B$; and

(e) $\mathrm{Hom}_R(B,C) = \mathrm{Hom}_{\bar{R}}(B,C) \simeq C$.

The first three remarks are evident; we provide clarification of the remaining two.

Proof of (d). Since B is a finite R-module, then $\mathrm{Hom}_{\bar{R}}(B, \coprod \bar{R}) \simeq \coprod \mathrm{Hom}_{\bar{R}}(B, \bar{R})$. Moreover, since B and $\bar{R}$ are local Gorenstein rings, then $\mathrm{Hom}_{\bar{R}}(B,\bar{R})$ is just the canonical module for B, which is naturally isomorphic to B. ∎

Proof of (e). Consider the maps

$$\mathrm{Hom}_B(B,C) \xrightarrow{i} \mathrm{Hom}_{\bar{R}}(B,C) \xrightarrow{j} C,$$

where i is the inclusion map and $j(\phi) = \phi(1)$. So $\mathrm{Hom}_B(B,C)$ is an $\bar{R}$-direct summand of $\mathrm{Hom}_{\bar{R}}(B,C)$, say $\mathrm{Hom}_{\bar{R}}(B,C) \simeq \mathrm{Hom}_B(B,C) \oplus M$. Since $B \otimes_{\bar{R}} K = \bar{R} \otimes_{\bar{R}} K$ (where K denotes the fraction field of $\bar{R}$ and B), the maps i and j become isomorphisms when $(\cdot) \otimes_{\bar{R}} K$ is applied. In particular, $M \otimes_{\bar{R}} K = 0$. But $\mathrm{Hom}_{\bar{R}}(B,C)$ is a torsion-free $\bar{R}$-module (since C is) so that M is torsion-free over $\bar{R}$ as well. Hence $M = 0$ and $\mathrm{Hom}_{\bar{R}}(B,C) \simeq \mathrm{Hom}_B(B,C) \simeq C$. ∎

Recalling the exact sequence (4.3) and utilizing these observations, we see that (4.3) has the form

$$0 \longrightarrow C \longrightarrow \coprod B \xrightarrow{\phi_*} \coprod B \longrightarrow \operatorname{Ext}^1_R(B,C) \longrightarrow 0,$$

where ϕ_* can be expressed by the same (column-finite) matrix as ϕ. We denote ϕ_* by ϕ_B, $\coprod B$ by F_B, and $\operatorname{Ext}^1_R(B,C)$ by T, and remark that T is naturally an A-module via the A-module structure from C. ∎

Remarks. (a) $\phi_B \not\equiv 0$. If $\phi_B \equiv 0$, then $C \simeq \coprod B$ and the entries of the matrix for ϕ are in the principal ideal $fR \subset R$. Applying $(\cdot) \otimes \bar{R}$ to the R-free resolution

$$0 \longrightarrow F \xrightarrow{\phi} F \longrightarrow C \longrightarrow 0$$

we obtain

$$\bar{F} \xrightarrow{\bar{\phi}} \bar{F} \longrightarrow C \longrightarrow 0$$

which is exact. But $\bar{\phi}$ is the 0-map, so that $C \simeq \bar{F}$. So as $\bar{R}$-modules, we have $\coprod B \simeq \bar{F}$, hence that B is $\bar{R}$-projective. But then $\bar{R}$ satisfies (R_1), since B does, and hence is integrally closed in K. This forces $\bar{R} = B$, a contradiction.

(b) $T = \operatorname{Ext}^1_R(B,C) \neq 0$. This follows immediately from Theorem 4 and the fact that $m_R C \neq C$.

B-Structure of T

We focus our attention on $T = \operatorname{Ext}^1_R(B,C)$ which has two B-module structures: one given through the B-action on the first variable; the other given via the B-action on C. Since C is free over the regular local ring R_0 and $R_0 \hookrightarrow B$ is module finite, then C is B-free in codimension k when B satisfies property (R_k) [19, Theorem 1.7]. Consequently, we first consider the case when $C = B$, that is, when $T = \operatorname{Ext}^1_R(B, B)$.

PROPOSITION 4.2. *There is a B-module isomorphism*

$$\Psi\colon \operatorname{Ext}^1_R(B,B) \xrightarrow{\sim} \operatorname{Ext}^1_R(B,B)$$

where the B-module structure of the left module is via the first variable, and the B-module structure on the right module is via the second variable.

Proof. Since $\operatorname{pd}_R(B) = 1$, let

$$0 \longrightarrow F \xrightarrow{\psi} F \longrightarrow B \longrightarrow 0 \qquad (4.4)$$

be a minimal free resolution of B over R, where F is a finite free R-module. Denote $\mathrm{Hom}_R(\cdot, R)$ by $(\cdot)^*$, $\mathrm{Hom}_{\bar{R}}(\cdot, \bar{R})$ by $(\cdot)^{\bar{*}}$, and (object) "mod f" by $\overline{(\cdot)}$. We proceed to construct a B-module commutative diagram with exact rows and with the indicated vertical maps isomorphisms:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\bar{R}}(B, \bar{F}) & \longrightarrow & \mathrm{Hom}_{\bar{R}}(B, \bar{F}) & \longrightarrow & \mathrm{Ext}^1_R(B, B) & \longrightarrow & 0 \\
\simeq\downarrow & & \simeq\downarrow & & & & \\
\mathrm{Hom}_{\bar{R}}(\bar{F}^{\bar{*}}, B^{\bar{*}}) & \longrightarrow & \mathrm{Hom}_{\bar{R}}(\bar{F}^{\bar{*}}, B^{\bar{*}}) & \longrightarrow & \mathrm{Ext}^1_R(B^{\bar{*}}, B^{\bar{*}}) & \longrightarrow & 0 \\
\simeq\downarrow & & \simeq\downarrow & & & & \\
\mathrm{Hom}_{\bar{R}}(\bar{F}, B) & \longrightarrow & \mathrm{Hom}_{\bar{R}}(\bar{F}, B) & \longrightarrow & \mathrm{Ext}^1_R(B, B) & \longrightarrow & 0
\end{array} \tag{4.5}$$

In the top row, the B-structure is via the domain of the homomorphism modules; in the remaining two rows, the B-structure is obtained via the codomain of the homomorphism modules. The conclusion is that the cokernels are isomorphic:

$$\mathrm{Ext}^1_R(B, B) \xrightarrow{\sim} \mathrm{Ext}^1_R(B^{\bar{*}}, B^{\bar{*}}) \xrightarrow{\sim} \mathrm{Ext}^1_R(B, B)$$

where the B-structures are as described. Hence, the proposition will be proved as soon as diagram (4.5) is constructed.

We make the following observations:

(a) Recall that $B^{\bar{*}} = \mathrm{Hom}_{\bar{R}}(B, \bar{R})$ is the canonical module for the Gorenstein ring B, hence there is a natural identification $B^{\bar{*}} \xrightarrow{\sim \gamma_B} B$.

(b) F is a finite free R-module and $\bar{F}$ is a finite free $\bar{R}$-module, so we have $F^* \xrightarrow{\sim \gamma_F} F$ and $\bar{F}^{\bar{*}} \xrightarrow{\sim \gamma_{\bar{F}}} \bar{F}$.

(c) f annihilates $\bar{F}$, B, and $B^{\bar{*}}$.

(d) $\overline{F^*} \xrightarrow{\sim} \bar{F}^{\bar{*}}$. This follow immediately by applying $(\cdot)^*$ to

$$0 \longrightarrow F \xrightarrow{f} F \longrightarrow \bar{F} \longrightarrow 0$$

and noting that $\mathrm{Ext}^1_R(\bar{F}, R) \simeq \mathrm{Hom}_{\bar{R}}(\bar{F}, \bar{R}) = (\bar{F})^{\bar{*}}$.

From the resolution

$$0 \longrightarrow F \xrightarrow{\psi} F \longrightarrow B \longrightarrow 0 \tag{4.6}$$

we obtain the exact sequence

$$0 \longrightarrow F^* \xrightarrow{\psi'} F^* \longrightarrow \bar{B}^* \longrightarrow 0 \tag{4.7}$$

by applying $(\cdot)^*$, where we use the information that $B^* = 0$ and that $\mathrm{Ext}_R^1(B, R) \simeq \mathrm{Hom}_{\bar{R}}(B, \bar{R}) = \bar{B}^*$. We apply $\mathrm{Hom}_R(B, \cdot)$ to (4.6) and $\mathrm{Hom}_R(\cdot, \bar{B}^*)$ to (4.7) to obtain the following exact B-sequences:

$$\mathrm{Ext}_R^1(B, F) \xrightarrow{\psi_*} \mathrm{Ext}_R^1(B, F) \longrightarrow \mathrm{Ext}_R^1(B, B) \longrightarrow 0 \tag{4.8}$$

$$\mathrm{Hom}_R(F^*, \bar{B}^*) \xrightarrow{(\psi')^*} \mathrm{Hom}_R(F^*, \bar{B}^*) \longrightarrow \mathrm{Ext}_R^1(\bar{B}^*, \bar{B}^*) \longrightarrow 0. \tag{4.9}$$

Notice that

$$\mathrm{Ext}_R^1(B, F) \simeq \mathrm{Hom}_{\bar{R}}(B, \bar{F}),$$

$$\mathrm{Hom}_R(F^*, \bar{B}^*) \simeq \mathrm{Hom}_{\bar{R}}(\bar{F}^*, \bar{B}^*),$$

and that $\mathrm{pd}_R(B) = \mathrm{pd}_R(\bar{B}^*) = 1$ provides exactness at the right hand end of the sequences. Rewriting with respect to this information, we have the exact B-sequences

$$\mathrm{Hom}_{\bar{R}}(B, \bar{F}) \xrightarrow{\psi_*} \mathrm{Hom}_{\bar{R}}(B, \bar{F}) \longrightarrow \mathrm{Ext}_R^1(B, B) \longrightarrow 0 \tag{4.10}$$

$$\mathrm{Hom}_{\bar{R}}(\bar{F}^*, \bar{B}^*) \xrightarrow{(\psi')^*} \mathrm{Hom}_{\bar{R}}(\bar{F}^*, \bar{B}^*) \longrightarrow \mathrm{Ext}_R^1(\bar{B}^*, \bar{B}^*) \longrightarrow 0 \tag{4.11}$$

with domain B-structure in (4.10) and codomain B-structure in (4.11).

The natural correspondence of maps $F \to B$ with maps $F^* \to \bar{B}^*$ via the isomorphisms γ_F and γ_B

$$\begin{array}{ccc} F & \xrightarrow{\quad \eta \quad} & B \\ \gamma_F \downarrow \simeq & & \gamma_B \downarrow \simeq \\ F^* & \xrightarrow{\gamma_B \eta \gamma_F^{-1}} & \bar{B}^* \end{array}$$

gives a B-isomorphism ε: $\mathrm{Hom}_R(F, B) \xrightarrow{\sim} \mathrm{Hom}_R(F^*, \bar{B}^*)$. To the exact sequence

$$0 \longrightarrow F \xrightarrow{\gamma_F^{-1}\psi'\gamma_F} F \longrightarrow B \longrightarrow 0$$

we apply $\operatorname{Hom}_R(\cdot, B)$ and obtain, after appropriate identifications, the following diagram with exact rows:

$$\begin{array}{ccccccc}
\operatorname{Hom}_{\overline{R}}(\overline{F}, B) & \xrightarrow{(\gamma_F^{-1}\psi^t\gamma_F)^*} & \operatorname{Hom}_{\overline{R}}(\overline{F}, B) & \longrightarrow & \operatorname{Ext}^1_R(B, B) & \longrightarrow & 0 \\
\varepsilon \downarrow \simeq & & \varepsilon \downarrow \simeq & & & & \\
\operatorname{Hom}_{\overline{R}}(\overline{\overline{F}^*}, \overline{B^*}) & \xrightarrow{(\psi^t)^*} & \operatorname{Hom}_{\overline{R}}(\overline{\overline{F}^*}, \overline{B^*}) & \longrightarrow & \operatorname{Ext}^1_R(\overline{B^*}, \overline{B^*}) & \longrightarrow & 0
\end{array}$$

Notice that the square commutes, hence that $\operatorname{Ext}^1_R(B, B) \simeq \operatorname{Ext}^1_R(\overline{B^*}, \overline{B^*})$ where all the B-structures are obtained from the codomain (refer to (4.5)).

For a B-module M and an $\overline{R}$-module N, we define the map $\Phi^{M,N}$: $\operatorname{Hom}_{\overline{R}}(M, N) \to \operatorname{Hom}_{\overline{R}}(N^*, M^*)$ by $g \mapsto g^*$, where $g^*(\varepsilon) = \varepsilon \circ g$ for $\varepsilon \in N^*$. Here $\operatorname{Hom}_{\overline{R}}(M, N)$ obtains its B-structure from the domain, and $\operatorname{Hom}_{\overline{R}}(N^*, M^*)$ from the codomain. $\Phi^{M,N}$ is B-linear: for $g \in \operatorname{Hom}_{\overline{R}}(M, N)$, $b \in B$, and any $\alpha \in N^*$ we have $(bg)^*(\alpha) = \alpha \circ bg = b(\alpha \circ g) = b(g^*(\alpha))$ since both $\operatorname{Hom}_{\overline{R}}(M, N)$ and M^* use the domain B-structure.

Consider the diagram (refer to (4.5))

$$\begin{array}{ccc}
\operatorname{Hom}_{\overline{R}}(B, \overline{F}) & \xrightarrow{\psi_*} & \operatorname{Hom}_{\overline{R}}(B, \overline{F}) \\
\downarrow \Phi^{B,\overline{F}} & & \downarrow \Phi^{B,\overline{F}} \\
\operatorname{Hom}_{\overline{R}}(\overline{\overline{F}^*}, \overline{B^*}) & \xrightarrow{(\psi^t)^*} & \operatorname{Hom}_{\overline{R}}(\overline{\overline{F}^*}, \overline{B^*})
\end{array}$$

Given $\phi \in \operatorname{Hom}_{\overline{R}}(B, \overline{F})$, it is clear that $(\psi \circ \phi)^* = \phi^* \circ \psi^t$ so that the square commutes. Finally we verify that $\Phi^{B,\overline{F}}$ is a bijection.

Let $g \in \operatorname{Hom}_{\overline{R}}(B, \overline{F}) - \{0\}$. Then $g(b) \neq 0$ in $\overline{F}$ for some $b \in B$, hence $\pi_j(g(b)) \neq 0$ for some j, where π_j: $\overline{F} \to \overline{R}$ is projection onto the jth factor. But then $g^*(\pi_j) = \pi_j \circ g \neq 0$, hence $g^* \neq 0$ and therefore $\Phi^{B,\overline{F}}$ is injective.

Since B and $\overline{F}$ are $\overline{R}$-reflexive, we have the following isomorphism of B-modules

$$\operatorname{Hom}_{\overline{R}}(\overline{B^{**}}, \overline{\overline{F}^{**}}) \underset{\eta}{\xrightarrow{\sim}} \operatorname{Hom}_{\overline{R}}(B, \overline{F})$$

which is evident from the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{\sigma_{\overline{F}}^{-1}\alpha\sigma_B} & \overline{F} \\ \sigma_B \downarrow \simeq & & \sigma_{\overline{F}} \downarrow \simeq \\ B^{\overline{**}} & \xrightarrow{\alpha} & \overline{F}^{\overline{**}} \end{array}$$

Take $g \in \operatorname{Hom}_{\overline{R}}(B, \overline{F})$. Applying $\Phi^{B,\overline{F}}$ and then $\Phi^{\overline{F}^{\overline{*}}, B^{\overline{*}}}$ we obtain $g^* \in \operatorname{Hom}_{\overline{R}}(\overline{F}^{\overline{*}}, B^{\overline{*}})$ and $(g^*)^* \in \operatorname{Hom}_{\overline{R}}(B^{\overline{**}}, \overline{F}^{\overline{**}})$. Under the above identification η, g^{**} corresponds to g: Let $x \in B$. Then,

$$\sigma_{\overline{F}}^{-1}(g^*)^* \sigma_B(x) = \sigma_{\overline{F}}^{-1} \circ (\sigma_B(x) \circ g^*).$$

For $\beta \in \overline{F}^{\overline{*}}$, $\sigma_B(x) \circ g^*(\beta) = \sigma_B(x) \circ (\beta \circ g) = \beta \circ g(x) = \beta(g(x))$. That is, $\sigma_B(x) \circ g^*$ is evaluation at $g(x) \in \overline{F}$. Therefore, $\sigma_{\overline{F}}^{-1}(g^*)^* \sigma_B(x) = g(x)$.

We have

$$\operatorname{Hom}_{\overline{R}}(B, \overline{F}) \xrightarrow{\Phi^{B,\overline{F}}} \operatorname{Hom}_{\overline{R}}(\overline{F}^{\overline{*}}, B^{\overline{*}}) \xrightarrow{\Phi^{\overline{F}^{\overline{*}}, B^{\overline{*}}}} \operatorname{Hom}_{\overline{R}}(B^{\overline{**}}, \overline{F}^{\overline{**}}) \xrightarrow{\sim} \operatorname{Hom}_{\overline{R}}(B, \overline{F})$$

where $g \mapsto g^* \mapsto g^{**} \xrightarrow{\sim} g$. Hence,

$$\operatorname{Hom}_{\overline{R}}(B, \overline{F}) \oplus M' \simeq_B \operatorname{Hom}_{\overline{R}}(\overline{F}^{\overline{*}}, B^{\overline{*}}).$$

Applying $(\cdot) \otimes_{\overline{R}} K$ (where K is the quotient field of $\overline{R}$ and B) we see that $\Phi^{B,\overline{F}}$ becomes an isomorphism (recall that $\overline{F}^{\overline{*}} \simeq \overline{F}$ and $B^{\overline{*}} \simeq B$). Since $\operatorname{Hom}_{\overline{R}}(\overline{F}^{\overline{*}}, B^{\overline{*}})$ is torsion-free, it must be that $M' = 0$ and that $\Phi^{B,\overline{F}}$ is an isomorphism.

So the commutative diagram (4.5) has been produced, hence the proposition is proved. ∎

COROLLARY 4.3. *With the notation as above, let $T = \operatorname{Ext}^1_R(B, \coprod B)$. Then two B-structures on T coincide.*

Proof. Since B is module finite over R, we have

$$\operatorname{Ext}^1_R\left(B, \coprod B\right) \simeq \coprod \operatorname{Ext}^1_R(B, B) \xrightarrow{\coprod \Psi} \coprod \operatorname{Ext}^1_R(B, B) \simeq \operatorname{Ext}^1_R\left(B, \coprod B\right),$$

where the left two modules use the B-structure from the first variable, the right two modules use the B-structure from the second variable, and the isomorphism Ψ (which "switches" B-structures) comes from the preceding proposition. ∎

We consider $T = \mathrm{Ext}^1_R(B,C)$ and denote by T^r the R-module T with B-action inherited from the action on C, and by T^l the R-module T with B-action inherited from the left variable. For $t \in T$ and $b \in B$, we write $t \cdot b$ for the action in T^r and $b \cdot t$ for the action in T^l.

PROPOSITION 4.4. *Let $T = \mathrm{Ext}^1_R(B,C)$ and assume B satisfies (R_2). Then, $T^l \simeq_B T^r$.*

Proof. Let $b \in B$ and define the map of abelian groups $\phi_b \colon T \to T$ by $\phi_b(t) = t \cdot b - b \cdot t$. Since the action of $\overline{R} = R/fR$ is the same through either variable (this is evident from the exactness of $\mathrm{Hom}_{\overline{R}}(B, \overline{F}) \to \mathrm{Hom}_{\overline{R}}(B, \overline{F}) \to T \to 0$), ϕ_b is an $\overline{R}$-linear, hence R_0-linear map. We assert that $0 = \mathrm{Im}(\phi_b) \subset T$. First, we remark that, for $p \in \mathrm{Spec}(R_0)$ with $\mathrm{codim}(p) \le 2$ we have $C_p \simeq \coprod B_p$: for B_p is regular semilocal, and by [19, Theorem 1.7], $\mathrm{pd}_{B_p}(C_p) = \mathrm{pd}_{(R_0)_p}(C_p) = 0$. So, C_p is B_p-projective, and hence B_p-free. Now we show that $\mathrm{Im}(\phi_b)$ has no R_0-support in codimension ≤ 2.

Let $p \in \mathrm{Spec}(R_0)$ such that $\mathrm{codim}(p) \le 2$. Put $\overline{S} := R_0 - p \subset \overline{R}$. Then $\overline{S}$ is a multiplicatively closed set in $\overline{R}$ which does not contain 0. Likewise put $S := \pi^{-1}(\overline{S}) \subset R$ where $\pi\colon R \to \overline{R} = R/fR$ is the natural map (recall that $R_0 \subset \overline{R} \subset B$), and notice that S is multiplicatively closed. Using the facts that $fT = 0$ and that B is module finite over R, we have

$$\begin{aligned} T_p = \mathrm{Ext}^1_R(B,C) \otimes_{\overline{R}} \overline{S}^{-1}\overline{R} &\xrightarrow[R]{\sim} \mathrm{Ext}^1_R(B,C) \otimes_R S^{-1}R \\ &\xrightarrow{\sim} \mathrm{Ext}^1_{S^{-1}R}(S^{-1}B, S^{-1}C) \\ &= \mathrm{Ext}^1_{S^{-1}R}(B_p, C_p) \xrightarrow{\sim} \mathrm{Ext}^1_{S^{-1}R}\left(B_p, \coprod B_p\right). \end{aligned}$$

By the previous corollary, $T^l_p \simeq_{B_p} T^r_p$ so that $\mathrm{Im}(\phi_b)_p = 0$. Suppose that $q \in \mathrm{Ass}_{R_0}(\mathrm{Im}(\phi_b))$. Since $\mathrm{Ass}_{R_0}(\mathrm{Im}(\phi_b)) \subset \mathrm{Supp}_{R_0}(\mathrm{Im}(\phi_b))$ then $\mathrm{codim}(q) \ge 3$. Localizing

$$0 \longrightarrow C \longrightarrow F_B \xrightarrow{\phi_B} F_B \longrightarrow T \longrightarrow 0$$

at q and applying the Depth Lemma gives that $\mathrm{depth}(C_q) = 2$, a contradiction (as $d = \dim(B) \ge 3$). Hence, $\mathrm{Ass}_{R_0}(\mathrm{Im}(\phi_b)) = \varnothing$, and so $\mathrm{Im}(\phi_b) = 0$. Since this holds for every $b \in B$, then $T^l \simeq_B T^r$. ∎

We close this section with some remarks regarding the divisibility properties of the A-modules C and T. By construction, $m_A C \ne C$, and we assert a similar result for T.

PROPOSITION 4.5. $k_B \otimes_B T \ne 0$ *where* $k_B = B/m_B$.

Proof. Let $\underline{x} = x_1, \dots, x_d \in m_{\overline{R}}$ form a system of parameters for B. C is balanced, hence $\underline{x}$ forms a C-sequence. We claim that $\underline{x}T \ne T$.

Applying $\operatorname{Ext}^1_R(B, \cdot)$ to the short exact sequence

$$0 \longrightarrow C \xrightarrow{x_1} C \longrightarrow C_1 \longrightarrow 0$$

we obtain the exact sequence

$$\operatorname{Ext}^1_R(B,C) \xrightarrow{x_1} \operatorname{Ext}^1_R(B,C) \longrightarrow \operatorname{Ext}^1_R(B,C_1) \longrightarrow 0,$$

where the zero at the right hand end appears because $\operatorname{pd}_R(B) = 1$. Therefore,

$$T/x_1T = \operatorname{Ext}^1_R(B,C)/x_1 \operatorname{Ext}^1_R(B,C) \xrightarrow{\sim} \operatorname{Ext}^1_R(B,C_1).$$

Assuming $T/x_1 \dots x_{r-1}T \simeq \operatorname{Ext}^1_R(B,C_{r-1})$ where $r - 1 < d$, it is clear how to proceed: from

$$0 \longrightarrow C_{r-1} \xrightarrow{x_{r-1}} C_{r-1} \longrightarrow C_r \longrightarrow 0$$

we find that

$$\operatorname{Ext}^1_R(B,C_{r-1})/x_r \operatorname{Ext}^1_R(B,C_{r-1}) \xrightarrow{\sim} \operatorname{Ext}^1_R(B,C_r),$$

that is, $T/x_1 \dots x_rT \simeq \operatorname{Ext}^1_R(B,C_r)$. Therefore, $T/\underline{x}T \simeq \operatorname{Ext}^1_R(B,C_d)$.

Notice that $m_AC \neq C$ implies that $m_RC \neq C$, and that $C_d/m_RC_d = (C/\underline{x}C)/(m_RC/\underline{x}C) \simeq C/m_RC \neq 0$, so that $m_RC_d \neq C_d$. By Theorem 4, $\operatorname{Ext}^1_R(B,C_d) \neq 0$, so that $T/\underline{x}T \neq 0$. Finally we claim that $T/m_BT \neq 0$. Otherwise, since $(\underline{x}) \subset m_B$ and $\underline{x}$ is a system of parameters, then $m_B^N \subset (\underline{x})$ for $N \gg 0$. In particular, $m_B^NT = m_B^{N-1}T = \cdots = T$, so that $\underline{x}T = T$, a contradiction. Hence, $T/m_BT = k_B \otimes_B T \neq 0$. ∎

5. THE MAIN THEOREM

We recall the general setting $(*)$ from Section 2.

$B \hookrightarrow A$ is a module finite ring extension,

B is an equicharacteristic local normal excellent domain of dimension d,

A is a normal ring.

The main result is:

THEOREM 5. *With the setting as above, assume that B is a Gorenstein ring of dimension $d \geq 5$ and that $B \hookrightarrow A$ is unramified in codimension one. Furthermore, in the event that* $\operatorname{char}(B) \neq 0$, *suppose the primitive element assumption is in force. If B satisfies (R_k), then A satisfies (S_{k-1}) where* $k \geq 4$.

COROLLARY 5.1. *Let $B \hookrightarrow A$ be as in the theorem. If B satisfies (R_k) with $k \geq \frac{1}{2}d + 2$ then A is a Gorenstein ring.*

Proof of Corollary. By the theorem, A satisfies (S_{k-1}). Since $k - 1 \geq \frac{1}{2}d + 1$, Theorem 3 applies and shows that A is Cohen–Macaulay.

But, referring to the Auslander–Goldman Theorem (Theorem 1 in Section 2) this is the same as A being Gorenstein. ▮

B Is a Hypersurface

As an illustration of the techniques, we recover a portion of a much stronger purity theorem for normal complete intersections which is due to the work of Grothendieck and Cutkosky. Grothendieck [23, Exposé 10, Theorem 3.4] proved that complete intersections of dimension ≥ 3 are "pure." From Cutkosky [10, Theorem 5] it follows that if the base ring is a normal complete intersection and the extension is unramified in codimension two, then the extension is unramified.

THEOREM 6 (see Grothendieck [23] and Cutkosky [10]). *Let the setting be as in* $(*)$. *Moreover, assume that*

B is a hypersurface ring of dimension $d \geq 3$,

A is a domain,

$B \hookrightarrow A$ is unramified in codimension one, and

A is locally free over B in codimension three.

Then $B \hookrightarrow A$ is an étale extension, that is, it is flat and unramified.

Proof. When $d = 3$, A is B-free and an appeal to [7, Corollary 3.7] or [35, Theorem 41.5] gives that $B \hookrightarrow A$ is étale. Let $p \in \operatorname{Spec}(B)$ be minimal with respect to the property that A_p is not unramified over B_p (or, what amounts to the same thing in this setting, that A_p is not B_p-free. Refer to the remarks following Proposition 3.1). Consider the extension $B_1 = B_p \hookrightarrow A_p = A_1$, noting that $\operatorname{codim}(p) \geq 4$. The various hypotheses for $B \hookrightarrow A$ remain in this new extension, with the addition that A_1 is unramified over B_1 at all nonmaximal prime ideals. It is enough to verify that A_1 is unramified over B_1, for then no such p exists.

We apply the reductions discussed in Section 2, observing that, since $(\cdot) \otimes_{B_1} \hat{B}_1$ is faithfully flat, it is enough to prove unramification in the complete case. In addition, we note that upon completion, excellence preserves unramification at all nonmaximal primes. Referring to the completed rings as A_1 and B_1 again, we consider the normal closure S_1 of $B_1 \hookrightarrow A_1$. In this setting (as discussed in Section 2), unramification in codimension i of $B_1 \hookrightarrow A_1$ implies the same for $B_1 \hookrightarrow S_1$ and vice versa (we also note that $A_1 \hookrightarrow S_1$ inherits the unramification in codimension i from $B_1 \hookrightarrow S_1$). Consequently, we consider the situation as in the theorem with the additions:

B_1 and A_1 are complete, with A_1 a local normal domain.

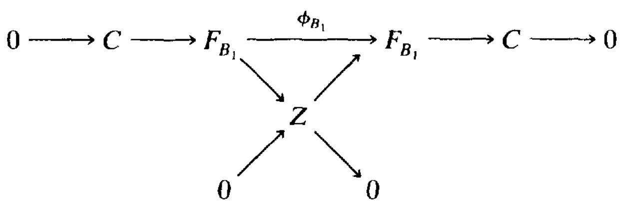
$B_1 \hookrightarrow A_1$ is unramified at all nonmaximal prime ideals—or equivalently (see the remarks following Proposition 3.1) A_1 is locally free at all nonmaximal prime ideals of B_1 (and likewise for the extension $B_1 \hookrightarrow S_1$).

Claim 1. A_1 is Cohen–Macaulay.

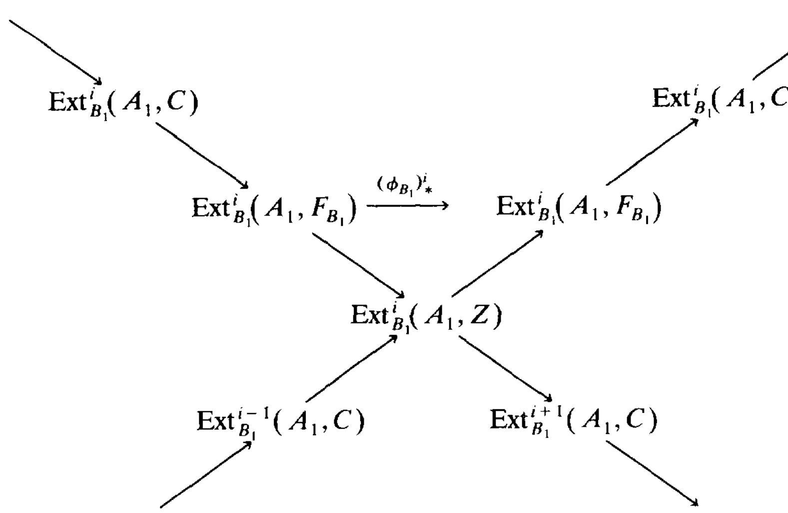
We consider the extensions $B_1 \hookrightarrow A_1 \hookrightarrow S_1$ and apply Proposition 4.1 to obtain an exact B_1-complex

$$0 \to C \to F_{B_1} \xrightarrow{\phi_{B_1}} F_{B_1} \to C \to 0,$$

where C is a countably generated balanced MCM S_1-module, free over a complete regular local ring $R_0 \subset B_1$ (where $R_0 \hookrightarrow B_1$ is module finite) and F_{B_1} is a countably generated free B_1-module. Writing this complex as two short exact sequences



we apply $\operatorname{Hom}_{B_1}(A_1, \cdot)$ and produce two long exact sequences which we write in the following commutative diagram:



(5.1)

Put $L^i := \operatorname{Ext}^i_{B_1}(A_1, B_1)$. We assert that if $L^i \neq 0$, then $(\phi_{B_1})^i_*$ is not a surjection. From the observation in Section 2, we have a commutative diagram,

$$\begin{array}{ccc} \operatorname{Ext}^i_{B_1}(A_1, F_{B_1}) & \xrightarrow{(\phi_{B_1})^i_*} & \operatorname{Ext}^i_{B_1}(A_1, F_{B_1}) \\ \cong \downarrow & & \cong \downarrow \\ F \otimes_R L^i & \xrightarrow{\phi \otimes_R 1_{L^i}} & F \otimes_R L^i \end{array}$$

where R, F, and ϕ are as in the proof of Proposition 4.1 (recall that $B_1 = R/fR$ and that

$$0 \longrightarrow F \xrightarrow{\phi} F \longrightarrow C \longrightarrow 0$$

is a free R-resolution of C), and the vertical maps are the natural identifications. Assume that $L^i \neq 0$. Then by Nakayama's Lemma, $L^i \neq m_{B_1} L^i$ since L^i is a finite B_1-module. Hence there is a map $L^i \to k_{B_1} \to 0$ where $k_{B_1} = B_1/m_{B_1}$. Applying $(\cdot) \otimes_R C$ we obtain the surjection

$$L^i \otimes_R C \longrightarrow k_{B_1} \otimes_R C \longrightarrow 0$$

and, noting that $k_{B_1} \otimes_R C \neq 0$, we have $L^i \otimes_R C \neq 0$. Applying $(\cdot) \otimes_R L^i$ to the free R-resolution of C,

$$0 \longrightarrow F \xrightarrow{\phi} F \longrightarrow C \longrightarrow 0$$

we obtain the exact sequence

$$F \otimes_R L^i \xrightarrow{\phi \otimes_R 1_{L^i}} F \otimes_R L^i \longrightarrow C \otimes_R L^i \longrightarrow 0$$

showing that $\phi \otimes_R 1_{L^i}$, hence $(\phi_{B_1})^i_*$ is not surjective.

Since A_1 is locally free on $\operatorname{Spec}^\circ(B_1)$ and C, because of Corollary 3.4, satisfies the hypotheses of Proposition 3.5, then the proposition gives that $\operatorname{Ext}^i_{B_1}(A_1, C) = 0$ for $i = 1, \ldots, d-2$. Referring to diagram (5.1), this implies that $(\phi_{B_1})^i_*$ is surjective for $i = 1, \ldots, d-3$, hence that $\operatorname{Ext}^i_{B_1}(A_1, B_1) = 0$ for $i = 1, \ldots, d-3$. From Auslander and Bridger [6] (also see [12, Theorem 3.8]), we get that $A_1^* = \operatorname{Hom}_{B_1}(A_1, B_1)$ satisfies (S_{d-1}). Referring to Theorem 1 and Theorem 3, since $\operatorname{depth}(A_1) + (d-1) - 2 \geq d$ when $d \geq 4$ (that is, $(d-1) + (d-3) = 2d - 4 \geq d$ when $d \geq 4$), then A_1^* is Cohen–Macaulay. Recalling that $A_1^* \simeq_{A_1} A_1$ by Auslander–Goldman (see Theorem 1), our first claim is proved.

Claim 2. A_1 is free over B_1.

We proceed in the manner of [22]. Since A_1 is a finite MCM B_1-module, and A_1 is locally free on $\operatorname{Spec}^\circ(B_1)$, we apply Proposition 3.5 to see that $\operatorname{Ext}^i_{B_1}(A_1, A_1) = 0$ for $i = 1, \ldots, d-2$. Viewing A_1 as an R-module (recall, $B_1 = R/fR$ where R is complete regular local ring) we obtain, from the Auslander–Buchsbaum theorem, that $\operatorname{pd}_R(A_1) = 1$. Let

$$0 \longrightarrow G_1 \longrightarrow G_1 \longrightarrow A_1 \longrightarrow 0$$

be a free R-resolution of A_1 and apply $(\cdot) \otimes_R B_1$ to this short exact sequence. Recalling that $\operatorname{Tor}_1^R(B_1, A_1) \simeq A_1$ (see the proof of Proposition 4.1 where the isomorphism $\operatorname{Tor}_1^R(B, C) \simeq C$ is explained), we arrive at the exact complex

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & G_{B_1} & & \longrightarrow & & G_{B_1} & \longrightarrow & A_1 \longrightarrow 0 \\ & & & & & \searrow & & \nearrow & & & \\ & & & & & & W & & & & \\ & & & & & \nearrow & & \searrow & & & \\ & & & & 0 & & & & 0 & & \end{array}$$

where W is the first syzygy of A_1 and G_{B_1} is a finite free B_1-module. Now

$$\operatorname{Ext}^1_{B_1}(W, A_1) \simeq \operatorname{Ext}^2_{B_1}(A_1, A_1) = 0 \qquad (\text{recall that } d \geq 4),$$

hence

$$0 \longrightarrow A_1 \longrightarrow G_{B_1} \longrightarrow W \longrightarrow 0$$

is split-exact. So A_1 is B_1-projective and hence B_1-free. But then [7, Corollary 3.7] or [35, Theorem 41.5] immediately gives that A_1 is unramified over B_1, and we are done. ▮

When B Is Not a Hypersurface

We proceed to the proof of Theorem 5. By the remarks in Section 2, we reduce the problem via completion and introduce the normal closure. Consequently, it is enough to prove the result in the setting of the theorem with the additional condition that B and A are complete local normal domains. We denote by S the normal closure of $B \hookrightarrow A$, and by G the corresponding group.

Proof of Main Theorem. Assume B satisfies (R_k) where $k \geq 4$. From Proposition 4.1, there is an exact B-complex

$$0 \longrightarrow C \longrightarrow F_B \xrightarrow{\phi_B} F_B \longrightarrow T \longrightarrow 0, \tag{5.2}$$

where C is a countably generated balanced MCM S-module, F_B is a countably generated free B-module, and $T = \operatorname{Ext}^1_R(B, C)$—hence T has an S-structure inherited from C. Since B satisfies (R_k) where $k \geq 4$, Proposition 4.4 gives that the two B-structures on T are isomorphic, the structure used in (5.2) coming from the first variable (that is, the domain structure). Since B is at least (R_4), the classical purity of branch locus theorem implies that A (and S) are locally free over B in codimension 4. Now $\operatorname{depth}_{B_p}(C_p) = \operatorname{codim}(p)$ for all $p \in \operatorname{Spec}(B)$ and from (5.2) and the Depth Lemma we have

$$\operatorname{depth}_{B_p}(T_p) \geq \operatorname{codim}(p) - 2 \qquad \text{for all } p \in \operatorname{Spec}(B) \text{ of codimension } > 4 \tag{5.3}$$

(here T_p is a B_p-module via the domain structure).

Since C is a balanced MCM S-module, Proposition 3.1, hence Proposition 3.3, applies to C. Since the B-structures coincide for T, then (5.3) holds when T is equipped with the codomain B-structure, hence Propositions 3.1 and 3.3 apply to T as well (the isomorphism in these propositions requires the S-structure of T, which comes from the second variable). Consequently, Corollary 3.4 gives that

$$\operatorname{depth}_{B_p}(C_p) = \operatorname{depth}_{B_p}\big(\operatorname{Hom}_{B_p}(A_p, C_p)\big)$$

and

$$\operatorname{depth}_{B_p}(T_p) = \operatorname{depth}_{B_p}\big(\operatorname{Hom}_{B_p}(A_p, T_p)\big)$$

for all $p \in \operatorname{Spec}(B)$.

Choose $q \in \operatorname{Spec}(B)$ so that q is minimal in the B-support of

$$\bigoplus_{i=1}^{k-1} \operatorname{Ext}^i_B(A, C) \oplus \bigoplus_{i=1}^{k-3} \operatorname{Ext}^i_B(A, T).$$

Then, $\operatorname{codim}(q) = h \geq 5$ (recall that A is locally free over B in codimension 4). By choice of q

$$\operatorname{Supp}_{B_q} \operatorname{Ext}^i_{B_q}(A_q, C_q) \subset \{qB_q\} \qquad \text{for } i = 1, \dots, k-1$$

and

$$\operatorname{Supp}_{B_q} \operatorname{Ext}^i_{B_q}(A_q, T_q) \subset \{qB_q\} \qquad \text{for } i = 1, \dots, k-3.$$

As noted above, we have $\operatorname{depth}_{B_q}(C_q) = \operatorname{depth}_{B_q}(\operatorname{Hom}_{B_q}(A_q, C_q))$ and $\operatorname{depth}_{B_q}(T_q) = \operatorname{depth}_{B_q}(\operatorname{Hom}_{B_q}(A_q, T_q))$. Now, $\operatorname{depth}_{B_q}(C_q) = h \geq k + 1$ and $\operatorname{depth}_{B_q}(T_q) \geq h - 2 \geq k - 1$. We apply Proposition 3.5 to $B_q \hookrightarrow A_q$ and the modules C_q, T_q to obtain

$$\operatorname{Ext}^j_{B_q}(A_q, C_q) = 0 \qquad \text{for } j = 1, \dots, (k+1) - 2$$

and

$$\operatorname{Ext}^j_{B_q}(A_q, T_q) = 0 \qquad \text{for } j = 1, \dots, (k-1) - 2.$$

Therefore,

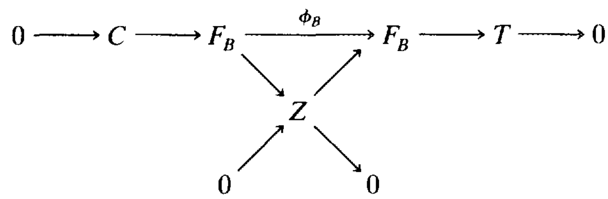
$$0 = \bigoplus_{i=1}^{k-1} \operatorname{Ext}^i_{B_q}(A_q, C_q) \oplus \bigoplus_{i=1}^{k-3} \operatorname{Ext}^i_{B_q}(A_q, T_q)$$
$$= \left(\bigoplus_{i=1}^{k-1} \operatorname{Ext}^i_B(A, C) \oplus \bigoplus_{i=1}^{k-3} \operatorname{Ext}^i_B(A, T) \right)_q$$

contrary to our choice of q. So the module

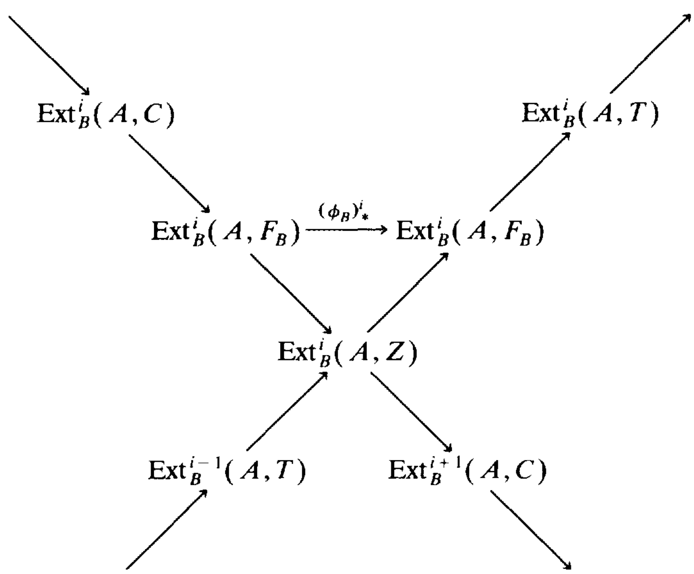
$$\bigoplus_{i=1}^{k-1} \operatorname{Ext}^i_B(A, C) \oplus \bigoplus_{i=1}^{k-3} \operatorname{Ext}^i_B(A, T)$$

has no B-associated primes, hence is the 0-module. That is, $\operatorname{Ext}^i_B(A, C) = 0$ for $i = 1, \dots, k - 1$, and $\operatorname{Ext}^i_B(A, T) = 0$ for $i = 1, \dots, k - 3$.

Recall the exact B-complex (5.2):



Applying $\operatorname{Hom}_B(A, \cdot)$ to the two short exact sequences which comprise (5.2), two long exact sequences are obtained, yielding the following commutative diagram:



(5.4)

Proceeding as in the case of a hypersurface B (refer to the previous subsection), we note that the following diagram commutes, where the vertical maps are the natural ones (also see the discussion in Section 2).

$$\begin{array}{ccc} \operatorname{Ext}^i_B(A, F_B) & \xrightarrow{(\phi_B)^i_*} & \operatorname{Ext}^i_B(A, F_B) \\ \simeq\downarrow & & \simeq\downarrow \\ \coprod \operatorname{Ext}^i_B(A, B) & \xrightarrow{(\phi_B)^i_*} & \coprod \operatorname{Ext}^i_B(A, B) \\ \simeq\downarrow & & \simeq\downarrow \\ F \otimes_B \operatorname{Ext}^i_B(A, B) & \xrightarrow{\phi_B \otimes_B 1} & F \otimes_B \operatorname{Ext}^i_B(A, B) \end{array} \tag{5.5}$$

Again, we assert that if $L^i := \operatorname{Ext}^i_B(A, B) \neq 0$ then $(\phi_B)^i_*$ is not surjective: Assume $L^i \neq 0$. We apply $(\cdot) \otimes_B L^i$ to the exact sequence (from (5.2))

$$F_B \xrightarrow{\phi_B} F_B \to T \to 0$$

and have the exact sequence

$$F \otimes_B L^i \xrightarrow{\phi_B \otimes_B 1} F \otimes_B L^i \to T \otimes_B L^i \to 0. \tag{5.6}$$

Since L^i is not zero and is finite over B, then $L^i \neq m_B L^i$ and we have the surjection $L^i \to k_B \to 0$, hence the surjection $T \otimes_B L^i \to T \otimes_B k_B \to 0$. By Proposition 4.5, $T \otimes_B k_B \neq 0$ so that $T \otimes_B L^i \neq 0$. Exact sequence (5.6) shows that $\phi_B \otimes_B 1$ is not a surjection, hence from (5.5) we conclude that $(\phi_B)^i_*$ is not a surjection.

Referring to diagram (5.4) in conjunction with the information

$$\operatorname{Ext}^i_B(A, C) = 0 \quad \text{for} \quad i = 1, \dots, k-1$$

and

$$\operatorname{Ext}^i_B(A, T) = 0 \quad \text{for} \quad i = 1, \dots, k-3,$$

we have that $(\phi_B)^i_*$ is a surjection for $i = 1, \dots, k-3$. Consequently, $0 = \operatorname{Ext}^i_B(A, B) = \operatorname{Ext}^i_B(A^{**}, B)$ for $i = 1, \dots, (k-1) - 2$ where $(\cdot)^*$ denotes $\operatorname{Hom}_B(\cdot, B)$ (recall that A is reflexive as a B-module). By [6] (also see [12, Theorem 3.8]), this gives that A^* satisfies (S_{k-1}). But again, $A \simeq_A A^*$ by the Auslander–Goldman result (see Theorem 1) so that A satisfies (S_{k-1}) and the theorem is proved. ∎

As a corollary to the above result, we consider the setting $(*)$ with the modification that the local normal excellent domain B is of mixed characteristic and is unramified—that is, $\operatorname{char}(B) = 0$, $\operatorname{char}(k_B) = p > 0$, and $p \in m_B - m_B^2$. Excellence implies that the singular locus of B is closed, say $\operatorname{Sing}(B) = V(I)$, where $I \subset B$ is an ideal. We have the following result:

COROLLARY 5.2. *Assume the setting is as in* $(*)$ *with the modification that B is unramified of mixed characteristic, and that* $[A:B]$ *is a unit in B. Suppose that B is Gorenstein of dimension* $d \geq 6$, *A satisfies* (S_3), $\operatorname{codim}(p, I) > k+1$, *and* $B \hookrightarrow A$ *is unramified in codimension one. Furthermore, suppose that the primitive element assumption holds for B/pB. If B satisfies* (R_k), *then A satisfies* (S_{k-1}) *where* $k \geq 4$.

COROLLARY 5.3. *Let* $B \hookrightarrow A$ *be as above. If B satisfies* (R_k) *with* $k \geq \frac{1}{2}d + 2$, *then A is a Gorenstein ring.*

Proof of Corollary 5.3. This follows from Corollary 5.2 in exactly the same fashion as Corollary 5.1 follows from Theorem 5. ▮

Proof of Corollary 5.2. Consider the extension $\bar{B} = B/pB \hookrightarrow A/pA = \bar{A}$ (the injectivity of $B/pB \hookrightarrow A/pA$ follows since p is regular on B, A, and —since B is a direct summand of A via the reduced trace map—on A/B as well). Then $\bar{B}$ is a Gorenstein local excellent ring—moreover $\bar{B}$ satisfies (R_k) (where $k \geq 4$), hence is a normal domain. To see that $\bar{B}$ satisfies the (R_k) property, let $\bar{Q} \in \operatorname{Spec}(\bar{B})$ such that $\operatorname{codim}(\bar{Q}) \leq k$. Then $\operatorname{codim}(Q) \leq k+1$ where $Q \in \operatorname{Spec}(B)$. So $Q \not\supset (p, I)$ and thus $Q \not\supset I$. As a result, B_Q is a regular local ring, and as $p \in QB_Q - (QB_Q)^2$, then $B_Q/pB_Q = (B/pB)_{\bar{Q}}$ is regular local also, showing that $\bar{B}$ satisfies (R_k). The extension $\bar{B} \hookrightarrow \bar{A}$ remains unramified in codimension one (since the original extension $B \hookrightarrow A$ is unramified in codimension four by the classical purity of branch locus theorem). Moreover, $\bar{B}$ being (R_k) $(k \geq 4)$ and the extension $\bar{B} \hookrightarrow \bar{A}$ being étale in low codimension forces $\bar{A}$ to satisfy (R_1) (in fact, to satisfy (R_k)). Since A is (S_3), then $\bar{A}$ is (S_2), hence is a normal ring. Finally, since $\bar{B}$ is an equicharacteristic domain whose dimension is $d - 1 \geq 5$, we find ourselves back in the setting of Theorem 5, and conclude that $A/pA = \bar{A}$ satisfies (S_{k-1}).

As in the proof of the main theorem, we can reduce to the case where the normal ring $\bar{A}$ is a local normal domain. Applying $\operatorname{Hom}_B(\cdot, B)$ to

$$0 \longrightarrow A \xrightarrow{p} A \longrightarrow \bar{A} \longrightarrow 0$$

we obtain a long exact sequence for Ext. Using standard identifications along with the facts that $\bar{A} \simeq_{\bar{A}} (\bar{A})^*$ (see Theorem 1) and that $\bar{A}$ satisfies (S_{k-1}), we conclude that A satisfies (S_{k-1}) as well (as $\operatorname{Ext}_B^j(A, B) = 0$ for $j = 1, \ldots, k-3$). ▮

Next, we consider an example which illustrates that a weak purity is all that we can hope for in this setting.

EXAMPLE (refer to Fossum [14, Example 16.5]). Let $A = k[X_1, \ldots, X_d]$, a polynomial ring over k, where $d \geq 2$, and assume that k contains ω, a primitive nth root of unity where n is prime to the characteristic of k. Consider the k-automorphism of A given by σ: $X_i \mapsto \omega X_i$, and put $G = \langle \sigma \rangle$, a cyclic group of order n. The invariant subring $A^G = B$ is generated as a k-algebra by the monomials of degree n. Moreover, if we take $d = n$, then we may consider G as a subgroup of $SL(n, k)$ since $\sigma^n = 1$. By Watanabe [41, Theorem 1], B is then a Gorenstein ring.

As discussed in [14], the extension $B \hookrightarrow A$ is unramified in codimension one. In fact, B is an isolated singularity, so that the extension is unramified at all nonmaximal prime ideals of A (or equivalently, at all nonmaximal prime ideals of B). To see this, note that $X_1, X_2, \ldots, X_n$ are all primitive elements for the corresponding extension of fraction fields $K \hookrightarrow L$, with minimum polynomials given by $f_i(T) = T^n - X_i^n$. Since $X_1^n, X_2^n, \ldots, X_n^n$ forms a system of parameters for B, the ideal $(X_1^n, X_2^n, \ldots, X_n^n)B$ is not contained in any prime ideal of B which is not maximal.

Given $p \in \operatorname{Spec}(B)$, p not maximal, we may assume that $X_1^n \notin p$. Then $f_1(T) = T^n - X_1^n$ is a separable polynomial in $B_p[T]$ (see [11, Chap. III, Sect. 4]), in particular, the extension $B_p \to B_p[T]/f_1(T)B_p[T] = B_1$ is étale. Consequently, the normality of B_p implies the normality of B_1. But $B_1 \hookrightarrow A_p$ is a finite extension with the fraction fields of B_1 and A_p being identical. Hence, $B_1 = A_p$ so that $B_p \hookrightarrow A_p$ is étale. Since A_p is regular, then B_p is also regular.

However, the extension $B \hookrightarrow A$ cannot be unramified, for A is regular yet B is not. To consider the complete local case, we refer to [16] and complete at the respective maximal ideals. We remark that in this case we have a weak purity: A is Gorenstein (in fact, regular) when B is Gorenstein satisfying (R_{d-1}), yet full purity is not achieved.

Properly Presented Modules

In the proof of the main results, we relied on the fact that when $\operatorname{Ext}_B^i(A, F_B) \neq 0$ then the induced map

$$\operatorname{Ext}_B^i(A, F_B) \xrightarrow{(\phi_B)_*^i} \operatorname{Ext}_B^i(A, F_B)$$

cannot be surjective—a result which we arrived at using some natural identifications and facts about the nondivisibility of the modules C and T. Another way to arrive at this conclusion is to obtain the exact B-complexes (refer to Proposition 4.1) from a special free R-resolution of C,

$$0 \to F \xrightarrow{\phi} F \to C \to 0,$$

one where the entries of the matrix giving ϕ are in m_R (we refer to such a resolution as a proper presentation of C). Since the $(\phi_B)^i_*$ can be given by the same matrix as ϕ, we see immediately that the $(\phi_B)^i_*$ cannot be surjective in the case of a proper presentation of C. A natural question to ask is whether, in our setting, such a presentation can be produced over R.

QUESTION. Consider the situation as described in the discussion of Section 2, that is,

$$\begin{array}{ccc} & B & \\ & \uparrow & \\ \dfrac{R}{fR} = & R_0[[\theta]] & \twoheadleftarrow R_0[[X]] = R \\ \uparrow & & \uparrow \\ R_0 & \twoheadleftarrow & R_0[[f]] \end{array}$$

and note that C has a nice resolution over $R_0[[f]]$:

$$0 \to \coprod R_0[[f]] \xrightarrow{f} \coprod R_0[[f]] \to C \to 0$$

(recall that $C \simeq \coprod R_0$ as an R_0-module). Using this information (or otherwise), is it possible to obtain a proper presentation

$$G \xrightarrow{\psi} F \to C \to 0$$

over R? (Recall that $\mathrm{pd}_R(C) = 1$ so that $\mathrm{Im}(\psi) \subset F$ is free.)

6. APPLICATIONS

When Cl(B) Has an Element of Finite Order

We begin this section by applying the main results in Section 5 to the situation considered in [21, Sect. 2].

THEOREM 7. *Let (B, m) be a complete local normal Gorenstein domain of dimension $d \geq 5$. Assume that I is a divisorial B-ideal of order n in $\mathrm{Cl}(B)$ where $(n, \mathrm{char}(B/m)) = 1$ and B contains the nth roots of unity. Furthermore, in the case $\mathrm{char}(B) \neq 0$ assume the primitive element assumption holds. If B satisfies (R_k) where $k \geq 4$, then the symbolic powers of I, $I^{(j)}$, satisfy (S_{k-1}).*

In exactly the same way that Corollary 5.1 follows Theorem 5, we obtain

COROLLARY 6.1. *Let (B, m) and I be as in Theorem 7. Assume B satisfies (R_k) where $k \geq \frac{1}{2}d + 2$. Then, the $I^{(j)}$ are Cohen–Macaulay.*

Proof of Corollary 6.1. This follows immediately from Theorem 7 and Theorem 3. ∎

Before embarking on the proof of Theorem 7, we offer some remarks:

(a) For information about divisorial ideals and class groups, we refer the reader to [9, Chap. VII; 14].

(b) The symbolic power of a divisorial ideal I is a generalization of the same notion for a prime. As a divisorial ideal has a primary decomposition $p_1^{(e_1)} \cap p_2^{(e_2)} \cap \cdots \cap p_t^{(e_t)}$ where $p_1, p_2, \ldots, p_t$ are codimension one primes of A which are associated to A/I, $I^{(l)}$ is defined by $I^{(l)} := p_1^{(le_1)} \cap p_2^{(le_2)} \cap \cdots \cap p_t^{(le_t)}$.

The proof of the theorem follows easily once the construction from [21, Theorem 2.4] is described.

Proof of Theorem 7. First, we outline the construction mentioned above. Let $a \in B$ be such that $I^{(n)} = (a)$ and denote by A the integral closure of B in $K[a^{1/n}]$, where K is the fraction field of B. The assumptions that the order of $[I]$ in $\mathrm{Cl}(B)$ is n and that B contains the nth roots of unity combine to ensure that $X^n - a$ is irreducible (over K and B), hence that the extension $K \hookrightarrow K(a^{1/n})$ is cyclic of order n. In [21] it is shown that $B \hookrightarrow A$ is unramified in codimension one and that, as a B-module,

$$A \simeq B \oplus It \oplus I^{(2)}t^2 \oplus \cdots \oplus I^{(n-1)}t^{n-1},$$

where $t^n = 1/a \in K$.

Now, as we are precisely in the situation of Theorem 5, we have that A satisfies (S_{k-1}). But then each $I^{(j)}$ must satisfy (S_{k-1}) as well, and we are done. ∎

We continue in a similar spirit to Section 5, and investigate module finite normal extensions $B \hookrightarrow A$, now in connection with the behavior of certain codimension two "Bourbaki" prime ideals of A under the action of the Galois group G (and the interpretation in terms of splitting). Specifically, the setting $(**)$ is as follows:

> $B \hookrightarrow A$ is a module finite extension of local normal domains, is unramified in codimension one, and is a normal extension with group G; and B is an equicharacteristic excellent Gorenstein ring of dimension d.

A significant ingredient which may be added to $(**)$ above is that A is locally free over B in some fixed low codimension, thus boosting the unramification to that same codimension (refer to the remarks following Proposition 3.1). First, we briefly discuss Bourbaki-exact sequences.

Comments on Bourbaki-Exact Sequences

Assume that M is a finitely generated torsion-free R-module where R is a normal domain. By [9, Chap. 7, Sect. 4.9, Theorem 6], there is a finite free R-module F so that

$$0 \longrightarrow F \longrightarrow M \longrightarrow I \longrightarrow 0$$

is exact and I is an ideal of R. We refer to such a sequence as a Bourbaki-exact sequence. We note that in Herzog and Kühl [24], the definition of Bourbaki-exact sequence requires that M be MCM and that either $\operatorname{codim}(I) \geq 2$ or $I = R$.

In suitable circumstances, it can be arranged that I has codimension 2, that I is actually prime, and that I is a normal prime (that is, R/I is integrally closed in its field of fractions). In particular, we have the following theorem due to Miller [33]:

THEOREM 8 [33]. *Let (R, m) be a factorial, (R_2), (S_3) excellent domain of dimension $d \geq 4$ containing an infinite field k. Let M be a non-free reflexive R-module of rank $d + 1$. Assume that one of the following holds:*

(1) $\operatorname{char}(k) = 0$, *or*

(2) $\operatorname{char}(k) = p > 0$, *$R/m$ is separable over k, and M is free at all primes in R of codimension three.*

Then there is a short exact sequence

$$0 \longrightarrow R^d \longrightarrow M \longrightarrow Q \longrightarrow 0,$$

where Q is a codimension two prime ideal. Furthermore, if R satisfies (R_3) and (S_4), and M is (S_3), then Q is a normal prime.

Also, we have the following (refer to [24, Proposition 1.8]):

THEOREM 9. *Let $P \in \operatorname{Spec}(A)$ be a Cohen–Macaulay prime ideal (that is, A/P is Cohen–Macaulay) of codimension two. Then there is a finite MCM A-module M with*

$$0 \longrightarrow F \longrightarrow M \longrightarrow P \longrightarrow 0$$

a Bourbaki-exact sequence. In the case of a normal extension $B \hookrightarrow A$, if P is normal, then so is $p = P \cap B$.

Remark. The normality of p follows since B/p is a ring of invariants of the normal domain A/P (see [35, Sect. 41]).

Referring to Theorem 8, the UFD property of (R, m) is used to guarantee that Q may be taken of codimension exactly two (codimension ≥ 3 primes are excluded because R is (S_3) and M is not free). We can generalize this statement slightly, replacing factoriality by the condition $[M]_R = 0$ in $\mathrm{Cl}(R)$. Here $[M]_R \in \mathrm{Cl}(R)$ denotes the divisor attached to the R-module M, and $\mathrm{Cl}(R)$ is the divisor class group of the normal domain R. For information about the divisor class group, attached divisors, and related ideas, we again refer the reader to Bourbaki [9, Chap. VII] and Fossum [14]. We will use elementary properties which can be found in the above references to sketch a proof that the "Bourbaki" ideal in Theorem 8 may be taken of codimension two when $[M]_R = 0$.

LEMMA 6.2. *In Theorem 8, assume that (R, m) is normal (but not necessarily factorial) and that $[M]_R = 0$. Then the two conclusions of the theorem hold—in particular, Q may be taken to have codimension two.*

Proof. Assume $[M]_R = 0$. First we show that if $\mathrm{codim}(Q) = 1$, then Q is contained in a principal ideal of R. From

$$0 \longrightarrow F \longrightarrow M \longrightarrow Q \longrightarrow 0$$

we obtain $0 = [M]_R = [F]_R + [Q]_R$. Since F is free, then $[F]_R = 0$ and so $[Q]_R = 0$. Denote by X^1 the set of all codimension one primes in R. Since Q_p is R_p-free for all $p \in X^1$, then $Q_p \simeq Q_p^{**}$ for $p \in X^1$ (here, $(\cdot)^*$ denotes the R-dual) and so $0 = [Q]_R = [Q^{**}]_R$. Now $Q^{**} \hookrightarrow R^{**} \simeq R$, so up to isomorphism, Q^{**} is a reflexive R-ideal with $[Q^{**}]_R = 0$. Hence, Q^{**} is principal and we may assume that $Q^{**} = xR$ with $x \in R - \{0\}$. Therefore, $Q \hookrightarrow Q^{**} \simeq xR$.

Now set $Q_1 = (1/x)Q \subseteq R$ and note that $Q_1 \simeq Q$. We show that $\mathrm{codim}(Q_1) > 1$. Since $((1/x)Q)^* = xQ^*$ we have that

$$(Q_1)^{**} = \left(\frac{1}{x}Q\right)^{**} = (xQ^*)^* = \frac{1}{x}Q^{**} = \frac{1}{x}(xR) = R.$$

Now $(Q_1)^{**} = \bigcap_{p \in X^1} (Q_1)_p$ so that $\bigcap_{p \in X^1} (Q_1)_p = R$. Hence, $Q_1 \not\subseteq p$ for every $p \in X^1$, that is, $\mathrm{codim}(Q_1) > 1$. Since $Q_1 \simeq Q$, we may replace Q by Q_1 and the lemma is proved. ∎

Results on Fixed Primes

In an integral extension of rings $B \hookrightarrow A$, primes (of A) split and ramify. In the particular case of codimension one primes in our usual setting (that is, $B \hookrightarrow A$ a normal extension of local normal domains with the extension of fraction fields separable), there is a formula which relates the splitting and ramification of $P \in \mathrm{Spec}(A)$ to the degree of the extension $K \hookrightarrow L$

(here K and L denote the fraction fields of B and A, respectively). More precisely, if $p = P \cap B$, and $pA = P_1^{(e_1)} \cap P_2^{(e_2)} \cap, \dots, P_g^{(e_g)}$ (the P_i are the primes lying over p, say $P_1 = P$), then $\Sigma_{i=1}^{g} e_i f_i = n$, where $[A : B] = [L : K] = n$ and $f_i = [A/P_i : B/p]$. Moreover, when the extension is normal (with Galois group G), then $e_1 = e_2 = \cdots = e_g$ and $f_1 = f_2 = \cdots = f_g$, and we have the formula $efg = n$ where $g =$ the number of G-conjugates of the prime P. In the case of codimension two primes in A, assuming our setting is sufficiently "good," a similar result will hold (this can be seen by using Bertini's Theorem (see Flenner [13] and also [12, Chap. 0, Sect. H]) to reduce to the codimension one case, or by considering splitting groups, inertia groups, etc. (refer to [35, Sect. 41]).

Our aim is to prove the following theorem, and along the way we are able to (under certain hypotheses) identify $\mathrm{Hom}_B(A, p)$ as $\sqrt{pA}$.

THEOREM 10. *Let $B \hookrightarrow A$ be as in $(**)$ with $d \geq 4$. Assume that B satisfies (R_3), A satisfies (S_4), and M is a finitely generated MCM A-module with $[M]_B = 0$. Then M^g (where $g = |G|$, $G = \mathrm{Gal}(A/B)$) has a Bourbaki codimension two prime ideal P which is normal and fixed under the action of G.*

We begin with a proposition and a lemma.

PROPOSITION 6.3. *Let $B \hookrightarrow A$ be as in $(**)$ with $d \geq 4$ and A locally free over B in codimension three. Suppose M is a finite A-module and that both M and A satisfy (S_4). Let $p \in \mathrm{Spec}(B)$ be normal of codimension two such that p fits into a Bourbaki-exact sequence (over B)*

$$0 \longrightarrow F \longrightarrow M \longrightarrow p \longrightarrow 0.$$

Then, for $P \in \mathrm{Spec}(A)$ such that $P \cap B = p$ we have that P is normal and fixed under the action of G.

Remark. In particular, it will be shown that $\sqrt{pA} = P$.

LEMMA 6.4. *Let $B \hookrightarrow A$ be as in $(**)$ and assume A is locally free over B in codimension i. Then, for $p \in \mathrm{Spec}(B)$ with $\mathrm{codim}(p) \leq i$, $\mathrm{Hom}_B(A, p) \simeq_A \sqrt{pA}$.*

Proof of Lemma 6.4. The assumption that $B \hookrightarrow A$ is unramified in codimension one implies that the extension is unramified in codimension zero, that is, the extension of fraction fields is separable, hence that the Auslander–Goldman result can be applied. Let $p \in \mathrm{Spec}(B)$ be such that $\mathrm{codim}(p) \leq i$. Applying $\mathrm{Hom}_B(A, \cdot\,)$ to the exact sequence

$$0 \longrightarrow p \longrightarrow B \longrightarrow B/p \longrightarrow 0$$

we obtain the A-exact sequence

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}_B(A,p) &\longrightarrow \operatorname{Hom}_B(A,B)\\ &\xrightarrow{\pi_*} \operatorname{Hom}_B(A,B/p) \xrightarrow{\delta} \operatorname{Ext}^1_B(A,p).\end{aligned} \tag{6.1}$$

From Auslander–Goldman (see Theorem 1), $\operatorname{Hom}_B(A,B) \simeq A \cdot \operatorname{tr}_{A/B} \simeq A$. We put $I = \operatorname{Hom}_B(A,p)$, and since $I \subset \operatorname{Hom}_B(A,B) \simeq_A A$, we identify I with an ideal of A. We assert that $pA \subset I \subset \sqrt{pA}$: Since $\operatorname{tr}_{A/B}$ is B-linear, then $\operatorname{tr}_{A/B}(pA) \subseteq pB$ so that $\pi_*(pA) = \pi \circ \operatorname{tr}_{A/B}(pA) = 0$. The exactness of

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\pi_*} \operatorname{Hom}_B(A,B/p)$$

shows that $pA \subseteq \ker \pi_* = I$.

Localizing (6.1) at $B - p$, we note that $\operatorname{Ext}^1_{B_p}(A_p, B_p/pB_p) = 0$, and $\operatorname{Hom}_{B_p}(A_p, B_p/pB_p) \simeq A_p/pA_p$ since A_p is B_p-free. So (6.1) yields

$$0 \longrightarrow I_p \longrightarrow A_p \longrightarrow A_p/pA_p \longrightarrow 0$$

after the appropriate identifications, and we may assume that $I_p = pA_p$. For $Q \in \operatorname{Spec}(A)$ such that $Q \supset pA$ we have that $(pA_p)_Q = pA_Q \neq A_Q$, so for each such Q, $I_Q \neq A_Q$, that is $I \subset Q$. In particular, $I \subset \sqrt{pA}$.

Claim. $\operatorname{depth}((\sqrt{pA})_Q)$ and $\operatorname{depth}(I_Q)$ are ≥ 2 for any $Q \in \operatorname{Spec}(A/\sqrt{pA})$ which is not minimal.

Since $A/\sqrt{pA}$ is reduced, it satisfies (S_1) as an $A/\sqrt{pA}$-module; since A is normal, it satisfies (S_2). From the exactness of

$$0 \longrightarrow \sqrt{pA} \longrightarrow A \longrightarrow A/\sqrt{pA} \longrightarrow 0$$

and the Depth Lemma (refer to Section 2), we conclude that $\operatorname{depth}((\sqrt{pA})_Q) \geq 2$ for any non-minimal $Q \in \operatorname{Spec}(A/\sqrt{pA})$.

From

$$0 \longrightarrow I \longrightarrow A \longrightarrow \operatorname{Hom}_B(A,B/p)$$

we see that $A/I \hookrightarrow \operatorname{Hom}_B(A,B/p)$. Since B/p is a domain, then $\operatorname{Hom}_B(A,B/p)$ and hence A/I are B/p-torsion free modules. Since $B \hookrightarrow A$ is unramified in codimension zero, the going down theorem holds between A and B. In particular, given $Q \in \operatorname{Spec}(A/\sqrt{pA})$ which is not minimal, then Q does not lie over p, that is, $Q \cap B \supset p$, but $Q \cap B \neq p$.

For any such Q, let $x \in Q \cap B - p \subseteq B$. Then $\bar{x} \in B/p \hookrightarrow A/I$ is A/I-regular and $(A/I)_Q$-regular as well. Consequently, for any such Q, $(A/I)_Q$ has positive depth. From the exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

and the Depth Lemma, we obtain $\operatorname{depth}(I_Q) \geq 2$ for any $Q \in \operatorname{Spec}(A/\sqrt{pA})$ which is not minimal.

Finally consider the exact sequence of A-modules

$$0 \longrightarrow I \longrightarrow \sqrt{pA} \longrightarrow W \longrightarrow 0$$

and let $Q \in \operatorname{Ass}_A(W)$. As $pA \subset I \subset \sqrt{pA}$ and by our assumptions, pA and $\sqrt{pA}$ agree when localized at any prime in A of codimension $\leq i$, then $\operatorname{codim}(Q) > i$ (≥ 2). We note that $Q \supset \sqrt{pA}$, for otherwise $I_Q = A_Q = (\sqrt{pA})_Q$. Hence, $Q \in \operatorname{Spec}(A/\sqrt{pA})$ and is not minimal. Applying the Depth Lemma to

$$0 \longrightarrow I_Q \longrightarrow \left(\sqrt{pA}\right)_Q \longrightarrow W_Q \longrightarrow 0$$

leads to a contradiction, as both $\operatorname{depth}(I_Q)$ and $\operatorname{depth}((\sqrt{pA})_Q)$ are ≥ 2. Therefore, $\operatorname{Ass}_A(W) = \varnothing$ so that $W = 0$ and $I \simeq \sqrt{pA}$. ∎

Proof of Proposition 6.3. Let $P \in \operatorname{Spec}(A)$ such that $P \cap B = p$. Consider

$$B/p \longrightarrow A/pA \xrightarrow{\pi} A/\sqrt{pA}$$

where π is the natural map. We claim that $A/\sqrt{pA}$ satisfies (R_1) and (S_2).

Since p is a normal codimension two prime ideal, then B/p is a normal domain, hence satisfies (R_1) and (S_2). The unramification of $B \hookrightarrow A$ in codimension three implies that $B/p \hookrightarrow A/pA$ is unramified in codimension one, because unramification is preserved under base change (refer to [7] or [32]). Let $\bar{Q} \in \operatorname{Spec}(A/pA)$ be of codimension one. Then $Q \cap B = q$ is a codimension three prime in B, so that $\bar{q} \in \operatorname{Spec}(B/p)$ is of codimension one. Then, $(B/p)_{\bar{q}} \to (A/pA)_{\bar{Q}}$ is an étale extension and, since $(B/p)_{\bar{q}}$ is regular local, then $(A/pA)_{\bar{Q}}$ is as well. That is, A/pA satisfies (R_1). By the hypotheses, pA and $\sqrt{pA}$ become equal upon localization at codimension three (or smaller) primes, so that A/pA and $A/\sqrt{pA}$ are equal when localized at primes (in $\operatorname{Spec}(A/pA) = \operatorname{Spec}(A/\sqrt{pA})$) of codimension ≤ 1. Therefore, $A/\sqrt{pA}$ satisfies (R_1).

We go on to demonstrate that $A/\sqrt{pA}$ satisfies (S_2). To the Bourbaki-exact sequence

$$0 \longrightarrow F \longrightarrow M \longrightarrow p \longrightarrow 0$$

apply $\mathrm{Hom}_B(A, \cdot\,)$ and produce the exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_B(A, F) \longrightarrow \mathrm{Hom}_B(A, M) \\ &\longrightarrow \mathrm{Hom}_B(A, p) \xrightarrow{\delta} \mathrm{Im}(\delta) \longrightarrow 0, \end{aligned}$$

where $\mathrm{Im}(\delta) \subset \mathrm{Ext}^1_B(A, F)$. Note that we have the following isomorphisms:

(a) $\mathrm{Hom}_B(A, F) = \mathrm{Hom}_B(A, \coprod B) \simeq \coprod \mathrm{Hom}_B(A, B) \simeq \coprod A$ since $\mathrm{Hom}_B(A, B) \simeq_A A$ by Auslander–Goldman.

(b) $\mathrm{Hom}_B(A, M) \simeq_A M^g$ where $g = [A : B]$—refer to Proposition 3.1.

(c) $\mathrm{Hom}_B(A, p) \simeq_A \sqrt{pA}$ from Lemma 6.4.

Claim. If A satisfies (S_3) then $\mathrm{Im}(\delta) = 0$.

Let $q \in \mathrm{Spec}(B)$ with $\mathrm{codim}(q) \leq 3$. Then

$$\mathrm{Ext}^1_B(A, F)_q \simeq \mathrm{Ext}^1_{B_q}(A_q, F_q) = 0$$

since A_q is a finite MCM B_q-module and $F_q = \coprod B_q$ is of finite injective dimension. Therefore, $\mathrm{Im}(\delta)$ is supported at primes in B of codimension ≥ 4.

Let $q' \in \mathrm{Ass}_B(\mathrm{Im}(\delta))$. Then $\mathrm{codim}(q') \geq 4$ and applying the Depth Lemma to

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_B(A, F)_{q'} \longrightarrow \mathrm{Hom}_B(A, M)_{q'} \\ &\longrightarrow \mathrm{Hom}_B(A, p)_{q'} \xrightarrow{\delta} \mathrm{Im}(\delta)_{q'} \longrightarrow 0 \end{aligned}$$

and using the identification mentioned above, we conclude that $\mathrm{depth}(\mathrm{Hom}_B(A, F)_{q'}) = \mathrm{depth}(A^r_{q'}) = 2$ (here r denotes $\mathrm{rank}(F)$), which is a contradiction. Therefore, $\mathrm{Im}(\delta) = 0$ and the following is A-exact:

$$0 \longrightarrow A^r \longrightarrow M^g \longrightarrow \sqrt{pA} \longrightarrow 0. \tag{6.2}$$

Let $Q \in \mathrm{Spec}(A)$ such that $\mathrm{codim}(Q) \geq 4$ and such that $Q \supset \sqrt{pA}$. Localizing (6.2) at Q we have

$$0 \longrightarrow A^r_Q \longrightarrow M^g_Q \longrightarrow \sqrt{pA}_Q \longrightarrow 0.$$

Since $\operatorname{depth}(A'_Q) \geq 4$ and $\operatorname{depth}(M^g_Q) \geq 4$, the Depth Lemma guarantees that $\operatorname{depth}((\sqrt{pA})_Q) \geq 3$.

Finally consider the exact sequence

$$0 \longrightarrow \sqrt{pA} \longrightarrow A \longrightarrow A/\sqrt{pA} \longrightarrow 0.$$

Since $A/\sqrt{pA}$ is reduced, it satisfies (S_1) as a ring. Let $Q \supset \sqrt{pA}$ be a prime in A of codimension ≥ 4. Since $\operatorname{depth}((\sqrt{pA})_Q) \geq 3$ and A satisfies (S_4), it must be that $\operatorname{depth}((A/\sqrt{pA})_Q) \geq 2$. Therefore, $A/\sqrt{pA}$ satisfies (S_2) as a ring.

Since $A/\sqrt{pA}$ satisfies (R_1) and (S_2), it is a normal ring, so that

$$A/\sqrt{pA} \simeq A/Q_1 \times A/Q_2 \times \cdots \times A/Q_m,$$

where the A/Q_i are normal domains (here the Q_i are the minimal primes in $A/\sqrt{pA}$, say $Q_1 = P$). But A, hence $A/\sqrt{pA}$ is local so that $m = 1$ and hence $A/\sqrt{pA} = A/P$. Therefore $\sqrt{pA} = P$ and P is a normal prime in A. Since P lies over p and is the unique minimal prime lying over p, it is the only prime that contracts to p (for $B \hookrightarrow A$ is a finite extension). As any G-conjugate of P also lies over p, it must be that P is fixed under the action of the Galois group G. ∎

We return to Theorem 10:

Proof of Theorem 10. Viewing M as a B-module, we have that M is a finite MCM B-module, and by Theorem 8 (with Lemma 6.2), there is a Bourbaki-exact sequence

$$0 \longrightarrow F \longrightarrow M \longrightarrow p \longrightarrow 0, \tag{6.3}$$

where F is a finite free B-module, and p is a normal codimension two prime in B. We apply $\operatorname{Hom}_B(A, \cdot)$ to (6.3) to obtain

$$0 \longrightarrow \operatorname{Hom}_B(A, F) \longrightarrow \operatorname{Hom}_B(A, M) \longrightarrow \operatorname{Hom}_B(A, p) \longrightarrow 0. \tag{6.4}$$

The exactness at the right-hand end follows as in Proposition 6.3. Using the identifications as in the proposition, (6.4) becomes

$$0 \longrightarrow F_A \longrightarrow M^g \longrightarrow \sqrt{pA} \longrightarrow 0,$$

where F_A is a finitely generated free A-module. Moreover, the proposition gives that $\sqrt{pA} = P$, a normal prime ideal in A which is fixed under the action of G. We add that $\operatorname{codim}(P) = 2$ since $\operatorname{codim}(p) = 2$. ∎

The following corollary involves a result from linkage theory. For appropriate definitions and clarification of this topic, we refer the reader to Peskine and Szpiro [37] and Huneke and Ulrich [29].

COROLLARY 6.5. *Let $B \hookrightarrow A$ be as in Theorem 10, and M a finite MCM A-module such that $[M]_B = 0$. Let $Q \in \operatorname{Spec}(A)$ be a codimension two Bourbaki prime ideal for M^g. Then Q is evenly linked to a normal codimension two prime P which is fixed under the action of G.*

Proof. Let

$$0 \longrightarrow F' \longrightarrow M^g \longrightarrow Q \longrightarrow 0 \tag{6.5}$$

be a Bourbaki-exact sequence of A-modules for Q. As in the proof of the above theorem, we view M as a B-module, produce the Bourbaki-exact sequence (over B)

$$0 \longrightarrow F \longrightarrow M \longrightarrow p \longrightarrow 0,$$

where p is a normal codimension two prime ideal, and apply $\operatorname{Hom}_B(A, \cdot)$ to obtain (after appropriate identifications) the exact sequence

$$0 \longrightarrow F_A \longrightarrow M^g \longrightarrow P \longrightarrow 0. \tag{6.6}$$

Here, F_A is a finite free A-module and P a codimension two normal prime which is fixed under the action of G. Considering (6.5) and (6.6) in light of the result [24, Theorem 2.1] immediately gives that Q and P are evenly linked. ∎

In general, it is difficult to determine when a codimension two prime ideal of A is fixed under the action of G (that is, when the prime is nonsplit). Primes in $\operatorname{Spec}(A)$ which are generated by two-sequences x_1, x_2 (where x_1, x_2 are chosen in m_B to preserve the (R_1) and (S_2) properties of $\bar{A} = A/(x_1, x_2)A$ as permitted by Bertini's Theorem (see [13; 12, Chap. 0, Sect. H]) give a trivial sort of example of this phenomenon, because $(x_1, x_2)A \cap B = (x_1, x_2)B$. The codimension two primes which are generated from the above results are not of this type, since the finite MCM modules which are considered are not free.

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