Europ. J. Combinatorics (2001) **22**, 297–301
doi:10.1006/eujc.2000.0467
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Fissioned Triangular Schemes via the Cross-ratio

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A construction of association schemes is presented; these are fission schemes of the triangular schemes $T(n)$ where $n = q + 1$ with q any prime power. The key observation is quite elementary, being that the natural action of $PGL(2, q)$ on the 2-element subsets of the projective line $PG(1, q)$ is generously transitive. In addition, some observations on the intersection parameters and fusion schemes of these association schemes are made.

1. THE CONSTRUCTION

This paper is a sequel to [4]. In that paper, it was observed that almost all known self-dual classical association schemes have natural fission schemes (fissioning the maximum-distance relation only); whereas in the non-self-dual case there seemed to be no analogous fission schemes. Subsequently, we found that there is at least one such non-self-dual classical association scheme that admits an interesting fission scheme, namely the triangular scheme $T(n) = J(n, 2)$ where $n = q + 1$ with q any prime power; this is the object of the present work. For terminology and background, we refer to Bannai and Ito [2] for association schemes and to Hirschfeld [9] for finite geometry. Recall that the group $PGL(2, q)$ acts (as Möbius transformations) on the projective line $PG(1, q)$; this action is (sharply) 3-transitive. There is a natural induced action on the 2-element subsets of the projective line, namely $M(\{x, y\}) := \{M(x), M(y)\}$ for each M in $PGL(2, q)$. In the proof below we apply the basic fact (cf. [9, p. 135]) that the cross-ratio

$$\rho(a, b, c, d) := \frac{(a - c)(b - d)}{(a - d)(b - c)}$$

is a complete invariant for ordered quadruples of distinct points on the projective line, i.e., one quadruple may be mapped to another quadruple (via a Möbius transformation) if and only if they have the same cross-ratio.

THEOREM. *The action of $PGL(2, q)$ on the 2-element subsets of $PG(1, q)$ is generously transitive.*

PROOF. Given intersecting 2-sets $\{a, b\}$ and $\{a, c\}$, there is some M in $PGL(2, q)$ that swaps them, since the group is triply transitive. And given disjoint 2-sets $\{a, b\}$ and $\{c, d\}$, there is also some Möbius transformation that interchanges them, because the ordered quadruples (a, b, c, d) and (c, d, a, b) have the same cross-ratio. □

Given any transitive permutation group G acting on a set Ω, the orbitals are the orbits in $\Omega \times \Omega$ under the natural action of G on pairs. If G is generously transitive, then the orbitals form the relations (associate classes) of a symmetric association scheme (cf. [2, p. 54]). In our case, the relations can be described as follows. One relation, say R_1, is the line-graph of the complete graph (i.e., one relation of the triangular scheme $T(q + 1)$ has remained unfissioned). Next, for each reciprocal pair $\{s, s^{-1}\}$ of elements in $GF(q)\backslash\{0, 1\}$, there is a relation $R_{\{s,s^{-1}\}}$ where $\{a, b\}$ and $\{c, d\}$ are in this relation when $\rho(a, b, c, d)$ equals s or s^{-1}. Note that $\rho(b, a, c, d) = \rho(a, b, c, d)^{-1}$ so this makes sense as a definition for unordered pairs $\{a, b\}$. Henceforth we will write R_s instead of $R_{\{s,s^{-1}\}}$ for typographical reasons; note that since the field element 1 cannot occur as a cross-ratio, this notation will not conflict with that of relation R_1 above.

We now easily find that this fissioned triangular scheme, which we shall denote by $FT(q+1)$, has $\frac{1}{2}(q+1)$ associate classes if q is odd and $\frac{1}{2}q$ classes if q is even. When q is odd the field element -1 is equal to its own reciprocal; thus the relation R_{-1} has valency $\frac{1}{2}(q-1)$ which is half the valency of the other relations R_s with s in $GF(q)\backslash\{0, 1, -1\}$. The relation R_1 has valency $2(q-1)$.

We remark that for small odd q the relation R_{-1} is a familiar object: for $q = 5$ it is the line-graph of Petersen's graph; for $q = 7$ it is the Coxeter graph (this was apparently known to Coxeter himself, cf. [6, p. 122]); for $q = 9$ it is the line-graph of Tutte's 8-cage. There seem to be some other such 'sporadic isomorphisms': for example when $q = 11$ the relation $R_2 = R_{\{2,6\}}$ is the line-graph of the point-block incidence graph of the (unique) symmetric $(11, 6, 3)$-design; and when $q = 9$ and $\{s, s^{-1}\}$ is the pair of primitive fourth roots of unity, then R_s is the second subconstituent of the Gewirtz graph (cf. [5, p. 106]).

2. Intersection Parameters

It is possible to give explicit formulas for the intersection parameters p_{ij}^k of the association scheme $FT(q+1)$; we now sketch the main points of the derivation. The cases q odd and q even are similar, with the latter case being slightly cleaner since the exceptional case '$\rho = -1$' does not occur. Hence we will only present the case q even; besides, this case is the more pertinent one in the discussion of fusion schemes in Section 3.

Thus let $q = 2^e$ be any power of two. The scheme $FT(2^e+1)$ has 2^{e-1} classes. The relation R_1 has valency $2(q-1)$ and each of the other relations $R_s = R_{\{s,s^{-1}\}}$ (for s in $GF(q)\backslash\{0, 1\}$) has valency $q-1$. The intersection parameters involving R_1 are easy to work out and we list them without proof: for distinct r and s (and $s \neq r^{-1}$) in $GF(q)\backslash\{0, 1\}$, $p_{11}^1 = q-1$, $p_{11}^r = 4$, $p_{1r}^1 = 2$, $p_{rr}^1 = 1$, and $p_{rs}^1 = 2$.

Now let the symbols r, s and t represent three (not necessarily distinct) elements of $GF(q)\backslash\{0, 1\}$; we aim at a formula for p_{st}^r. What one has to do is fix a pair of 2-sets $\{a, b\}$ and $\{c, d\}$ in relation R_r, and count the number of 2-sets $\{x, y\}$ such that $\{a, b\}$ and $\{x, y\}$ are in relation R_s and $\{c, d\}$ and $\{x, y\}$ are in relation R_t. The triple transitivity of $PGL(2, q)$ is useful here, since it implies that we may take, without loss of generality, $\{a, b\} = \{\infty, 0\}$ and $\{c, d\} = \{1, r\}$. For the unknown pair $\{x, y\}$ we then obtain the two equations

$$s \text{ or } s^{-1} = \frac{(\infty - x)(0 - y)}{(\infty - y)(0 - x)} = \frac{y}{x} \tag{1}$$

and

$$t \text{ or } t^{-1} = \frac{(1 - x)(r - y)}{(1 - y)(r - x)}. \tag{2}$$

Equations (1) and (2) together involve two essentially different cases, not four, since $\{y, x\} = \{x, y\}$; thus we may fix the left-hand side of (1) as being s, and examine the two cases for (2) in turn. In the first case we have $y = sx$ and

$$t = \frac{(1 - x)(r - y)}{(1 - y)(r - x)} = \frac{(1 - x)(r - sx)}{(1 - sx)(r - x)}.$$

This leads to the following quadratic for x (after changing all minus signs to plus signs, as we may since we are in characteristic two):

$$s(t + 1)x^2 + (rst + r + s + t)x + r(t + 1) = 0. \tag{3}$$

The other case (when the left-hand side of (2) is t^{-1}) leads to the similar quadratic

$$s(t+1)x^2 + (rs + rt + st + 1)x + r(t+1) = 0. \tag{4}$$

Note that since r, s and t are all in $GF(q)\backslash\{0, 1\}$, Eqns (3) and (4) are genuine quadratics, with non-zero quadratic and constant terms. The linear coefficient $(rst+r+s+t)$ in (3) could equal 0, in which case the unique solution for x is the square root of $\frac{r}{s}$. If $rst+r+s+t \neq 0$, then (3) has (two) solutions x if and only if

$$\mathrm{Tr}\left[\frac{rs(t+1)^2}{(rst+r+s+t)^2}\right] = 0 \tag{5}$$

where $\mathrm{Tr}(z)$ is the trace map from $GF(2^e)$ onto $GF(2)$. Similarly, if $rs + rt + st + 1 \neq 0$, then (4) has (two) solutions x if and only if

$$\mathrm{Tr}\left[\frac{rs(t+1)^2}{(rs+rt+st+1)^2}\right] = 0. \tag{6}$$

Thus p^r_{st} has a value of anywhere from 0 to 4. A reasonably concise formula is the following: let $A = A(r, s, t)$ be the expression for the argument of the trace map in (5), and $B = B(r, s, t)$ be the one for (6). Then, when $rst + r + s + t \neq 0$ and $rs + rt + st + 1 \neq 0$

$$p^r_{st} = 2 + (-1)^{\mathrm{Tr}[A]} + (-1)^{\mathrm{Tr}[B]} \tag{7}$$

with the obvious modifications being made in the other cases. Incidentally, it is easy to check that $(rst + r + s + t)$ and $(rs + rt + st + 1)$ cannot simultaneously equal 0.

We make one more remark concerning the form of the intersection parameters. The expressions $A(r, s, t)$ and $B(r, s, t)$ are not symmetric in s and t, hence formula (7) for p^r_{st} appears not to be symmetric either. This may seem strange, since we know from general principles that $p^r_{st} = p^r_{ts}$. An explanation for this is the following. $A(r, s, t)$ has the same trace as $C(r, s, t) := \frac{rs+rt+st}{(rst+r+s+t)^2}$ since their sum is of the form $\frac{xy}{x^2+y^2}$ and such field elements, in characteristic two, must have trace 0 (exercise for the reader).

Similarly, $B(r, s, t)$ has the same trace as $D(r, s, t) := \frac{rst(r+s+t)}{(rs+rt+st+1)^2}$. Thus we may replace A by C and B by D in (7) without changing the value of the right-hand side; and C and D are both symmetric functions of the three variables r, s and t. This confirms the fact that, since the valencies n_r are the same for all r in $GF(q)\backslash\{0, 1\}$, the intersection parameter p^r_{st} is symmetric in all three variables.

It would be interesting to find explicit formulas for the entries of the eigenmatrix (character table) of $FT(q+1)$. One strategy for doing this (used by Bannai and his co-workers in several papers; see [1] for a survey) is the following. First calculate all of the intersection parameters; it is usually feasible to do this, at least in some reasonable algebraic form perhaps involving character sums. This tells us what the intersection matrices $B_i(k, j) := p^k_{ij}$ are. Secondly, from these B_i's (at small values of q) it may be possible to guess what the eigenmatrix P should be. Once the right guess has been made it is usually straightforward to actually prove the result, using Theorem II.4.1 in [2]. Unfortunately, we have been unable so far to guess the general shape of P for our schemes $FT(q + 1)$; we generated these character tables using a computer for all prime powers q less than 40, and they seem to have a very complicated form.

3. Fusion Schemes

Given any association scheme, it is of interest to determine all of its fusion schemes (also called subschemes). This is in general a very hard problem that has not been worked out completely even for quite classical examples such as the Johnson schemes (cf. [10]). In the case of the schemes $FT(q+1)$, there is of course the original two-class triangular scheme $T(q+1)$. Observe also that if $q = p^e$ is a proper power of p, then the Frobenius map $x \mapsto x^p$ (and its iterates) gives a fusion scheme. In other words, there is an overgroup ($P\Gamma L(2,q)$ in case p is prime) of $PGL(2,q)$, and the orbitals under this overgroup constitute a fusion scheme of $FT(q+1)$.

Limited computational evidence suggests that $FT(q+1)$ has no other non-trivial fusions, except maybe in some sporadic cases, and when $q = 4^f$ (f any integer at least 2) where there seems to be an interesting 4-class fusion scheme. We say 'seems' because we are lacking a proof that this is indeed an association scheme. To describe this (putative) scheme, let the ground-set be all 2-element subsets of the projective line $PG(1,4^f)$; the four possible relations for two distinct 2-sets $\{a,b\}$ and $\{c,d\}$ are:

S_1: $\{a,b\} \cap \{c,d\} \neq \emptyset$, i.e., R_1 in the earlier notation.
S_2: $\{a,b\} \cap \{c,d\} = \emptyset$ and the cross-ratio $\rho = \rho(a,b,c,d)$ satisfies $\rho^{2^f-1} = 1$, i.e., ρ lies in the subfield $GF(2^f)$.
S_3: $\{a,b\} \cap \{c,d\} = \emptyset$ and the cross-ratio $\rho = \rho(a,b,c,d)$ satisfies $\rho^{2^f+1} = 1$.
S_4: The remainder.

We have been able to show by computer that these four relations do indeed form a scheme when f is less than or equal to 6. In addition, we can prove in general that some of the intersection parameters, such as p^3_{23}, are well defined; but certain other parameters such as p^3_{33} have left us baffled. An explicit knowledge of the eigenmatrix of $FT(4^f+1)$ would theoretically settle this question (cf. [10, Lemma 1]), which is partly why we raised the issue of computing it earlier.

CONJECTURE. *The above relations S_i on the 2-subsets of $PG(1,4^f)$ do form a 4-class association scheme for all $f \geq 2$. The corresponding eigenmatrix is given by*

$$P = \begin{bmatrix} 1 & 2(4^f-1) & (2^{f-1}-1)(4^f-1) & 2^{f-1}(4^f-1) & 2^f(2^{f-1}-1)(4^f-1) \\ 1 & 4^f-3 & 2-2^f & -2^f & -2^f(2^f-2) \\ 1 & -2 & 1-2^f & 0 & 2^f \\ 1 & -2 & (2^{f-1}-1)(2^f-1) & 2^{f-1}(2^f-1) & -2^f(2^f-2) \\ 1 & -2 & 2^{f-1}(2^f-1)+1 & -2^{f-1}(2^f+1) & 2^f \end{bmatrix}.$$

We note finally that, granting this conjecture, one can merge S_2 and S_3 to obtain a 3-class scheme, and then further merge S_1 with S_2 and S_3 to obtain a 2-class scheme. The resulting graph $G = S_1 \cup S_2 \cup S_3$ is strongly regular with parameters $v = 2^{2f-1}(2^{2f}+1)$, $k = (2^f+1)(2^{2f}-1)$, $\lambda = (2^f-1)(3\cdot 2^f+2)$, $\mu = 2^{f+1}(2^f+1)$. Graphs with these parameters have already been constructed by Brouwer and Wilbrink (cf. [3, 7B]); it was checked that in the smallest case $f = 2$ ($v = 136$) the two constructions yield isomorphic strongly regular graphs. We know nothing for larger values; but the two constructions look totally different, so that it is a reasonable guess that they are not isomorphic in general.

ADDED IN PROOF. The above conjecture is proven by Tanaka [11] and independently by Ebert, Egner, Hollmann, and Xiang [7, 8]. Tanaka gives a group theoretic proof using characters, while Ebert *et al*. give a geometric proof using inversive planes in [7], and a direct proof from the intersection parameters in [8]. In [7] it is also proved that the strongly regular graph $G = S_1 \cup S_2 \cup S_3$ is isomorphic to the Brouwer-Wilbrink strongly regular graph.

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Received 6 October 1999 and accepted 3 August 2000

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