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On certain problem in the class of k-starlike functions

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ABSTRACT

In this paper we consider the classes of k-uniformly convex and k-starlike functions defined in Kanas and Wiśniowska (1999, 2000) [1,2] which generalize the class of uniformly convex functions introduced by Goodman (1991) [3]. We discuss the real part of $f(z)/z$, when f is k-starlike. We find the minimum of $\mathfrak{Re} f(z)/z$ improving the results obtained recently in Wiśniowska-Wajnryb (2009) [11].

1. Introduction

Let $\mathcal{U} = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the class of all functions f that are analytic in $\mathcal{U}$ and normalized by $f(0) = 0, f'(0) = 1$. By $\mathcal{S}$ we denote the class of functions $f \in \mathcal{A}$ that are univalent in $\mathcal{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order α, $0 \le \alpha < 1$, if and only if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in \mathcal{U}. \tag{1.1}$$

We denote by $\mathcal{ST}(\alpha)$ the subset of $\mathcal{A}$ consisting of all functions which satisfy (1.1). For $\alpha = 0$ we get the class $\mathcal{ST}$ of functions f that maps $\mathcal{U}$ onto a starlike domain with respect to the origin.

A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. A function $f \in \mathcal{S}$ maps $\mathcal{U}$ onto a convex domain E if and only if

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathcal{U}. \tag{1.2}$$

Such a function f is said to be convex in $\mathcal{U}$ (or briefly convex) and we denote by $\mathcal{CV}$ the set of all functions which satisfy (1.2). Let us recall the classes of k-uniformly convex and of k-starlike functions: for a fixed $k \ge 0$

$$k\text{-}\mathcal{UCV} := \left\{f \in \mathcal{S} : \mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in \mathcal{U}\right\},$$

$$k\text{-}\mathcal{ST} := \left\{f \in \mathcal{S} : \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in \mathcal{U}\right\}.$$

These classes were introduced by Kanas and Wiśniowska in [1,2], respectively, where their geometric definitions and connections with the conic domains were considered. For a fixed $k \ge 0$, the class k-$\mathcal{UCV}$ is defined purely geometrically

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as a subclass of univalent functions which map the intersection of $\mathcal{U}$ with any disk centered at ζ, $|\zeta| \le k$, onto a convex domain. The notion of k-uniform convexity is a natural extension of the classical convexity. Observe that, if $k = 0$ then the center ζ is the origin and the class k-$\mathcal{UCV}$ reduces to the class $\mathcal{CV}$. Moreover for $k = 1$ it coincides with the class of uniformly convex functions $\mathcal{UCV}$ introduced by Goodman [3] and studied extensively by Rønning [4] and independently by Ma and Minda [5]. The class k-$\mathcal{ST}$ is related to the class k-$\mathcal{UCV}$ via the well-known Alexander equivalence between the usual classes of convex and starlike functions (see also the works [2,4–8] concerning the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$). The class k-$\mathcal{ST}$ has the following geometric characterization (see [9]): If $f \in k$-$\mathcal{ST}$ than it maps a lens-like domain $U(\zeta, r) \cap U(0, R)$ onto a starlike domain, where $U(\zeta, r)$ is a disk of radius r with center ζ, and $0 < R \le 1$, $|\zeta| \le k$, $r \ge \sqrt{|\zeta|^2 + R^2}$.

In [10] it was proved that for every function $f \in \mathcal{CV}$

$$\mathfrak{Re}\sqrt{f'(z)} > \frac{1}{2}, \quad z \in \mathcal{U},$$

and the bound is the best possible. Equivalently, every function $f \in \mathcal{ST}$ satisfies

$$\mathfrak{Re}\sqrt{\frac{f(z)}{z}} > \frac{1}{2}, \quad z \in \mathcal{U}. \tag{1.3}$$

These results were generalized in [11] to k-uniformly convex functions for all $k \ge 0$. It was proved there that, for $k \ge 0$, every function $f \in k$-$\mathcal{ST}$ satisfies the condition

$$\mathfrak{Re}\sqrt{\frac{f(z)}{z}} > \exp\left\{\int_0^{-1} \frac{p_k(t) - 1}{2t}\,\mathrm{d}t\right\} \ge \frac{k+1}{k+2}, \quad z \in \mathcal{U}, \tag{1.4}$$

where

$$p_1(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad z \in \mathcal{U},$$

and if $0 \le k < 1$, then

$$p_k(z) = \frac{1}{1-k^2}\cosh\left\{\left(\frac{2}{\pi}\arccos k\right)\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2}{1-k^2}, \quad z \in \mathcal{U}, \tag{1.5}$$

and if $k > 1$, then

$$p_k(z) = \frac{1}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{u(z)}{\sqrt{\kappa}}} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}}\right) + \frac{k^2}{k^2-1}, \quad z \in \mathcal{U}, \tag{1.6}$$

where

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa} z}, \quad z \in \mathcal{U},$$

and $\kappa \in (0, 1)$ is chosen such that $k = \cosh(\pi K'(\kappa)/(4K(\kappa)))$. Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and $K'(\kappa) = K(\sqrt{1-\kappa^2})$.

The first inequality in (1.4) is the best possible, moreover for $k = 0$ it becomes (1.3), while for $k = 1$ it becomes the recent result [12] of Mannino. The function p_k maps $\mathcal{U}$ onto domain bounded by the conic curve (see [1]) and more precisely $f \in k$-$\mathcal{ST}$ iff $zf'(z)/f(z) \prec p_k$, where $\prec$ denotes the subordination in the unit disk $\mathcal{U}$. The extremal function for (1.4) is $f_k \in k$-$\mathcal{ST}$, defined by the conditions

$$\frac{zf_k'(z)}{f_k(z)} = p_k(z), \quad z \in \mathcal{U} \quad \text{and} \quad f_k(0) = f_k'(0) - 1 = 0. \tag{1.7}$$

The function f_k is extremal for various problems in the class k-$\mathcal{ST}$. Note that the first inequality in (1.4) is the best possible because

$$\sqrt{\frac{f_k(-1)}{-1}} = \exp\left\{\int_0^{-1} \frac{p_k(t) - 1}{2t}\,\mathrm{d}t\right\}$$

so any increase in the smaller side makes the assertion false. It seems there is no way to obtain explicitly the value of the integral on the right-hand side of (1.4) except in the special case when $k = \sqrt{2}/2$ and then the extremal value was computed explicitly in [11, p. 2639], and it is not far from the estimate $(k+1)/(k+2)$.

It is an interesting and important problem to find

$$q(r) = \min\left\{\mathfrak{Re}\frac{f(z)}{z} : f \in \mathcal{F},\ |z| = r < 1\right\}, \tag{1.8}$$

in the given class $\mathcal{F}$, or to find the largest radius $\rho(\beta)$ of the disk $|z| < \rho(\beta) < 1$ in which $\mathfrak{Re}\,(f(z)/z) > \beta$ over all $f \in \mathcal{F}$.

Reade and Silverman [13] obtained the solution of the above problem in the classes $\mathcal{S}$ and $\mathcal{ST}(\alpha)$ for $\alpha = 0$ and $1/2 \leq \alpha < 1$. They left the case of $\alpha \in (0, 1/2)$ as an open problem, which was solved by Wiśniowska [14]. In [11, pp. 2639–40] the number $q(r)$ was computed in the class k-$\mathcal{ST}$ for all $k \geq 1$. The remaining case $k \in (0, 1)$ is more difficult similarly to the case $\alpha \in (0, 1/2)$ omitted by Reade and Silverman in the class $\mathcal{ST}(\alpha)$. In the present paper a more precise investigation of the properties of the functions p_k permit us to fill partially the gap $0 < k < 1$.

2. Preliminary results

Theorem 2.1. *Suppose that $0 \leq k < 1$ and p_k is given by (1.5). Then*

$$\mathfrak{Re}\,\frac{zp_k'(z)}{p_k(z)-1} > \frac{1}{\pi}\sqrt{\frac{1+k}{1-k}}\arccos k \tag{2.1}$$

for all $z \in \mathcal{U}$. The result is sharp.

Proof. For a fixed $0 \leq k < 1$ we have

$$p_k(z) = 1 + \frac{2}{1-k^2}\left[\sinh\left(\frac{A}{2}\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2,$$

where $A = A(k) = \frac{2}{\pi}\arccos k$. Hence

$$\frac{zp_k'(z)}{p_k(z)-1} = \frac{A\sqrt{z}}{1-z}\coth\left(\frac{A}{2}\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)$$

and then

$$\begin{aligned}\frac{e^{it}p_k'(e^{it})}{p_k(e^{it})-1} &= \frac{Ai}{2\sin\frac{t}{2}}\coth\left(\frac{A}{2}\log\left(i\cot\frac{t}{4}\right)\right)\\ &= \frac{A}{2\sin\frac{t}{2}}\,\frac{2\sin\frac{A\pi}{2}\left(\cot\frac{t}{4}\right)^A + i\left[\left(\cot\frac{t}{4}\right)^{2A}-1\right]}{\left(\cot\frac{t}{4}\right)^{2A} - 2\cos\frac{A\pi}{2}\left(\cot\frac{t}{4}\right)^A + 1}.\end{aligned}$$

Therefore

$$\mathfrak{Re}\,\frac{e^{it}p_k'(e^{it})}{p_k(e^{it})-1} = \frac{A\sin\frac{A\pi}{2}\left(\cot\frac{t}{4}\right)^A}{\sin\frac{t}{2}\left[\left(\cot\frac{t}{4}\right)^{2A} - 2\cos\frac{A\pi}{2}\left(\cot\frac{t}{4}\right)^A + 1\right]}. \tag{2.2}$$

Moreover,

$$\begin{aligned}\frac{-p_k'(-1)}{p_k(-1)-1} &= \frac{e^{i\pi}p_k'(e^{i\pi})}{p_k(e^{i\pi})-1} = \frac{A\sin\frac{A\pi}{2}}{2\left(1-\cos\frac{A\pi}{2}\right)}\\ &= \frac{A}{2}\cot\frac{A\pi}{4} = \left(\frac{1}{\pi}\arccos k\right)\cot\left(\frac{1}{2}\arccos k\right)\\ &= \frac{1}{\pi}\sqrt{\frac{1+k}{1-k}}\arccos k = \frac{A}{2}\sqrt{\frac{1+k}{1-k}},\end{aligned} \tag{2.3}$$

that is the right-hand side of (2.1). Therefore in order to prove (2.1) it suffices to show

$$\mathfrak{Re}\,\frac{zp_k'(z)}{p_k(z)-1} > \frac{-p_k(-1)}{p_k(-1)-1}, \tag{2.4}$$

for all $z = e^{it}$ and since the function p_k has the real coefficients we may assume $t \in [0, \pi]$. Note that $k = \cos\frac{\pi A}{2}$, thus from (2.2) and (2.3) the condition (2.4) becomes

$$\frac{A\sqrt{1-k^2}\left(\cot\frac{t}{4}\right)^A}{\sin\frac{t}{2}\left[\left(\cot\frac{t}{4}\right)^{2A} - 2k\left(\cot\frac{t}{4}\right)^A + 1\right]} \geq \frac{A}{2}\sqrt{\frac{1+k}{1-k}} \quad \text{for all } t \in [0, \pi], \tag{2.5}$$

or equivalently

$$\frac{x^A + x^{A+2}}{x^{2A+1} - 2kx^{A+1} + x} \geq \frac{1}{1-k} \quad \text{for all } x \geq 1, \tag{2.6}$$

where $x = \cot \frac{t}{4}$, $t \in [0, \pi]$. Since $0 \le k < 1$ we have

$$x^{2A+1} - 2kx^{A+1} + x > x^{2A+1} - 2x^{A+1} + x = x\left(x^A - 1\right)^2 \ge 0$$

and the required inequality (2.6) is equivalent to

$$(1-k)(1+x^2) - x^{A+1} + 2kx - x^{1-A} \ge 0 \quad \text{for all } x \ge 1.$$

Let

$$h(x) = (1-k)(1+x^2) - x^{A+1} + 2kx - x^{1-A}, \quad x \ge 1.$$

Then

$$\begin{aligned} h'(x) &= 2(1-k)x - (A+1)x^A + 2k + (A-1)x^{-A}, \\ h''(x) &= 2(1-k) - A(A+1)x^{A-1} + (1-A)Ax^{-A-1}, \\ h'''(x) &= (1-A)A(1+A)x^{A-2} - (1-A)A(1+A)x^{-A-2}. \end{aligned}$$

Since $x \ge 1$ we get $h'''(x) \ge 0$ and h'' is increasing. From $h''(1) = 2(1-k) - 2A^2 \ge 0$ (see Lemma 2.2) we conclude that $h''(x) \ge 0$ and h' is also increasing for $x \ge 1$. Note that

$$h'(1) = 2(1-k) - (A+1) + 2k + A - 1 = 0$$

and so $h'(x) > 0$ for $x > 1$. Hence h increases and since $h(1) = 0$ we get $h(x) \ge 0$ for $x \ge 1$, as desired. The bound (2.1) is the best possible because

$$\frac{-p_k(-1)}{p_k(-1) - 1} = \frac{1}{\pi}\sqrt{\frac{1+k}{1-k}} \arccos k$$

so any increase in the smaller side makes the assertion false, which ends the proof. □

Lemma 2.2. *For $0 \le A \le 1$ we have $1 - k \ge A^2$, where $k = \cos \frac{\pi A}{2}$.*

Proof. Note that $1 - k = 1 - \cos \frac{\pi A}{2} = 2\sin^2 \frac{\pi A}{4}$. Therefore it suffices to prove that

$$\sqrt{2} \sin \frac{\pi A}{4} \ge A.$$

Let

$$g(A) = \sqrt{2} \sin \frac{\pi A}{4} - A, \, 0 \le A \le 1.$$

Then

$$g'(A) = \frac{\pi\sqrt{2}}{4} \cos \frac{\pi A}{4} - 1,$$

and

$$g''(A) = -\frac{\sqrt{2}\pi^2}{16} \sin \frac{\pi A}{4} < 0 \quad \text{for } 0 \le A \le 1.$$

Moreover

$$g'(A) = 0 \Leftrightarrow A = \frac{4}{\pi} \arccos \frac{2\sqrt{2}}{\pi} := A_0.$$

Hence the function g attains its unique local extremum, namely maximum, at the point A_0. It follows from $g(0) = g(1) = 0$ that $g(A) \ge 0$ for all $0 \le A \le 1$. □

Lemma 2.3. *The function*

$$\psi(k) = \frac{1}{\pi}\sqrt{\frac{1+k}{1-k}} \arccos k, \quad k \in [0, 1) \tag{2.7}$$

strictly increases from $\psi(0) = 1/2$ to

$$\lim_{k \to 1^-} \psi(k) = \frac{2}{\pi} \approx 0.6366.$$

Proof. After some calculations we obtain

$$\psi'(k) = \frac{\sqrt{1-k}\left[\arccos k - \sqrt{1-k^2}\right]}{\pi\sqrt{1+k}(1-k)^2}, \quad k \in [0,1).$$

Thus

$$\psi'(k) > 0 \Leftrightarrow \arccos k - \sqrt{1-k^2} > 0. \tag{2.8}$$

Let

$$V(k) = \arccos k - \sqrt{1-k^2}.$$

We get

$$\begin{aligned} &V(0) = \pi/2 - 1 > 0,\\ &V(1) = 0,\\ &V'(k) = \frac{k-1}{\sqrt{1-k^2}} < 0 \quad \text{for all } k \in [0,1). \end{aligned}$$

Thus $V(k) > 0$ for all $k \in [0,1)$ and by (2.8) we get $\psi'(k) > 0$ for all $k \in [0,1)$. Therefore $\psi(k)$ increases in $[0,1)$. □

The above Lemma 2.3 and Theorem 2.1 show that for $k \in [0,1)$

$$\mathfrak{Re}\,\frac{zp_k'(z)}{p_k(z)-1} > \frac{1}{2}, \quad z \in \mathcal{U}. \tag{2.9}$$

The function f_k defined by the differential equation (1.7) has the form

$$f_k(z) = z\exp\left\{\int_0^z \frac{p_k(t)-1}{t}\,\mathrm{d}t\right\}, \quad z \in \mathcal{U} \tag{2.10}$$

(see [2]). We shall make use of Theorem 2.1 to show when $f_k(z)/z$ is convex univalent.

Theorem 2.4. *Let $r \in (0,1]$ be a given number. Then the function $f_k(z)/z$ is convex univalent in $|z| < r$ whenever k satisfies the inequality*

$$\frac{k+3}{k+1} - \frac{2}{\pi}\sqrt{\frac{1+k}{1-k}}\arccos k < \frac{1}{r}. \tag{2.11}$$

Proof. It is known that $f_k(z)/z$ is convex univalent in $|z| < r \le 1$ if and only if the function

$$H_k(z) = \frac{1}{p_k'(0)}\left(\frac{f_k(z)}{z} - 1\right), \quad z \in \mathcal{U} \tag{2.12}$$

satisfies the inequality (1.2) in $|z| < r \le 1$. In order to show that $H_k \in \mathcal{CV}$ note that

$$1 + \frac{zH_k''(z)}{H_k'(z)} = \frac{zp_k'(z)}{p_k(z)-1} + p_k(z) - 1. \tag{2.13}$$

Since $\mathfrak{Re}\,p_k(z) > p_k(-1) = k/(k+1)$ in $\mathcal{U}$ (see [1]), we get

$$p_k(z) \prec \frac{1-(2\varphi(k)-1)z}{1-z}, \quad \varphi(k) := \frac{k}{k+1}.$$

Moreover from (2.1) we have

$$\frac{zp_k'(z)}{p_k(z)-1} \prec \frac{1-[2\psi(k)-1]z}{1-z}, \quad \psi(k) := \frac{1}{\pi}\sqrt{\frac{1+k}{1-k}}\arccos k,$$

where $1/2 \le \psi(k) < 2/\pi \approx 0.6366$ by Lemma 2.3. Therefore, by the subordination principle, we have

$$\mathfrak{Re}\{p_k(z)\} > \frac{1+[2\varphi(k)-1]r}{1+r} \quad \text{for } |z| < r \tag{2.14}$$

and

$$\mathfrak{Re}\left\{\frac{zp_k'(z)}{p_k(z)-1}\right\} > \frac{1+[2\psi(k)-1]r}{1+r} \quad \text{for } |z| < r. \tag{2.15}$$

From (2.13), by (2.14) and by (2.15) we obtain

$$\begin{aligned}\mathfrak{Re}\left\{1+\frac{zH_k''(z)}{H_k'(z)}\right\} &> \frac{1+[2\psi(k)-1]\,r}{1+r}+\frac{1+[2\varphi(k)-1]r}{1+r}-1\\ &= \frac{1+r[2\psi(k)+2\varphi(k)-3]}{1+r} \quad \text{for } |z|<r. \end{aligned} \tag{2.16}$$

It is easy to see that the right-hand side of (2.16) is positive whenever k satisfies the inequality (2.11). □

In [11] it was shown that $f_k(z)/z$ is convex univalent for $k \geq 1$. Now we shall consider the radius of convexity of $f_k(z)/z$ for $k \in [0, 1)$. We will need the following lemma.

Lemma 2.5. *The function*

$$L(k)=\frac{k+3}{k+1}-\frac{2}{\pi}\sqrt{\frac{1+k}{1-k}}\arccos k, \quad k\in[0,1) \tag{2.17}$$

strictly decreases from $L(0) = 2$ to

$$\lim_{k\to 1^-} L(k) = 2 - 4/\pi \approx 0.72676.$$

Proof. After some calculations we obtain

$$L'(k) = 2\frac{\sqrt{1-k^2}[k^2+k(2+\pi)+1-\pi]-(k+1)^2\arccos k}{\pi\sqrt{1-k^2}(1-k)(1+k)^2}.$$

Thus

$$L'(k)<0 \Leftrightarrow \frac{\sqrt{1-k}[k^2+k(2+\pi)+1-\pi]}{(1+k)^{3/2}}-\arccos k<0. \tag{2.18}$$

Let us denote

$$W(k)=\frac{\sqrt{1-k}[k^2+k(2+\pi)+1-\pi]}{(1+k)^{3/2}}-\arccos k.$$

Hence after calculations we get

$$\begin{aligned} &W(0) = 1-3\pi/2<0,\\ &W(1)=0,\\ &W'(k)=\frac{\sqrt{1-k}\left[(k+1)^2+3\pi\right]}{(1+k)^{5/2}}>0 \quad \text{for all } k\in[0,1). \end{aligned}$$

Thus $W(k) < 0$ for all $k \in [0, 1)$ and by (2.18) we get $L'(k) < 0$ for all $k \in [0, 1)$. Therefore $L(k)$ decreases in $[0, 1)$. □

Theorem 2.6. *Each function $f_k(z)/z$, $k \geq 0$, is convex univalent in $|z| < 1/2$.*

Proof. For $k = 0$ the function $f_k(z)/z$ becomes $f_0(z)/z = 1/(1-z)^2$ which is convex precisely in $|z| \leq 1/2$. By Theorem 2.4 the function $f_k(z)/z$ is convex univalent in $|z| < 1/2$ whenever k satisfies the inequality

$$L(k)=\frac{k+3}{k+1}-\frac{2}{\pi}\sqrt{\frac{1+k}{1-k}}\arccos k<2,$$

but for $k \in (0, 1)$ it is true by Lemma 2.5. For $k \geq 1$ the function $f_k(z)/z$ is convex in the whole $\mathcal{U}$ (see the proof of Theorem 2.1 in [11]). □

Theorem 2.7. *The function $f_k(z)/z$ is convex univalent in $|z| < 1$ for $k > k_0(1)$, where $k_0(1)$ is the unique in $(1/2, \sqrt{2}/2)$ solution of the equation*

$$\frac{(1+k)^2}{\sqrt{1-k^2}}\arccos k=\pi. \tag{2.19}$$

Proof. Substituting $r = 1$ in (2.11) we see that the function $f_k(z)/z$ is convex univalent in $|z| < 1$ whenever

$$\frac{k+3}{k+1}-\frac{2}{\pi}\sqrt{\frac{1+k}{1-k}}\arccos k<1,$$

which yields

$$L(k) < 1,$$

where $L(k)$ is given by (2.17). With this notation, after some calculations, the equation $L(k) = 1$ becomes (2.19). By Lemma 2.5 the function $L(k)$ is strictly decreasing in $[0, 1)$ and it decreases from 2 to $2 - 4/\pi \approx 0.72676$. Therefore, the equation $L(k) = 1$ has a unique solution $k_0(1) \in (0, 1)$ such that

$$L(k_0(1)) = 1 \quad \text{and} \quad L(k) < 1 \quad \text{for } k > k_0(1).$$

It is easy to check that $1/2 < k_0(1) < \sqrt{2}/2$. □

Theorem 2.8. *Let $k \in (0, k_0(1))$ be a given number. Then the function $f_k(z)/z$ is convex univalent in $|z| < r_0(k)$, where $r_0(k) \in (1/2, 1)$ is the value of r given by the equation*

$$\frac{k+3}{k+1} - \frac{2}{\pi}\sqrt{\frac{1+k}{1-k}} \arccos k = \frac{1}{r}. \tag{2.20}$$

Proof. By Theorem 2.4 the function $f_k(z)/z$ is convex univalent in $|z| < r$ whenever k satisfies the inequality (2.11). It can be written as

$$L(k) < \frac{1}{r},$$

where $L(k)$, given by (2.17), is a strictly decreasing function in $[0, 1)$ and it decreases from 2 to $2 - 4/\pi \approx 0.72676$, while in $(0, k_0(1))$ it decreases from 2 to 1. Therefore, there exists a unique $r_0(k) \in [1/2, 1)$ such that

$$L(k) = \frac{1}{r_0(k)} \quad \text{and} \quad L(k) < \frac{1}{r} \quad \text{for } r < r_0(k). \quad \square$$

3. Concluding results

Using the results proved in the previous section we shall find

$$q(r) = \min\left\{\mathfrak{Re}\,\frac{f(z)}{z} : f \in k\text{-}\mathcal{ST},\ |z| = r < 1\right\}. \tag{3.1}$$

This problem is partially solved. In [13] Reade and Silverman it was proved that if $f \in \mathcal{ST}$ (which gives the case $k = 0$) then

$$\min\left\{\mathfrak{Re}\,\frac{f(z)}{z} : |z| = r \le 1/2\right\} = \frac{1}{(1+r)^2} \tag{3.2}$$

and

$$\min\left\{\mathfrak{Re}\,\frac{f(z)}{z} : |z| = r > 1/2\right\} = \frac{1-2r^2}{2(1-r^2)^2}. \tag{3.3}$$

Moreover, it was proved in [11, pp. 2639–40], that if $k \ge 1$, then

$$\min\left\{\mathfrak{Re}\,\frac{f(z)}{z} : f \in k\text{-}\mathcal{ST},\ |z| = r < 1\right\} = \exp\left\{\int_0^{-r} \frac{p_k(t) - 1}{t}\,\mathrm{d}t\right\}. \tag{3.4}$$

Now we are going to consider the case $0 < k < 1$.

It follows from [1, Theorem 3.2, p. 333], that

$$f \in k\text{-}\mathcal{ST} \Longrightarrow \frac{f(z)}{z} \prec \frac{f_k(z)}{z} = \exp\left\{\int_0^{z} \frac{p_k(t) - 1}{t}\,\mathrm{d}t\right\} \quad \text{in } \mathcal{U}. \tag{3.5}$$

Hence we get

$$\min\left\{\mathfrak{Re}\,\frac{f(z)}{z} : f \in k\text{-}\mathcal{ST},\ |z| = r < 1\right\} = \min_{|z|=r} \mathfrak{Re}\,\frac{f_k(z)}{z}. \tag{3.6}$$

Theorem 3.1. *Consider $k \in (0, 1)$. Let $k_0(1)$ be the unique in $(1/2, \sqrt{2}/2)$ solution of the Eq. (2.19) and let $r_0(k) \in (1/2, 1)$ be the value of r given by the Eq. (2.20). Then*

$$\min\left\{\mathfrak{Re}\,\frac{f(z)}{z} : f \in k\text{-}\mathcal{ST},\ |z| = r < \frac{1}{2}\right\} = \frac{f_k(-r)}{-r}. \tag{3.7}$$

If $0 \le k < k_0(1)$, *then*

$$\min\left\{\mathfrak{Re}\,\frac{f(z)}{z} : f \in k\text{-}\mathcal{ST},\ |z| = r_0(k) < 1\right\} = \frac{f_k(-r_0(k))}{-r_0(k)}. \tag{3.8}$$

If $k_0(1) \le k < 1$, *then*

$$\min\left\{\mathfrak{Re}\,\frac{f(z)}{z} : f \in k\text{-}\mathcal{ST},\ |z| = r < 1\right\} = \frac{f_k(-r)}{-r}. \tag{3.9}$$

Proof. In view of (3.5) and (3.6) to get the bounds (3.7)–(3.9) it suffices to find

$$\min_{|z|=r} \mathfrak{Re}\,\frac{f_k(z)}{z} = \min_{|z|=r}\left[\exp\left\{\int_0^z \frac{p_k(t)-1}{t}\,\mathrm{d}t\right\}\right]. \tag{3.10}$$

It seems to be difficult to obtain explicitly the value of (3.10) except for the special case when the function $f_k(z)/z$ maps $|z| < r$ onto a convex domain. It is known that the function $f_k(z)/z$ maps $\mathcal{U}$ onto a domain symmetric with respect to the real axis thus if this domain is also convex, then (3.10) is attained at $z = -r$. The investigations in the previous chapter make it possible to determine when $f_k(z)/z$ is convex. From Theorem 2.6 we obtain (3.7), by Theorem 2.8 we get (3.8) and Theorem 2.7 yields (3.9). □

Corollary 3.2. *If $f \in k$-$\mathcal{ST}$ for some $k \ge k_0$, where k_0 is the unique in $(1/2, \sqrt{2}/2)$ solution of the Eq.* (2.19), *then*

$$\mathfrak{Re}\,\frac{f(z)}{z} > -f_k(-1) \quad \text{for } z \in \mathcal{U}. \tag{3.11}$$

Notice that for $k \ge 0$

$$-f_k(-1) = \mathcal{K}(k\text{-}\mathcal{ST}),$$

where $\mathcal{K}(k\text{-}\mathcal{ST})$ denote the Koebe constant for the class k-$\mathcal{ST}$ (see [2]). For example, if $k = \sqrt{2}/2$, then

$$-f_{\sqrt{2}/2}(-1) = 16(\sqrt{2}-1)^4 \approx 0.47.$$

For $k \ge 1$ and for every function $f \in k$-$\mathcal{ST}$ we have (see [11]) the following not sharp result

$$\mathfrak{Re}\,\frac{f(z)}{z} > \frac{k+1}{k+3}, \quad z \in \mathcal{U}.$$

Also in [11], the sharp inequality of the form (3.11) was proved for all $f \in k$-$\mathcal{ST}$ with $k \ge 1$, so Corollary 3.2 somewhat improves this result.

From Theorem 3.1 we can directly obtain the following results.

Theorem 3.3. *Let $k \in [k_0(1), 1)$, where $k_0(1)$ is the unique in $(1/2, \sqrt{2}/2)$ solution of the Eq.* (2.19) *and let $\mathcal{K}(k\text{-}\mathcal{ST}) \le \beta < 1$. Then for $f \in k$-$\mathcal{ST}$ we have*

$$\rho(\beta) = \max\{r : \mathfrak{Re}\,f(z)/z > \beta, |z| < r\} = \min\{r_0, 1\},$$

where r_0 is given by the equation

$$\log(1/\beta) = \int_0^{r_0} \frac{1-p_k(-x)}{x}\,\mathrm{d}x,$$

Theorem 3.4. *If $f \in k$-$\mathcal{UCV}$ for some $k \le k_0(1)$, where $k_0(1)$ is the unique in $(1/2, \sqrt{2}/2)$ solution of the Eq.* (2.19), *then*

$$\mathfrak{Re}\,f'(z) > \frac{f_k(-r)}{-r} = \exp\left\{\int_0^{-r} \frac{p_k(t)-1}{t}\,\mathrm{d}t\right\}, \quad |z| < r.$$

The result is sharp.

Remark. Suppose that $0 < k < k_0(1)$, where $k_0(1)$ is a unique in $(1/2, \sqrt{2}/2)$ solution of the Eq. (2.19). For $0 < k < k_0(1)$ the function $f_k(z)/z$ may be not convex in $|z| < r$ with $1/2 < r < 1$. The problem of determining (3.1) for such k and r remains still open.

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