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Theory of Finite Pseudoalgebras

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1. INTRODUCTION

Since the seminal papers of Belavin *et al.* [BPZ] and of Borcherds [Bo1] there has been a great deal of work towards understanding the algebraic structures underlying the notion of operator product expansion (OPE) of chiral fields in conformal field theory.

In physics literature the OPE of local chiral fields φ and ψ is written in the form [BPZ]

$$\varphi(z)\,\psi(w) = \sum_{j \ll \infty} \frac{\varphi(w)_{(j)}\,\psi(w)}{(z-w)^{j+1}}, \tag{1.1}$$

where $\varphi(w)_{(j)}\,\psi(w)$ are some new fields, which may be viewed as bilinear products of fields φ and ψ for all $j \in \mathbb{Z}$ (see, e.g., [K2] for a rigorous interpretation of (1.1)). If now V is a space of pairwise local chiral fields which contains 1, is invariant with respect to the derivative $\partial = \partial_w$, and is closed under all jth products, $j \in \mathbb{Z}$, we obtain an algebraic structure which physicists (respectively mathematicians) call a chiral (respectively vertex) algebra. In more abstract terms, V is a module over $\mathbb{C}[\partial]$ with a marked element 1 and infinitely many bilinear over $\mathbb{C}$ products $\varphi_{(j)}\psi$, $j \in \mathbb{Z}$, satisfying a certain system of identities, first written down by Borcherds [Bo1]. (An equivalent system of axioms, which is much easier to verify, may be found in [K2].)

One of the important features of the OPE (1.1) is that its singular part encodes the commutation relations of fields, namely one has (see, e.g., [K2])

$$[\varphi(z), \psi(w)] = \sum_{j \geqslant 0} (\varphi(w)_{(j)}\, \psi(w))\, \partial_w^j \delta(z-w)/j!, \tag{1.2}$$

where $\delta(z-w) = \sum_{n \in \mathbb{Z}} z^n w^{-n-1}$ is the delta-function. This leads to the notion of a Lie conformal algebra, which is a $\mathbb{C}[\partial]$-module with $\mathbb{C}$-bilinear products $\varphi_{(j)}\psi$ for all non-negative integers j, subject to certain identities [K2]. In order to write down these identities in a compact form, it is convenient to consider the formal Fourier transform of (1.2), called the λ-bracket (where λ is an indeterminate):

$$[\varphi_\lambda \psi] = \sum_{j \geqslant 0} \frac{\lambda^j}{j!} (\varphi_{(j)}\psi).$$

Then a *Lie conformal algebra* L is defined as a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map

$$L \otimes L \to \mathbb{C}[\lambda] \otimes L, \qquad a \otimes b \mapsto [a_\lambda b]$$

satisfying the following axioms [DK] ($a, b, c \in L$):

(sesquilinearity) $\quad [\partial a_\lambda b] = -\lambda[a_\lambda b],\ [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b],$

(skew-commutativity) $\quad [b_\lambda a] = -[a_{-\lambda-\partial} b],$

(Jacobi identity) $\quad [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]].$

In the past few years a structure theory [DK], representation theory [CK, CKW], and cohomology theory [BKV] of finite (i.e., finitely generated as $\mathbb{C}[\partial]$-modules) Lie conformal algebras have been worked out. For example, one of the main results of [DK] states that any finite simple Lie conformal algebra is isomorphic either to the *Virasoro* conformal algebra:

$$\mathrm{Vir} = \mathbb{C}[\partial]\, \ell, \qquad [\ell_\lambda \ell] = (\partial + 2\lambda)\, \ell \tag{1.3}$$

or to the *current* conformal algebra associated to a simple finite-dimensional Lie algebra $\mathfrak{g}$,

$$\operatorname{Cur}\mathfrak{g} = \mathbb{C}[\partial] \otimes \mathfrak{g}, \qquad [a_\lambda b] = [a, b], \qquad a, b \in \mathfrak{g}. \tag{1.4}$$

The objective of the present paper is to develop a theory of "multi-dimensional" Lie conformal algebras, i.e., a theory where the algebra of polynomials $\mathbb{C}[\partial]$ is replaced by a "multi-dimensional" associative algebra H. In order to explain the definition, let us return to the singular part (1.2) of the OPE. Choosing a set of generators a^i of the $\mathbb{C}[\partial]$-module L, we can write

$$[a^i_\lambda a^j] = \sum_k Q^{ij}_k(\lambda, \partial)\, a^k,$$

where Q^{ij}_k are some polynomials in λ and ∂. The corresponding singular part of the OPE is

$$[a^i(z), a^j(w)] = \sum_k Q^{ij}_k(\partial_w, \partial_t)(a^k(t)\, \delta(z-w))|_{t=w}.$$

Letting $P^{ij}_k(x, y) = Q^{ij}_k(-x, x+y)$, we can rewrite this in a more symmetric form

$$[a^i(z), a^j(w)] = \sum_k P^{ij}_k(\partial_z, \partial_w)(a^k(w)\, \delta(z-w)). \tag{1.5}$$

We thus obtain an $H = \mathbb{C}[\partial]$-bilinear map (i.e., a map of $H \otimes H$-modules):

$$L \otimes L \to (H \otimes H) \otimes_H L, \qquad a \otimes b \mapsto [a * b]$$

(where H acts on $H \otimes H$ via the comultiplication map $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$), defined by

$$[a^i * a^j] = \sum_k P^{ij}_k(\partial \otimes 1, 1 \otimes \partial) \otimes_H a^k.$$

Hence the notion of λ-bracket $[a_\lambda b]$ is equivalent to the notion of the $*$-bracket $[a * b]$ introduced by Beilinson and Drinfeld [BD], the relation between the two brackets being given by letting $\lambda = -\partial \otimes 1$. For example, the Virasoro conformal algebra (1.3) corresponds to the *Virasoro pseudoalgebra*

$$\operatorname{Vir} = \mathbb{C}[\partial]\, \ell, \qquad [\ell * \ell] = (1 \otimes \partial - \partial \otimes 1) \otimes_{\mathbb{C}[\partial]} \ell. \tag{1.6}$$

It is natural to introduce the general notion of a conformal algebra as a $\mathbb{C}[\partial]$-module L endowed with a $\mathbb{C}$-linear map $L \otimes L \to \mathbb{C}[\lambda] \otimes L$, $a \otimes b \mapsto a_\lambda b$ satisfying the sesquilinearity property

$$(\partial a)_\lambda b = -\lambda(a_\lambda b), \qquad a_\lambda(\partial b) = (\partial + \lambda)(a_\lambda b).$$

Such a conformal algebra is called associative (respectively commutative) if

$$a_\lambda (b_\mu c) = (a_\lambda b)_{\lambda+\mu}\, c \qquad \text{(respectively } b_\lambda a = a_{-\lambda-\partial} b\text{)},$$

and the λ-product of an associative conformal algebra defines a λ-bracket

$$[a_\lambda b] = a_\lambda b - b_{-\lambda-\partial} a,$$

making it a Lie conformal algebra [K4, DK].

As above, we have the equivalent notion of a $*$-product on an $H = \mathbb{C}[\partial]$-module L, which is an H-bilinear map

$$L \otimes L \to (H \otimes H) \otimes_H L, \quad a \otimes b \mapsto a * b. \tag{1.7}$$

Now it is clear that the notion of a $*$-product can be defined by (1.7) for any Hopf algebra H by making use of the comultiplication $\Delta \colon H \to H \otimes H$ to define $(H \otimes H) \otimes_H L$. A *pseudoalgebra* is a (left) H-module L endowed with an H-bilinear map (1.7). The name is motivated by the fact that this is an algebra in a pseudotensor category (introduced in [L, BD]). Accordingly, the $*$-product will be called a *pseudoproduct*.

One is able to define a pseudoproduct as soon as a structure of a bialgebra is given on H. However, in order to generalize the equivalence of a pseudoalgebra and an H-conformal algebra structure on an H-module L, we need H to be a Hopf algebra. In this case any element of $H \otimes H$ can be uniquely written as a (finite) sum:

$$\sum_i (h_i \otimes 1)\, \Delta(f_i), \qquad \text{where } h_i \text{ are linearly independent.}$$

Hence the pseudoproduct on L can be written in the form

$$a * b = \sum_i (h_i \otimes 1) \otimes_H c_i\,. \tag{1.8}$$

The corresponding *H-conformal algebra* structure is then a $\mathbb{C}$-linear map $L \otimes L \to H \otimes L$ given by

$$ab = \sum_i h_i \otimes c_i\,. \tag{1.9}$$

Every element x of H^* then defines an *x-product* $L \otimes L \to L$,

$$a_x b = \sum_i \langle x, S(h_i) \rangle \, c_i , \tag{1.10}$$

where S is the antipode of H.

The H-bilinearity property of the pseudoproduct (1.8) is, of course, easily translated to certain sesquilinearity properties of the products (1.9) and (1.10). In particular, in the case $H = \mathbb{C}[\partial]$, the product (1.9) is the λ-product if we let $\lambda = -\partial$, and the product (1.10) for $x = t^j$ is the jth product described above, where $H^* \simeq \mathbb{C}[[t]]$, $\langle t, \partial \rangle = 1$. The equivalence of these three structures (discussed in Section 9) is very useful in the study of pseudoalgebras.

In order to define associativity of a pseudoproduct, we extend it from $L \otimes L \to H^{\otimes 2} \otimes_H L$ to $(H^{\otimes 2} \otimes_H L) \otimes L \to H^{\otimes 3} \otimes_H L$ and to $L \otimes (H^{\otimes 2} \otimes_H L) \to H^{\otimes 3} \otimes_H L$ by letting

$$(f \otimes_H a) * b = \sum_i (f \otimes 1)(\Delta \otimes \mathrm{id})(g_i) \otimes_H c_i ,$$

$$a * (f \otimes_H b) = \sum_i (1 \otimes f)\,(\mathrm{id} \otimes \Delta)(g_i) \otimes_H c_i ,$$

$$\text{where} \quad a * b = \sum_i g_i \otimes_H c_i .$$

Then the associativity property is given by the usual equality (in $H^{\otimes 3} \otimes_H L$):

$$(a * b) * c = a * (b * c).$$

The easiest example of a pseudoalgebra is a current pseudoalgebra, defined as follows. Let H' be a Hopf subalgebra of H and let A be an H'-pseudoalgebra (for example, if $H' = \mathbb{C}$, then A is an ordinary algebra over $\mathbb{C}$). Then the associated *current* H-pseudoalgebra is $\operatorname{Cur} A = H \otimes_{H'} A$ with the pseudoproduct

$$(f \otimes_{H'} a) * (g \otimes_{H'} b) = ((f \otimes g) \otimes_H 1)\,(a * b).$$

The H-pseudoalgebra Cur A is associative iff the H'-pseudoalgebra A is.

The most important example of an associative H-pseudoalgebra is the pseudoalgebra of all pseudolinear endomorphisms of a finitely generated H-module V, which is denoted by Cend V (see Section 10). A *pseudolinear endomorphism* of V is a $\mathbb{C}$-linear map $\phi \colon V \to (H \otimes H) \otimes_H V$ such that

$$\phi(hv) = ((1 \otimes h) \otimes_H 1)\, \phi(v), \qquad h \in H,\ v \in V.$$

The space Cend V of all such ϕ becomes a (left) H-module if we define

$$(h\phi)(v) = ((h \otimes 1) \otimes_H 1)\, \phi(v).$$

The definition of a pseudoproduct on Cend V is especially simple when V is a free H-module, $V = H \otimes V_0$, where V_0 is a finite-dimensional vector space over $\mathbb{C}$ with a trivial action of H. Then Cend V is isomorphic to $H \otimes H \otimes \operatorname{End} V_0$, with H acting by left multiplication on the first factor, with the pseudoproduct

$$(f \otimes a \otimes A) * (g \otimes b \otimes B) = \sum_i (f \otimes g a_i') \otimes_H (1 \otimes b a_i'' \otimes AB),$$

where $\Delta(a) = \sum_i a_i' \otimes a_i''$.

The main objects of our study are Lie pseudoalgebras. The corresponding pseudoproduct is conventionally called a *pseudobracket* and denoted by $[a * b]$. Given an associative pseudoalgebra with pseudoproduct $a * b$ we may give it a structure of a Lie pseudoalgebra by defining the pseudobracket

$$[a * b] = a * b - (\sigma \otimes_H \mathrm{id})\, b * a,$$

where $\sigma \colon H \otimes H \to H \otimes H$ is the permutation of factors. It is immediate to see that this pseudobracket satisfies the following skew-commutativity and Jacobi identity axioms:

$$[b * a] = -(\sigma \otimes_H \mathrm{id})\, [a * b], \tag{1.11}$$

$$[a * [b * c]] = [[a * b] * c] + ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\, [b * [a * c]]. \tag{1.12}$$

It is important to point out here that the above pseudobracket and both identities are well defined, provided that the Hopf algebra H is cocommutative. A pseudoalgebra with pseudoproduct $[a * b]$ satisfying identities (1.11) and (1.12) is called a *Lie pseudoalgebra*. We will always assume that H is cocommutative when talking about Lie pseudoalgebras. Of course, the simplest examples of Lie pseudoalgebras are Cur A, where A is a $H'(\subset H)$ Lie pseudoalgebra ($=$ Lie algebra if $H' = \mathbb{C}$). It is needless to say that in the case $H = \mathbb{C}[\partial]$, $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$, the H-conformal algebras associated to Lie pseudoalgebras are nothing else but the Lie conformal algebras discussed above.

We will explain now the connection of the notion of a Lie pseudoalgebra to the more classical notion of a differential Lie algebra studied in [R1–R4], [C, NW] and many other papers (see Section 7). Let Y be a commutative associative algebra over $\mathbb{C}$ with compatible left and right actions of the Hopf algebra H. Then, given a Lie pseudoalgebra L, we let

$\mathscr{A}_Y L = Y \otimes_H L$ with the obvious left H-module structure and the following Lie algebra (over $\mathbb{C}$) structure:

$$[(x \otimes_H a), (y \otimes_H b)] = \sum_i (xf_i)(yg_i) \otimes_H c_i$$

$$\text{if} \quad [a * b] = \sum_i (f_i \otimes g_i) \otimes_H c_i .$$

Provided that L is a free H-module, the Lie algebra $\mathscr{A}_Y L$ is a free Y-module, hence $\mathscr{A}_Y L$ is a differential Lie algebra in the sense of [NW]. The most classical case is again $H = \mathbb{C}[\partial]$, when Y is simply a commutative associative algebra with a (left and right) derivation ∂, and we get the differential Lie algebras of Ritt [R1–R4]. Thus, the notion of a Lie pseudoalgebra is reminiscent of the notion of a group scheme: each Lie pseudoalgebra L, which is free as an H-module, gives rise to a functor $\mathscr{A}$ from the category of commutative associative algebras with compatible left and right actions of H to the category of differential Lie algebras (= category of formal differential groups).

For example, the functor $\mathscr{A}$ corresponding to the Virasoro pseudoalgebra (1.6) associates to any commutative associative algebra Y with a derivation $'$ the differential Lie algebra Y with bracket $[u, v] = uv' - u'v$, called the substitutional Lie algebra by Ritt. The current pseudoalgebra $\operatorname{Cur} \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$, associates to Y the obvious differential Lie algebra $Y \otimes \mathfrak{g}$. Thus, a result of [DK] asserts that any simple finite differential Lie algebra with "constant coefficients" is isomorphic either to the substitutional Lie algebra or to $Y \otimes \mathfrak{g}$ where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra. In the rank 1 case, but without the constant coefficients assumption, this is the main result of [R1].

The main tool in the study of pseudoalgebras is the *annihilation algebra* $\mathscr{A}_X L$, where $X = H^*$ is the associative algebra dual to the coalgebra H. We find it remarkable that the annihilation algebra of the associative pseudoalgebra $\operatorname{Cend} H = H \otimes H$ is nothing else but the Drinfeld double (with the obvious comultiplication) of the Hopf algebra H. Note that in the associative case Y need not be commutative in order to define the functor $\mathscr{A}_Y$, but in the Lie algebra case it must be. So, in order to construct the annihilation Lie algebra we again use cocommutativity of H.

Recall that, by Kostant's theorem (Theorem 2.1), any cocommutative Hopf algebra H is a smash product of a group algebra $\mathbb{C}[\Gamma]$ and the universal enveloping algebra $U(\mathfrak{d})$ of a Lie algebra $\mathfrak{d}$. In Sections 5 and 13.7 we show that the theory of pseudoalgebras over a smash product of $\mathbb{C}[\Gamma]$ and any Hopf algebra H reduces to that over H. This allows us in many cases to assume, without loss of generality, that H is the universal enveloping algebra of a Lie algebra $\mathfrak{d}$.

However, for most of our results we have to assume that $\mathfrak{d}$ is finite dimensional. In this case the algebra $H = U(\mathfrak{d})$ is Noetherian, and the annihilation algebra $\mathscr{A}_X L$ is linearly compact, provided that L is *finite* (i.e., finitely generated as an H-module). Recall that a topological Lie algebra is called *linearly compact* if its underlying topological vector space is a topological product of finite-dimensional vector spaces with the discrete topology (see Section 6).

In Section 11 we prove "reconstruction" theorems, which claim that, under some mild assumptions, a Lie pseudoalgebra is completely determined by its annihilation Lie algebra along with the action of $\mathfrak{d}$. This reduces the classification of finite simple Lie pseudoalgebras to the well developed structure theory of linearly compact Lie algebras, which goes back to E. Cartan (see [G1, G2] and Section 6).

We turn now to examples of finite Lie pseudoalgebras beyond the rather obvious examples of current Lie pseudoalgebras. The first example is the generalization of the Virasoro pseudoalgebra (1.6) defined for $H = \mathbb{C}[\partial]$ (which is the universal enveloping algebra of the 1-dimensional Lie algebra) to the case $H = U(\mathfrak{d})$, where $\mathfrak{d}$ is any finite-dimensional Lie algebra. This is the Lie pseudoalgebra $W(\mathfrak{d}) = H \otimes \mathfrak{d}$ with pseudobracket

$$
\begin{aligned}
&[(1 \otimes a) * (1 \otimes b)] \\
&\qquad = (1 \otimes 1) \otimes_H (1 \otimes [a, b]) + (b \otimes 1) \otimes_H (1 \otimes a) - (1 \otimes a) \otimes_H (1 \otimes b).
\end{aligned}
$$

Since the associated annihilation algebra $\mathscr{A}_X W(\mathfrak{d}) \simeq X \otimes \mathfrak{d}$ is isomorphic to the Lie algebra of formal vector fields on the Lie group D with Lie algebra $\mathfrak{d}$, it is natural to call $W(\mathfrak{d})$ the pseudoalgebra of all vector fields. In fact we develop (in Section 8) a formalism of pseudoforms similar to the usual formalism of differential forms, which may be viewed as the beginning of a "pseudo differential geometry."

This allows us to define the remaining three series of finite simple Lie pseudoalgebras: $S(\mathfrak{d}, \chi)$, $H(\mathfrak{d}, \chi, \omega)$, and $K(\mathfrak{d}, \theta)$. The annihilation algebras of the simple Lie pseudoalgebras $W(\mathfrak{d})$, $S(\mathfrak{d}, \chi)$, $H(\mathfrak{d}, \chi, \omega)$ and $K(\mathfrak{d}, \theta)$ are isomorphic to the four series of Lie–Cartan linearly compact Lie algebras W_N, S_N, P_N (which is an extension of H_N by a 1-dimensional center) and K_N, where $N = \dim \mathfrak{d}$. However, the Lie pseudoalgebras $S(\mathfrak{d}, \chi)$, $H(\mathfrak{d}, \chi, \omega)$ and $K(\mathfrak{d}, \theta)$ depend on certain parameters χ, ω and θ, due to inequivalent actions of $\mathfrak{d}$ on the annihilation algebra. The parameter χ is a 1-dimensional representation of $\mathfrak{d}$, i.e., $\chi \in \mathfrak{d}^*$ such that $\chi([\mathfrak{d}, \mathfrak{d}]) = 0$. The parameter ω is an element of $\mathfrak{d}^* \wedge \mathfrak{d}^*$ such that $\omega^{N/2} \neq 0$ and $\mathrm{d}\omega + \chi \wedge \omega = 0$ in the case $H(\mathfrak{d}, \chi, \omega)$, when N is even. The parameter $\theta \in \mathfrak{d}^*$ is such that $\theta \wedge (\mathrm{d}\theta)^{(N-1)/2} \neq 0$ in the case $K(\mathfrak{d}, \theta)$, when N is odd. In the cases $H(\mathfrak{d}, \chi, \omega)$, $K(\mathfrak{d}, \theta)$, these parameters are in one-to-one correspondence with "nondegenerate" skew-symmetric solutions $\alpha = r + s \otimes 1 - 1 \otimes s$ $(r \in \mathfrak{d} \wedge \mathfrak{d}$, $s \in \mathfrak{d})$ of a modification of the classical Yang–Baxter equation, which is a special case of the dynamical classical Yang–Baxter equation (see [Fe, ES]).

The central result of the paper is the classification of finite simple Lie pseudoalgebras over the Hopf algebra $H = U(\mathfrak{d})$. As usual, a Lie pseudoalgebra L is called *simple* if it is nonabelian (i.e., $[L * L] \neq 0$) and its only ideals are 0 and L. Our Theorem 13.2 states that any such Lie pseudoalgebra is isomorphic either to a current pseudoalgebra $\mathrm{Cur}\,\mathfrak{g} = \mathrm{Cur}_{\mathbb{C}}^{H}\,\mathfrak{g}$ over a simple finite-dimensional Lie algebra $\mathfrak{g}$, or to a current pseudoalgebra $\mathrm{Cur}_{H'}^{H} L'$ over one of the Lie pseudoalgebras $L' = W(\mathfrak{d}')$, $S(\mathfrak{d}', \chi')$, $H(\mathfrak{d}', \chi', \omega')$ or $K(\mathfrak{d}', \theta')$, where $H' = U(\mathfrak{d}')$ and $\mathfrak{d}'$ is a subalgebra of $\mathfrak{d}$.

A Lie pseudoalgebra L is called *semisimple* if it contains no nonzero abelian ideals. One also defines in the usual way the derived pseudoalgebra, solvable and nilpotent pseudoalgebras, and for a finite Lie pseudoalgebra L one has the solvable radical $\mathrm{Rad}\,L$ (so that $L/\mathrm{Rad}\,L$ is semisimple).

Our Theorem 13.3 states that any finite semisimple Lie $U(\mathfrak{d})$-pseudoalgebra is a direct sum of finite simple Lie pseudoalgebras and of Lie pseudoalgebras of the form $A \ltimes \mathrm{Cur}\,\mathfrak{g}$, where A is a subalgebra of $W(\mathfrak{d})$ and $\mathfrak{g}$ is a simple finite-dimensional Lie algebra. In addition, in Theorem 13.4 we show that any subalgebra of $W(\mathfrak{d})$ is simple, and in Corollary 13.6 we give a complete list of all these subalgebras. (A more concise formulation of Theorem 13.2 is that any finite simple Lie pseudoalgebra over $U(\mathfrak{d})$ is either a current pseudoalgebra $\mathrm{Cur}\,\mathfrak{g}$ over a simple finite-dimensional Lie algebra $\mathfrak{g}$, or a nonzero subalgebra of $W(\mathfrak{d})$.)

Note, however, that Levi's theorem on L being a semidirect sum of $L/\mathrm{Rad}\,L$ and $\mathrm{Rad}\,L$ is not true even in the case $\dim \mathfrak{d} = 1$. This stems from the fact that the cohomology of simple Lie pseudoalgebras with nontrivial coefficients is (highly) nontrivial (see Section 15 and [BKV]), in a sharp contrast with the Lie algebra case. For example, it follows from [BKV] that there are precisely five cases (three isolated examples and two families) of non-split extensions of Vir by $\mathrm{Cur}\,\mathbb{C}$. Translated into the language of differential Lie algebras, this result goes back to Ritt [R3].

Closely related to the present paper are the papers [Ki, NW], where (in our terminology) the annihilation algebras of rank 1 over H Lie pseudoalgebras, and of simple Lie pseudoalgebras of arbitrary finite rank, respectively, are studied. In fact, our Theorems 13.2 and 13.3 provide a completed form of the classification results of [NW] (in the "constant coefficients" case).

The structural results of the present paper in the simplest case $\dim \mathfrak{d} = 1$ reproduce the results of [DK]. However, this case is much easier than the case $\dim \mathfrak{d} > 1$, mainly due to the fact that only in this case is any finite torsionless H-module free.

Note also the close connection of our work to Hamiltonian formalism in the theory of nonlinear evolution equations (see the review [DN2], the book [Do] and references there, and also [GD, DN1, Z, M, X], and many other papers). In Section 16 we derive, as a corollary of Theorems 13.2 and 13.3, a classification of simple and semisimple linear Poisson brackets in any finite number of indeterminates.

In Section 14 we develop a representation theory of finite Lie pseudoalgebras. First, we prove an analogue of Lie's Lemma that any weight space for an ideal of a Lie pseudoalgebra L acting on a finite module is an L-submodule (Proposition 14.1). This implies an analogue of Lie's Theorem that a solvable Lie pseudoalgebra has an eigenvector in any finite module (Theorem 14.1), and an analogue of Cartan–Jacobson Theorem that describes all finite Lie pseudoalgebras which have a finite faithful irreducible module (Theorem 14.2). Finally, we reduce the classification and construction of finite irreducible modules over semisimple Lie pseudoalgebras to that of irreducible modules over linearly compact Lie algebras of the type studied by Rudakov [Ru1, Ru2] (the complete classification will appear in a future publication). Note that complete reducibility fails already in the simplest case of Lie pseudoalgebras with $\dim \mathfrak{d} = 1$ [CKW].

In Section 15 we define cohomology of Lie pseudoalgebras and show that it describes module extensions, abelian pseudoalgebra extensions, and pseudoalgebra deformations. We also relate this cohomology to the Gelfand–Fuchs cohomology [Fu]. These results generalize those of [BKV] in the $\dim \mathfrak{d} = 1$ case.

Note that in the case $\dim \mathfrak{d} = 1$ Lie pseudoalgebras are closely related to vertex algebras in a way similar to the relation of Lie algebras to universal enveloping algebras [K2]. We expect that, under certain conditions, there is a similar relation of "multi-dimensional" Lie pseudoalgebras to "multi-dimensional" vertex algebras defined in [Bo2]. In the case of a commutative Lie algebra $\mathfrak{d}$ the Lie pseudoalgebras encode the OPE between ultralocal fields (as well as the linear Poisson brackets). However, it is not clear how Lie pseudoalgebras are related to the OPE of realistic quantum field theories.

In order to end the introduction on a more optimistic note, we would like to point out that in the definition of a Lie pseudoalgebra one may replace the permutation σ by the map $f \otimes g \mapsto (g \otimes f)\, R$ where R is an R-matrix for H, hence one can take for H any quasi-triangular Hopf algebra (defined in [D]). This observation, the appearance of the classical Yang–Baxter equation, and the fact that the annihilation algebra of the associative pseudoalgebra $\operatorname{Cend} H$ is the Drinfeld double of H, lead us to believe that there should be a deep connection between the theories of pseudoalgebras and quantum groups.

Unless otherwise specified, all vector spaces, linear maps, and tensor products are considered over an algebraically closed field $\mathbf{k}$ of characteristic 0.

2. PRELIMINARIES ON HOPF ALGEBRAS

The goal of this section is to gather some facts and notation which will be used throughout the paper. The material in Sections 2.1 and 2.2 is standard and can be found, for example, in Sweedler's book [Sw]. The material in Section 2.3 seems new.

2.1. *Notation and Basic Identities*

Let H be a Hopf algebra with a coproduct Δ, a counit ε, and an antipode S. We will use the following notation (cf. [Sw]):

$$\Delta(h) = h_{(1)} \otimes h_{(2)}, \tag{2.1}$$

$$(\Delta \otimes \mathrm{id})\, \Delta(h) = (\mathrm{id} \otimes \Delta)\, \Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}, \tag{2.2}$$

$$(S \otimes \mathrm{id})\, \Delta(h) = h_{(-1)} \otimes h_{(2)}, \qquad \text{etc.} \tag{2.3}$$

Note that notation (2.2) uses the coassociativity of Δ. The axioms of the antipode and the counit can be written as

$$h_{(-1)} h_{(2)} = h_{(1)} h_{(-2)} = \varepsilon(h), \tag{2.4}$$

$$\varepsilon(h_{(1)})\, h_{(2)} = h_{(1)} \varepsilon(h_{(2)}) = h, \tag{2.5}$$

while the fact that Δ is a homomorphism of algebras translates as

$$(fg)_{(1)} \otimes (fg)_{(2)} = f_{(1)} g_{(1)} \otimes f_{(2)} g_{(2)}. \tag{2.6}$$

Equations (2.4) and (2.5) imply the following useful relations:

$$h_{(-1)} h_{(2)} \otimes h_{(3)} = 1 \otimes h = h_{(1)} h_{(-2)} \otimes h_{(3)}. \tag{2.7}$$

Let $\mathrm{G}(H)$ be the subset of group-like elements of H, i.e., $g \in H$ such that $\Delta(g) = g \otimes g$. Then $\mathrm{G}(H)$ is a group, because $S(g)\, g = g S(g) = 1$ for $g \in \mathrm{G}(H)$. Let $\mathrm{P}(H)$ be the subspace of primitive elements of H, i.e., $p \in H$ such that $\Delta(p) = p \otimes 1 + 1 \otimes p$. This is a Lie algebra with respect to the commutator $[p, q] = pq - qp$. Note that $\mathrm{G}(H)$ acts on $\mathrm{P}(H)$ by inner automorphisms: $gpg^{-1} \in \mathrm{P}(H)$ for $p \in \mathrm{P}(H)$, $g \in \mathrm{G}(H)$.

The proof of the following theorem may be found in [Sw].

THEOREM 2.1 (Kostant). *Let H be a cocommutative Hopf algebra over* $\mathbf{k}$ (*an algebraically closed field of characteristic* 0). *Then H is isomorphic* (*as* *a Hopf algebra) to the smash product of the universal enveloping algebra* $U(\mathrm{P}(H))$ *and the group algebra* $\mathbf{k}[\mathrm{G}(H)]$.

An associative algebra A is called an *H-differential algebra* if it is also a left H-module such that the multiplication $A \otimes A \to A$ is a homomorphism of H-modules. In other words,

$$h(xy) = (h_{(1)}x)(h_{(2)}y) \tag{2.8}$$

for $h \in H$, $x, y \in A$. The *smash product* $A \sharp H$ of an H-differential algebra A with H is the tensor product $A \otimes H$ of vector spaces but with a new multiplication:

$$(a \sharp g)(b \sharp h) = a(g_{(1)}b) \sharp g_{(2)}h. \tag{2.9}$$

If both A and H are Hopf algebras, then $A \sharp H$ is a Hopf algebra if we consider it as a tensor product of coalgebras. In the theorem above, $U(\mathrm{P}(H))$ is a $\mathbf{k}[\mathrm{G}(H)]$-differential algebra with respect to the adjoint action of $\mathrm{G}(H)$ on $\mathrm{P}(H)$.

It is worth mentioning that as a byproduct of Kostant's Theorem, we obtain that the antipode of a cocommutative Hopf algebra is an involution, i.e., $S^2 = \mathrm{id}$.

We will often be working with the Hopf algebra $H = U(\mathfrak{d})$, where $\mathfrak{d}$ is a finite-dimensional Lie algebra. It is well known that this is a Noetherian domain, and any two nonzero elements $f, g \in H$ have a nonzero left (respectively right) common multiple. In particular, $H = U(\mathfrak{d})$ has a skew-field of fractions K.

Lemma 2.1. *Let H be a Noetherian domain which has a skew-field of fractions K, and let L be a finite H-module. Then there is a homomorphism $i: L \to F$ from L to a free H-module F, whose kernel is the torsion submodule of L. If L is torsion-free, then the module F can be chosen in such a way that $hF \subset i(L)$ for some nonzero $h \in H$ and $i(L)/hF$ is torsion.*

Proof. The kernel of the natural map $\imath: L \to L_K := K \otimes_H L$ is the torsion of L. The image of L under this map is contained inside a free H-submodule of L_K. In order to see this, let us consider a set of H-generators $\{l_1, ..., l_n\}$ of L, and a K-basis $\{v_1, ..., v_k\}$ of L_K. We can express the elements $\imath(l_j)$ as K-linear combinations of the v_i's, and by rescaling elements of this basis by a common multiple of the denominators, we can assume the $\imath(l_j)$'s to be H-linear combinations of the v_i's. Hence the image $\imath(L)$ is contained in the H-module F spanned by the v_i's, which is free by construction.

The fact that F/L is torsion is clear because there exist nonzero elements $h_i \in H$ such that $h_i v_i \in L$. If h is a common multiple of the h_i's, then hF

is contained in L. On the other hand, the inclusion $L \subset F$ implies $hL \subset hF$, hence $h(L/hF) = 0$ and L/hF is torsion. ∎

2.2. *Filtration and Topology*

We define an increasing sequence of subspaces of a Hopf algebra H inductively by

$$\mathrm{F}^n H = 0 \qquad \text{for} \quad n < 0, \qquad \mathrm{F}^0 H = \mathbf{k}[\mathrm{G}(H)], \tag{2.10}$$

$$\mathrm{F}^n H = \mathrm{span}_{\mathbf{k}} \{ h \in H \mid \Delta(h) \in \mathrm{F}^0 H \otimes h + h \otimes \mathrm{F}^0 H + \sum_{i=1}^{n-1} \mathrm{F}^i H \otimes \mathrm{F}^{n-i} H \} \qquad \text{for} \quad n \geqslant 1. \tag{2.11}$$

It has the following properties (which are immediate from definitions):

$$(\mathrm{F}^m H)(\mathrm{F}^n H) \subset \mathrm{F}^{m+n} H, \tag{2.12}$$

$$\Delta(\mathrm{F}^n H) \subset \sum_{i=0}^{n} \mathrm{F}^i H \otimes \mathrm{F}^{n-i} H, \tag{2.13}$$

$$S(\mathrm{F}^n H) \subset \mathrm{F}^n H. \tag{2.14}$$

When H is cocommutative, using Theorem 2.1, one can show that:

$$\bigcup_n \mathrm{F}^n H = H. \tag{2.15}$$

(This condition is also satisfied when H is a quantum universal enveloping algebra.) Provided that (2.15) holds, we say that a nonzero element $a \in H$ has degree n if $a \in \mathrm{F}^n H \backslash \mathrm{F}^{n-1} H$.

When H is a universal enveloping algebra, we get its canonical filtration. Later in some instances we will also impose the following finiteness condition on H:

$$\dim \mathrm{F}^n H < \infty \qquad \forall n. \tag{2.16}$$

It is satisfied when H is a universal enveloping algebra of a finite-dimensional Lie algebra, or its smash product with the group algebra of a finite group.

Now let $X = H^* := \mathrm{Hom}_{\mathbf{k}}(H, \mathbf{k})$ be the dual of H. Recall that H acts on X by the formula ($h, f \in H$, $x \in X$):

$$\langle hx, f \rangle = \langle x, S(h) f \rangle, \tag{2.17}$$

so that X is an associative H-differential algebra (see (2.8)). Moreover, X is commutative when H is cocommutative. Similarly, one can define a right action of H on X by

$$\langle xh, f\rangle = \langle x, fS(h)\rangle, \tag{2.18}$$

and then we have

$$(xy)\,h = (xh_{(1)})(yh_{(2)}). \tag{2.19}$$

Associativity of H implies that X is an H-bimodule, i.e.,

$$f(xg) = (fx)\,g, \qquad f, g \in H, \quad x \in X. \tag{2.20}$$

Let $X = \mathrm{F}_{-1}X \supset \mathrm{F}_0 X \supset \cdots$ be the decreasing sequence of subspaces of X dual to $\mathrm{F}^n H$: $\mathrm{F}_n X = (\mathrm{F}^n H)^\perp$. It has the following properties:

$$(\mathrm{F}_m X)(\mathrm{F}_n X) \subset \mathrm{F}_{m+n+1} X, \tag{2.21}$$

$$(\mathrm{F}^m H)(\mathrm{F}_n X) \subset \mathrm{F}_{n-m} X, \tag{2.22}$$

and

$$\bigcap_n \mathrm{F}_n X = 0, \qquad \text{provided that (2.15) holds.} \tag{2.23}$$

We define a topology of X by considering $\{\mathrm{F}_n X\}$ as a fundamental system of neighborhoods of 0. We will always consider X with this topology, while H with the discrete topology. It follows from (2.23) that X is Hausdorff, provided that (2.15) holds. By (2.21) and (2.22), the multiplication of X and the action of H on it are continuous; in other words, X is a topological H-differential algebra.

We define an antipode $S: X \to X$ as the dual of that of H:

$$\langle S(x), h\rangle = \langle x, S(h)\rangle. \tag{2.24}$$

Then we have

$$S(ab) = S(b)\,S(a) \qquad \text{for} \quad a, b \in X \text{ or } H. \tag{2.25}$$

We will also define a comultiplication $\Delta: X \to X \mathbin{\hat{\otimes}} X$ as the dual of the multiplication $H \otimes H \to H$, where $X \mathbin{\hat{\otimes}} X := (H \otimes H)^*$ is the completed tensor product. Formally, we will use the same notation for X as for H (see

(2.1)–(2.3)), writing for example, $\Delta(x) = x_{(1)} \otimes x_{(2)}$ for $x \in X$. By definition, for $x, y \in X$, $f, g \in H$, we have:

$$\langle xy, f\rangle = \langle x \otimes y, \Delta(f)\rangle = \langle x, f_{(1)}\rangle \langle y, f_{(2)}\rangle, \tag{2.26}$$

$$\langle x, fg\rangle = \langle \Delta(x), f \otimes g\rangle = \langle x_{(1)}, f\rangle \langle x_{(2)}, g\rangle. \tag{2.27}$$

We have

$$S(\mathrm{F}_n X) \subset \mathrm{F}_n X, \tag{2.28}$$

$$\Delta(\mathrm{F}_n X) \subset \sum_{i=-1}^{n} \mathrm{F}_i X \mathbin{\hat{\otimes}} \mathrm{F}_{n-1-i} X. \tag{2.29}$$

If H satisfies the finiteness condition (2.16), then the filtration of X satisfies

$$\dim X/\mathrm{F}_n X < \infty \qquad \forall n, \tag{2.30}$$

which implies that X is linearly compact (see Section 6 below).

By a basis of X we will always mean a topological basis $\{x_i\}$ which tends to 0, i.e., such that for any n all but a finite number of x_i belong to $\mathrm{F}_n X$. Let $\{h_i\}$ be a basis of H (as a vector space) compatible with the filtration. Then the set of elements $\{x_i\}$ of X defined by $\langle x_i, h_j\rangle = \delta_{ij}$ is called the dual basis of X. If H satisfies (2.16), then $\{x_i\}$ is a basis of X in the above sense, i.e., it tends to 0. We have for $g \in H$, $y \in X$,

$$g = \sum_i \langle g, x_i\rangle h_i, \qquad y = \sum_i \langle y, h_i\rangle\, x_i,$$

where the first sum is finite, and the second one is convergent in X.

EXAMPLE 2.1. Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of an N-dimensional Lie algebra $\mathfrak{d}$. Fix a basis $\{\partial_i\}$ of $\mathfrak{d}$, and for $I = (i_1, ..., i_N) \in \mathbb{Z}_+^N$ let $\partial^{(I)} = \partial_1^{i_1} \cdots \partial_N^{i_N} / i_1! \cdots i_N!$. Then $\{\partial^{(I)}\}$ is a basis of H (the Poincaré–Birkhoff–Witt basis). Moreover, it is easy to see that

$$\Delta(\partial^{(I)}) = \sum_{J+K=I} \partial^{(J)} \otimes \partial^{(K)}. \tag{2.31}$$

If $\{t_I\}$ is the dual basis of X, defined by $\langle t_I, \partial^{(J)}\rangle = \delta_{I,J}$, then (2.31) implies $t_J t_K = t_{J+K}$. Therefore, X can be identified with the ring $\mathcal{O}_N = \mathbf{k}[[t_1, ..., t_N]]$ of formal power series in N indeterminates. Then the action of H on $\mathcal{O}_N$ is given by differential operators.

LEMMA 2.2. *If $\{h_i\}$, $\{x_i\}$ are dual bases in H and X, then*

$$\Delta(x) = \sum_i xS(h_i) \otimes x_i = \sum_i x_i \otimes S(h_i)\, x \tag{2.32}$$

for any $x \in X$.

Proof. For $f, g \in H$, we have

$$\left\langle \sum_i xS(h_i) \otimes x_i, f \otimes g \right\rangle = \sum_i \langle xS(h_i), f \rangle \langle x_i, g \rangle$$
$$= \langle xS(g), f \rangle = \langle x, fg \rangle = \langle \Delta(x), f \otimes g \rangle,$$

which proves the first identity. The second one is proved in the same way. ▮

2.3. *Fourier Transform*

For an arbitrary Hopf algebra H, we introduce a map $\mathscr{F} : H \otimes H \to H \otimes H$, called the *Fourier transform*, by the formula

$$\mathscr{F}(f \otimes g) = (f \otimes 1)\,(S \otimes \mathrm{id})\,\Delta(g) = fg_{(-1)} \otimes g_{(2)}. \tag{2.33}$$

It follows from (2.7) that $\mathscr{F}$ is a vector space isomorphism with an inverse given by

$$\mathscr{F}^{-1}(f \otimes g) = (f \otimes 1)\,\Delta(g) = fg_{(1)} \otimes g_{(2)}. \tag{2.34}$$

Indeed, using the coassociativity of Δ and (2.7), we compute

$$\mathscr{F}^{-1}(fg_{(-1)} \otimes g_{(2)}) = fg_{(-1)}(g_{(2)})_{(1)} \otimes (g_{(2)})_{(2)} = fg_{(-1)}g_{(2)} \otimes g_{(3)} = f \otimes g.$$

The significance of $\mathscr{F}$ is in the identity

$$f \otimes g = \mathscr{F}^{-1}\mathscr{F}(f \otimes g) = (fg_{(-1)} \otimes 1)\,\Delta(g_{(2)}), \tag{2.35}$$

which, together with properties (2.12)–(2.14) of the filtration of H, implies the next result.

LEMMA 2.3. (i) *Every element of $H \otimes H$ can be uniquely represented in the form $\sum_i (h_i \otimes 1)\,\Delta(l_i)$, where $\{h_i\}$ is a fixed $\mathbf{k}$-basis of H and $l_i \in H$. In other words, $H \otimes H = (H \otimes \mathbf{k})\,\Delta(H)$.*

(ii) *We have*

$$(\mathrm{F}^n H \otimes \mathbf{k})\,\Delta(H) = \mathrm{F}^n(H \otimes H)\,\Delta(H) = (\mathbf{k} \otimes \mathrm{F}^n H)\,\Delta(H), \tag{2.36}$$

where $\mathrm{F}^n(H \otimes H) = \sum_{i+j=n} \mathrm{F}^i H \otimes \mathrm{F}^j H$.

In particular, for any H-module W, we have

$$(\mathrm{F}^n H \otimes \mathbf{k}) \otimes_H W = \mathrm{F}^n (H \otimes H) \otimes_H W = (\mathbf{k} \otimes \mathrm{F}^n H) \otimes_H W. \tag{2.37}$$

Proof. For $h \in H \otimes H$ we have

$$h = \sum_i (h_i \otimes 1)\, \Delta(l_i) = \mathscr{F}^{-1}\Big(\sum_i h_i \otimes l_i\Big) \qquad \text{iff} \quad \sum_i h_i \otimes l_i = \mathscr{F}(h).$$

This proves (i).

To prove (2.36), it is enough to show that $\mathrm{F}^n(H \otimes H) \subset (\mathrm{F}^n H \otimes \mathbf{k})\, \Delta(H)$. This follows from the above equation and the fact that $\mathscr{F}(\mathrm{F}^n(H \otimes H)) \subset \mathrm{F}^n(H \otimes H) \subset \mathrm{F}^n H \otimes H$. ∎

The Fourier transform $\mathscr{F}$ has the following properties (which are easy to check using (2.4)–(2.6)):

$$\mathscr{F}((f \otimes g)\, \Delta(h)) = \mathscr{F}(f \otimes g)\,(1 \otimes h), \tag{2.38}$$

$$\mathscr{F}(hf \otimes g) = (h \otimes 1)\, \mathscr{F}(f \otimes g), \tag{2.39}$$

$$\mathscr{F}(f \otimes hg) = (1 \otimes h_{(2)})\, \mathscr{F}(f \otimes g)\,(h_{(-1)} \otimes 1), \tag{2.40}$$

$$\mathscr{F}_{12}\mathscr{F}_{13}\mathscr{F}_{23} = \mathscr{F}_{23}\mathscr{F}_{12}. \tag{2.41}$$

Here in (2.41), we use the standard notation $\mathscr{F}_{12} = \mathscr{F} \otimes \mathrm{id}$ acting on $H \otimes H \otimes H$.

3. PSEUDOTENSOR CATEGORIES AND PSEUDOALGEBRAS

In this section, we review some definitions of Beilinson and Drinfeld [BD]; we also use the exposition in [BKV, Sect. 12].

The theory of conformal algebras [K2] is in many ways analogous to the theory of Lie algebras. The reason is that in fact conformal algebras can be considered as Lie algebras in a certain "pseudotensor" category, instead of the category of vector spaces. A pseudotensor category [BD] is a category equipped with "polylinear maps" and a way to compose them (such categories were first introduced by Lambek [L] under the name multicategories). This is enough to define the notions of Lie algebra, representations, cohomology, etc.

As an example, consider first the category $\mathcal{VEC}$ of vector spaces (over $\mathbf{k}$). For a finite nonempty set I and a collection of vector spaces $\{L_i\}_{i \in I}$, M, we can define the space of *polylinear maps* from $\{L_i\}_{i \in I}$ to M as

$$\operatorname{Lin}(\{L_i\}_{i \in I}, M) = \operatorname{Hom}\left(\bigotimes_{i \in I} L_i, M\right).$$

The symmetric group S_I acts among these spaces by permuting the factors in $\bigotimes_{i \in I} L_i$.

For any surjection of finite sets $\pi: J \twoheadrightarrow I$ and a collection $\{N_j\}_{j \in J}$, we have the obvious compositions of polylinear maps

$$\operatorname{Lin}(\{L_i\}_{i \in I}, M) \otimes \bigotimes_{i \in I} \operatorname{Lin}(\{N_j\}_{j \in J_i}, L_i) \to \operatorname{Lin}(\{N_j\}_{j \in J}, M), \tag{3.1}$$

$$\phi \times \{\psi_i\}_{i \in I} \mapsto \phi \circ \left(\bigotimes_{i \in I} \psi_i\right) \equiv \phi(\{\psi_i\}_{i \in I}), \tag{3.2}$$

where $J_i = \pi^{-1}(i)$ for $i \in I$.

The compositions have the following properties:

Associativity. If $K \twoheadrightarrow J$, $\{P_k\}_{k \in K}$ is a family of objects and $\chi_j \in \operatorname{Lin}(\{P_k\}_{k \in K_j}, N_j)$, then $\phi(\{\psi_i(\{\chi_j\}_{j \in J_i})\}_{i \in I}) = (\phi(\{\psi_i\}_{i \in I}))(\{\chi_j\}_{j \in J}) \in \operatorname{Lin}(\{P_k\}_{k \in K}, M)$.

Unit. For any object M there is an element $\mathrm{id}_M \in \operatorname{Lin}(\{M\}, M)$ such that for any $\phi \in \operatorname{Lin}(\{L_i\}_{i \in I}, M)$ one has $\mathrm{id}_M(\phi) = \phi(\{\mathrm{id}_{L_i}\}_{i \in I}) = \phi$.

Equivariance. The compositions (3.1) are equivariant with respect to the natural action of the symmetric group.

DEFINITION 3.1 [BD]. *A pseudotensor category* is a class of objects $\mathcal{M}$ together with vector spaces $\operatorname{Lin}(\{L_i\}_{i \in I}, M)$, equipped with actions of the symmetric groups S_I among them and composition maps (3.1), satisfying the above three properties.

Remark 3.1. For a pseudotensor category $\mathcal{M}$ and objects $L, M \in \mathcal{M}$, let $\operatorname{Hom}(L, M) = \operatorname{Lin}(\{L\}, M)$. This gives a structure of an ordinary (additive) category on $\mathcal{M}$ and all Lin are functors $(\mathcal{M}^\circ)^I \times \mathcal{M} \to \mathcal{VEC}$, where $\mathcal{M}^\circ$ is the dual category of $\mathcal{M}$.

Remark 3.2. The notion of pseudotensor category is a straightforward generalization of the notion of operad. By definition, an *operad* is a pseudotensor category with only one object.

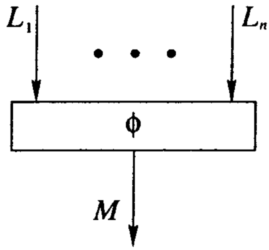

FIG. 1. A polylinear map from $\{L_i\}_{i=1}^n$ to M.

DEFINITION 3.2. A *Lie algebra in a pseudotensor category* $\mathcal{M}$ is an object L equipped with $\beta \in \mathrm{Lin}(\{L, L\}, L)$ satisfying the following properties.

Skew-commutativity. $\beta = -\sigma_{12}\,\beta$, where $\sigma_{12} = (12) \in S_2$.

Jacobi identity. $\beta(\beta(\cdot, \cdot), \cdot) = \beta(\cdot, \beta(\cdot, \cdot)) - \sigma_{12}\,\beta(\cdot, \beta(\cdot, \cdot))$, where now $\sigma_{12} = (12)$ is viewed as an element of S_3.

It is instructive to think of a polylinear map $\phi \in \mathrm{Lin}(\{L_i\}_{i=1}^n, M)$ as an operation with n inputs and 1 output, as depicted in Fig. 1. The skew-commutativity and Jacobi identity for a Lie algebra (L, β) are represented pictorially in Figs. 2 and 3.

DEFINITION 3.3. A *representation* of a Lie algebra (L, β) is an object M together with $\rho \in \mathrm{Lin}(\{L, M\}, M)$ satisfying

$$\rho(\beta(\cdot, \cdot), \cdot) = \rho(\cdot, \rho(\cdot, \cdot)) - \sigma_{12}\,\rho(\cdot, \rho(\cdot, \cdot)).$$

Similarly, one can define cohomology of a Lie algebra (L, β) with coefficients in a module (M, ρ) (cf. [BKV]).

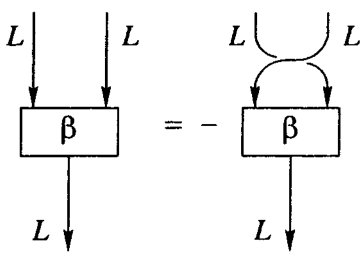

FIG. 2. Skew-commutativity.

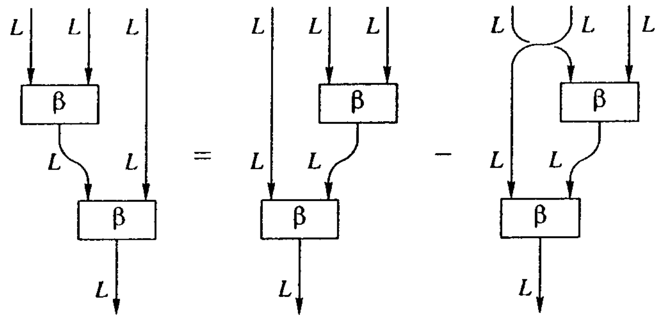

FIG. 3. Jacobi identity.

Definition 3.4. An *n-cochain* of a Lie algebra (L, β), with coefficients in a module (M, ρ) over it, is a polylinear operation

$$\gamma \in \operatorname{Lin}(\{\underbrace{L, \ldots, L}_{n}\}, M)$$

which is skew-symmetric, i.e., satisfying for all $i = 1, \ldots, n-1$ the identity shown in Fig. 4. The differential $d\gamma$ of a cochain γ is defined by Fig. 5. The same computation as in the ordinary Lie algebra case shows that $d^2 = 0$. The cohomology of the resulting complex is called the *cohomology of L with coefficients in M* and is denoted by $\mathrm{H}^{\bullet}(L, M)$.

Here are two simple examples.

Example 3.1. A Lie algebra in the category of vector spaces $\mathscr{V}\mathscr{E}\mathscr{C}$ is just an ordinary Lie algebra. The same is true for representations and cohomology.

Example 3.2. Let H be a cocommutative bialgebra. Then the category $\mathscr{M}^l(H)$ of left H-modules is a symmetric tensor category. Hence, $\mathscr{M}^l(H)$

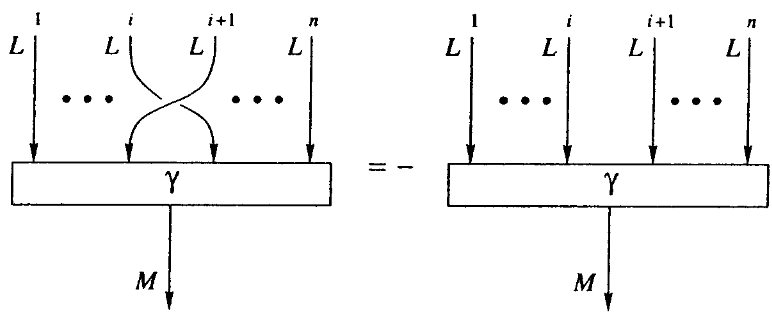

FIG. 4. Skew-symmetry of a cochain.

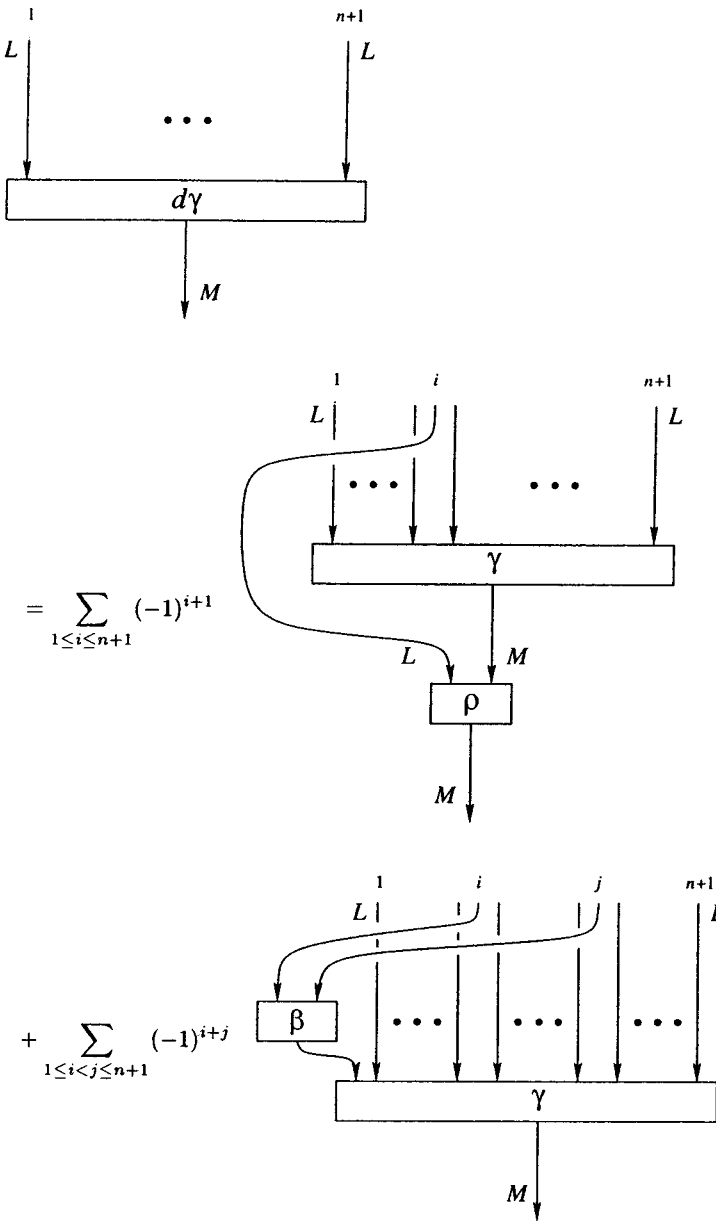

FIG. 5. Differential of a cochain.

is a pseudotensor category with the obvious pseudotensor structure

$$\operatorname{Lin}(\{L_i\}_{i\in I}, M) = \operatorname{Hom}_H\left(\bigotimes_{i\in I} L_i, M\right). \tag{3.3}$$

The composition of polylinear maps is given by (3.2). An algebra (e.g., Lie or associative) in the category $\mathcal{M}^l(H)$ will be called an *H-differential algebra*: this is an ordinary algebra which is also a left H-module and such that the product (or the bracket) is a homomorphism of H-modules; see (2.8).

One can also define the notions of associative algebra or commutative algebra in a pseudotensor category, their representations and analogues of the Hochschild, cyclic, or Harrison cohomology.

DEFINITION 3.5. An *associative algebra in a pseudotensor category* $\mathcal{M}$ is an object A and a product $\mu \in \operatorname{Lin}(\{A, A\}, A)$ satisfying

Associativity. $\mu(\mu(\cdot, \cdot), \cdot) = \mu(\cdot, \mu(\cdot, \cdot))$,

see Fig. 6. The algebra (A, μ) is called *commutative* if, in addition, μ satisfies

Commutativity. $\mu = \sigma_{12}\mu$, where $\sigma_{12} = (12) \in S_2$.

Remark 3.3. In order to define the notion of an associative algebra in a pseudotensor category, one does not use the actions of the symmetric groups among the spaces of polylinear maps. One can relax the definition of a pseudotensor category by forgetting these actions. Then what we call a "pseudotensor category" should be termed a "symmetric pseudotensor category," while there is a more general notion of a "braided" one (cf. [So]).

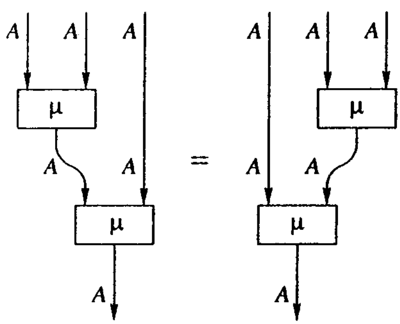

FIG. 6. Associativity.

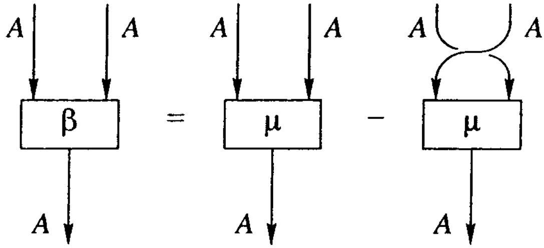

FIG. 7. Commutator.

PROPOSITION 3.1. *Let (A, μ) be an associative algebra in a pseudotensor category $\mathscr{M}$. Define $\beta \in \mathrm{Lin}(\{A, A\}, A)$ as the commutator $\beta := \mu - \sigma_{12}\,\mu$, see Fig. 7. Then (A, β) is a Lie algebra in $\mathscr{M}$.*

Proof is straightforward.

Now we turn to our main example of a pseudotensor category. Let H be a cocommutative bialgebra with a comultiplication Δ. We introduce a pseudotensor category $\mathscr{M}^*(H)$ with the same objects as $\mathscr{M}^l(H)$ (i.e., left H-modules) but with another pseudotensor structure [BD]:

$$\mathrm{Lin}(\{L_i\}_{i \in I}, M) = \mathrm{Hom}_{H^{\otimes I}}\Big(\boxtimes_{i \in I} L_i,\ H^{\otimes I} \otimes_H M\Big). \tag{3.4}$$

Here $\boxtimes_{i \in I}$ is the tensor product functor $\mathscr{M}^l(H)^I \to \mathscr{M}^l(H^{\otimes I})$. For a surjection $\pi\colon J \twoheadrightarrow I$, the composition of polylinear maps is defined as follows:

$$\phi(\{\psi_i\}_{i \in I}) = \Delta^{(\pi)}(\phi) \circ \Big(\boxtimes_{i \in I} \psi_i\Big). \tag{3.5}$$

Here $\Delta^{(\pi)}$ is the functor $\mathscr{M}^l(H^{\otimes I}) \to \mathscr{M}^l(H^{\otimes J})$, $M \mapsto H^{\otimes J} \otimes_{H^{\otimes I}} M$, where $H^{\otimes I}$ acts on $H^{\otimes J}$ via the iterated comultiplication determined by π.

Explicitly, let $n_j \in N_j$ $(j \in J)$, and write

$$\psi_i\Big(\bigotimes_{j \in J_i} n_j\Big) = \sum_r g_i^r \otimes_H l_i^r, \qquad g_i^r \in H^{\otimes J_i}, \quad l_i^r \in L_i, \tag{3.6}$$

where, as before, $J_i = \pi^{-1}(i)$ for $i \in I$. Let

$$\phi\Big(\bigotimes_{i \in I} l_i^r\Big) = \sum_s f^{rs} \otimes_H m^{rs}, \qquad f^{rs} \in H^{\otimes I}, \quad m^{rs} \in M. \tag{3.7}$$

Then, by definition,

$$(\phi(\{\psi_i\}_{i \in I}))\Big(\bigotimes_{j \in J} n_j\Big) = \sum_{r,s}\Big(\bigotimes_{i \in I} g_i^r\Big)\,\Delta^{(\pi)}(f^{rs}) \otimes_H m^{rs}, \tag{3.8}$$

where $\Delta^{(\pi)}: H^{\otimes I} \to H^{\otimes J}$ is the iterated comultiplication determined by π. For example, if $\pi: \{1, 2, 3\} \to \{1, 2\}$ is given by $\pi(1) = \pi(2) = 1$, $\pi(3) = 2$, then $\Delta^{(\pi)} = \Delta \otimes \mathrm{id}$; if $\pi(1) = 1$, $\pi(2) = \pi(3) = 2$, then $\Delta^{(\pi)} = \mathrm{id} \otimes \Delta$.

The symmetric group S_I acts among the spaces $\mathrm{Lin}(\{L_i\}_{i \in I}, M)$ by simultaneously permuting the factors in $\boxtimes_{i \in I} L_i$ and $H^{\otimes I}$. This is the only place where we need the cocommutativity of H; for example, the permutation $\sigma_{12} = (12) \in S_2$ acts on $(H \otimes H) \otimes_H M$ by

$$\sigma_{12}((f \otimes g) \otimes_H m) = (g \otimes f) \otimes_H m,$$

and this is well defined only when H is cocommutative.

One can generalize the above construction for (quasi)triangular bialgebras as follows.

Remark 3.4. Let H be a triangular bialgebra with a universal R-matrix R. Recall that R is an invertible element of $H \otimes H$ satisfying the following equations:

$$\sigma(R) = R^{-1}, \tag{3.9}$$

$$\sigma(\Delta(h))\, R = R\Delta(h) \qquad \forall h \in H, \tag{3.10}$$

$$(\mathrm{id} \otimes \Delta)\, R = R_{13} R_{12}, \tag{3.11}$$

$$(\Delta \otimes \mathrm{id})\, R = R_{13} R_{23}, \tag{3.12}$$

where σ is the permutation $\sigma(f \otimes g) = g \otimes f$, and we use the standard notation $R_{12} = R \otimes \mathrm{id} \in H \otimes H \otimes H$, etc. Then we define a pseudotensor category $\mathscr{M}^*(H)$ as above but with a modified action of the symmetric groups. It is enough to describe the action of the transposition $\sigma_{12} = (12) \in S_2$ on $(H \otimes H) \otimes_H M$; it is given by

$$\sigma_{12}((f \otimes g) \otimes_H m) = (g \otimes f)\, R \otimes_H m.$$

This is well defined because of (3.10), and $\sigma_{12}^2 = \mathrm{id}$ because of (3.9). Since any permutation is a product of transpositions, this can be extended to an action of the symmetric group among the spaces of polylinear maps; due to (3.11), (3.12), this action is compatible with compositions.

If H is quasitriangular, i.e., if we drop relation (3.9), we will get an action of the braid group instead of the symmetric one and a "braided" pseudotensor category (cf. Remark 3.3).

The following notion will be the main object of our study.

DEFINITION 3.6. A *Lie H-pseudoalgebra* (or just a Lie pseudoalgebra) is a Lie algebra (L, β) in the pseudotensor category $\mathscr{M}^*(H)$ defined above.

Examples of Lie pseudoalgebras will be given in Sections 4 and 8 below. One can also define *associative H-pseudoalgebras* as associative algebras (A, μ) in the pseudotensor category $\mathcal{M}^*(H)$. It is convenient to define the general notion of an algebra in $\mathcal{M}^*(H)$ as follows.

DEFINITION 3.7. An *H-pseudoalgebra* (or just a pseudoalgebra) is a left H-module A together with an operation $\mu \in \mathrm{Hom}_{H\otimes H}(A\otimes A, (H\otimes H) \otimes_H A)$, called the *pseudoproduct*.

We will denote the pseudoproduct $\mu(a\otimes b) \in (H\otimes H)\otimes_H A$ of two elements $a, b\in A$ by $a * b$. It has the following defining property:

H-bilinearity. For $a, b \in A$, $f, g\in H$, one has

$$fa * gb = ((f\otimes g)\otimes_H 1)(a * b). \tag{3.13}$$

Explicitly, if

$$a * b = \sum_i (f_i \otimes g_i)\otimes_H e_i , \tag{3.14}$$

then $fa * gb = \sum_i (ff_i \otimes gg_i)\otimes_H e_i$.

To describe explicitly the associativity condition for a pseudoproduct μ, we need to compute the compositions $\mu(\mu(\cdot, \cdot), \cdot)$ and $\mu(\cdot, \mu(\cdot, \cdot))$ in $\mathcal{M}^*(H)$. Let $a * b$ be given by (3.14), and let

$$e_i * c = \sum_{i,j} (f_{ij} \otimes g_{ij})\otimes_H e_{ij} . \tag{3.15}$$

Then $(a * b) * c \equiv \mu(\mu(a\otimes b)\otimes c)$ is the following element of $H^{\otimes 3}\otimes_H A$ (cf. (3.8)):

$$(a * b) * c = \sum_{i,j} (f_i f_{ij(1)} \otimes g_i f_{ij(2)} \otimes g_{ij})\otimes_H e_{ij} . \tag{3.16}$$

Similarly, if we write

$$b * c = \sum_i (h_i \otimes l_i)\otimes_H d_i , \tag{3.17}$$

$$a * d_i = \sum_{i,j} (h_{ij} \otimes l_{ij})\otimes_H d_{ij} , \tag{3.18}$$

then

$$a * (b * c) = \sum_{i,j} (h_{ij} \otimes h_i l_{ij(1)} \otimes l_i l_{ij(2)})\otimes_H d_{ij} . \tag{3.19}$$

Now a pseudoproduct $a * b$ is associative iff it satisfies

Associativity,

$$a * (b * c) = (a * b) * c \tag{3.20}$$

in $H^{\otimes 3} \otimes_H A$, where the compositions $(a * b) * c$ and $a * (b * c)$ are given by the above formulas.

The pseudoproduct $a * b$ is commutative iff it satisfies

Commutativity,

$$b * a = (\sigma \otimes_H \mathrm{id})\,(a * b), \tag{3.21}$$

where $\sigma\colon H \otimes H \to H \otimes H$ is the permutation $\sigma(f \otimes g) = g \otimes f$. Explicitly,

$$b * a = \sum_i (g_i \otimes f_i) \otimes_H e_i, \tag{3.22}$$

if $a * b$ is given by (3.14). Note that the right-hand side of (3.21) is well defined due to the cocommutativity of H.

In the case of a Lie pseudoalgebra (L, β), we will call the pseudoproduct β a *pseudobracket*, and we will denote it by $[a * b]$. Let us spell out its properties $(a, b, c \in L,\ f, g \in H)$:

H-bilinearity,

$$[fa * gb] = ((f \otimes g) \otimes_H 1)\,[a * b]. \tag{3.23}$$

Skew-commutativity,

$$[b * a] = -(\sigma \otimes_H \mathrm{id})\,[a * b]. \tag{3.24}$$

Jacobi identity,

$$[a * [b * c]] - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\,[b * [a * c]] = [[a * b] * c] \tag{3.25}$$

in $H^{\otimes 3} \otimes_H L$, where the compositions $[[a * b] * c]$ and $[a * [b * c]]$ are defined as above.

PROPOSITION 3.2. *Let (A, μ) be an associative H-pseudoalgebra. Define a pseudobracket β as the commutator $[a * b] = a * b - (\sigma \otimes_H \mathrm{id})\,(b * a)$. Then (A, β) is a Lie H-pseudoalgebra* (*cf. Proposition* 3.1).

The definitions of representations of Lie pseudoalgebras or associative pseudoalgebras are obvious modifications of the above.

DEFINITION 3.8. A *representation* of an associative H-pseudoalgebra A is a left H-module M together with an operation $\rho \in \mathrm{Lin}(\{A, M\}, M)$, written as $a * c \equiv \rho(a \otimes c) \in (H \otimes H) \otimes_H M$, which satisfies (3.20) for $a, b \in A$, $c \in M$.

DEFINITION 3.9. A *representation* of a Lie H-pseudoalgebra L is a left H-module M together with an operation $\rho \in \mathrm{Lin}(\{L, M\}, M)$, written as $a * c \equiv \rho(a \otimes c)$, which satisfies

$$a * (b * c) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\,(b * (a * c)) = [a * b] * c \tag{3.26}$$

for $a, b \in L$, $c \in M$.

4. SOME EXAMPLES OF LIE PSEUDOALGEBRAS

In this section we give some examples of Lie pseudoalgebras, and discuss their relationship with previously known objects. Other important examples—the pseudoalgebras of vector fields—are treated in detail in Section 8.

4.1. *Conformal Algebras*

The (Lie) conformal algebras introduced by Kac [K2] are exactly the (Lie) $\mathbf{k}[\partial]$-pseudoalgebras, where $\mathbf{k}[\partial]$ is the Hopf algebra of polynomials in one variable ∂. The explicit relation between the λ-bracket of [DK] and the pseudobracket of Section 3 is:

$$[a_\lambda b] = \sum_i p_i(\lambda)\, c_i \Leftrightarrow [a * b] = \sum_i (p_i(-\partial) \otimes 1) \otimes_{\mathbf{k}[\partial]} c_i\,.$$

This correspondence has been explained in detail in the introduction.

Similarly, for $H = \mathbf{k}[\partial_1, ..., \partial_N]$ we get conformal algebras in N indeterminates, see [BKV, Sect. 10]. We may say that for $N = 0$, H is $\mathbf{k}$; then a $\mathbf{k}$-conformal algebra is the same as a Lie algebra, cf. Example 3.1.

On the other hand, when $H = \mathbf{k}[\Gamma]$ is the group algebra of a group Γ, one obtains the Γ-conformal algebras studied in [GK]. This is a special case of a more general construction described in Section 5 below.

4.2. *Current Pseudoalgebras*

Let H' be a Hopf subalgebra of H, and let A be an H'-pseudoalgebra. Then we define the *current* H-pseudoalgebra $\mathrm{Cur}_{H'}^{H} A \equiv \mathrm{Cur}\, A$ as $H \otimes_{H'} A$

by extending the pseudoproduct $a * b$ of A using the H-bilinearity. Explicitly, for $a, b \in A$, we define

$$(f \otimes_{H'} a) * (g \otimes_{H'} b) = ((f \otimes g) \otimes_H 1)(a * b)$$
$$= \sum_i (ff_i \otimes gg_i) \otimes_H (1 \otimes_{H'} e_i),$$

if $a * b = \sum_i (f_i \otimes g_i) \otimes_{H'} e_i$. Then $\mathrm{Cur}_{H'}^H A$ is an H-pseudoalgebra which is Lie or associative when A is so.

An important special case is when $H' = \mathbf{k}$: given a Lie algebra $\mathfrak{g}$, let $\mathrm{Cur}\,\mathfrak{g} = H \otimes \mathfrak{g}$ with the following pseudobracket

$$[(f \otimes a) * (g \otimes b)] = (f \otimes g) \otimes_H (1 \otimes [a, b]).$$

Then $\mathrm{Cur}\,\mathfrak{g}$ is a Lie H-pseudoalgebra.

4.3. *H-Pseudoalgebras of Rank* 1

Let $L = He$ be a Lie pseudoalgebra which is a free H-module of rank 1. Then, by H-bilinearity, the pseudobracket on L is determined by $[e * e]$, or equivalently, by an $\alpha \in H \otimes H$ such that $[e * e] = \alpha \otimes_H e$.

PROPOSITION 4.1. *$L = He$ with the pseudobracket $[e * e] = \alpha \otimes_H e$ is a Lie H-pseudoalgebra iff $\alpha \in H \otimes H$ satisfies the following equations*:

$$\alpha = -\sigma(\alpha), \tag{4.1}$$

$$(\alpha \otimes 1)(\Delta \otimes \mathrm{id})(\alpha) = (1 \otimes \alpha)(\mathrm{id} \otimes \Delta)(\alpha) - (\sigma \otimes \mathrm{id})((1 \otimes \alpha)(\mathrm{id} \otimes \Delta)(\alpha)). \tag{4.2}$$

Similarly, $A = Ha$ with a pseudoproduct $a * a = \alpha \otimes_H a$ is an associative H-pseudoalgebra iff α satisfies

$$(\alpha \otimes 1)(\Delta \otimes \mathrm{id})(\alpha) = (1 \otimes \alpha)(\mathrm{id} \otimes \Delta)(\alpha).$$

Proof. Follows immediately from definitions. Indeed, if $[e * e] = \alpha \otimes_H e$, then:

$$[[e * e] * e] = (\alpha \otimes 1)(\Delta \otimes \mathrm{id})(\alpha) \otimes_H e,$$
$$[e * [e * e]] = (1 \otimes \alpha)(\mathrm{id} \otimes \Delta)(\alpha) \otimes_H e. \quad \blacksquare$$

LEMMA 4.1. *Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{d}$. Then any solution $\alpha \in H \otimes H$ of Eqs.* (4.1), (4.2) *is of the form $\alpha = r + s \otimes 1 - 1 \otimes s$, where $r \in \mathfrak{d} \wedge \mathfrak{d}$, $s \in \mathfrak{d}$.*

In this case (4.2) *is equivalent to the following system of equations*:

$$[r, \Delta(s)] = 0, \tag{4.3}$$

$$([r_{12}, r_{13}] + r_{12}s_3) + \text{cyclic} = 0. \tag{4.4}$$

(As usual, $r_{12} = r \otimes 1$, $s_3 = 1 \otimes 1 \otimes s$, etc., and "cyclic" here and further means applying the two nontrivial cyclic permutations on $H \otimes H \otimes H$.)

Proof. Using an argument similar to that of [Ki], we will show that if α satisfies (4.2) then $\alpha \in H \otimes (\mathfrak{d} + \mathbf{k})$. Then (4.1) will imply the first claim, that $\alpha \in (\mathfrak{d} + \mathbf{k}) \otimes (\mathfrak{d} + \mathbf{k})$.

Let $\{\partial_1, ..., \partial_N\}$ be a basis of $\mathfrak{d}$, and let us consider the corresponding Poincaré–Birkhoff–Witt basis of $H = U(\mathfrak{d})$ given by elements $\partial^{(I)} := \partial_1^{i_1} \cdots \partial_N^{i_N} / i_1! \cdots i_N!$, where $I = (i_1, ..., i_N) \in \mathbb{Z}_+^N$. In this basis the comultiplication takes the simple form (2.31). We can write $\alpha = \sum_I \alpha_I \otimes \partial^{(I)}$, $\alpha_I \in H$. Equation (4.2) then becomes

$$\sum_I \alpha \Delta(\alpha_I) \otimes \partial^{(I)} = \sum_{I, J, K} (\alpha_{J+K} \otimes \alpha_I \partial^{(J)} - \alpha_I \partial^{(J)} \otimes \alpha_{J+K}) \otimes \partial^{(I)} \partial^{(K)}. \tag{4.5}$$

Let p be the maximal value of $|I| = i_1 + \cdots + i_N$ for I such that $\alpha_I \neq 0$. We want to show that $p \leqslant 1$. Among all I such that $|I| = p$ there will be some (nonzero) α_I of maximal degree d. Then without loss of generality we can change the basis $\partial_1, ..., \partial_N$ and assume that the coefficient $\alpha_{(p, 0, ..., 0)}$ is nonzero and of degree d. If $p > 1$, then no nonzero term in the left-hand side of (4.5) has a third tensor factor of degree $2p$ or $2p - 1$ since $2p - 1 > p$. Hence, terms from the right-hand side of degree $2p$ (respectively $2p - 1$) in the third tensor factor must cancel against each other.

Terms having degree $2p$ in the third tensor factor cancel, since they give the sum

$$\sum_{|I| = |K| = p} \alpha_K \otimes \alpha_I \otimes [\partial^{(I)}, \partial^{(K)}], \tag{4.6}$$

which in the third tensor factor has degree $2p - 1$ and lower. Note also that their coefficients have total degree at most $2d$.

Terms having third tensor factors of degree $2p - 1$, besides (4.6), arise when we choose $|I + K| = 2p - 1$. Those with $|I| = p - 1$, $|K| = p$ can be expressed in terms of commutators as above, and hence only contribute to lower degree. So, we only need to account for terms with $|I| = p$, $|K| = p - 1$.

Let us focus on such terms having a third tensor factor proportional to ∂_1^{2p-1}, whose coefficient must be zero. They occur in (4.5) only when

$I=(p, 0, ..., 0)$, $K=(p-1, 0, ..., 0)$. When $J=0$, things cancel as above. The only other nonzero terms are

$$\sum_j (\alpha_{K+\varepsilon_j} \otimes \alpha_I \partial_j - \alpha_I \partial_j \otimes \alpha_{K+\varepsilon_j}) \otimes \partial^{(I)} \partial^{(K)},$$

where $\{\varepsilon_j\}$ is the standard basis of $\mathbb{Z}^N$.

We have seen that all other contributions have coefficients of degree at most $2d$, so the sum $\sum_j (\alpha_{K+\varepsilon_j} \otimes \alpha_I \partial_j - \alpha_I \partial_j \otimes \alpha_{K+\varepsilon_j})$ must lie inside $\mathrm{F}^{2d}(H \otimes H)$. All $\alpha_{K+\varepsilon_j}$ are of degree at most d and $\alpha_I \partial_j$ are of degree exactly $d+1$, hence $\sum_j \alpha_{K+\varepsilon_j} \otimes \alpha_I \partial_j$ must lie in $\mathrm{F}^{2d}(H \otimes H)$ too. But this implies that $\alpha_{K+\varepsilon_j} \in \mathrm{F}^{d-1} H$ for all j, so in particular $\alpha_I \in \mathrm{F}^{d-1} H$, which is a contradiction.

This proves that $\alpha \in (\mathfrak{d} + \mathbf{k}) \otimes (\mathfrak{d} + \mathbf{k})$. Now if $\alpha = r + s_1 - s_2$, where $r \in \mathfrak{d} \wedge \mathfrak{d}$, $s \in \mathfrak{d}$, then we have

$$(\Delta \otimes \mathrm{id})(\alpha) = r_{13} + r_{23} + s_1 + s_2 - s_3,$$

and (4.2) becomes

$$([r_{12}, r_{13} + s_1 + s_2] + r_{12} s_3) + \text{cyclic} = 0. \tag{4.7}$$

Comparing the terms in $\mathfrak{d} \otimes \mathfrak{d} \otimes \mathbf{k}$, we see that (4.7) is equivalent to the system (4.3, 4.4). ∎

Note that when $\alpha = r \in \mathfrak{d} \wedge \mathfrak{d}$, $s=0$, (4.4) is exactly the *classical Yang–Baxter equation*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{4.8}$$

Equation (4.4) is a special case of the *dynamical* classical Yang–Baxter equation (see [Fe, ES]).

5. $(H \sharp \mathbf{k}[\Gamma])$-PSEUDOALGEBRAS

Let again H be a cocommutative Hopf algebra. Let Γ be a group acting on H by automorphisms, and let $\tilde{H} = H \sharp \mathbf{k}[\Gamma]$ be the smash product of H with the group algebra of Γ. As an associative algebra this is the semidirect product of H with $\mathbf{k}[\Gamma]$, while as a coalgebra it is the tensor product of coalgebras.

We will denote the action of Γ on H by $g \cdot f$ for $g \in \Gamma$, $f \in H$; then $g \cdot f = g f g^{-1}$. Then a left $\tilde{H}$-module L is the same as an H-module together with an action of Γ on it which is compatible with that of H, i.e., such that $(g \cdot f)\, l = g(f(g^{-1} l))$ for $g \in \Gamma$, $f \in H$, $l \in L$.

In this section we will study the relationship between the pseudotensor categories $\mathcal{M}^*(\tilde{H})$ and $\mathcal{M}^*(H)$. In particular, we will show that an $\tilde{H}$-pseudoalgebra is the same as an H-pseudoalgebra on which the group Γ acts by preserving the pseudoproduct.

Let us start by defining maps $\delta_I \colon \tilde{H}^{\otimes I} \to H^{\otimes I} \otimes_H \tilde{H}$ for each finite nonempty set I. It is enough to define δ_I on elements of the form $\bigotimes_{i\in I} f_i g_i$ where $f_i \in H$, $g_i \in \Gamma$, in which case we let

$$\delta_I\Big(\bigotimes_{i\in I} f_i g_i\Big) = \begin{cases} \Big(\bigotimes_{i\in I} f_i\Big) \otimes_H g, & \text{if all } g_i \text{ are equal to some } g, \\ 0, & \text{if some of } g_i \text{ are different.} \end{cases}$$

It is easy to see that δ_I is a homomorphism of both left $H^{\otimes I}$-modules and of right $\tilde{H}$-modules.

This allows us to define a pseudotensor functor $\delta \colon \mathcal{M}^*(\tilde{H}) \to \mathcal{M}^*(H)$ as follows. For an object L (a left $\tilde{H}$-module), we let $\delta(L) \equiv L$ be the left H-module obtained by restricting the action of $\tilde{H}$ to $H \subset \tilde{H}$. For a polylinear map $\phi \in \operatorname{Lin}(\{L_i\}_{i\in I}, M)$ in $\mathcal{M}^*(\tilde{H})$, i.e., for a homomorphism of left $\tilde{H}^{\otimes I}$-modules

$$\phi \colon \boxtimes_{i\in I} L_i \to \tilde{H}^{\otimes I} \otimes_{\tilde{H}} M,$$

we let $\delta(\phi)$ be the composition

$$\delta(\phi) \colon \boxtimes_{i\in I} L_i \xrightarrow{\phi} \tilde{H}^{\otimes I} \otimes_{\tilde{H}} M \xrightarrow{\delta_I \otimes_{\tilde{H}} \mathrm{id}} (H^{\otimes I} \otimes_H \tilde{H}) \otimes_{\tilde{H}} M \simeq H^{\otimes I} \otimes_H M.$$

This is a homomorphism of left $H^{\otimes I}$-modules, i.e., a polylinear map in $\mathcal{M}^*(H)$. Moreover, since the maps δ_I are compatible with the actions of the symmetric groups and with the comultiplication of $\tilde{H}$, it follows that δ is compatible with the actions of the symmetric groups and with compositions of polylinear maps, i.e., it is a pseudotensor functor.

As usual, the action of Γ on H can be extended to an action of Γ on $H^{\otimes I}$ by using the comultiplication $\Delta^{(I)}(g) = \otimes_{i\in I}\, g$. Hence, Γ also acts on $H^{\otimes I} \otimes_H M$ by the formula

$$g \cdot \Big(\Big(\bigotimes_{i\in I} f_i\Big) \otimes_H m\Big) = \Big(\bigotimes_{i\in I} g\cdot f_i\Big) \otimes_H gm, \qquad g \in \Gamma, \quad f_i \in H, \quad m \in M.$$

Then it is easy to see that $\psi = \delta(\phi)$ has the property

$$\psi\left(\bigotimes_{i\in I} gl_i\right) = g\cdot\psi\left(\bigotimes_{i\in I} l_i\right), \qquad g\in\Gamma,\ \ l_i\in L_i\,; \tag{5.1}$$

in other words, it commutes with the action of Γ.

We let $\mathcal{M}^*_\Gamma(H)$ be the subcategory of $\mathcal{M}^*(H)$ with objects left $\tilde{H}$-modules, and with polylinear maps those polylinear maps ψ of $\mathcal{M}^*(H)$ that commute with the action of Γ (see (5.1)). This is a pseudotensor category, and δ is a pseudotensor functor from $\mathcal{M}^*(\tilde{H})$ to $\mathcal{M}^*_\Gamma(H)$.

THEOREM 5.1. *If Γ is a finite group, the functor $\delta\colon \mathcal{M}^*(\tilde{H}) \to \mathcal{M}^*_\Gamma(H)$ constructed above is an equivalence of pseudotensor categories.*

Proof. We will construct a pseudotensor functor Σ from $\mathcal{M}^*_\Gamma(H)$ to $\mathcal{M}^*(\tilde{H})$. On objects L we let $\Sigma(L) = L$. In order to define Σ on polylinear maps, we need to find out how ϕ can be recovered from $\delta(\phi)$ and the action of Γ.

Denote by ι the embedding $H \hookrightarrow \tilde{H}$, and let π_I be the composition

$$\pi_I\colon H^{\otimes I} \otimes_H \tilde{H} \xrightarrow{\iota^{\otimes I}\otimes_H \mathrm{id}} \tilde{H}^{\otimes I} \otimes_H \tilde{H} \twoheadrightarrow \tilde{H}^{\otimes I} \otimes_{\tilde{H}} \tilde{H} \xrightarrow{\sim} \tilde{H}^{\otimes I}.$$

Explicitly, π_I is given by the formula

$$\pi_I\left(\left(\bigotimes_{i\in I} f_i\right)\otimes_H g\right) = \bigotimes_{i\in I} f_i g, \qquad f_i\in H,\ g\in\Gamma.$$

This is a homomorphism of both left $H^{\otimes I}$-modules and of right $\tilde{H}$-modules. Moreover, for $f_i \in H$, $g_i \in \Gamma$, we have

$$\pi_I\delta_I\left(\bigotimes_{i\in I} f_i g_i\right) = \begin{cases} \bigotimes_{i\in I} f_i g_i\,, & \text{if all } g_i \text{ are equal,}\\ 0, & \text{otherwise.}\end{cases}$$

The crucial observation, which will allow us to invert δ_I, is that for any $h_i \in \tilde{H}$, $g_i \in \Gamma$, we have:

$$\sum_{(g_i)\in\Gamma^I/\Gamma} \left(\bigotimes_{i\in I} g_i\right)(\pi_I\delta_I)\left(\bigotimes_{i\in I} g_i^{-1} h_i\right) = \bigotimes_{i\in I} h_i\,. \tag{5.2}$$

Here Γ acts diagonally on Γ^I from the right; the left-hand side of (5.2) is invariant under $(g_i)\mapsto(g_i g)$.

Given a polylinear map $\psi \in \mathrm{Lin}(\{L_i\}_{i\in I}, M)$ in $\mathcal{M}^*_\Gamma(H)$, we can extend it to a map

$$\tilde{\psi}\colon \boxtimes_{i\in I} L_i \xrightarrow{\psi} H^{\otimes I} \otimes_H M \simeq (H^{\otimes I} \otimes_H \tilde{H}) \otimes_{\tilde{H}} M \xrightarrow{\pi_I \otimes_{\tilde{H}} \mathrm{id}} \tilde{H}^{\otimes I} \otimes_{\tilde{H}} M.$$

(Note, however, that $\tilde{\psi}$ is not $\tilde{H}^{\otimes I}$-linear.) Now we define $\Sigma\psi\colon \boxtimes_{i\in I} L_i \to \tilde{H}^{\otimes I} \otimes_{\tilde{H}} M$ by the formula

$$(\Sigma\psi)\Big(\bigotimes_{i\in I} l_i\Big) = \sum_{(g_i)\in\Gamma^I/\Gamma} \Big(\Big(\bigotimes_{i\in I} g_i\Big) \otimes_{\tilde{H}} 1\Big)\, \tilde{\psi}\Big(\bigotimes_{i\in I} g_i^{-1} l_i\Big). \tag{5.3}$$

It is easy to check that $\Sigma\psi$ is $\tilde{H}^{\otimes I}$-linear, so it is a polylinear map in $\mathcal{M}^*(\tilde{H})$. Moreover, $\delta\Sigma\psi = \psi$. For a polylinear map $\phi \in \mathrm{Lin}(\{L_i\}_{i\in I}, M)$ in $\mathcal{M}^*(\tilde{H})$, it is immediate from (5.2) and the $\tilde{H}^{\otimes I}$-linearity of ϕ that $\Sigma\delta\phi = \phi$. Therefore, $\Sigma\colon \mathcal{M}^*_\Gamma(H) \to \mathcal{M}^*(\tilde{H})$ is a pseudotensor functor inverse to δ. ∎

Remark 5.1. The above theorem holds also for infinite groups Γ if we restrict ourselves to polylinear maps ψ of $\mathcal{M}^*_\Gamma(H)$ satisfying the following finiteness condition:

$$\psi\Big(\bigotimes_{i\in I} g_i l_i\Big) \neq 0 \qquad \text{for only a finite number of } (g_i) \in \Gamma\backslash\Gamma^I \tag{5.4}$$

for any fixed $l_i \in L$. (Note that, by (5.1), this condition does not depend on the choice of representatives (g_i).) Indeed, the only place in the proof where we used the finiteness of Γ was to insure that the right-hand side of (5.3) is a finite sum.

If $\psi = \delta(\phi)$ comes from a polylinear map ϕ of $\mathcal{M}^*(\tilde{H})$, then it satisfies (5.4), because ϕ is $\tilde{H}^{\otimes I}$-linear and for any element $h \in \tilde{H}^{\otimes I}$ one has $\delta_I((\bigotimes_{i\in I} g_i)\, h) \neq 0$ for only a finite number of $(g_i) \in \Gamma\backslash\Gamma^I$.

Therefore, $\delta\colon \mathcal{M}^*(\tilde{H}) \to \mathcal{M}^*_{\Gamma,\mathrm{fin}}(H)$ is an equivalence of pseudotensor categories, where $\mathcal{M}^*_{\Gamma,\mathrm{fin}}(H)$ is the subcategory of $\mathcal{M}^*_\Gamma(H)$ consisting of polylinear maps ψ satisfying (5.4).

COROLLARY 5.1. *A Lie $\tilde{H} = (H \sharp \mathbf{k}[\Gamma])$-pseudoalgebra L is the same as a Lie H-pseudoalgebra L on which the group Γ acts in a way compatible with the action of H, by preserving the H-pseudobracket*:

$$[ga * gb] = g \cdot [a * b] \qquad \textit{for} \quad g \in \Gamma,\ a,b \in L, \tag{5.5}$$

and satisfying the following finiteness condition:

$$\textit{given } a, b \in L,\ [ga * b] \neq 0 \textit{ for only a finite number of } g \in \Gamma. \tag{5.6}$$

The $\tilde{H}$-pseudobracket of L is given by the formula

$$[a \mathbin{\tilde{*}} b] = \sum_{g \in \Gamma} ((g^{-1} \otimes 1) \otimes_{\tilde{H}} 1)[ga * b], \qquad a, b \in L. \tag{5.7}$$

A similar statement holds for representations, as well as for associative pseudoalgebras.

This result, combined with Kostant's Theorem 2.1, will allow us in many cases to reduce the study of H-pseudoalgebras to the case when H is a universal enveloping algebra (see Section 13.7).

EXAMPLE 5.1. Let Γ be a subgroup of $\mathbf{k}^*$ and let

$$H = \mathbf{k}[\partial] \sharp \mathbf{k}[\Gamma] = \bigoplus_{m \in \mathbb{Z}_+, \alpha \in \Gamma} \mathbf{k}\, \partial^m T_\alpha$$

with multiplication $T_\alpha T_\beta = T_{\alpha\beta}$, $T_1 = 1$, $T_\alpha \partial T_\alpha{}^{-1} = \alpha\partial$ and comultiplication $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$, $\Delta(T_\alpha) = T_\alpha \otimes T_\alpha$. Then the notion of a Lie H-pseudoalgebra is equivalent to that of a Γ-conformal algebra (cf. [K4]).

EXAMPLE 5.2. Let now $H = \mathbf{k}[\partial] \times F(\Gamma)$, where $F(\Gamma)$ is the function algebra of a finite abelian group Γ. In other words, $H = \bigoplus_{m \in \mathbb{Z}_+, \alpha \in \Gamma} \mathbf{k}\partial^m \pi_\alpha$ with multiplication $\pi_\alpha \pi_\beta = \delta_{\alpha,\beta} \pi_\alpha$, $\partial \pi_\alpha = \pi_\alpha \partial$ and comultiplication $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$, $\Delta(\pi_\alpha) = \sum_{\gamma \in \Gamma} \pi_{\alpha\gamma^{-1}} \otimes \pi_\gamma$. Then one gets the notion of a Γ-twisted conformal algebra (cf. [K4]).

6. A DIGRESSION TO LINEARLY COMPACT LIE ALGEBRAS

We will view the base field $\mathbf{k}$ as a topological field with discrete topology. A topological vector space $\mathscr{L}$ over $\mathbf{k}$ is called *linearly compact* if it is the space of all linear functionals on a vector space $\mathscr{V}$ with discrete topology, with the topology on $\mathscr{L}$ defined by taking all subspaces $\{U^\perp \subset \mathscr{L} \mid U \subset \mathscr{V}, \dim U < \infty\}$ as a fundamental system of neighborhoods of 0 in $\mathscr{L}$. Here, as usual, $U^\perp$ denotes the subspace of $\mathscr{L}$ consisting of all linear functionals vanishing on U.

In general, given a topological vector space $\mathscr{W}$, we define a topology on $\mathscr{W}^*$ by taking for the fundamental system of neighborhoods of 0 the subspaces $U^\perp$ where U is a linearly compact subspace of $\mathscr{W}$.

Several equivalent definitions of linear-compactness are provided by the next proposition.

PROPOSITION 6.1. *For a topological vector space $\mathscr{L}$ over the topological field $\mathbf{k}$ the following statements are equivalent*:

(1) *$\mathscr{L}$ is the dual of a discrete vector space.*

(2) *The topological dual $\mathscr{L}^*$ of $\mathscr{L}$ is a discrete topological space.*

(3) *$\mathscr{L}$ is the topological product of finite-dimensional discrete vector spaces.*

(4) *$\mathscr{L}$ is the projective limit of finite-dimensional discrete vector spaces.*

(5) *$\mathscr{L}$ has a collection of finite-codimensional open subspaces whose intersection is $\{0\}$, with respect to which it is complete.*

Proof. Can be found in [G1]. ∎

Remark 6.1. For both discrete and linearly compact vector spaces, the canonical map from $\mathscr{L}$ to $\mathscr{L}^{**}$ is an isomorphism.

A *linearly compact* (associative or Lie) *algebra* is a topological (associative or Lie) algebra for which the underlying topological space is linearly compact.

The basic example of a linearly compact associative algebra is the algebra $\mathscr{O}_N = \mathbf{k}[[t_1, ..., t_N]]$ of formal power series over $\mathbf{k}$ in $N \geqslant 1$ indeterminates $t_1, ..., t_N$, with the usual formal topology for which $(t_1, ..., t_N)^j$, the powers of the ideal $(t_1, ..., t_N)$, form a fundamental system of neighborhoods of $\mathscr{O}_N$.

Remark 6.2. The topological vector spaces $\mathscr{O}_N$ are isomorphic and characterized among linearly compact vector spaces by each of the following properties:

(1) $\mathscr{O}_N^*$ is countable-dimensional.

(2) $\mathscr{O}_N$ has a filtration by open subspaces.

Remark 6.3. (i) One defines a *completed tensor product* of two linearly compact vector spaces $\mathscr{V}$, $\mathscr{W}$ by $\mathscr{V} \mathbin{\hat{\otimes}} \mathscr{W} = (\mathscr{V}^* \otimes \mathscr{W}^*)^*$ where we put the discrete topology on $\mathscr{V}^* \otimes \mathscr{W}^*$. Then $\mathscr{V} \mathbin{\hat{\otimes}} \mathscr{W}$ is linearly compact.

(ii) With this definition, $\mathscr{O}_{M+N} \simeq \mathscr{O}_M \mathbin{\hat{\otimes}} \mathscr{O}_N$ as topological algebras.

(iii) Given a commutative associative linearly compact algebra $\mathscr{O}$ and a linearly compact Lie algebra $\mathscr{L}$, their completed tensor product $\mathscr{O} \mathbin{\hat{\otimes}} \mathscr{L}$ is again a linearly compact Lie algebra.

The basic example of a linearly compact Lie algebra is the Lie algebra W_N of continuous derivations of the topological algebra $\mathscr{O}_N$. The filtration

$$\mathrm{F}_j \mathscr{O}_N = (t_1, ..., t_N)^{j+1}, \qquad j = -1, 0, 1, ...$$

of $\mathcal{O}_N$ induces the *canonical filtration* $\mathrm{F}_j W_N$ of W_N, where

$$\mathrm{F}_j W_N = \{D \in W_N \mid D(\mathrm{F}_i \mathcal{O}_N) \subset \mathrm{F}_{i+j} \mathcal{O}_N \ \forall i\}, \qquad j = -1, 0, 1, \ldots.$$

It is clear that W_N consists of all linear differential operators of the form

$$D = \sum_{i=1}^{N} P_i(t) \frac{\partial}{\partial t_i}, \qquad \text{where} \quad P_i(t) \in \mathcal{O}_N,$$

and that $\mathrm{F}_j W_N$ $(j \geqslant -1)$ consists of those D for which all $P_i(t)$ lie in $\mathrm{F}_j \mathcal{O}_N$.

Let $E = \sum_{i=1}^{N} t_i(\partial/\partial t_i)$ be the Euler operator. The spectrum of ad E consists of all integers $j \geqslant -1$, and, denoting by $W_{N;\, j}$ the jth eigenspace of ad$E$ we obtain the *canonical $\mathbb{Z}$-gradation*:

$$W_N = \prod_{j \geqslant -1} W_{N;\, j}, \qquad [W_{N;\, i}, W_{N;\, j}] \subset W_{N;\, i+j}.$$

The following fact is well known.

Lemma 6.1. *$W_{N;\, 0} \simeq \mathfrak{gl}_N(\mathbf{k})$ and one has the following isomorphism of $\mathfrak{gl}_N(\mathbf{k})$-modules:*

$$W_{N;\, j} \simeq \mathbf{k}^N \otimes (\mathrm{S}^{j+1} \mathbf{k}^N)^*.$$

Furthermore, one has a decomposition into a direct sum of irreducible submodules: $W_{N;\, j} = W'_{N;\, j} + W''_{N;\, j}$, where $W'_{N;\, j} \simeq (\mathrm{S}^j \mathbf{k}^N)^*$ $(=0$ if $j = -1)$ and $W''_{N;\, j} \simeq$ the highest component of $\mathbf{k}^N \otimes (\mathrm{S}^{j+1} \mathbf{k}^N)^*$. The subspace $\mathfrak{p} = W_{N;\, -1} + W_{N;\, 0} + W'_{N;\, 1}$ is a subalgebra of W_N isomorphic to $\mathfrak{sl}_{N+1}(\mathbf{k})$.

Let $\Omega_N = \bigoplus_{j=0}^{N} \Omega_{N;\, j}$ denote the algebra of differential forms over $\mathcal{O}_N$. The defining representation of W_N on $\mathcal{O}_N$ extends uniquely to a representation on Ω_N commuting with the differential d.

Recall that a *volume form* is a differential N-form $v = f(t_1, \ldots, t_N)\, \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_N$ such that $f(0) \neq 0$, a *symplectic form* is a closed 2-form $s = \sum_{i,j=1}^{N} s_{ij}(t_1, \ldots, t_N)\, \mathrm{d}t_i \wedge \mathrm{d}t_j$ such that $\det(s_{ij}(0)) \neq 0$, and a *contact form* is a 1-form c such that $c \wedge (\mathrm{d}c)^{(N-1)/2}$ is a volume form. The following facts are well known.

Lemma 6.2. (i) *Any volume form can be transformed by an automorphism of $\mathcal{O}_N$ to the standard volume form $v_0 = \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_N$.*

(ii) *A symplectic form exists iff N is even, $N = 2n$, and by an automorphism of $\mathcal{O}_N$ it can be transformed to the standard symplectic form $s_0 = \sum_{i=1}^{n} \mathrm{d}t_i \wedge \mathrm{d}t_{n+i}$.*

(iii) *A contact form exists iff N is odd, $N=2n+1$, and by an automorphism of $\mathcal{O}_N$ it can be brought to the standard contact form* $c_0 = \mathrm{d}t_N + \sum_{i=1}^{n} t_i \,\mathrm{d}t_{n+i}$.

Consider the following (closed) subalgebras of the Lie algebra W_N:

$$S_N(v) = \{D \in W_N \mid Dv = 0\} \qquad (N \geqslant 2),$$

$$H_N(s) = \{D \in W_N \mid Ds = 0\} \qquad (N \text{ even } \geqslant 2),$$

$$K_N(c) = \{D \in W_N \mid Dc = fc \text{ for some } f \in \mathcal{O}_N\} \qquad (N \text{ odd } \geqslant 3).$$

Let also $S_N = S_N(v_0)$, $H_N = H_N(s_0)$, $K_N = K_N(c_0)$. Lemma 6.2 implies isomorphisms: $S_N(v) \simeq S_N$, $H_N(s) \simeq H_N$, $K_N(c) \simeq K_N$, $S_2 \simeq H_2$.

The canonical filtration of W_N induces *canonical filtrations* $\mathrm{F}_j S_N(v) := \mathrm{F}_j W_N \cap S_N(v)$, etc. Note that $\dim W_N/\mathrm{F}_{-1} W_N = N$. A Lie subalgebra $\mathscr{L}$ of W_N is called *transitive* if $\dim \mathscr{L}/(\mathscr{L} \cap \mathrm{F}_{-1} W_N) = N$. It is known that the Lie algebras W_N, S_N, H_N and K_N are transitive. In addition, the canonical filtrations $\mathrm{F}_j \mathscr{L}$ of these Lie algebras have the following transitivity property:

$$\mathrm{F}_{j+1} \mathscr{L} = \{a \in \mathrm{F}_j \mathscr{L} \mid [a, \mathscr{L}] \subset \mathrm{F}_j \mathscr{L}\}. \tag{6.1}$$

Noting that $Ev_0 = Nv_0$ and $Es_0 = 2s_0$, we conclude that $\operatorname{ad} E$ is an (outer) derivation of S_N and H_N, hence the canonical gradation of W_N induces *canonical $\mathbb{Z}$-gradations* $S_N = \prod_{j \geqslant -1} S_{N;\,j}$ and $H_N = \prod_{j \geqslant -1} H_{N;\,j}$.

Let $E' = 2t_N(\partial/\partial t_N) + \sum_{i=1}^{N-1} t_i(\partial/\partial t_i)$. Then $E' c_0 = 2c_0$, hence $E' \in K_N$ and the eigenspace decomposition of $\operatorname{ad} E'$ defines the *canonical $\mathbb{Z}$-gradation* $K_N = \prod_{j \geqslant -2} K_{N;\,j}$. The following facts are well known.

LEMMA 6.3. (i) $S_{N;\,0} \simeq \mathfrak{sl}_N(\mathbf{k})$, $H_{N;\,0} \simeq \mathfrak{sp}_N(\mathbf{k})$, $K_{N;\,0} \simeq \mathfrak{csp}_{N-1}(\mathbf{k})$ $(\simeq \mathfrak{sp}_{N-1}(\mathbf{k}) \oplus \mathbf{k})$.

(ii) *The $S_{N;\,0}$-module $S_{N;\,j}$ is isomorphic to the highest component of the $\mathfrak{sl}_N(\mathbf{k})$-module $\mathbf{k}^N \otimes (\mathrm{S}^{j+1}\mathbf{k}^N)^*$.*

(iii) *The $H_{N;\,0}$-module $H_{N;\,j}$ is isomorphic to the (irreducible) $\mathfrak{sp}_N(\mathbf{k})$-module $\mathrm{S}^{j+2}\mathbf{k}^N$.*

(iv) *$K_{N;\,0} = \mathfrak{sp}_{N-1}(\mathbf{k}) \oplus \mathbf{k}E'$ and the $\mathfrak{sp}_{N-1}(\mathbf{k})$-module $K_{N;\,j}$ decomposes into the following direct sum of irreducible modules:*

$$K_{N;\,j} = \bigoplus_{i=0}^{[j/2]+1} K_{N;\,j}^{(i)}, \qquad \text{where} \quad K_{N;\,j}^{(i)} \simeq \mathrm{S}^{j+2-2i}\mathbf{k}^{N-1}.$$

The subspace $\mathfrak{p} = K_{N;\,-2} + K_{N;\,-1} + K_{N;\,0} + K_{N;\,1}^{(1)} + K_{N;\,2}^{(2)}$ is a subalgebra of K_N isomorphic to $\mathfrak{sp}_{N+1}(\mathbf{k})$.

The following celebrated theorem goes back to E. Cartan (see [G2] for a relatively simple proof).

THEOREM 6.1. *Any infinite-dimensional simple linearly compact Lie algebra is isomorphic to one of the topological Lie algebras* W_N, S_N, H_N, *or* K_N.

Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{h}$ be its subalgebra of codimension N. Then $F = \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), \mathbf{k})$, with the product $(f_1 f_2)(u) = f_1(u_{(1)}) f_2(u_{(2)})$, is (canonically) isomorphic to the algebra of formal power series on $(\mathfrak{g}/\mathfrak{h})^*$ [B2], which is (non-canonically) isomorphic to the linearly compact algebra $\mathcal{O}_N$. The Lie algebra D of continuous derivations of F is then isomorphic to W_N. F has a canonical $\mathfrak{g}$-action induced by the left-multiplication $\mathfrak{g}$-action on $U(\mathfrak{g})$, which gives us a homomorphism γ of $\mathfrak{g}$ to W_N. (This is non-canonical since the identification of F with $\mathcal{O}_N$ is not canonical.)

We will use in the sequel the following theorem of Guillemin and Sternberg [GS] (see [B2] for a simple proof).

PROPOSITION 6.2. *Let* $\mathfrak{g}$ *be a Lie algebra, and let* $\mathfrak{h}$ *be its subalgebra of codimension* N. *Provided that* $\mathfrak{h}$ *contains no nonzero ideals of* $\mathfrak{g}$, *the above-defined* γ *is a Lie algebra isomorphism of* $\mathfrak{g}$ *with a subalgebra of* W_N *which maps* $\mathfrak{h}$ *into* $\mathrm{F}_0 W_N$.

Conversely, if the inclusion $\mathfrak{g} \hookrightarrow W_N$ *maps* $\mathfrak{h}$ *into* $\mathrm{F}_0 W_N$, *then* $\mathfrak{h}$ *doesn't contain nonzero ideals of* $\mathfrak{g}$. *Every Lie algebra homomorphism of* $\mathfrak{g}$ *to* W_N, *which coincides with* γ *modulo* $\mathrm{F}_0 W_N$, *is conjugated to* γ *via a unique automorphism of* $\mathcal{O}_N$.

We have the following important property of the filtrations on H and $X = H^*$, defined in Section 2.2.

LEMMA 6.4. *Let* $H = U(\mathfrak{g}) \sharp \mathbf{k}[\Gamma]$ *be a cocommutative Hopf algebra, and* $X = H^*$. *If* $h \in \mathrm{F}^i U(\mathfrak{g}) \subset H$ *but* $h \notin \mathrm{F}^{i-1} U(\mathfrak{g})$, *then* $h \mathrm{F}_n X = \mathrm{F}_{n-i} X$. *In particular, for any* $h \in \mathfrak{g} \backslash \{0\}$ *and for every open subspace* $U \subset X$, *there is some* n *such that* $h^n U = X$. *Similar statements hold for the right action of* h *as well.*

Proof. By the construction of the filtrations it is evident that we can assume $H = U(\mathfrak{g})$. Then $X \simeq \mathcal{O}_N$ ($N = \dim \mathfrak{g}$), and $\mathfrak{g} \hookrightarrow W_N$ acts on it by linear differential operators. The rest of the proof is clear.

The following result from [G1, G2] will be essential for our purposes.

PROPOSITION 6.3. (i) *A linearly compact Lie algebra $\mathscr{L}$ satisfies the descending chain condition on closed ideals if and only if it has a fundamental subalgebra, i.e., an open subalgebra containing no ideals of $\mathscr{L}$.*

(ii) *When either of the assumptions of* (i) *holds, the noncommutative minimal closed ideals of $\mathscr{L}$ are of the form $\mathcal{O}_r \hat{\otimes} \mathfrak{s}$ where $\mathfrak{s}$ is a simple linearly compact Lie algebra and $r \in \mathbb{Z}_+$.*

We will also need the following examples of non-simple linearly compact Lie algebras:

$$CS_N(v) = \{D \in W_N \mid Dv = av,\ a \in \mathbf{k}\},$$
$$CH_N(s) = \{D \in W_N \mid Ds = as,\ a \in \mathbf{k}\}.$$

As before, we have isomorphisms $CS_N(v) \simeq CS_N \equiv CS_N(v_0)$ and $CH_N(s) \simeq CH_N \equiv CH_N(s_0)$. Note also that $CS_N = \mathbf{k}E \ltimes S_N$ and $CH_N = \mathbf{k}E \ltimes SH_N$. Another important example of a non-simple linearly compact Lie algebra is the Poisson algebra P_N, which is $\mathcal{O}_N$ ($N = 2n$) endowed with the Poisson bracket:

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial t_i} \frac{\partial g}{\partial t_{n+i}} - \frac{\partial f}{\partial t_{n+i}} \frac{\partial g}{\partial t_i}.$$

It is a nontrivial central extension of H_N,

$$0 \to \mathbf{k} \to P_N \xrightarrow{\varphi} H_N \to 0,$$

where $\varphi(f) = \sum_{i=1}^{n} (\partial f/\partial t_i)(\partial/\partial t_{n+i}) - (\partial f/\partial t_{n+i})(\partial/\partial t_i)$.

We can describe also K_N in a more explicit way, similar to the above description of P_N. For $f, g \in \mathcal{O}_N$, define

$$\{f, g\}' = \{f, g\}_{2n} + \frac{\partial f}{\partial t_{2n+1}} (E_{2n} g - 2g) - (E_{2n} f - 2f) \frac{\partial g}{\partial t_{2n+1}},$$

where $\{f, g\}_{2n}$ is the Poisson bracket taken with respect to the variables $t_1, ..., t_{2n}$ and E_{2n} is the Euler operator $\sum_{i=1}^{2n} t_i (\partial/\partial t_i)$. If we define

$$\psi(f) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial t_i} \frac{\partial}{\partial t_{n+i}} - \frac{\partial f}{\partial t_{n+i}} \frac{\partial}{\partial t_i} \right) + \frac{\partial f}{\partial t_{2n+1}} E_{2n} - (E_{2n} f - 2f) \frac{\partial}{\partial t_{2n+1}},$$

then we have $\psi(f)\, g = \{f, g\}' + 2(\partial f/\partial t_{2n+1})\, g$. It is easy to see that $[\psi(f), \psi(g)] = \psi(\{f, g\}')$ and $\psi(f)\, c_0 = 2(\partial f/\partial t_{2n+1})\, c_0$. Thus K_N is isomorphic to $\mathcal{O}_N$ with the bracket $\{\, , \}'$.

For a linearly compact Lie algebra $\mathscr{L}$ denote by Der $\mathscr{L}$ the Lie algebra of its continuous derivations and by $\hat{\mathscr{L}}$ the universal central extension of $\mathscr{L}$. Then we have:

PROPOSITION 6.4. (i) $\mathrm{Der}\, W_N = W_N$, $\mathrm{Der}\, S_N = CS_N$, $\mathrm{Der}\, H_N = CH_N$, $\mathrm{Der}\, K_N = K_N$.

(ii) $\mathrm{Der}(\mathcal{O}_r \hat{\otimes} \mathcal{L}) = W_r \otimes 1 + \mathcal{O}_r \hat{\otimes} \mathrm{Der}\, \mathcal{L}$ *for any simple linearly compact Lie algebra* $\mathcal{L}$.

(iii) *The Lie algebras* $\mathcal{O}_r \hat{\otimes} W_N$, $\mathcal{O}_r \hat{\otimes} S_N$ (*for* $N > 2$) *and* $\mathcal{O}_r \hat{\otimes} K_N$ *have no nontrivial central extensions. The universal central extension of* $\mathcal{O}_r \hat{\otimes} H_N$ *is* $\mathcal{O}_r \hat{\otimes} P_N$.

(iv) *If* $\mathfrak{g}$ *is a simple finite-dimensional Lie algebra, then* $(\mathcal{O}_r \otimes \mathfrak{g})^\wedge = (\mathcal{O}_r \otimes \mathfrak{g}) + (\Omega_{r;\,1}/\mathrm{d}\mathcal{O}_r)$ *with the bracket*

$$[f \otimes a, g \otimes b]^\wedge = fg \otimes [a, b] + (a \mid b) f\, \mathrm{d}g \qquad \mathrm{mod}\ \mathrm{d}\mathcal{O}_r,$$

where $(a \mid b)$ *is the Killing form on* $\mathfrak{g}$.

Proof. For a proof of (iv) see [Ka].

In order to prove (ii), notice that if d is a derivation of $\mathcal{O}_r \hat{\otimes} \mathcal{L}$, then its action on $1 \otimes \mathcal{L}$ is given by $d(1 \otimes x) = \sum_i a_i \otimes d_i(x)$ for all $x \in \mathcal{L}$, where the a_i form a topological basis of $\mathcal{O}_r$ and the d_i are continuous derivations of $\mathcal{L}$. Subtracting $\sum_i a_i \otimes d_i$ from d, we get a derivation $\tilde{d}$ acting trivially on $1 \otimes \mathcal{L}$. We are going to show that if $\mathcal{L}$ is simple then $\tilde{d}$ is of the form $D \otimes 1$ where $D \in \mathrm{Der}\, \mathcal{O}_r = W_r$.

Let us fix $P \in \mathcal{O}_r$. Then $\tilde{d}(P \otimes x)$ can be written as $\sum_i a_i \otimes f_i(x)$, where f_i are continuous $\mathbf{k}$-endomorphisms of $\mathcal{L}$. From $\tilde{d}([P \otimes x, 1 \otimes y]) = [\tilde{d}(P \otimes x), 1 \otimes y]$ we see that $[f_i(x), y] = f_i([x, y])$ for every $x, y \in \mathcal{L}$. This means that f_i commutes with $\mathrm{ad}\, y$ for all $y \in \mathcal{L}$. By Schur's lemma from [G1, Proposition 4.4] we conclude that the f_i are multiples of the identity map, hence $\tilde{d}(P \otimes x) = a_P \otimes x$ for some $a_P \in \mathcal{O}_r$ and all $x \in \mathcal{L}$. It is now immediate to check that the mapping $D\colon P \mapsto a_P$ is indeed a derivation of $\mathcal{O}_r$, proving (ii).

In order to prove the rest of the statements, denote by $\mathfrak{a}$ the 0th component of the canonical $\mathbb{Z}$-gradation of $\mathcal{L} = W_N$, S_N, H_N or K_N. This is a reductive subalgebra of $\mathcal{L}$, hence $\mathrm{Der}\, \mathcal{L} = V \oplus \mathcal{L}$, where $[\mathfrak{a}, V] \subset V$. But $[V, \mathcal{L}] \subset \mathcal{L}$, hence $[\mathfrak{a}, V] = 0$, i.e., any element $D \in V$ defines an endomorphism of $\mathcal{L}$ viewed as an $\mathfrak{a}$-module. Since $E \in W_N$ and $E' \in K_N$, we conclude that D also preserves the canonical gradation of these Lie algebras and we may assume that D acts trivially on the (-1)st component. Using the transitivity of W_N and K_N, we conclude that $D = 0$. By Lemma 6.3, all components of the canonical $\mathbb{Z}$-gradation of S_N and H_N are inequivalent $\mathfrak{a}$-modules, hence D preserves this gradation in this case as well. Subtracting from D a multiple of E, we may assume that D acts trivially on the (-1)st component and, using transitivity, we conclude that D is a multiple of E. Thus (i) is proved.

Since $\mathfrak{a}$ acts completely reducibly on the space Z^2 of 2-cocycles on $\mathcal{O}_r \hat{\otimes} \mathcal{L}$ with values in $\mathbf{k}$, and since $\mathfrak{a}$ acts trivially on cohomology, we may choose a subspace U of Z^2, complementary to the space of trivial 2-cocycles, on which $\mathfrak{a}$ acts trivially. Hence for any 2-cocycle $\alpha \in U$ we have: $\alpha(a, b) = 0$ if $a \in M_1$, $b \in M_2$ and M_i are irreducible non-contragredient $\mathfrak{a}$-submodules of $\bar{L} := \mathcal{O}_r \hat{\otimes} \mathcal{L}$. Let $\bar{L}_j = \mathcal{O}_r \otimes \mathcal{L}_j$ for short, where $\mathcal{L}_j$ is the jth component of the canonical gradation.

It follows from Lemma 6.3(ii) that all pairs of $\mathfrak{a}$-submodules in $\bar{L} = \mathcal{O}_r \hat{\otimes} S_N$ are non-contragredient, except for the adjoint $\mathfrak{a}$-submodules in $\bar{L}_0 = \mathcal{O}_r \otimes S_{N;\,0}$. Thus, we have

$$\alpha(a, b) = 0 \qquad \text{if} \quad a \in \bar{S}_{N;\,i}, \ b \in \bar{S}_{N;\,j}, \qquad i \neq 0 \quad \text{or} \quad j \neq 0.$$

Taking now $a \in \bar{S}_{N;\,-1}$, $b \in \bar{S}_{n;\,1}$ and $c \in \bar{S}_{N;\,0}$, the cocycle condition

$$\alpha([a, b], c) + \alpha([b, c], a) + \alpha([c, a], b) = 0$$

gives $\alpha([a, b], c) = 0$. Since $\bar{S}_{N;\,0} = [\bar{S}_{N;\,-1}, \bar{S}_{N;\,1}]$, we conclude that $\alpha = 0$. Hence all central extensions of $\bar{S}_N$ are trivial.

Likewise, α is zero on any pair of subspaces $\bar{W}_{N;\,i}$, $\bar{W}_{N;\,j}$, unless $i + j = 0$, and on the pair $\bar{W}_{N;\,-1}$, $\bar{W}''_{N;\,1}$ (see Lemma 6.1). Choosing $a \in \bar{W}_{N;\,-1}$, $b \in \bar{W}''_{N;\,1}$, $c \in \bar{W}_{N;\,0}$, we obtain, as above, from the cocycle condition, that α is zero on the pair $\bar{W}_{N;\,0}$, $[\bar{W}_{N;\,0}, \bar{W}_{N;\,0}]$. It follows from (iv) applied to the subalgebra $\mathcal{O}_r \otimes \mathfrak{sl}_{N+1}(\mathbf{k})$ of $\bar{W}_N$ (see Lemma 6.1) that α is zero on this subalgebra if $N > 1$. Thus any cocycle on $\bar{W}_N$ $(N > 1)$ is trivial. In the case of $\bar{W}_1$ the cocycle α is trivial. The case of $\bar{K}_N$ is similar.

In the remaining case of $\bar{H}_N$ we show, as above, that the cocycle α is trivial on any pair $\bar{H}_{N;\,i}$, $\bar{H}_{N;\,j}$ if $i \neq j$. Using the cocycle condition for $a \in \bar{H}_{N;\,k}$, $b \in \bar{H}_{N;\,k+1}$ and $c \in \bar{H}_{N;\,-1}$, and the fact that $\bar{H}_{N;\,k} = [\bar{H}_{N;\,k+1}, \bar{H}_{N;\,-1}]$, we conclude that α is trivial on any pair $\bar{H}_{N;\,i}$, $\bar{H}_{N;\,i}$ as well, unless $i = -1$. It is easy to see that this implies that $\hat{\bar{H}}_N = \bar{P}_N$.

7. H-PSEUDOALGEBRAS AND H-DIFFERENTIAL ALGEBRAS

In this section, H will be a cocommutative Hopf algebra with a comultiplication Δ and a counit ε, and as before, $X = H^*$.

7.1. *The Annihilation Algebra*

Let Y be an H-bimodule which is a commutative associative H-differential algebra both for the left and for the right action of H (see (2.8), (2.19); for example, $Y = X := H^*$.

For a left H-module L, let $\mathscr{A}_Y L = Y \otimes_H L$. We define a left action of H on $\mathscr{A}_Y L$ in the obvious way:

$$h(x \otimes_H a) = hx \otimes_H a, \qquad h \in H,\ x \in Y,\ a \in L. \tag{7.1}$$

If, in addition, L is an H-pseudoalgebra with a pseudoproduct $a * b$, we can define a product on $\mathscr{A}_Y L$ by the formula

$$\begin{gathered}(x \otimes_H a)(y \otimes_H b) = \sum_i (xf_i)(yg_i) \otimes_H e_i\,, \\ \text{if} \quad a * b = \sum_i (f_i \otimes g_i) \otimes_H e_i\,.\end{gathered} \tag{7.2}$$

By (2.19) and the H-bilinearity (3.23) of the pseudoproduct, it is clear that (7.2) is well defined.

PROPOSITION 7.1. *If L is a Lie H-pseudoalgebra, then $\mathscr{A}_Y L$ is a Lie H-differential algebra, i.e., a Lie algebra which is also a left H-module so that*

$$h[\alpha, \beta] = [h_{(1)}\alpha, h_{(2)}\beta], \qquad \textit{for} \quad h \in H,\ \alpha, \beta \in \mathscr{A}_Y L. \tag{7.3}$$

Similarly, if L is an associative H-pseudoalgebra, then $\mathscr{A}_Y L$ is an associative H-differential algebra. A similar statement holds for modules as well: if M is an L-module, then $\mathscr{A}_Y M$ is an $\mathscr{A}_Y L$-module with a compatible H-action so that

$$h(am) = (h_{(1)}a)(h_{(2)}m) \qquad \textit{for} \quad h \in H,\ a \in \mathscr{A}_Y L,\ m \in \mathscr{A}_Y M. \tag{7.4}$$

Proof. Equation (7.3) follows from (2.8). The skew-commutativity of the bracket (7.2) follows immediately from that of $[a * b]$. The proof of the Jacobi identity is straightforward by using (3.25). Let us check for example that the associativity of L is equivalent to that of $\mathscr{A}_Y L$; the case of the Jacobi identity is similar.

We will use the notation from (3.14)–(3.19), and we will write $a_x \equiv x \otimes_H a$ for $a \in L$, $x \in Y$. Then we want to compute the products $a_x(b_y c_z)$ and $(a_x b_y)\, c_z$. By definition, if we have (3.17) and (3.18), then

$$b_y c_z = \sum_i (yh_i)(zl_i) \otimes_H d_i$$

and

$$
\begin{aligned}
a_x(b_y c_z) &= \sum_{i,\,j} (xh_{ij})((yh_i)(zl_i))\, l_{ij}) \otimes_H d_i \\
&= \sum_{i,\,j} (xh_{ij})(yh_i\, l_{ij(1)})(zl_i\, l_{ij(2)}) \otimes_H d_i\,.
\end{aligned}
$$

Similarly, if we have (3.14) and (3.15), then

$$
(a_x b_y)\, c_z = \sum_{i,\,j} (xf_i\, f_{ij_{(1)}})(yg_i\, f_{ij_{(2)}})(zg_{ij}) \otimes_H e_i\,.
$$

Now recalling (3.16) and (3.19), we see that the associativity of L is equivalent to that of $\mathscr{A}_Y L$. ∎

DEFINITION 7.1. The H-differential algebra $\mathscr{A}(L) \equiv \mathscr{A}_X L := X \otimes_H L$ is called the *annihilation algebra* of the pseudoalgebra L. We will write $a_x \equiv x \otimes_H a$ for $a \in L$, $x \in X$.

Remark 7.1. When L is an associative H-pseudoalgebra, one does not need the cocommutativity of H or the commutativity of Y in order to define $\mathscr{A}_Y L$ (cf. Remark 3.3).

LEMMA 7.1. *Let $H = U(\mathfrak{d}) \sharp \mathbf{k}[\Gamma]$, and let M be a left H-module. If an element $a \in M$ is $U(\mathfrak{d})$-torsion, i.e., if $ha = 0$ for some $h \in U(\mathfrak{d}) \setminus \{0\}$, then $X \otimes_H a = 0$. In particular, for $H = U(\mathfrak{d})$, we have $\mathscr{A}(M) \simeq \mathscr{A}(M/\mathrm{Tor}\, M)$, where* Tor M *is the torsion submodule of M.*

Proof. We have $0 = x \otimes_H ha = xh \otimes_H a$ for every $x \in X$. Since the right action of h on X is surjective (see Lemma 6.4), it follows that $x \otimes_H a = 0$ for any $x \in X$.

7.2. *The Functor $\mathscr{A}_Y$*

Analyzing the proof of Proposition 7.1, one can notice that the definition of $\mathscr{A}_Y$ is a special case of a more general construction which we describe below.

First, recall that a commutative associative H-differential algebra Y is the same as a commutative associative algebra in the pseudotensor category $\mathscr{M}^l(H)$ from Example 3.2. We denote by $\mathscr{M}^b(H)$ the category of H-bimodules, provided with a pseudotensor structure given by (3.3), but with Hom_H there replaced by Hom_{H-H} which means homomorphisms of H-bimodules. The composition of polylinear maps in $\mathscr{M}^b(H)$ is given again by (3.2). Then H-differential algebras Y considered above are exactly the commutative associative algebras in $\mathscr{M}^b(H)$.

Instead of one H-bimodule Y one can use several: for any collections of objects $Y_i \in \mathscr{M}^b(H)$ and $L_i \in \mathscr{M}^*(H)$ $(i \in I)$ we can consider the left H-modules $\mathscr{A}_{Y_i} L_i = Y_i \otimes_H L_i$ as objects of $\mathscr{M}^l(H)$. Assume we are given polylinear maps $f \in \mathrm{Lin}(\{Y_i\}_{i\in I}, Z)$ in $\mathscr{M}^b(H)$ and $\phi \in \mathrm{Lin}(\{L_i\}_{i\in I}, M)$ in $\mathscr{M}^*(H)$. Then we define a polylinear map $f \otimes_H \phi \in \mathrm{Lin}(\{Y_i \otimes_H L_i\}_{i\in I}, Z \otimes_H M)$ in $\mathscr{M}^l(H)$ as the following composition:

$$\bigotimes_{i\in I} (Y_i \otimes_H L_i) \xrightarrow{\sim} \left(\boxtimes_{i\in I} Y_i\right) \otimes_{H^{\otimes I}} \left(\boxtimes_{i\in I} L_i\right) \xrightarrow{\mathrm{id}\otimes\phi} \left(\boxtimes_{i\in I} Y_i\right) \otimes_{H^{\otimes I}} (H^{\otimes I} \otimes_H M)$$
$$\xrightarrow{\sim} \left(\bigotimes_{i\in I} Y_i\right) \otimes_H M \xrightarrow{f\otimes\mathrm{id}} Z \otimes_H M.$$

PROPOSITION 7.2. *The above definition is compatible with compositions of polylinear maps in $\mathscr{M}^b(H)$, $\mathscr{M}^*(H)$, and $\mathscr{M}^l(H)$:*

$$f(\{g_i\}_{i\in I}) \otimes_H \phi(\{\psi_i\}_{i\in I}) = (f \otimes_H \phi)(\{g_i \otimes_H \psi_i\}_{i\in I}).$$

The proof of this proposition is straightforward and is left to the reader.

COROLLARY 7.1. *Let (Y, ν) be a commutative associative algebra in $\mathscr{M}^b(H)$. For a finite nonempty set I, let $\nu^{(I)}\colon Y^{\otimes I} \to Y$ be the iterated multiplication $\nu(\nu\otimes\mathrm{id})\cdots(\nu\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id})$. Recall that for an object L in $\mathscr{M}^*(H)$, we define $\mathscr{A}_Y(L) := Y \otimes_H L$. For a polylinear map $\phi \in \mathrm{Lin}(\{L_i\}_{i\in I}, M)$ in $\mathscr{M}^*(H)$, let $\mathscr{A}_Y(\phi) := \nu^{(I)} \otimes_H \phi$. Then $\mathscr{A}_Y$ is a pseudotensor functor from $\mathscr{M}^*(H)$ to $\mathscr{M}^l(H)$.*

As a special case of this corollary, we obtain Proposition 7.1.

Let us give another application of Proposition 7.2. An instance of an H-bimodule is H itself (however, H is not an H-differential algebra!). The coproduct $\Delta\colon H \to H \otimes H$, the evaluation map $ev\colon X \otimes H \to \mathbf{k}$, and the isomorphism $\mathbf{k} \otimes H \xrightarrow{\sim} H$ are all homomorphisms of H-bimodules, so the composition

$$\eta\colon X \otimes H \xrightarrow{\mathrm{id}\otimes\Delta} X \otimes H \otimes H \xrightarrow{ev\otimes\mathrm{id}} \mathbf{k} \otimes H \xrightarrow{\sim} H \tag{7.5}$$

is a polylinear map in $\mathscr{M}^b(H)$. Let again L be a (Lie) pseudoalgebra and (M, ρ) be an L-module, where $\rho \in \mathrm{Lin}(\{L, M\}, M)$ in $\mathscr{M}^*(H)$. Then $\eta \otimes_H \rho \in \mathrm{Lin}(\{X \otimes_H L, H \otimes_H M\}, H \otimes_H M)$ is a polylinear map in $\mathscr{M}^l(H)$. In other words, we get a homomorphism of H-modules $\eta \otimes_H \rho\colon \mathscr{A}(L) \otimes M \to M$. Proposition 7.2 now implies:

COROLLARY 7.2. *The above map $\eta \otimes_H \rho$ provides M with the structure of an $\mathscr{A}(L)$-module, and this structure is compatible with that of an H-module* (*cf.* (7.4)).

For $a \in L$, $x \in X$, the action of $a_x \equiv x \otimes_H a$ on an element $m \in M$ will be denoted by $a_x \cdot m$. This defines *x-products* $a_x m := a_x \cdot m \in M$. When $M = L$ is the Lie pseudoalgebra with the adjoint action, these will be called *x-brackets* and denoted as $[a_x b]$. Then all the axioms of (Lie or associative) pseudoalgebras, representations, etc., can be reformulated in terms of the properties of the x-brackets or products—this will be done in Section 9. Although this may seem a mere tautology, it is more explicit and convenient in some cases.

Finally, let us give two more constructions.

EXAMPLE 7.1. The base field $\mathbf{k}$, with the action $h \cdot 1 = \varepsilon(h)$ $(h \in H)$, is a commutative associative H-differential algebra. Then for any Lie H-pseudoalgebra L, $\mathscr{A}_{\mathbf{k}} L = \mathbf{k} \otimes_H L$ is a Lie H-differential algebra. Explicitly, $\mathscr{A}_{\mathbf{k}} L \simeq L/H_+ L$, where $H_+ = \{h \in H \mid \varepsilon(h) = 0\}$ is the augmentation ideal. The Lie bracket in $L/H_+ L$ is given by (cf. (7.2))

$$[a \bmod H_+ L, b \bmod H_+ L] = \sum_i \varepsilon(f_i)\, \varepsilon(g_i)\, e_i \qquad \bmod H_+ L, \tag{7.6}$$

if

$$[a * b] = \sum_i (f_i \otimes g_i) \otimes_H e_i\,. \tag{7.7}$$

In the case when $\mathfrak{d} = \mathbf{k}\partial$ is 1-dimensional, we recover the usual construction $L \mapsto L/\partial L$ that assigns a Lie algebra to any Lie conformal algebra [K2].

Remark 7.2. Let Y be a commutative associative H-differential algebra with a right action of H, and let L be a Lie H-pseudoalgebra. We provide $Y \otimes L$ with the following structure of a left H-module:

$$h(x \otimes a) = x h_{(-1)} \otimes h_{(2)} a, \qquad h \in H,\ x \in Y,\ a \in L. \tag{7.8}$$

Then define a Lie pseudobracket on $Y \otimes L$ by the formula

$$[(x \otimes a) * (y \otimes b)] = \sum_i (f_{i(1)} \otimes g_{i(1)}) \otimes_H ((x f_{i(2)})(y g_{i(2)}) \otimes e_i), \tag{7.9}$$

if $[a * b]$ is given by (7.7). It is easy to check that (7.9) is well defined and provides $Y \otimes L$ with the structure of a Lie H-pseudoalgebra. Moreover, $\mathscr{A}_Y L \simeq (Y \otimes L)/H_+(Y \otimes L)$ as a Lie algebra (cf. Example 7.1).

In the case $\mathfrak{d}=\mathbf{k}\partial$, $Y=\mathbf{k}[t,t^{-1}]$, $\partial=-\partial_t$, the Lie $\mathbf{k}[\partial]$-pseudoalgebra ($=$conformal algebra) $Y\otimes L$ is known as an *affinization* of the conformal algebra L [K2].

7.3. *Relation to Differential Lie Algebras*

Fix two positive integers N, r and let $\mathcal{O}_N=\mathbf{k}[[t_1,\ldots,t_N]]$, $\mathscr{L}=\mathcal{O}_N\otimes\mathbf{k}^r$. A structure of a Lie algebra on $\mathscr{L}$ is called *local* (and $\mathscr{L}$ is called a *local Lie algebra* [Ki]) if the Lie bracket is given by matrix bi-differential operators. More explicitly, let $\{e^i\}$ be a basis of $\mathbf{k}^r$. Then for any $x, y\in\mathcal{O}_N$, the bracket in $\mathscr{L}$ is given by

$$[x\otimes e^i, y\otimes e^j]=\sum_{k,l}(P^{ij}_{kl}\cdot x)(Q^{ij}_{kl}\cdot y)\otimes e^k, \tag{7.10}$$

where P^{ij}_{kl}, Q^{ij}_{kl} are differential operators with coefficients in $\mathcal{O}_N$. The number r is called the *rank* of $\mathscr{L}$.

A related notion is that of a *differential Lie algebra* [R1–R4] (see also [C]). This is a Lie algebra structure on $\mathscr{L}=Y\otimes\mathbf{k}^r$, where Y is any commutative associative $H=\mathbf{k}[\partial_1,\ldots,\partial_N]$-differential algebra, given by (7.10) for $x, y\in Y$, $P^{ij}_{kl}, Q^{ij}_{kl}\in Y\otimes H$. One can allow a universal enveloping algebra $H=U(\mathfrak{d})$ ($\dim\mathfrak{d}=N$) in place of $\mathbf{k}[\partial_1,\ldots,\partial_N]$, cf. [NW].

Recall that for $H=U(\mathfrak{d})$, $X=H^*$ is a commutative associative H-differential algebra that can be identified with $\mathcal{O}_N$ for $N=\dim\mathfrak{d}$. Moreover, the action of H (and of $X\otimes H$) on X is given by differential operators in this identification. Therefore a differential Lie algebra for $Y=X$ is the same as a local Lie algebra.

Then the results of Section 7.1 immediately imply:

PROPOSITION 7.3. *Let $L=H\otimes\mathbf{k}^r$ be a Lie pseudoalgebra which is a free H-module of rank r. Let Y be an H-bimodule which is a commutative associative H-differential algebra both for the left and for the right action of H (see (2.8), (2.19)). Then $\mathscr{A}_Y L\simeq Y\otimes\mathbf{k}^r$ is a differential Lie algebra. In particular, $\mathscr{A}(L)=\mathscr{A}_X L$ is a local Lie algebra.*

Note that the differential Lie algebras $\mathscr{A}_Y L$ that we get are with "constant coefficients": in (7.10) all $P^{ij}_{kl}, Q^{ij}_{kl}\in H$.

7.4. *Topology on the Annihilation Algebra*

Now let us discuss the problem of defining a topology on $\mathscr{A}(M)=X\otimes_H M$ where M is any finite H-module. Recall that X has a decreasing filtration $X=\mathrm{F}_{-1}X\supset\mathrm{F}_0X\supset\cdots$ defined in Section 2. We can use this filtration to construct an induced filtration on $\mathscr{A}(M)$ as follows. Choose a finite-dimensional (over $\mathbf{k}$) subspace M_0 of M which generates M over H, and set

$$\mathrm{F}_i\,\mathscr{A}(M) = \{x \otimes_H m \mid x \in \mathrm{F}_i\,X,\ m \in M_0\}. \tag{7.11}$$

Note that since H is cocommutative, its filtration satisfies (2.15), hence $\bigcap \mathrm{F}_i\,X = 0$. This implies

$$\bigcap_i \mathrm{F}_i\,\mathscr{A}(M) = 0. \tag{7.12}$$

The filtration (7.11) will in general depend on the choice of M_0, but the topology induced by it will not, as any two such filtrations are equivalent by the next lemma.

LEMMA 7.2. *Let M_0 and M'_0 be two finite-dimensional subspaces of M generating it over H, and let $\{\mathrm{F}_i\,\mathscr{A}(M)\}$, $\{\mathrm{F}'_i\,\mathscr{A}(M)\}$ be the corresponding filtrations on $\mathscr{A}(M)$. Then there exist integers a, b such that $\mathrm{F}_{i+a}\,\mathscr{A}(M) \subset \mathrm{F}'_i\,\mathscr{A}(M) \subset \mathrm{F}_{i+b}\,\mathscr{A}(M)$ for all values of i.*

Proof. Let us choose bases of M_0 and M'_0, and let us fix expressions of elements from the first basis as H-linear combinations of elements from the second basis. Denote by a the highest degree of the coefficients of all these expressions. Using (2.22), we see that $\mathrm{F}_i\,\mathscr{A}(M) \subset \mathrm{F}'_{i-a}\mathscr{A}(M)$ for all i. Repeating the same reasoning after switching the roles of M_0 and M'_0, we get $\mathrm{F}'_i\,\mathscr{A}(M) \subset \mathrm{F}_{i-b}\mathscr{A}(M)$ for some b and all i. ∎

PROPOSITION 7.4. *Let H be a cocommutative Hopf algebra which satisfies* (2.16).

(i) *If M is a finite H-module, then $\mathscr{A}(M)$ is a linearly compact topological vector space when provided with the filtration* (7.11). *The action of H on $\mathscr{A}(M)$ is continuous if we endow H with the discrete topology.*

(ii) *If L is a finite Lie H-pseudoalgebra, then its annihilation algebra $\mathscr{A}(L)$ is a linearly compact Lie H-differential algebra, i.e., it is a linearly compact topological vector space and both the Lie bracket and the action of H are continuous.*

A similar statement holds for representations and for associative pseudoalgebras as well.

Proof. (i) The linear-compactness follows from Proposition 6.1, since (7.11) is a filtration by finite-codimensional subspaces with trivial intersection and $\mathscr{A}(M)$ is complete with respect to this filtration. The continuity of the H-action follows from (2.22):

$$\mathrm{F}^i H \cdot \mathrm{F}_j \mathscr{A}(M) \subset \mathrm{F}_{j-i} \mathscr{A}(M) \qquad \text{for all} \quad i, j. \tag{7.13}$$

(ii) It only remains to check that the Lie bracket of $\mathscr{A}(L)$ is continuous. Let L_0 be a finite-dimensional (over $\mathbf{k}$) subspace of L which generates it over H. For $a, b \in L_0$, we can write

$$[a * b] = \sum_i (f_i \otimes g_i) \otimes_H e_i$$

for some f_i, $g_i \in H$ and $e_i \in L_0$. Then the Lie bracket in $\mathscr{A}(L)$, for $x, y \in X$, is given by

$$[x \otimes_H a, y \otimes_H b] = \sum_i (xf_i)(yg_i) \otimes_H e_i .$$

We can find a number p such that all coefficients f_i, $g_i \in H$ occurring in pseudobrackets of any elements $a, b \in L_0$ belong to $\mathrm{F}^p H$. Then equations (2.21), (2.22) imply

$$[\mathrm{F}_i \mathscr{A}(L), \mathrm{F}_j \mathscr{A}(L)] \subset \mathrm{F}_{i+j-s} \mathscr{A}(L) \qquad \text{for all} \quad i, j, \tag{7.14}$$

where $s = 2p - 1$. This shows that the Lie bracket is continuous. ▮

LEMMA 7.3. *Let $H = U(\mathfrak{d}) \sharp \mathbf{k}[\Gamma]$. Then for any nonzero $h \in \mathfrak{d}$ and for every open subspace U of $\mathscr{A}(M)$ there is some n such that $h^n U = \mathscr{A}(M)$. In particular, each such h acts surjectively on $\mathscr{A}(M)$.*

Proof. Follows immediately from Lemma 6.4. ▮

7.5. *Growth of the Annihilation Algebra*

Let M be a finite H-module. Then any choice of a finite-dimensional subspace M_0 generating M over H provides $\mathscr{M} = \mathscr{A}(M)$ with a filtration $\mathscr{M}_n := \mathrm{F}_n X \otimes_H M_0$.

DEFINITION 7.2. For a filtered vector space $\mathscr{M} = \mathscr{M}_{-1} \supset \mathscr{M}_0 \supset \cdots$ we define its *growth* $\mathrm{gw}\mathscr{M}$ to be d if the function $n \mapsto \dim \mathscr{M}/\mathscr{M}_n$ can be bounded from above and below by polynomials of degree d.

By Lemma 7.2, a different choice of M_0 would give a uniformly equivalent filtration of the same growth as $\{\mathscr{M}_n\}$. Hence, we can speak of the growth of $\mathscr{A}(M)$ independently of the choice of M_0.

Proposition 7.5. *Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$, and M be a finitely generated H-module. Then the growth of $\mathscr{A}(M)$ is equal to the dimension of $\mathfrak{d}$.*

Proof. First of all, notice that we can assume M is torsion-free, since by Lemma 7.1, $\mathscr{A}(M) \simeq \mathscr{A}(M/\mathrm{Tor}\, M)$ where Tor M is the torsion submodule of M. The proof of the proposition is then based on Lemma 2.1 and the following two lemmas.

Lemma 7.4. *The map $\mathscr{A}(f)\colon \mathscr{A}(M) \to \mathscr{A}(F)$ induced by the inclusion $f\colon M \hookrightarrow F$ constructed in Lemma 2.1 is uniformly continuous, i.e., for every i we have*

$$\mathrm{F}_{i-a}\, \mathscr{A}(F) \subset \mathscr{A}(f)(\mathrm{F}_i\, \mathscr{A}(M)) \subset \mathrm{F}_{i+b}\, \mathscr{A}(F),$$

where a and b are independent of i.

The same is true for $\mathscr{A}(g)\colon \mathscr{A}(F) \to \mathscr{A}(M)$ where g is the embedding $g\colon hF \hookrightarrow M$ from Lemma 2.1.

Proof. Let us choose finite-dimensional vector subspaces F_0 of F generating F over H, and M_0 of M generating M over H and containing hF_0. Let us also choose a second finite-dimensional vector subspace F'_0 of F containing M_0 and generating F over H. We will denote the filtrations induced by these subspaces by $\{\mathscr{F}_i\}$, $\{\mathscr{M}_i\}$, and $\{\mathscr{F}'_i\}$, respectively.

Up to identifying hF with F, we have constructed injective maps $F \xrightarrow{g} M \xrightarrow{f} F$ such that the composition fg is a multiplication by h. These maps induce maps $\mathscr{A}(F) \xrightarrow{\mathscr{A}(g)} \mathscr{A}(M) \xrightarrow{\mathscr{A}(f)} \mathscr{A}(F)$ which are surjective, as one can see by tensoring by X and using that $\mathscr{A}(T) = 0$ if T is a torsion H-module (see Lemma 7.1).

The above maps are also continuous with respect to the common topology defined by any of the above constructed filtrations. In fact, by construction, one has

$$\mathscr{A}(g)(\mathscr{F}_i) \subset \mathscr{M}_i \qquad \text{and} \qquad \mathscr{A}(f)(\mathscr{M}_i) \subset \mathscr{F}'_i.$$

The second inclusion proves that $\mathscr{A}(f)(\mathscr{M}_i) \subset \mathscr{F}_{i+b}$ for some b independent of i, because the filtrations $\{\mathscr{F}_i\}$ and $\{\mathscr{F}'_i\}$ are uniformly equivalent by Lemma 7.2.

Applying $\mathscr{A}(f)$ to the first inclusion, we get $\mathscr{A}(f)\, \mathscr{A}(g)(\mathscr{F}_i) \subset \mathscr{A}(f)(\mathscr{M}_i)$. On the other hand, $\mathscr{A}(f)\, \mathscr{A}(g) = \mathscr{A}(fg) = h \otimes_H \mathrm{id}_F$, and Lemma 6.4 implies that $\mathscr{A}(f)\, \mathscr{A}(g)(\mathscr{F}_i) = \mathscr{F}_{i-a}$ where a is such that $h \in \mathrm{F}^a H$ but $h \notin \mathrm{F}^{a-1} H$. Therefore $\mathscr{F}_{i-a} \subset \mathscr{A}(f)(\mathscr{M}_i)$ for all i.

A similar argument works for $\mathscr{A}(g)$. ∎

LEMMA 7.5. *If* $\varphi: \mathscr{M} \to \mathscr{N}$ *is a surjective uniformly continuous map of filtered modules, then* $\operatorname{gw} \mathscr{M} \geqslant \operatorname{gw} \mathscr{N}$.

Proof. By assumption $\varphi(\mathscr{M}_i) \subset \mathscr{N}_{i+b}$ for all i and some b independent of i. The induced map $\mathscr{M}/\mathscr{M}_i \to \mathscr{N}/\mathscr{N}_{i+b}$ is a surjective map of finite-dimensional vector spaces. Hence $\operatorname{gw} \mathscr{M} \geqslant \operatorname{gw} \mathscr{N}$. ∎

Using the above lemmas, now we can complete the proof of Proposition 7.5. We have constructed an embedding $f: M \hookrightarrow F$ of M into a free H-module F, and we have shown that the induced map $\mathscr{A}(f): \mathscr{A}(M) \to \mathscr{A}(F)$ is surjective and uniformly continuous. This implies $\operatorname{gw} \mathscr{A}(M) \geqslant \operatorname{gw} \mathscr{A}(F)$. Similarly, the inclusion $hF \hookrightarrow M$ gives us $\operatorname{gw} \mathscr{A}(F) = \operatorname{gw} \mathscr{A}(hF) \geqslant \operatorname{gw} \mathscr{A}(M)$. Therefore, $\operatorname{gw} \mathscr{A}(M) = \operatorname{gw} \mathscr{A}(F)$. It remains to note that, for any (nonzero) free H-module F of finite rank, one has $\operatorname{gw} \mathscr{A}(F) = \dim \mathfrak{d}$. This follows from the fact that $\operatorname{gw} X = N = \dim \mathfrak{d}$ because $X \simeq \mathcal{O}_N$. ∎

Remark 7.3. The growth of a linearly compact Lie algebra $\mathscr{L}$ satisfying the descending chain condition can be defined as follows. Take a fundamental subalgebra $A \subset \mathscr{L}$, and build a filtration of $\mathscr{L}$ by

$$\mathscr{L}_0^A = A, \qquad \mathscr{L}_{i+1}^A = \{x \in \mathscr{L}_i^A \mid [x, \mathscr{L}] \in \mathscr{L}_i^A\}, \qquad i \geqslant 0.$$

Taking $A' = \mathscr{L}_k^A$, we have $\mathscr{L}_i^{A'} = \mathscr{L}_{k+i}^A$, hence replacing A by A' does not change the growth. Therefore, by the Chevalley principle [G1], the growth of this filtration does not depend on the choice of A. We will denote this common growth by $\operatorname{gw} \mathscr{L}$.

Notice that all simple linearly compact Lie algebras satisfy the descending chain condition, and therefore have a well defined growth which equals N for W_N, S_N, H_N, and K_N, and 0 for finite-dimensional Lie algebras.

8. PRIMITIVE PSEUDOALGEBRAS OF VECTOR FIELDS

In this section, $\mathfrak{d}$ will be a (finite-dimensional) Lie algebra and $H = U(\mathfrak{d})$ will be its universal enveloping algebra. As usual, we will identify $\mathfrak{d}$ with its image in H. Then $X := H^*$ is the algebra of formal power series on $\mathfrak{d}^*$, which is isomorphic as a topological algebra to $\mathcal{O}_N$ for $N = \dim \mathfrak{d}$. In this section we are going to define H-pseudoalgebra analogues of the primitive linearly compact Lie algebras W_N, S_N, H_N, K_N, which will be called *primitive pseudoalgebras of vector fields*.

8.1. $W(\mathfrak{d})$

Let Y be a commutative associative algebra on which $\mathfrak{d}$ acts by derivations from the right (i.e., Y is an H-differential algebra). One can define a left action of $Y \otimes \mathfrak{d}$ on Y using the right action of $\mathfrak{d}$ on Y:

$$(x \otimes a)\, z = -x(za), \qquad x, z \in Y,\ a \in \mathfrak{d}. \tag{8.1}$$

This will define a representation of $Y \otimes \mathfrak{d}$ in Y if the Lie bracket of $\mathfrak{d}$ is extended to $Y \otimes \mathfrak{d}$ by the formula

$$[x \otimes a, y \otimes b] = xy \otimes [a, b] - x(ya) \otimes b + (xb)\, y \otimes a. \tag{8.2}$$

In particular, for $Y = X = H^*$, this gives the Lie algebra of all vector fields on X, which is isomorphic to W_N for $N = \dim \mathfrak{d}$.

Comparing (8.2) with (7.2), we are led to define the pseudoalgebra $W(\mathfrak{d}) = H \otimes \mathfrak{d}$ with pseudobracket

$$\begin{aligned} [(f \otimes a) * (g \otimes b)] = {} & (f \otimes g) \otimes_H (1 \otimes [a, b]) \\ & - (f \otimes ga) \otimes_H (1 \otimes b) + (fb \otimes g) \otimes_H (1 \otimes a). \end{aligned} \tag{8.3}$$

It is easy to check that $W(\mathfrak{d})$ is indeed a Lie pseudoalgebra, and that the Lie algebra $\mathscr{A}_Y W(\mathfrak{d})$ defined in Section 7 is isomorphic to $Y \otimes \mathfrak{d}$ with bracket defined by (8.2). In a similar fashion, the module Y over $Y \otimes \mathfrak{d}$, defined by (8.1), leads to a structure of a $W(\mathfrak{d})$-module on H:

$$(f \otimes a) * g = -(f \otimes ga) \otimes_H 1. \tag{8.4}$$

8.2. *Differential Forms*

We can think of $X = H^*$ as the space of functions on $\mathfrak{d}$, and of the elements of $X \otimes \mathfrak{d}$ as vector fields. Then the space of n-forms ($n = 0, ..., \dim \mathfrak{d}$) is

$$\Omega^n_X := \operatorname{Hom}_{\mathbf{k}} \left(\bigwedge\nolimits^n \mathfrak{d}, X \right) \simeq X \otimes \bigwedge\nolimits^n \mathfrak{d}^*.$$

It is convenient to extend the elements $\omega \in \Omega^n_X$ to functions from $\bigwedge^n (X \otimes \mathfrak{d})$ to X, polylinear over X,

$$\omega(x_1 \otimes a_1 \wedge \cdots \wedge x_n \otimes a_n) = x_1 \cdots x_n\, \omega(a_1 \wedge \cdots \wedge a_n),$$

so that

$$\Omega^n_X = \operatorname{Hom}_X \left(\bigwedge\nolimits^n (X \otimes \mathfrak{d}), X \right).$$

We view X as a left $(X\otimes \mathfrak{d})$-module using the right action of $\mathfrak{d}$; see (8.1). There is a differential $\mathrm{d}\colon \Omega^n_X \to \Omega^{n+1}_X$ satisfying $\mathrm{d}^2=0$; this is just the usual differential for the cohomology of $\mathfrak{d}$ with coefficients in X where X is viewed as a right $\mathfrak{d}$-module:

$$
\begin{aligned}
(\mathrm{d}\omega)(a_1\wedge\cdots\wedge a_{n+1})
&= \sum_{i<j}(-1)^{i+j}\,\omega([a_i,a_j]\wedge a_1\wedge\cdots\wedge \hat{a}_i\wedge\cdots\wedge\hat{a}_j\wedge\cdots\wedge a_{n+1})\\
&\quad+\sum_i(-1)^i\,\omega(a_1\wedge\cdots\wedge\hat{a}_i\wedge\cdots\wedge a_{n+1})\cdot a_i.
\end{aligned}
$$

The following analogue of the Poincaré Lemma is very useful.

LEMMA 8.1. *The complex* $(\Omega^\bullet_X,\mathrm{d})$ *is acyclic, i.e., its* n*th cohomology is trivial for* $n>0$ *and* 1*-dimensional for* $n=0$.

Proof. It is well known that $\mathrm{H}^n(\mathfrak{d}, U(\mathfrak{d})^*)\simeq \mathrm{H}_n(\mathfrak{d}, U(\mathfrak{d}))^*$ is trivial for $n>0$ and 1-dimensional for $n=0$; see, e.g., [Fu]. ∎

For a vector field $A\in X\otimes\mathfrak{d}$, we have the contraction operator $\iota_A\colon\Omega^n_X\to\Omega^{n-1}_X$ given by

$$(\iota_A\omega)(a_1\wedge\cdots\wedge a_{n-1})=\omega(A\wedge a_1\wedge\cdots\wedge a_{n-1}).$$

We define the Lie derivative $L_A\colon\Omega^n_X\to\Omega^n_X$ by Cartan's formula $L_A=\mathrm{d}\iota_A+\iota_A\mathrm{d}$. Explicitly, for $x\otimes a\in X\otimes\mathfrak{d}$, we have

$$
\begin{aligned}
(L_{x\otimes a}\omega)(a_1\wedge\cdots\wedge a_n) &= -x(\omega(a_1\wedge\cdots\wedge a_n)\cdot a)\\
&\quad+\sum_i(-1)^i\,(x\cdot a_i)\,\omega(a\wedge a_1\wedge\cdots\wedge\hat{a}_i\wedge\cdots\wedge a_n)\\
&\quad+\sum_i(-1)^i\,x\omega([a,a_i]\wedge a_1\wedge\cdots\wedge\hat{a}_i\wedge\cdots\wedge a_n).
\end{aligned}
\tag{8.5}
$$

The Lie derivative provides each Ω^n_X with the structure of a module over the Lie algebra of vector fields $X\otimes\mathfrak{d}$.

For $n=0$, $\Omega^0_X=X$ and this is the usual action (8.1) of $X\otimes\mathfrak{d}$ on X. When $n=N=\dim\mathfrak{d}$, we have $\Omega^N_X=Xv_0$ where $v_0\in\bigwedge^N\mathfrak{d}^*$, $v_0\neq 0$ is a volume form. An easy calculation shows that

$$L_{x\otimes a}(yv_0)=-((xy)(a+\operatorname{tr}\operatorname{ad}a))\,v_0,\qquad x,y\in X,\ a\in\mathfrak{d}. \tag{8.6}$$

8.3. *Pseudoforms*

The module Ω^n_X over the Lie algebra $X \otimes \mathfrak{d}$ leads to a module $\Omega^n(\mathfrak{d})$ over the Lie pseudoalgebra $W(\mathfrak{d})$ which we now define. We let

$$\Omega^n(\mathfrak{d}) = H \otimes \bigwedge\nolimits^n \mathfrak{d}^*, \qquad \Omega(\mathfrak{d}) = \bigoplus_{n=0}^{N} \Omega^n(\mathfrak{d}) \quad (N = \dim \mathfrak{d}).$$

The elements of $\Omega(\mathfrak{d})$ are called *pseudoforms.*

$\Omega^n(\mathfrak{d})$ is a free H-module, so that $\mathscr{A}(\Omega^n(\mathfrak{d})) = X \otimes_H \Omega^n(\mathfrak{d}) \simeq X \otimes \bigwedge^n \mathfrak{d}^* = \Omega^n_X$. The action of $W(\mathfrak{d}) = H \otimes \mathfrak{d}$ on $\Omega^n(\mathfrak{d})$ is obtained by comparing (7.2) with (8.5). To write an explicit formula, we identify $\Omega^n(\mathfrak{d})$ with the space of linear maps from $\bigwedge^n \mathfrak{d}$ to H, and $(H \otimes H) \otimes_H \Omega^n(\mathfrak{d})$ with the space of linear maps from $\bigwedge^n \mathfrak{d}$ to $H \otimes H$. Then for $f \otimes a \in W(\mathfrak{d})$, $w \in \Omega^n(\mathfrak{d})$, and $a_i \in \mathfrak{d}$, we have

$$\begin{aligned} ((f \otimes a) * w)(a_1 \wedge \cdots \wedge a_n) &= -f \otimes w(a_1 \wedge \cdots \wedge a_n)\, a \\ &\quad + \sum_i (-1)^i fa_i \otimes w(a \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n) \\ &\quad + \sum_i (-1)^i f \otimes w([a, a_i] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n). \end{aligned} \tag{8.7}$$

When $n = 0$, $\Omega^0(\mathfrak{d}) = H$, and we recover (8.4). In the other extreme case, when $n = N := \dim \mathfrak{d}$, $\Omega^N(\mathfrak{d}) = Hv_0$ is again a free H-module of rank one, where $v_0 \in \bigwedge^N \mathfrak{d}^*$, $v_0 \neq 0$ is any volume form on $\mathfrak{d}$. We have (cf. (8.6))

$$(f \otimes a) * v_0 = -(f(a + \operatorname{tr} \operatorname{ad} a) \otimes 1 + f \otimes a) \otimes_H v_0. \tag{8.8}$$

Define polylinear maps $*_\iota \in \operatorname{Lin}(\{W(\mathfrak{d}), \Omega^n(\mathfrak{d})\}, \Omega^{n-1}(\mathfrak{d}))$ by

$$((f \otimes a) *_\iota w)(a_1 \wedge \cdots \wedge a_{n-1}) = f \otimes w(a \wedge a_1 \wedge \cdots \wedge a_{n-1}). \tag{8.9}$$

Also define a differential $\mathrm{d}\colon \Omega^n(\mathfrak{d}) \to \Omega^{n+1}(\mathfrak{d})$ by

$$\begin{aligned} (\mathrm{d}w)(a_1 \wedge \cdots \wedge a_{n+1}) &= \sum_{i<j} (-1)^{i+j} w([a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_{n+1}) \\ &\quad + \sum_i (-1)^i w(a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_{n+1})\, a_i \qquad \text{if} \quad n \geqslant 1, \\ (\mathrm{d}w)(a) &= -wa \qquad \text{if} \quad n = 0, \end{aligned}$$

so that d is H-linear and $\mathrm{d}^2 = 0$. For any pseudoform w and $x \in X$, we have a differential form $x \otimes_H w$ and the relation $\mathrm{d}(x \otimes_H w) = x \otimes_H \mathrm{d}w$.

Remark 8.1. The nth cohomology of the complex $(\Omega^{\bullet}\mathfrak{d}^*(\mathfrak{d}), \mathrm{d})$ is equal to $\mathrm{H}^n(\mathfrak{d}, U(\mathfrak{d}))$. This space is trivial for $n \neq N = \dim \mathfrak{d}$ and 1-dimensional for $n = N$. This follows from the Poincaré duality $\mathrm{H}^n(\mathfrak{d}, U(\mathfrak{d})) \simeq \mathrm{H}^{N-n}(\mathfrak{d}, U(\mathfrak{d})^*)$; see, e.g., [Fu].

We have the following analogue of Cartan's formula for the action of $W(\mathfrak{d})$ on $\Omega(\mathfrak{d})$

$$\alpha * w = ((\mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{d})(\alpha *_\iota w) + \alpha *_\iota (\mathrm{d}w) \in (H \otimes H) \otimes_H \Omega(\mathfrak{d}). \tag{8.10}$$

This implies that the action of $W(\mathfrak{d})$ commutes with d:

$$\alpha * (\mathrm{d}w) = ((\mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{d})(\alpha * w). \tag{8.11}$$

We note that the maps $\alpha *_\iota$ anticommute with each other,

$$\alpha *_\iota (\beta *_\iota w) + ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\, \beta *_\iota (\alpha *_\iota w) = 0 \tag{8.12}$$

for $\alpha, \beta \in W(\mathfrak{d})$, $w \in \Omega(\mathfrak{d})$.

The wedge product on $\bigwedge^{\bullet} \mathfrak{d}^*$ can be extended to a pseudoproduct $*$ on $\Omega(\mathfrak{d}) = H \otimes \bigwedge^{\bullet} \mathfrak{d}^*$, so that it becomes a current pseudoalgebra. Then it is easy to check that for $\alpha \in W(\mathfrak{d})$, $v \in \bigwedge^m \mathfrak{d}^*$, $w \in \bigwedge^n \mathfrak{d}^*$, one has

$$\alpha * (v * w) = (\alpha * v) * w + ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\, v * (\alpha * w), \tag{8.13}$$

and similarly

$$\alpha *_\iota (v * w) = (\alpha *_\iota v) * w + (-1)^m ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\, v * (\alpha *_\iota w). \tag{8.14}$$

This can be interpreted as saying that $\alpha *$ and $\alpha *_\iota$ are superderivations of $\Omega(\mathfrak{d})$, see Example 10.2 below.

8.4. $S(\mathfrak{d}, \chi)$

The divergence of a vector field $\sum_i x_i \otimes a_i \in X \otimes \mathfrak{d}$ is defined by $\operatorname{div}(\sum_i x_i \otimes a_i) = \sum_i x_i a_i \in X$. Then one easily checks

$$\operatorname{div}([A, B]) = A \cdot \operatorname{div}(B) - B \cdot \operatorname{div}(A), \qquad A, B \in X \otimes \mathfrak{d}, \tag{8.15}$$

so that the divergence zero vector fields form a Lie subalgebra S_N of W_N. Let χ be a trace form on $\mathfrak{d}$, i.e., a linear functional from $\mathfrak{d}$ to $\mathbf{k}$ which vanishes on $[\mathfrak{d}, \mathfrak{d}]$. Then we can define

$$\operatorname{div}^\chi\left(\sum_i x_i \otimes a_i\right) := \sum_i x_i(a_i + \chi(a_i)),$$

which still satisfies (8.15).

Remark 8.2. Let χ be as above, and let $\psi = \chi - \operatorname{tr}\operatorname{ad}$, which is again a trace form on $\mathfrak{d}$. We can consider ψ as an element of $\Omega^1_X = X \otimes \mathfrak{d}^*$; then $d\psi = 0$ and by Lemma 8.1 we have $\psi = -dz$ for some $z \in X$. This means that $\psi(a) = za$ for all $a \in \mathfrak{d}$. Let $y = e^z$; then $ya = y\psi(a)$ for $a \in \mathfrak{d}$. Consider the volume form $v = yv_0$, where $v_0 \in \bigwedge^N \mathfrak{d}^*$, $v_0 \neq 0$. Equation (8.6) gives

$$L_A v = -\operatorname{div}^\chi(A)\, v \qquad \text{for} \quad A \in X \otimes \mathfrak{d}. \tag{8.16}$$

Therefore, the Lie algebra of vector fields A with $\operatorname{div}^\chi(A) = 0$ coincides with the Lie algebra $S_N(v)$ of vector fields annihilating the volume form v.

Using the notation $\alpha_x \equiv x \otimes_H \alpha \in \mathscr{A}(W(\mathfrak{d})) \simeq X \otimes \mathfrak{d}$ for $\alpha \in W(\mathfrak{d})$, $x \in X$, we find for $\alpha = h \otimes a$:

$$\alpha_x = x \otimes_H (h \otimes a) = xh \otimes_H (1 \otimes a) \equiv xh \otimes a \in X \otimes \mathfrak{d},$$

hence, $\operatorname{div}^\chi(\alpha_x) = xh(a + \chi(a))$. Define the divergence operator $\operatorname{div}^\chi \colon W(\mathfrak{d}) \to H$ by the formula

$$\operatorname{div}^\chi\left(\sum_i h_i \otimes a_i\right) = \sum_i h_i\,(a_i + \chi(a_i)). \tag{8.17}$$

Then we have

$$\operatorname{div}^\chi(\alpha_x) = x \cdot \operatorname{div}^\chi \alpha \qquad \text{for} \quad \alpha \in W(\mathfrak{d}),\ x \in X. \tag{8.18}$$

Since div^χ is H-linear, we can define

$$\operatorname{div}^\chi_2 \colon (H \otimes H) \otimes_H W(\mathfrak{d}) \xrightarrow{\operatorname{id} \otimes_H \operatorname{div}^\chi} (H \otimes H) \otimes_H H \xrightarrow{\sim} H \otimes H.$$

Similarly to (8.15), one has

$$\operatorname{div}^\chi_2([\alpha * \beta]) = (\operatorname{div}^\chi \alpha \otimes 1)\, \sigma(\beta) - (1 \otimes \operatorname{div}^\chi \beta)\, \alpha, \qquad \alpha, \beta \in W(\mathfrak{d}), \tag{8.19}$$

where $\sigma \colon H \otimes H \to H \otimes H$ is the transposition.

Equation (8.19) implies that

$$S(\mathfrak{d}, \chi) := \{\alpha \in W(\mathfrak{d}) \mid \operatorname{div}^\chi \alpha = 0\} \tag{8.20}$$

is a subalgebra of the Lie pseudoalgebra $W(\mathfrak{d})$. By Eq. (8.18) and Remark 8.2, its annihilation algebra

$$\mathscr{A}(S(\mathfrak{d}, \chi)) = \{A \in W_N \mid \operatorname{div}^\chi A = 0\} \simeq S_N. \tag{8.21}$$

The rank of $S(\mathfrak{d}, \chi)$ as an H-module is $N-1$; however, it is free only for $N=2$.

PROPOSITION 8.1. *$S(\mathfrak{d}, \chi)$ is generated over H by elements*

$$e_{ab} := (a+\chi(a)) \otimes b - (b+\chi(b)) \otimes a - 1 \otimes [a, b] \qquad \textit{for} \quad a, b \in \mathfrak{d}. \tag{8.22}$$

These elements satisfy $e_{ab} = -e_{ba}$ and the relations (for $\chi = 0$)

$$ae_{bc} + be_{ca} + ce_{ab} = e_{[a, b], c} + e_{[b, c], a} + e_{[c, a], b}. \tag{8.23}$$

For $\chi = 0$, their pseudobrackets are given by

$$\begin{aligned}[][e_{ab} * e_{cd}] = {} & (a \otimes d) \otimes_H e_{bc} + (b \otimes c) \otimes_H e_{ad} \\ & - (a \otimes c) \otimes_H e_{bd} - (b \otimes d) \otimes_H e_{ac} \\ & + (a \otimes 1) \otimes_H e_{b, [c, d]} - (b \otimes 1) \otimes_H e_{a, [c, d]} \\ & - (1 \otimes c) \otimes_H e_{d, [a, b]} + (1 \otimes d) \otimes_H e_{c, [a, b]} \\ & - (1 \otimes 1) \otimes_H e_{[a, b], [c, d]}.\end{aligned} \tag{8.24}$$

For arbitrary χ, replace everywhere in (8.23), (8.24) *all $h \in \mathfrak{d}$ by $h + \chi(h)$.*

Remark 8.3. Equation (8.24) implies that for $\chi = 0$

$$\begin{aligned}[][e_{ab} * e_{ab}] = {} & (b \otimes a - a \otimes b) \otimes_H e_{ab} \\ & + (1 \otimes b - b \otimes 1) \otimes_H e_{a, [a, b]} + (a \otimes 1 - 1 \otimes a) \otimes_H e_{b, [a, b]}.\end{aligned} \tag{8.25}$$

(Again, for any χ, replace a, b with $a + \chi(a), b + \chi(b)$.) In particular, when the elements a, b span a Lie algebra, He_{ab} is a Lie pseudoalgebra.

In the proof of Proposition 8.1 we are going to use the following lemma.

LEMMA 8.2. *Let $H = U(\mathfrak{d})$, and let $\{\partial_1, ..., \partial_N\}$ be a basis of $\mathfrak{d}$. If elements $h_i \in \mathrm{F}^d H$ are such that $\sum_i h_i \partial_i \in \mathrm{F}^d H$, then there exist $f_{ij} \in \mathrm{F}^{d-1} H$ such that*

$$\sum_i h_i \otimes \partial_i = \sum_{i, j} (f_{ij} \otimes 1)(\partial_i \otimes \partial_j - \partial_j \otimes \partial_i) \qquad \mathrm{mod}\ \mathrm{F}^{d-1} H \otimes \mathfrak{d}.$$

Proof. The proof is by induction on the number of h_i not contained in $\mathrm{F}^{d-1} H$, the basis of induction being trivial. Consider $\sum_{i=1}^n h_i \partial_i \in \mathrm{F}^d H$, with all $h_i \notin \mathrm{F}^{d-1} H$. We can write $h_i = f_i \partial_1 + k_i$ so that $k_i \in \mathrm{F}^d H$ is a linear combination of Poincaré–Birkhoff–Witt basis elements of H not containing ∂_1 in their expression, and $f_i \in \mathrm{F}^{d-1}H$. Then

$$\begin{aligned}\sum_{i=1}^{n} h_i \partial_i &= h_1 \partial_1 + \sum_{i=2}^{n} (f_i \partial_1 \partial_i + k_i \partial_i) \\ &= \left(h_1 + \sum_{i=2}^{n} f_i \partial_i\right) \partial_1 + \sum_{i=2}^{n} k_i \partial_i + \sum_{i=2}^{n} f_i [\partial_1, \partial_i].\end{aligned}$$

Since the third summand in the right-hand side belongs to $\mathrm{F}^d H$, it follows that the first and second summands lie in $\mathrm{F}^d H$ too. This implies: $h_1 + \sum_{i=2}^{n} f_i \partial_i \in \mathrm{F}^{d-1}H$. Hence

$$\sum_{i=1}^{n} h_i \otimes \partial_i = \sum_{i=2}^{n} (f_i \partial_1 \otimes \partial_i - f_i \partial_i \otimes \partial_1) + \sum_{i=2}^{n} k_i \otimes \partial_i \qquad \mathrm{mod}\ \mathrm{F}^{d-1} H \otimes \mathfrak{d},$$

and we can apply the inductive assumption. ∎

Proof of Proposition 8.1. First of all, it is easy to check that the elements (8.22) indeed belong to $S(\mathfrak{d}, \chi)$. Equation (8.23) is easy, and the computation of the pseudobrackets is straightforward using (8.3), reformulated as

$$\begin{aligned}[(1 \otimes a) * (1 \otimes b)] = ((a + \chi(a)) \otimes 1) \otimes_H (1 \otimes b) \\ - (1 \otimes (b + \chi(b))) \otimes_H (1 \otimes a) - (1 \otimes 1) \otimes_H e_{ab}.\end{aligned} \tag{8.26}$$

Now let us consider an element $\alpha = \sum_i h_i \otimes \partial_i \in S(\mathfrak{d}, \chi)$, $h_i \in H$. We will prove that α can be expressed as H-linear combination of the above elements (8.22) by induction on the maximal degree d of the h_i. Since $\alpha \in S(\mathfrak{d}, \chi)$, then $\sum_i h_i (\partial_i + \chi(\partial_i)) = 0$, hence $\sum_i h_i \partial_i \in \mathrm{F}^d H$.

By Lemma 8.2, we can find elements $f_{ij} \in \mathrm{F}^{d-1} H$ such that

$$\alpha = \sum_{i,j} (f_{ij} \otimes 1)(\partial_i \otimes \partial_j - \partial_j \otimes \partial_i) \qquad \mathrm{mod}\ \mathrm{F}^{d-1} H \otimes \mathfrak{d}.$$

Therefore the difference

$$\alpha - \sum_{i,j} (f_{ij} \otimes 1)((\partial_i + \chi(\partial_i)) \otimes \partial_j - (\partial_j + \chi(\partial_j)) \otimes \partial_i - 1 \otimes [\partial_i, \partial_j])$$

still lies in $S(\mathfrak{d}, \chi)$ and its first tensor factor terms have degree strictly less than d. By inductive assumption, we are done. ∎

Remark 8.4. (i) Let, as before, $\chi \in \mathfrak{d}^*$ be such that $\chi([\mathfrak{d}, \mathfrak{d}]) = 0$. For any $\lambda \in \mathbf{k}$, let $V_{\lambda,\chi} = Hv$ be a free H-module of rank 1 with the following action of $W(\mathfrak{d})$ on it

$$\alpha * v = (\lambda \operatorname{div}^{\chi} \alpha \otimes 1 - \alpha) \otimes_H v. \tag{8.27}$$

Using (8.19), it is easy to check that this indeed defines a representation of $W(\mathfrak{d})$.

For $\lambda = 0$ we get the action (8.4), while for $\lambda = -1$, $\chi = \operatorname{tr} \operatorname{ad}$ we get (8.8). One can show that all representations of $W(\mathfrak{d})$ on a free H-module of rank 1 are given by

$$(1 \otimes a) * v = ((\lambda a + \chi'(a)) \otimes 1 - 1 \otimes a) \otimes_H v, \tag{8.28}$$

where $a \in \mathfrak{d}$, $\lambda \in \mathbf{k}$ and χ' is a trace form on $\mathfrak{d}$. This can be rewritten as in (8.27), for $\chi = \chi'/\lambda$ whenever $\lambda \neq 0$.

(ii) More generally, let M be any $W(\mathfrak{d})$-module, equipped with a compatible action of $H = \operatorname{Cur} \mathbf{k}$. Here $H = \operatorname{Cur} \mathbf{k}$ is the associative pseudoalgebra with a pseudoproduct $f * g = (f \otimes g) \otimes_H 1$, and compatibility of the actions of $W(\mathfrak{d})$ and H means that

$$\alpha * (h * m) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\, h * (\alpha * m) = (\alpha * h) * m \tag{8.29}$$

for $\alpha \in W(\mathfrak{d})$, $h \in H$, $m \in M$, where $\alpha * h = -(1 \otimes h)\, \alpha \otimes_H 1$ is the action (8.4) of $W(\mathfrak{d})$ on H.

Then, for any λ, χ as above,

$$\alpha *_{\lambda,\chi} m = \lambda(\operatorname{div}^{\chi} \alpha) * m + \alpha * m \tag{8.30}$$

is an action of $W(\mathfrak{d})$ on M.

8.5. *Pseudoalgebras of Rank* 1

All Lie pseudoalgebras that are free of rank one over H were described by Proposition 4.1 and Lemma 4.2. The next lemma implies that all of them are subalgebras of $W(\mathfrak{d})$.

LEMMA 8.3. *Let $\alpha \in H \otimes H$ be a solution of Eqs.* (4.1), (4.2). *Write $\alpha = r + s \otimes 1 - 1 \otimes s$ with a skew-symmetric $r \in \mathfrak{d} \otimes \mathfrak{d}$ and $s \in \mathfrak{d}$, as in Lemma* 4.1. *Consider $e = -r + 1 \otimes s \in H \otimes \mathfrak{d}$ as an element of $W(\mathfrak{d})$. Then $[e * e] = \alpha \otimes_H e$ in $W(\mathfrak{d})$.*

Proof. Straightforward computation, using the definition (8.3) and Eqs. (4.3), (4.4). ∎

Let us study Eqs. (4.3), (4.4) in more detail. We can write

$$r=\sum_i (a_i\otimes b_i-b_i\otimes a_i) \tag{8.31}$$

for some linearly independent set $a_i, b_i \in \mathfrak{d}$. Denote by $\mathfrak{d}_1$ their linear span, and let $\mathfrak{d}_0=\mathfrak{d}_1+\mathbf{k}s$.

LEMMA 8.4. *$\mathfrak{d}_0$ is a Lie subalgebra of $\mathfrak{d}$, and $\mathfrak{d}_1$ is* ad *s-invariant. Moreover, $[a_i, a_j]$, $[b_i, b_j]$, $[a_i, b_j]$ and $[a_i, b_i]+s$ belong to $\mathfrak{d}_1$ for $i\neq j$.*

Proof. Similar to that of Proposition 2.2.6 in [CP]. If $\mathfrak{d}_1=\mathfrak{d}$, there is nothing to prove. Let $\{c_j\}$ be elements that complement $\{a_i, b_i\}$ to a basis of $\mathfrak{d}$. If s is not in $\mathfrak{d}_1$ we take it to be one of the c_j's.

Write out (4.3) as

$$\sum_i ([a_i, s]\otimes b_i-[b_i, s]\otimes a_i+a_i\otimes[b_i, s]-b_i\otimes[a_i, s])=0.$$

Now, if $[a_i, s]$ involves some c_j's, there is no way to cancel out the terms $c_j\otimes b_i$. This proves that $[s, \mathfrak{d}_1]\subset\mathfrak{d}_1$.

Similarly, (4.4) reads

$$\begin{aligned}\sum_{i,j} ([a_i, a_j]\otimes b_i\otimes b_j-[b_i, a_j]\otimes a_i\otimes b_j\\ +[b_i, b_j]\otimes a_i\otimes a_j-[a_i, b_j]\otimes b_i\otimes a_j+\text{cyclic})\\ +\sum_i (a_i\otimes b_i\otimes s-b_i\otimes a_i\otimes s+\text{cyclic})=0.\end{aligned}$$

If, for example, $[a_i, a_j]$ involves c_k's, then the terms $c_k\otimes b_i\otimes b_j$ cannot be cancelled. Therefore $[a_i, a_j]\in\mathfrak{d}_1$. If $[a_i, b_j]$ involves c_k's, then the terms $c_k\otimes b_i\otimes a_j$ can be cancelled only with terms coming from $s\otimes r$. This shows that $[a_i, b_j]+\delta_{ij}s\in\mathfrak{d}_1$. ∎

The universal enveloping algebra $H_0=U(\mathfrak{d}_0)$ is a Hopf subalgebra of $H=U(\mathfrak{d})$. Since $\alpha\in H_0\otimes H_0$, we can consider the Lie pseudoalgebra H_0e with pseudobracket $[e*e]=\alpha\otimes_{H_0}e$. Then our pseudoalgebra He is a current pseudoalgebra over H_0e.

Clearly, $\mathfrak{d}_1$ is even dimensional. There are two cases which are treated in detail in the next two subsections: when $\mathfrak{d}_0=\mathfrak{d}_1$ and when $\mathfrak{d}_0=\mathfrak{d}_1\oplus\mathbf{k}s$. They give rise to Lie pseudoalgebras $H(\mathfrak{d}, \chi, \omega)$, $K(\mathfrak{d}, \theta)$ whose annihilation Lie algebras are of hamiltonian and contact type, respectively. The following theorem summarizes some of the results of Sections 4.3 and 8.5–8.7.

THEOREM 8.1. *Any Lie pseudoalgebra which is free of rank one is either abelian or isomorphic to a current pseudoalgebra over one of the Lie pseudoalgebras* $H(\mathfrak{d}, \chi, \omega)$, $K(\mathfrak{d}, \theta)$ *defined in Sections* 8.6, 8.7, *respectively.*

8.6. $H(\mathfrak{d}, \chi, \omega)$

This is defined as a Lie H-pseudoalgebra of rank 1 (see Section 8.5) corresponding to a solution (r, s) of equations (4.3), (4.4) with a nondegenerate $r \in \mathfrak{d} \wedge \mathfrak{d}$ (i.e., $\mathfrak{d}_1 = \mathfrak{d}$), in which case $N = \dim \mathfrak{d}$ is even. The parameters χ and ω are defined as follows.

Since r is nondegenerate, the linear map $\mathfrak{d}^* \to \mathfrak{d}$ induced by it is invertible; its inverse gives rise to a 2-form $\omega \in \bigwedge^2 \mathfrak{d}^*$. Explicitly, if $r = \sum r^{ij} \partial_i \otimes \partial_j$ where $\{\partial_i\}$ is a basis of $\mathfrak{d}$, then $\omega(\partial_i \wedge \partial_j) = \omega_{ij}$ is the matrix inverse to r^{ij}. We also define a 1-form $\chi := \iota_s \omega \in \mathfrak{d}^*$.

Conversely, given a nondegenerate skew-symmetric 2-form ω and a 1-form χ, we can define uniquely $r \in \mathfrak{d} \wedge \mathfrak{d}$ as the dual to ω and $s \in \mathfrak{d}$ so that $\chi = \iota_s \omega$.

LEMMA 8.5. *When* $r \in \mathfrak{d} \wedge \mathfrak{d}$ *is nondegenerate, Eqs.* (4.3), (4.4) *are equivalent to the following identities for the above-defined* ω, χ,

$$\mathrm{d}\omega + \chi \wedge \omega = 0, \tag{8.32}$$

$$\mathrm{d}\chi = 0, \tag{8.33}$$

which simply mean that ω *is a* 2-*cocycle for* $\mathfrak{d}$ *in the* 1-*dimensional* $\mathfrak{d}$-*module defined by* χ. *This establishes a one-to-one correspondence between solutions* (r, s) *of* (4.3), (4.4) *with nondegenerate* r *and solutions* (ω, χ) *of* (8.32), (8.33) *with nondegenerate* ω.

Proof. Let us write $[\partial_i, \partial_j] = \sum c_{ij}^k \partial_k$ and $s = \sum s^k \partial_k$ (summation over repeated indices). Then (4.4) is equivalent to

$$\left(\sum r^{ij} r^{kl} c_{ik}^m + r^{mj} s^l \right) + \text{cyclic} = 0, \tag{8.34}$$

where "cyclic" means summing over cyclic permutations of the indices m, j, l. Multiply this equation by $\omega_{jn} \omega_{lp} \omega_{mq}$ and sum over m, j, l. Using that $\sum r^{ij} \omega_{jn} = \delta_n^i$, we get

$$\left(\sum c_{np}^m \omega_{mq} + \sum s^l \omega_{lp} \omega_{nq} \right) + \text{cyclic} = 0, \tag{8.35}$$

where now the cyclic permutations are over n, p, q. This is exactly Eq. (8.32). Conversely, multiplying (8.35) by $r^{in} r^{jp} r^{kq}$ and summing over n, p, q, we get (8.34).

Similarly, since $[s, \mathfrak{d}_1] \subset \mathfrak{d}_1$, we can write $[s, \partial_i] = \sum_k c_i^k \partial_k$. Then (4.3) is equivalent to

$$\sum r^{ij} c_i^k + \sum r^{kl} c_l^j = 0, \tag{8.36}$$

which after multiplying by $\omega_{jm}\omega_{kn}$ and summing over j, k becomes

$$\sum c_m^k \,\omega_{kn} + \sum c_n^j \,\omega_{mj} = 0, \tag{8.37}$$

or $L_s\omega = 0$. Conversely, (8.37) gives (8.36) after multiplying by $r^{pm} r^{qn}$ and summing over m, n.

Now start with a solution (r, s) of (4.3), (4.4). Above we have deduced (8.32) and $L_s\omega = 0$. On the other hand, since $\iota_s \chi = 0$, we have $\iota_s(\chi \wedge \omega) = 0$, and (8.32) implies $\iota_s \,\mathrm{d}\omega = 0$. Together with $L_s\omega = 0$ this gives $\mathrm{d}\iota_s\omega = 0$, which is (8.33).

If we start with a solution (ω, χ) of (8.32), (8.33), the above arguments can be inverted to show that $L_s\omega = 0$, and we get (4.3), (4.4). ∎

In the basis $\{a_i, b_i\}$ of $\mathfrak{d}$ we have (8.31) and $\omega(a_i \wedge b_i) = -\omega(b_i \wedge a_i) = -1$, all other values of ω are zero. For $e = -r + 1 \otimes s$ and any $x \in X$, the element $e_x := x \otimes_H e$ of the annihilation algebra $\mathscr{A}(W(\mathfrak{d})) \simeq X \otimes \mathfrak{d}$ is equal to $-\sum (xa_i \otimes b_i - xb_i \otimes a_i) + x \otimes s$, and it is easy to check that

$$\omega(e_x \wedge a) = x(-a + \chi(a)), \qquad a \in \mathfrak{d}. \tag{8.38}$$

Since $\mathrm{d}\chi = 0$, Lemma 8.1 implies that $\chi = \mathrm{d}y$ for some $y \in \Omega_X^0 = X$, i.e., $\chi(a) = -ya$. Then $\tilde{\omega} := e^y \omega$ satisfies $\tilde{\omega}(e_x \wedge a) = -(xe^y)\, a$ for any $x \in X$, $a \in \mathfrak{d}$. This is equivalent to $\iota_{e_x} \tilde{\omega} = \mathrm{d}(xe^y)$. Moreover, (8.32) implies $\mathrm{d}\tilde{\omega} = 0$. Therefore, $L_{e_x}\tilde{\omega} = 0$, and we have the following proposition.

Proposition 8.2. *Let $H(\mathfrak{d}, \chi, \omega) := He$ be a Lie H-pseudoalgebra of rank 1 corresponding to a solution (r, s) of Eqs. (4.3), (4.4) with a nondegenerate $r \in \mathfrak{d} \otimes \mathfrak{d}$. Define the 2-form $\tilde{\omega}$ as above. Then $\tilde{\omega}$ is a symplectic form, and the subalgebra $X \otimes_H H(\mathfrak{d}, \chi, \omega)$ of $X \otimes_H W(\mathfrak{d}) \simeq X \otimes \mathfrak{d}$ is the Lie algebra $H_N(\tilde{\omega})$ of vector fields annihilating $\tilde{\omega}$ (which is isomorphic to H_N).*

Proof. It remains to show that, conversely, any vector field that preserves the form $\tilde{\omega}$ is equal to e_x for some $x \in X$. Indeed, let $A \in X \otimes \mathfrak{d}$ be such that $L_A\tilde{\omega} = 0$. Since $\mathrm{d}\tilde{\omega} = 0$ and $\tilde{\omega} = e^y\omega$, this is equivalent to $\mathrm{d}(e^y \iota_A \omega) = 0$ which implies $e^y \iota_A \omega = \mathrm{d}z$ for some $z \in X$. In other words, $e^y \omega(A \wedge a) = -za$ for any $a \in \mathfrak{d}$. Using $\chi(a) = -ya$, we get $\omega(A \wedge a) = x(-a + \chi(a))$ for $x = e^{-y}z$. This, together with (8.38), implies $A = e_x$ since the 2-form ω is nondegenerate. ∎

Remark 8.5. Let $r \in \mathfrak{d} \otimes \mathfrak{d}$ be given by (8.31), and let $x = \sum_i [a_i, b_i]$, $\phi = -\chi + \iota_x \omega = \iota_{x-s} \omega$. Then it is easy to check that $\operatorname{div}^\phi(-r + 1 \otimes s) = 0$, so we have: $H(\mathfrak{d}, \chi, \omega) \subset S(\mathfrak{d}, \phi)$.

EXAMPLE 8.1. Let the Lie algebra $\mathfrak{d}$ be 2-dimensional with basis $\{a, b\}$ and commutation relations $[a, b] = \lambda b$. Then up to multiplition by a scalar, all nondegenerate solutions (r, s) of (4.3) are given by: $r = a \otimes b - b \otimes a$, any s in case $\lambda = 0$, and by the same r, and $s \in \mathbf{k}b$ when $\lambda \neq 0$. It is immediate to see that in both cases s can be written as $-\phi(a)\, b + \phi(b)\, a + [a, b]$ for some trace form $\phi \in \mathfrak{d}^*$. Then $r - 1 \otimes s = e_{ab}$ is a free generator of $S(\mathfrak{d}, \phi)$, since $\dim \mathfrak{d} = 2$ (see Proposition 8.1). This shows that the above pairs (r, s) also satisfy (4.4). We have: $H(\mathfrak{d}, \chi, \omega) = S(\mathfrak{d}, \phi)$, where $\chi = \iota_s \omega = -\phi + \operatorname{tr} \operatorname{ad}$. (Note that $\operatorname{tr} \operatorname{ad} = \iota_x \omega$ for $x = [a, b] = \lambda b$.)

EXAMPLE 8.2. When $\mathfrak{d}$ is abelian of dimension $N = 2n > 2$, then (8.32) and (8.33) become $\chi \wedge \omega = 0$, hence $\chi = 0$ and ω is any nondegenerate skew-symmetric 2-form. In this case all solutions of (4.3), (4.4) are $s = 0$ and r given by (8.31) in some basis $\{a_i, b_i\}$ of $\mathfrak{d}$.

EXAMPLE 8.3. When the Lie algebra $\mathfrak{d}$ is simple, there are no solutions (ω, χ) of (8.32), (8.33) with a nondegenerate ω. Indeed, since $[\mathfrak{d}, \mathfrak{d}] = \mathfrak{d}$, we have $\chi = 0$, and ω is a 2-cocycle: $\mathrm{d}\omega = 0$. Any 2-cocycle $\omega \in \bigwedge^2 \mathfrak{d}^*$ for a simple Lie algebra $\mathfrak{d}$ is degenerate, since $\omega = \mathrm{d}\alpha$ for some $\alpha \in \mathfrak{d}^*$ and the stabilizer $\mathfrak{d}_\alpha$ of α is always nonzero.

8.7. $K(\mathfrak{d}, \theta)$

This is defined as a Lie H-pseudoalgebra of rank 1 (see Section 8.5) corresponding to a solution (r, s) of Eqs. (4.3), (4.4) with $\mathfrak{d} = \mathfrak{d}_1 \oplus \mathbf{k}s$ and nondegenerate $r \in \mathfrak{d}_1 \wedge \mathfrak{d}_1$; in this case $N = \dim \mathfrak{d}$ is odd. The parameter θ is defined below.

Let $\{\partial_i\}$ be a basis of $\mathfrak{d}_1$, and $r = \sum r^{ij} \partial_i \otimes \partial_j$. As before, we define a 2-form ω on $\mathfrak{d}_1$ by $\omega(\partial_i \wedge \partial_j) = \omega_{ij}$, where (ω_{ij}) is the matrix inverse to (r^{ij}). Let us write $[\partial_i, \partial_j] = \sum c_{ij}^k \partial_k + c_{ij} s$ and $[s, \partial_j] = \sum c_j^k \partial_k$. Then we have

LEMMA 8.6. *With the above notation, Eqs.* (4.3), (4.4) *are equivalent to the identities*

$$\mathrm{d}\omega = 0 \qquad on \quad \bigwedge\nolimits^3 \mathfrak{d}_1, \tag{8.39}$$

$$c_{ij} = \omega_{ij}, \tag{8.40}$$

$$L_s \omega = 0. \tag{8.41}$$

If we extend ω to a 2-form on $\mathfrak{d}$ by defining $\iota_s \omega = 0$, then ω is closed: $\mathrm{d}\omega = 0$.

Proof. The proof is very similar to that of Lemma 8.5. There we showed that $L_s\omega = 0$ is equivalent to (4.3), and the same argument applies here. Similarly, (4.4) is equivalent to (8.39, 8.40). Now if $\iota_s \omega = 0$, then $L_s\omega = 0$ implies $\iota_s \,\mathrm{d}\omega = 0$, which together with (8.39) leads to $\mathrm{d}\omega = 0$. ∎

Let ω be extended to a 2-form on $\mathfrak{d}$ by defining $\iota_s\omega = 0$, so that $\mathrm{d}\omega = 0$. We define a 1-form $\theta \in \mathfrak{d}^*$ by $\theta(s) := -1$, $\theta|_{\mathfrak{d}_1} := 0$. Then we have $\mathrm{d}\theta = \omega$; indeed

$$(\mathrm{d}\theta)(\partial_i \wedge \partial_j) = -\theta([\partial_i, \partial_j]) = c_{ij} = \omega_{ij} = \omega(\partial_i \wedge \partial_j),$$
$$(\mathrm{d}\theta)(s \wedge \partial_j) = -\theta([s, \partial_j]) = 0 = \omega(s \wedge \partial_j),$$

using (8.40) and the fact that $[s, \mathfrak{d}_1] \subset \mathfrak{d}_1$.

LEMMA 8.7. *There is a one-to-one correspondence between contact forms θ, i.e. 1-forms $\theta \in \mathfrak{d}^*$ such that $\theta \wedge (\mathrm{d}\theta)^{(N-1)/2} \neq 0$ $(N = \dim \mathfrak{d})$, and solutions (r, s) of (4.3), (4.4) with $\mathfrak{d} = \mathfrak{d}_1 \oplus \mathbf{k}s$ and nondegenerate $r \in \mathfrak{d}_1 \otimes \mathfrak{d}_1$.*

Proof. Given (r, s), above we have defined the 1-form θ such that $\theta(s) = -1$, $\theta|_{\mathfrak{d}_1} = 0$ and $\mathrm{d}\theta = \omega$. Since $\omega \in \bigwedge^2 \mathfrak{d}_1^*$ is nondegenerate, we have $\theta \wedge \omega^{(N-1)/2} \neq 0$. Conversely, starting with a contact 1-form $\theta \in \mathfrak{d}^*$, we can define s and ω satisfying (8.39)–(8.41). ∎

EXAMPLE 8.4. When $\mathfrak{d}$ is the Heisenberg Lie algebra with a basis $\{a_i, b_i, c\}$ and the only nonzero commutation relations $[a_i, b_i] = c$ $(1 \leqslant i \leqslant n,\ N = 2n+1)$, then

$$r = \sum_{i=1}^{n} (a_i \otimes b_i - b_i \otimes a_i), \qquad s = -c$$

is a solution of (4.3), (4.4).

EXAMPLE 8.5. When $\mathfrak{d}$ is abelian and $\dim \mathfrak{d} = 2n+1 > 1$, then Eqs. (4.3), (4.4) have no solutions (r, s) with $\mathfrak{d} = \mathfrak{d}_1 \oplus \mathbf{k}s$ and a nondegenerate $r \in \mathfrak{d}_1 \wedge \mathfrak{d}_1$, because $\mathrm{d}\theta = 0$ and therefore there are no contact forms.

EXAMPLE 8.6. When the Lie algebra $\mathfrak{d}$ is simple, a solution (r, s) of (4.3), (4.4) with $\mathfrak{d} = \mathfrak{d}_1 \oplus \mathbf{k}s$ and a nondegenerate $r \in \mathfrak{d}_1 \wedge \mathfrak{d}_1$ exists iff $\mathfrak{d} = \mathfrak{sl}_2$, and it is

$$r = e \wedge f := e \otimes f - f \otimes e, \qquad s = -h.$$

Only $\mathfrak{d} = \mathfrak{sl}_2$ is possible since the dimension of the stabilizer of θ equals 1.

Now let us compute $L_{e_x}\theta$. Recall that, as in Section 8.6, for any $x \in X$ we identify $e_x := x \otimes_H e$ with $-\sum (xa_i \otimes b_i - xb_i \otimes a_i) + x \otimes s$. Similarly to (8.38), it is easy to see that $\omega(e_x \wedge a) = -xa$ for $a \in \mathfrak{d}_1$ (in this case $\chi = \iota_s \omega := 0$). On the other hand, $\iota_{e_x}\theta = \theta(e_x) = -x$, and hence $(\mathrm{d}\iota_{e_x}\theta)(a) = -(\mathrm{d}x)(a) = xa$ for any $a \in \mathfrak{d}$. Therefore $(L_{e_x}\theta)(a) = 0$ for $a \in \mathfrak{d}_1$, and $(L_{e_x}\theta)(s) = xs$. In other words,

$$L_{e_x}\theta = -(xs)\,\theta,$$

and we have the following proposition.

PROPOSITION 8.3. *Let $K(\mathfrak{d}, \theta) := He$ be a Lie H-pseudoalgebra of rank 1 corresponding to a solution (r, s) of Eqs.* (4.3), (4.4) *with $\mathfrak{d} = \mathfrak{d}_1 \oplus \mathbf{k}s$ and a nondegenerate $r \in \mathfrak{d}_1 \otimes \mathfrak{d}_1$, where the 1-form $\theta \in \mathfrak{d}^*$ is defined by $\theta(s) = -1$, $\theta|_{\mathfrak{d}_1} = 0$. Then θ is a contact form, and the subalgebra $X \otimes_H K(\mathfrak{d}, \theta)$ of $X \otimes_H W(\mathfrak{d}) \simeq X \otimes \mathfrak{d}$ is the Lie algebra $K_N(\theta)$ of vector fields that preserve θ up to a multiplication by a function* (*which is isomorphic to K_N*).

Proof. It remains to show that, conversely, any vector field from $K_N(\theta)$ is equal to e_x for some $x \in X$. Indeed, let $A \in X \otimes \mathfrak{d}$ be such that $L_A\theta = f\theta$ for some $f \in X$. Let us write $A = \sum_i (x_i \otimes a_i + y_i \otimes b_i) + x \otimes s$ for some $x_i, y_i, x \in X$. Then $\omega(A \wedge a_i) = y_i$ and $\omega(A \wedge b_i) = -x_i$, while $\theta(A) = -x$. Therefore $(L_A\theta)(a) = \omega(A \wedge a) + xa$, which implies $y_i + xa_i = 0$, $-x_i + xb_i = 0$, and $xs = -f$. ∎

Remark 8.6. To any H-type Lie pseudoalgebra, i.e., to any triple $(\mathfrak{d}, \omega, \chi)$ where $\mathfrak{d}$ is a finite-dimensional Lie algebra, $\omega \in \bigwedge^2 \mathfrak{d}^*$ is a nondegenerate 2-form and $\chi \in \mathfrak{d}^*$ satisfying (8.32) and (8.33), we can associate a K-type Lie pseudoalgebra as follows. Set on the vector space $\mathfrak{d}' = \mathfrak{d} \oplus \mathbf{k}c$ the Lie bracket $[\ ,\ ]'$ defined as

$$[g, h]' = [g, h] + \omega(g, h)\, c, \qquad [g, c]' = \chi(g)\, c,$$

for $g, h \in \mathfrak{d}$. Then $c + s \in \mathfrak{d}'$ stabilizes $\mathfrak{d}$, where $s \in \mathfrak{d}$ is the unique element such that $\chi = \iota_s\omega$; indeed,

$$[g, s + c]' = [g, s] + \omega(g, s)\, c + \chi(g)\, c = [g, s] \in \mathfrak{d}.$$

Define $\theta \in (\mathfrak{d}')^*$ as the unique element restricting to 0 on $\mathfrak{d}$ such that $\theta(c) = -1$.

Note that not all K-type data are obtained in this way, since the Lie algebra $\mathfrak{d}'$ just constructed always has a one dimensional ideal $\mathbf{k}c$, and this fails in Example 8.6.

8.8. *Annihilation Algebras of Pseudoalgebras of Vector Fields*

To conclude this section, we determine the annihilation algebras of the primitive pseudoalgebras of vector fields defined above, and of current pseudoalgebras over them.

THEOREM 8.2. (i) *If L is one of the Lie $H=U(\mathfrak{d})$-pseudoalgebras $W(\mathfrak{d})$, $S(\mathfrak{d},\chi)$, $H(\mathfrak{d},\chi,\omega)$ or $K(\mathfrak{d},\theta)$, then its annihilation algebra $\mathscr{A}(L)$ is isomorphic to W_N, S_N, P_N or K_N, respectively.*

(ii) *If $L=\operatorname{Cur} L'$ is a current pseudoalgebra over the Lie H'-pseudoalgebra L', then its annihilation algebra $\mathscr{A}(L)$ is isomorphic to a current Lie algebra $\mathscr{O}_r \mathbin{\hat{\otimes}} \mathscr{A}(L')$ over $\mathscr{A}(L')$, where $H'=U(\mathfrak{d}')$ and $\mathfrak{d}'$ is a codimension r subalgebra of $\mathfrak{d}$.*

Proof. (i) We have seen in Section 8.1 that $\mathscr{A}(W(\mathfrak{d}))\simeq W_N$. Let L be one of the pseudoalgebras $S(\mathfrak{d},\chi)$, $H(\mathfrak{d},\chi,\omega)$, or $K(\mathfrak{d},\theta)$ and i be its natural embedding in $W(\mathfrak{d})$. We have shown in Sections 8.4, 8.6, and 8.7 that in this case the image of $\mathscr{A}(L)$ in W_N under $\mathscr{A}(i)$ is S_N, H_N, or K_N, respectively.

Lemma 11.4 below implies that $\mathscr{A}(L)$ is a central extension of its image in W_N. Moreover, since L is simple, it is equal to its derived subalgebra, and therefore $\mathscr{A}(L)$ is equal to its derived subalgebra (see Section 13.1). Hence, $\mathscr{A}(L)$ is an irreducible central extension of its image in W_N.

Now Proposition 6.4(iii) implies that $\mathscr{A}(L)\simeq S_N$, K_N in the cases $L=S(\mathfrak{d},\chi)$, $K(\mathfrak{d},\theta)$ respectively, and $\mathscr{A}(L)$ is a quotient of P_N in the case $L=H(\mathfrak{d},\chi,\omega)$. However, since $L=H(\mathfrak{d},\chi,\omega)$ is a free H-module of rank one, $\mathscr{A}(L)$ is isomorphic to X as a topological H-module. Therefore, $\mathscr{A}(L)\simeq P_N$.

(ii) Note that $X=H^*$ maps surjectively to $X'=(H')^*$ with kernel isomorphic to $\mathscr{O}_r$. Moreover $X\simeq\mathscr{O}_N$, $X'\simeq\mathscr{O}_{N'}$ ($N'=N-r$), and hence $X\simeq\mathscr{O}_r \mathbin{\hat{\otimes}} X'$. We have: $\mathscr{A}(L'):=X'\otimes_{H'}L'$ and $\mathscr{A}(L):=X\otimes_H L= X\otimes_H(H\otimes_{H'}L')\simeq X\otimes_{H'}L'\simeq(\mathscr{O}_r \mathbin{\hat{\otimes}} X')\otimes_{H'}L'\simeq\mathscr{O}_r \mathbin{\hat{\otimes}} (X'\otimes_{H'}L')$.

Remark 8.7. Let us assume that the base field $\mathbf{k}=\mathbb{C}$, and let L be as in Theorem 8.2(i). Then the action of $\mathfrak{d}$ on $\mathscr{A}(L)$ can be constructed via the embedding of $\mathfrak{d}$ in W_N as follows.

(i) Any N-dimensional Lie algebra $\mathfrak{d}$ can be embedded in W_N: every $a\in\mathfrak{d}$ defines a left-invariant vector field on the connected simply-connected Lie group D with Lie algebra $\mathfrak{d}$, and we take the corresponding formal vector field in the formal neighborhood of the identity element. (See also Proposition 6.2.)

(ii) If we have a homomorphism of Lie algebras $\chi\colon\mathfrak{d}\to\mathbb{C}$, it defines a homomorphism $\tilde{\chi}$ of D to $\mathbb{C}^\times$. Consider a volume form v on D defined, up to a constant factor, by the property $g \cdot v_0 = \tilde{\chi}(g)\, v_0$, $g \in D$, where v_0 is the value of v at the identity element. Then we get an embedding of $\mathfrak{d}$ in $CS_N(v) = \operatorname{Der} S_N(v) \simeq \operatorname{Der} S_N$.

(iii) Given χ and $\omega \in \bigwedge^2 \mathfrak{d}^*$ such that $\mathrm{d}\omega + \chi \wedge \omega = 0$, consider a 2-form s on D whose value at the identity element is ω and such that $g \cdot s = \tilde{\chi}(g)\, s$, $g \in D$. Then s is a symplectic form on D, and we get an embedding of $\mathfrak{d}$ in $CH_N(s) = \operatorname{Der} H_N(s) \simeq \operatorname{Der} P_N$.

(iv) Given a contact form $\theta \in \mathfrak{d}^*$, consider the left-invariant 1-form c on D with the value θ at the identity element. Then c is a contact form on D, and we get an embedding of $\mathfrak{d}$ in $K_N(c) \simeq K_N$.

9. *H*-CONFORMAL ALGEBRAS

The goal of this section is to reformulate the definition of a Lie (or associative) H-pseudoalgebra in terms of the properties of the x-brackets (or products) introduced in Section 7.2. The resulting notion, equivalent to that of an H-pseudoalgebra, will be called an H-conformal algebra.

Let us start by deriving explicit formulas for the x-brackets. We will use the notation of Section 7.2. Let (L, β) be a Lie H-pseudoalgebra with a pseudobracket

$$[a * b] \equiv \beta(a \otimes b) = \sum_i (f_i \otimes g_i) \otimes_H e_i. \tag{9.1}$$

Then for $x \in X$, $h \in H$ we have $\eta(x \otimes h) = \langle x, h_{(1)} \rangle h_{(2)}$ (see (7.5)), and

$$\begin{aligned}(\eta \otimes_H \beta)((x \otimes_H a) \otimes (h \otimes_H b)) &= \sum_i \eta(xf_i \otimes hg_i) \otimes_H e_i \\ &= \sum_i \langle xf_i, (hg_i)_{(1)} \rangle\, (hg_i)_{(2)} \otimes_H e_i.\end{aligned}$$

Taking $h = 1$, we get the following expression for the *x-bracket* in L

$$[a_x b] = \sum_i \langle S(x), f_i g_{i(-1)} \rangle\, g_{i(2)} e_i, \qquad \text{if} \quad [a * b] = \sum_i (f_i \otimes g_i) \otimes_H e_i. \tag{9.2}$$

Here we can recognize the Fourier transform $\mathscr{F}$, defined by (2.33):

$$\mathscr{F}(f \otimes g) = f g_{(-1)} \otimes g_{(2)}.$$

The identity (2.35)

$$f \otimes g = (f g_{(-1)} \otimes 1)\, \Delta(g_{(2)}),$$

implies

$$[a * b] = \sum_i (f_i g_{i(-1)} \otimes 1) \otimes_H g_{i(2)} e_i . \tag{9.3}$$

Hence $[a * b]$ can be written uniquely in the form $\sum_i (h_i \otimes 1) \otimes_H c_i$, where $\{h_i\}$ is a fixed $\mathbf{k}$-basis of H (cf. Lemma 2.3).

We introduce another bracket $[a, b] \in H \otimes L$ as the Fourier transform of $[a * b]$:

$$[a, b] = \sum_i \mathscr{F}(f_i \otimes g_i)(1 \otimes e_i) = \sum_i f_i g_{i(-1)} \otimes g_{i(2)} e_i . \tag{9.4}$$

In other words,

$$[a, b] = \sum_i h_i \otimes c_i \qquad \text{if} \quad [a * b] = \sum_i (h_i \otimes 1) \otimes_H c_i . \tag{9.5}$$

Then we have

$$[a_x b] = (\langle S(x), \cdot \rangle \otimes \mathrm{id})\, [a, b] = \sum_i \langle S(x), h_i \rangle c_i . \tag{9.6}$$

Using properties (2.38)–(2.41) of the Fourier transform, it is straightforward to derive the properties of the bracket (9.5). Then the definition of a Lie pseudoalgebra can be equivalently reformulated as follows.

DEFINITION 9.1. A *Lie H-conformal algebra* is a left H-module L equipped with a bracket $[\cdot, \cdot]: L \otimes L \to H \otimes L$, satisfying the following properties ($a, b, c \in L$, $h \in H$)

H-sesqui-linearity,

$$[ha, b] = (h \otimes 1)\, [a, b], \tag{9.7}$$

$$[a, hb] = (1 \otimes h_{(2)})\, [a, b]\, (h_{(-1)} \otimes 1). \tag{9.8}$$

Skew-commutativity. If $[a, b]$ is given by (9.5), then

$$[b, a] = -\sum_i h_{i(-1)} \otimes h_{i(2)} c_i . \tag{9.9}$$

Jacobi identity,

$$[a, [b, c]] - (\sigma \otimes \mathrm{id})[b, [a, c]] = (\mathscr{F}^{-1} \otimes \mathrm{id})[[a, b], c] \tag{9.10}$$

in $H \otimes H \otimes L$, where $\sigma\colon H \otimes H \to H \otimes H$ is the permutation $\sigma(f \otimes g) = g \otimes f$, and

$$[a, [b, c]] = (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes [a, \cdot])[b, c], \tag{9.11}$$

$$[[a, b], c] = (\mathrm{id} \otimes [\cdot, c])[a, b]. \tag{9.12}$$

EXAMPLE 9.1. (i) For the current Lie pseudoalgebra $\operatorname{Cur} \mathfrak{g} = H \otimes \mathfrak{g}$ with the pseudobracket (4.2), the bracket (9.5) is given by

$$[f \otimes a, g \otimes b] = fg_{(-1)} \otimes (g_{(2)} \otimes [a, b]).$$

(ii) For the Lie pseudoalgebra $W(\mathfrak{d}) = H \otimes \mathfrak{d}$ with pseudobracket defined by (8.3), the bracket (9.5) is given by

$$[1 \otimes a, 1 \otimes b] = 1 \otimes (1 \otimes [a, b]) + a \otimes (1 \otimes b) + b \otimes (1 \otimes a) - 1 \otimes (a \otimes b).$$

One can also reformulate Definition 9.1 in terms of the x-brackets (9.6).

DEFINITION 9.2. A *Lie H-conformal algebra* is a left H-module L equipped with x-brackets $[a_x b] \in L$ for $a, b \in L$, $x \in X$, satisfying the following properties

Locality,

$$\operatorname{codim}\{x \in X \mid [a_x b] = 0\} < \infty \qquad \text{for any} \quad a, b \in L. \tag{9.13}$$

Equivalently, for any basis $\{x_i\}$ of X,

$$[a_{x_i} b] \neq 0 \qquad \text{for only a finite number of } i. \tag{9.14}$$

H-sesqui-linearity,

$$[ha_x b] = [a_{xh} b], \tag{9.15}$$

$$[a_x hb] = h_{(2)} [a_{h_{(-1)} x} b]. \tag{9.16}$$

Skew-commutativity. Choose dual bases $\{h_i\}$, $\{x_i\}$ in H and X. Then

$$[a_x b] = -\sum_i \langle x, h_{i(-1)} \rangle \, h_{i(-2)} [b_{x_i} a]. \tag{9.17}$$

Jacobi identity,

$$[a_x [b_y c]] - [b_y [a_x c]] = [[a_{x_{(2)}} b]_{y x_{(1)}} c]. \tag{9.18}$$

Lemma 2.2 implies that (9.18) can be rewritten as

$$[a_x[b_y c]] - [b_y[a_x c]] = \sum_i [[a_{x_i} b]_{y(xS(h_i))} c]. \tag{9.19}$$

In particular, the right-hand side of (9.18) is well defined: the sum is finite because of (9.14).

The definitions of an *associative H-conformal algebra* or of *representations* of H-conformal algebras are obvious modifications of the above. For example, in terms of x-products, the associativity looks as (cf. (9.18))

$$a_x(b_y c) = (a_{x_{(2)}} b)_{yx_{(1)}} c. \tag{9.20}$$

The same argument as the one used for $\mathscr{F}$ shows that the map $x \otimes y \mapsto x_{(2)} \otimes yx_{(1)}$ has an inverse given by $x \otimes y \mapsto x_{(2)} \otimes yx_{(-1)}$. Therefore, (9.20) is equivalent to the equation

$$a_{x_{(2)}}(b_{yx_{(-1)}} c) = (a_x b)_y\, c. \tag{9.21}$$

Note that when considering associative H-conformal algebras, H need not be cocommutative, so X may be noncommutative.

We also note that there is a simple relationship between the x-bracket (9.6) of a Lie H-conformal algebra (or, equivalently, pseudoalgebra) L and the commutator in its annihilation Lie algebra $\mathscr{A}(L)$ defined in Section 7. Let $\{h_i\}$, $\{x_i\}$ again be dual bases in H and X. Then in (9.5) one has $c_i = [a_{S^{-1}(x_i)} b]$; therefore

$$[a, b] = \sum_i S(h_i) \otimes [a_{x_i} b] \qquad \text{and} \qquad [a * b] = \sum (S(h_i) \otimes 1) \otimes_H [a_{x_i} b]. \tag{9.22}$$

Recall that we denote the element $x \otimes_H a$ of $\mathscr{A}(L) := X \otimes_H L$ by a_x. Then the definition (7.2) and (9.22) imply

$$[a_x, b_y] = \sum_i [a_{x_i} b]_{(xS(h_i))\, y} = [a_{x_{(2)}} b]_{x_{(1)}\, y}, \tag{9.23}$$

using (2.32). This is also equivalent to

$$[a_x b]_y = [a_{x_{(2)}}, b_{x_{(-1)}\, y}] = \sum_i [a_{x_i}, b_{(h_i S(x))\, y}]. \tag{9.24}$$

Comparing these formulas with Eq. (9.18), we obtain the following important result.

PROPOSITION 9.1. *Any module M over a Lie pseudoalgebra L has a natural structure of an $\mathcal{A}(L)$-module, given by $(x \otimes_H a) \cdot m = a_x m$, where*

$$a_x m = \sum_i \langle S(x), f_i g_{i(-1)} \rangle \, g_{i(2)} v_i , \qquad \text{if} \quad a * m = \sum_i (f_i \otimes g_i) \otimes_H v_i \quad (9.25)$$

for $a \in L$, $x \in X$, $m \in M$. This action is compatible with the action of H (see (7.4)) and satisfies the locality condition

$$\operatorname{codim}\{x \in X \mid a_x m = 0\} < \infty, \qquad a \in L,\ m \in M, \qquad (9.26)$$

or equivalently, for any basis $\{x_i\}$ of X,

$$a_{x_i} m \neq 0 \qquad \text{for only a finite number of } i. \qquad (9.27)$$

(The above conditions on M mean that, when endowed with the discrete topology, M is a topological $\mathcal{A}(L)$-module in the category $\mathcal{M}^l(H)$.)

Conversely, any $\mathcal{A}(L)$-module M satisfying the above conditions has a natural structure of an L-module, given by

$$a * m = \sum_i (S(h_i) \otimes 1) \otimes_H a_{x_i} \cdot m, \qquad (9.28)$$

where $\{h_i\}$, $\{x_i\}$ are dual bases in H and X, and we use the notation $a_x \equiv x \otimes_H a$.

This proposition provides the main tool for constructing modules over Lie pseudoalgebras. Of course, there is an analogous result in the case of associative algebras as well.

Finally, let us give two more expressions for the bracket in $\mathcal{A}(L)$ which will be useful later. Recall that, by Proposition 9.1, we have an action of $\mathcal{A}(L)$ on L given by $a_x \cdot b = [a_x b]$. Recall also that the action of H on $\mathcal{A}(L)$ is defined by $h(a_x) = a_{hx}$. Let $\alpha \in \mathcal{A}(L)$, $b \in L$, $y \in X$. Then

$$(\alpha \cdot b)_y = \sum_i [h_i \alpha, b_{x_i y}], \qquad (9.29)$$

$$[\alpha, b_y] = \sum_i ((S(h_i)\, \alpha) \cdot b)_{x_i y} . \qquad (9.30)$$

Note that the infinite sums on the right-hand sides make sense since they are convergent in the complete topology of $\mathcal{A}(L)$. It is enough to prove both statements for α of the form $a_x = x \otimes_H a$ since such elements span $\mathcal{A}(L)$. Equation (9.30) then follows from (9.23) and (2.32). Analogously, (9.29) derives from (9.24) by noticing that $x_{(-1)} \otimes x_{(2)} = \sum_i x_i \otimes h_i x$.

10. H-PSEUDOLINEAR ALGEBRA

The definition of a module over a pseudoalgebra motivates the following definition of a pseudolinear map.

DEFINITION 10.1. Let V and W be two H-modules. An *H-pseudolinear map* from V to W is a $\mathbf{k}$-linear map $\phi\colon V \to (H \otimes H) \otimes_H W$ such that

$$\phi(hv) = ((1 \otimes h) \otimes_H 1)\, \phi(v), \qquad h \in H,\, v \in V. \tag{10.1}$$

We denote the space of all such ϕ by $\operatorname{Chom}(V, W)$. We define a left action of H on $\operatorname{Chom}(V, W)$ by

$$(h\phi)(v) = ((h \otimes 1) \otimes_H 1)\, \phi(v). \tag{10.2}$$

When $V = W$, we set $\operatorname{Cend} V = \operatorname{Chom}(V, V)$.

EXAMPLE 10.1. Let A be an H-pseudoalgebra and V be an A-module. Then for any $a \in A$ the map $m_a\colon V \to (H \otimes H) \otimes_H V$ defined by $m_a(v) = a * v$ is an H-pseudolinear map. Moreover, we have $hm_a = m_{ha}$ for $h \in H$.

Consider the map $\rho\colon \operatorname{Chom}(V, W) \otimes V \to (H \otimes H) \otimes_H W$ given by $\rho(\phi \otimes v) = \phi(v)$. By definition it is H-bilinear, so it is a polylinear map in $\mathcal{M}^*(H)$. We will also use the notation $\phi * v := \phi(v)$ and consider this as a pseudoproduct (or rather action, see Proposition 10.1 below).

The corresponding x-products are called *Fourier coefficients* of ϕ and are given by a formula analogous to (9.2):

$$\phi_x v = \sum_i \langle S(x), f_i g_{i(-1)} \rangle\, g_{i(2)} w_i, \qquad \text{if} \quad \phi(v) = \sum_i (f_i \otimes g_i) \otimes_H w_i. \tag{10.3}$$

They satisfy a locality relation and an H-sesqui-linearity relation similar to (9.13) and (9.16)

$$\operatorname{codim}\{x \in X \mid \phi_x v = 0\} < \infty \qquad \text{for any} \quad v \in V, \tag{10.4}$$

$$\phi_x(hv) = h_{(2)}(\phi_{h_{(-1)}x} v). \tag{10.5}$$

Conversely, any collection of maps $\phi_x \in \operatorname{Hom}(V, W)$, $x \in X$, satisfying relations (10.4), (10.5) comes from an H-pseudolinear map $\phi \in \operatorname{Chom}(V, W)$. Explicitly

$$\phi(v) = \sum_i (S(h_i) \otimes 1) \otimes_H \phi_{x_i} v, \tag{10.6}$$

where $\{h_i\}$, $\{x_i\}$ are dual bases in H and X (cf. (9.28)).

Remark 10.1. It follows from (10.5) that for $\phi \in \mathrm{Chom}(V, W)$, the map $\phi_1 \colon V \to W$ is H-linear, where $1 \in X$ is the unit element. This establishes an isomorphism

$$\mathrm{Hom}_H(V, W) \simeq \mathbf{k} \otimes_H \mathrm{Chom}(V, W) \simeq \mathrm{Chom}(V, W)/H_+ \mathrm{Chom}(V, W),$$

where $H_+ = \{h \in H \mid \varepsilon(h) = 0\}$ is the augmentation ideal.

LEMMA 10.1. *Let U, V, W be three H-modules, and assume that U is finite. Then there is a unique polylinear map*

$$\mu \in \mathrm{Lin}(\{\mathrm{Chom}(V, W), \mathrm{Chom}(U, V)\}, \mathrm{Chom}(U, W))$$

in $\mathcal{M}^*(H)$, denoted as $\mu(\phi \otimes \psi) = \phi * \psi$, such that

$$(\phi * \psi) * u = \phi * (\psi * u) \tag{10.7}$$

in $H^{\otimes 3} \otimes_H W$ for $\phi \in \mathrm{Chom}(V, W)$, $\psi \in \mathrm{Chom}(U, V)$, $u \in U$.

Proof. We define $\phi * \psi$ in terms of its Fourier coefficients—the x-products $\phi_x \psi$. We have already seen, when we discussed associativity, that (10.7) is equivalent to the equation (cf. (9.20))

$$\phi_x(\psi_y u) = (\phi_{x_{(2)}} \psi)_{y x_{(1)}} u.$$

This can be inverted to find (cf. (9.21))

$$(\phi_x \psi)_y u = \phi_{x_{(2)}}(\psi_{y x_{(-1)}} u) = \sum_i \phi_{x_i}(\psi_{y(h_i S(x))} u).$$

The H-sesqui-linearity properties of $(\phi_x \psi)_y u$ with respect to x and y are easy to check by a direct calculation. By properties (2.21), (2.28), (2.29) of the filtration $\{\mathrm{F}_n X\}$, and locality of ψ, it follows that for each fixed $x \in X$, $u \in U$ there is an n such that $(\phi_x \psi)_y u = 0$ for $y \in \mathrm{F}_n X$. Therefore, for each $x \in X$ we have defined $\phi_x \psi \in \mathrm{Chom}(U, W)$.

In order that $\phi * \psi$ be well defined, we need to check that $\phi_x \psi$ satisfies locality, i.e., that $\phi_x \psi = 0$ when $x \in \mathrm{F}_n X$ with $n \gg 0$. By the locality of ϕ and ψ, for each $u \in U$ there is an n such that $(\phi_x \psi)_y u = 0$ for $x \in \mathrm{F}_n X$ and all $y \in X$. Since U is finite, we can choose an n that works for all u belonging to a set of generators of U over H. Now the H-sesqui-linearity of $(\phi_x \psi)_y u$ with respect to y (for fixed x) implies that $(\phi_x \psi)_y u = 0$ for all y and u. Hence $\phi_x \psi = 0$ for $x \in \mathrm{F}_n X$. ∎

Specifying to the case $U = V = W$, we obtain a pseudoproduct μ on Cend V, and an action ρ of Cend V on V.

PROPOSITION 10.1. (i) *For any finite H-module V, the above pseudoproduct provides* Cend V *with the structure of an associative H-pseudoalgebra; V has a natural structure of a* Cend V*-module given by* $\phi * v \equiv \phi(v)$.

(ii) *For an associative H-pseudoalgebra A, giving a structure of an A-module on V is equivalent to giving a homomorphism of associative H-pseudoalgebras from A to* Cend V.

Proof. Part (i) is an immediate consequence of Lemma 10.1. Indeed, the only thing that remains to be checked is the associativity of Cend V, and it follows from (10.7):

$$((\phi * \psi) * \chi) * v = (\phi * \psi) * (\chi * v) = \phi * (\psi * (\chi * v))$$
$$= \phi * ((\psi * \chi) * v) = (\phi * (\psi * \chi)) * v.$$

To prove part (ii), we associate with each $a \in A$ the H-pseudolinear map $m_a \in \mathrm{Cend}\, V$ given by $m_a(v) = a * v$. Then

$$(m_a * m_b) * v = m_a * (m_b * v) = a * (b * v) = (a * b) * v = m_{a*b} * v,$$

which shows that $m_a * m_b = m_{a*b}$. ∎

We denote by gc$V$ the Lie H-pseudoalgebra obtained from the associative one Cend V by the construction of Proposition 3.2. Then V is a gc$V$-module, and one has a statement analogous to part (ii) above.

PROPOSITION 10.2. *Let V be a finite H-module. Then, for a Lie H-pseudoalgebra L, giving a structure of an L-module on V is equivalent to giving a homomorphism of Lie H-pseudoalgebras from L to* gc V.

Remark 10.2. Let L be a Lie H-pseudoalgebra, and U, V be finite L-modules. Then the formula ($a \in L$, $u \in U$, $\phi \in \mathrm{Chom}(U, V)$)

$$(a * \phi)(u) = a * (\phi * u) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\, \phi * (a * u) \tag{10.8}$$

provides Chom(U, V) with the structure of an L-module.

DEFINITION 10.2. Let A be an H-pseudoalgebra. A *derivation* of A is an H-pseudolinear map $\phi \in \operatorname{gc} A$ which satisfies

$$\phi * (a * b) = (\phi * a) * b + ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\, a * (\phi * b), \qquad a, b \in A. \quad (10.9)$$

We denote the space of all such ϕ by $\operatorname{Der} A$.

The next result follows easily from definitions.

LEMMA 10.2. (i) *For any H-pseudoalgebra A,* $\operatorname{Der} A$ *is a subalgebra of* $\operatorname{gc} A$.

(ii) *When A is associative (respectively Lie), we have a homomorphism of pseudoalgebras $i\colon A \to \operatorname{Der} A$ given by $i(a)(b) = a * b - (\sigma \otimes_H \mathrm{id})\, b * a$ (respectively $i(a)(b) = [a * b]$), whose kernel is the center of A.*

(iii) *For any $x \in X$ and $\phi \in \operatorname{Der} A$, ϕ_x is a derivation of the annihilation algebra of A. In other words, we have: $\mathscr{A}(\operatorname{Der} A) \subset \operatorname{Der} \mathscr{A}(A)$.*

(iv) *Let A be an associative H-pseudoalgebra and L be the corresponding Lie pseudoalgebra with pseudobracket given by commutator. Then* $\operatorname{Der} A \subset \operatorname{Der} L$.

EXAMPLE 10.2. Recall that in Section 8.3 we defined the $W(\mathfrak{d})$-module of pseudoforms $\Omega(\mathfrak{d}) = H \otimes \bigwedge^{\bullet} \mathfrak{d}^*$. Since $\bigwedge^{\bullet} \mathfrak{d}^*$ is an associative algebra with respect to the wedge product, we can consider $\Omega(\mathfrak{d})$ as an associative pseudoalgebra: the current pseudoalgebra over $\bigwedge^{\bullet} \mathfrak{d}^*$. Then, as in the case of usual differential forms, for any $\alpha \in W(\mathfrak{d})$, $\alpha\, *$ and $\alpha\, *_\iota$ are *superderivations* of $\Omega(\mathfrak{d})$; see (8.13), (8.14).

In the case when V is a free H-module of finite rank, one can give an explicit description of $\operatorname{Cend} V$, and hence of $\operatorname{gc} V$, as follows.

PROPOSITION 10.3. *Let $V = H \otimes V_0$, where H acts trivially on V_0 and $\dim V_0 < \infty$. Then* $\operatorname{Cend} V$ *is isomorphic to $H \otimes H \otimes \operatorname{End} V_0$, with H acting by left multiplication on the first factor, and with the pseudoproduct*

$$(f \otimes a \otimes A) * (g \otimes b \otimes B) = (f \otimes g a_{(1)}) \otimes_H (1 \otimes b a_{(2)} \otimes AB). \quad (10.11)$$

The action of $\operatorname{Cend} V$ *on $V = H \otimes V_0$ is given by*

$$(f \otimes a \otimes A) * (h \otimes v) = (f \otimes ha) \otimes_H (1 \otimes Av). \quad (10.12)$$

The pseudobracket in gc V *is given by*

$$[(f\otimes a\otimes A)*(g\otimes b\otimes B)]=(f\otimes ga_{(1)})\otimes_H(1\otimes ba_{(2)}\otimes AB)$$
$$-(fb_{(1)}\otimes g)\otimes_H(1\otimes ab_{(2)}\otimes BA). \quad (10.13)$$

Proof. Since $(H\otimes H)\otimes_H V\simeq H\otimes H\otimes V_0$, we can identify Cend V with $H\otimes H\otimes \operatorname{End} V_0$ so that its action on V is given by (10.12). To prove (10.11), we use (10.7) and the explicit definition of associativity from Section 3. Due to H-bilinearity, we can assume that $f=g=h=1$. Then

$$(1\otimes a\otimes A)*((1\otimes b\otimes B)*(1\otimes v))=(1\otimes a\otimes A)*((1\otimes b)\otimes_H(1\otimes Bv))$$
$$=(1\otimes a_{(1)}\otimes ba_{(2)})\otimes_H(1\otimes ABv).$$

On the other hand, we have

$$((1\otimes a_{(1)})\otimes_H(1\otimes ba_{(2)}\otimes AB))*(1\otimes v)=(1\otimes a_{(1)}\otimes ba_{(2)})\otimes_H(1\otimes ABv).$$

Now (10.11) follows from the uniqueness provided by Lemma 10.1. ∎

Remark 10.3. Let $V=H\otimes V_0$, where H acts trivially on V_0 and $\dim V_0<\infty$. Then Cur End V_0 can be identified with $H\otimes 1\otimes \operatorname{End} V_0\subset \operatorname{Cend} V$. Similarly, Cur $\mathfrak{gl}\, V_0$ is a subalgebra of gc V.

When $V=H\otimes \mathbf{k}^n$, we will denote Cend V by Cend_n, and gc V by gc_n. Of course, the essential case is when $V=H$ is of rank one. Let us describe the associative algebra $\mathscr{A}_Y\,\mathrm{Cend}_1$, where $\mathscr{A}_Y$ is as in Section 7. As an H-module it is isomorphic to $Y\otimes_H \mathrm{Cend}_1\simeq Y\otimes H$ with H acting on the first factor. We have $a_x=x\otimes_H(1\otimes a)\equiv x\otimes a$ for $x\in Y$, $a\in H$. Comparing (7.2) with (10.11), we see that the product in $Y\otimes H$ is given by

$$(x\otimes a)(y\otimes b)=x(ya_{(1)})\otimes ba_{(2)}. \quad (10.14)$$

Hence $\mathscr{A}_Y\,\mathrm{Cend}_1$ is isomorphic to the smash product $Y\sharp H$ (see Section 2). The annihilation algebra $\mathscr{A}(\mathrm{Cend}_1)\equiv\mathscr{A}_X\,\mathrm{Cend}_1\simeq X\sharp H$ is isomorphic as an associative algebra to the Drinfeld double of H (see [D]). For $H=U(\mathfrak{d})$, $\mathscr{A}(\mathrm{Cend}_1)$ can be identified with the associative algebra of all differential operators on X, while $\mathscr{A}(\mathrm{gc}_1)$ with the corresponding Lie algebra.

Example 10.3. Let $H=U(\mathfrak{d})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{d}$. We identify $\mathfrak{d}$ with its image in H, so that $\mathrm{gc}_1=H\otimes H$ contains $H\otimes\mathfrak{d}$. We claim that $f\otimes a\mapsto -f\otimes a$ ($f\in H$, $a\in\mathfrak{d}$) is an embedding of Lie pseudoalgebras $W(\mathfrak{d})\hookrightarrow \mathrm{gc}_1$, compatible with their actions on H. This is immediate by comparing (10.13) with (8.3) and (10.12) with (8.4).

Consider H as $\mathrm{Cur}\,\mathbf{k}$, i.e., as an associative H-pseudoalgebra with a pseudoproduct $f * g = (f \otimes g) \otimes_H 1$. Then $W(\mathfrak{d}) = \mathrm{Der}\, H \subset \mathrm{gc}_1$.

EXAMPLE 10.4. Let $H = \mathbf{k}[\Gamma]$ be the group algebra of a group Γ. Then for $V = H$ and $f, g, a, b \in \Gamma$, (10.11) takes the form

$$(f \otimes a) * (g \otimes b) = (f \otimes ga) \otimes_H (1 \otimes ba).$$

We end this section with two lemmas that will be useful in representation theory.

LEMMA 10.3. *For* $\phi \in \mathrm{Chom}(V, W)$, *let*

$$\ker_n \phi = \{ v \in V \mid \phi_x v = 0 \ \forall x \in \mathrm{F}_n X \},$$

so that, for example, $\ker_{-1} \phi = \ker \phi$. *If V is a finite H-module and* $\mathrm{F}^n H$ *is finite dimensional, then* $\ker_n \phi / \ker \phi$ *is finite dimensional.*

Proof. Since $\ker \phi$ is an H-submodule of V, after replacing V by $V/\ker \phi$, we can assume that $\ker \phi = 0$.

By definition, $\ker_n \phi = \phi^{-1}((\mathrm{F}^n H \otimes \mathbf{k}) \otimes_H W)$. Since, by Lemma 2.37, $(\mathrm{F}^n H \otimes \mathbf{k}) \otimes_H W = (\mathbf{k} \otimes \mathrm{F}^n H) \otimes_H W$, we have $\phi(\ker_n \phi) \subset (\mathbf{k} \otimes \mathrm{F}^n H) \otimes_H W$.

On the other hand, since V is finite over H and ϕ satisfies (10.1), there exists a finite-dimensional subspace W' of W such that $\phi(\ker_n \phi) \subset (\mathbf{k} \otimes H) \otimes_H W'$. It follows that $\phi(\ker_n \phi) \subset (\mathbf{k} \otimes \mathrm{F}^n H) \otimes_H W'$, which is finite dimensional. Since ϕ is injective, $\ker_n \phi$ is finite dimensional. ∎

LEMMA 10.4. *Let* $\phi \in \mathrm{Chom}(V, W)$ *and* $h \in H$. *If h is not a divisor of zero, then*:

(i) $h\phi = 0$ *implies* $\phi = 0$;

(ii) $hv \in \ker \phi$ *implies* $v \in \ker \phi$.

Proof. Part (i) follows from Eq. (10.2): if $\phi(v) = \sum_i (f_i \otimes 1) \otimes_H w_i$ with linearly independent w_i, then $(h\phi)(v) = \sum_i (hf_i \otimes 1) \otimes_H w_i$ can be zero only if all $hf_i = 0$, which implies $f_i = 0$.

Similarly, part (ii) follows from (10.1), since we can write $\phi(v)$ uniquely in the form $\sum_i (1 \otimes g_i) \otimes_H w_i$. ∎

COROLLARY 10.1. *Let L be a pseudoalgebra (Lie or associative) and M be an L-module. Then any torsion element from L acts trivially on M, and any torsion element from M is acted on trivially by L. In particular, the torsion of a Lie pseudoalgebra is central.*

11. RECONSTRUCTION OF AN H-PSEUDOALGEBRA FROM AN H-DIFFERENTIAL ALGEBRA

11.1. *The Reconstruction Functor* $\mathscr{C}$

Let, as before, H be a cocommutative Hopf algebra and $X = H^*$. Given a topological left H-module $\mathscr{L}$ (where H is endowed with the discrete topology), let

$$\mathscr{C}(\mathscr{L}) = \mathrm{Hom}_H^{\mathrm{cont}}(X, \mathscr{L}) \tag{11.1}$$

be the space of continuous H-homomorphisms. We define a structure of a left H-module on $\mathscr{C}(\mathscr{L})$ by

$$(h\alpha)(x) = \alpha(xh). \tag{11.2}$$

Then $\mathscr{C}$ is a covariant functor from the category of topological H-modules to the category of H-modules.

LEMMA 11.1. (i) *The functor $\mathscr{C}$ is left exact: $\mathscr{C}(i)$ is injective if i is injective.*

(ii) *The functor $\mathscr{C}$ preserves direct sums: $\mathscr{C}(\mathscr{L}_1 \oplus \mathscr{L}_2) = \mathscr{C}(\mathscr{L}_1) \oplus \mathscr{C}(\mathscr{L}_2)$.*

(iii) *Assume that the Hopf algebra H contains nonzero primitive elements. If $\mathscr{L}$ is finite dimensional over* $\mathbf{k}$ *with discrete topology, then $\mathscr{C}(\mathscr{L}) = 0$.*

(iv) *If $H = U(\mathfrak{d})$, then $\mathscr{C}(\mathscr{L})$ has no torsion as an H-module.*

Proof. Parts (i) and (ii) are obvious.

(iii) By Kostant's Theorem 2.1, $H = U(\mathfrak{d}) \sharp \mathbf{k}[\Gamma]$ with $\mathfrak{d} \neq 0$. If $\mathscr{L}$ is finite dimensional, any continuos homomorphism $\alpha\colon X \to \mathscr{L}$ must contain some $\mathrm{F}_n X$ in its kernel. Let $h \in \mathrm{F}^{n-1} U(\mathfrak{d})$ but $h \notin \mathrm{F}^{n-2} U(\mathfrak{d})$. Then, by Lemma 6.4, $h \mathrm{F}_n X = X$ so that for each $x \in X$, $x = hy$ for some $y \in \mathrm{F}_n X$. This implies $\alpha(x) = \alpha(hy) = h(\alpha(y)) = 0$, since $\alpha(y) = 0$, proving part (iii).

Similarly, part (iv) follows from the fact that $Xh = X$ for any nonzero $h \in U(\mathfrak{d})$. ∎

If, in addition, $\mathscr{L}$ is a topological Lie H-differential algebra, we define x-brackets in $\mathscr{C}(\mathscr{L})$ by the formula (cf. (9.24)):

$$[\alpha_x \beta](y) = [\alpha(x_{(2)}), \beta(yx_{(-1)})] = \sum_i [\alpha(x_i), \beta(y(h_i S(x)))]. \tag{11.3}$$

This is well defined because the infinite sum in the right-hand side converges in $\mathscr{L}$. Equation (11.3) is also equivalent to (cf. (9.23))

$$[\alpha(x), \beta(y)] = [\alpha_{x_{(2)}} \beta](yx_{(1)}) = \sum_i [\alpha_{x_i} \beta](y(xS(h_i))). \tag{11.4}$$

Proposition 11.1. *For any topological Lie H-differential algebra $\mathscr{L}$, $\mathscr{C}(\mathscr{L})$ satisfies properties* (9.15)–(9.18).

Proof. This can be verified by straightforward but rather tedious computations. To illustrate them, let us check (9.15). By definition, we have

$$[h\alpha_x \beta](y) = \sum_i [(h\alpha)(x_i), \beta(y(h_i S(x)))] = \sum_i [\alpha(x_i h), \beta(y(h_i S(x)))],$$

while

$$[\alpha_{xh} \beta](y) = \sum_i [\alpha(x_i), \beta(y(h_i S(xh)))].$$

Hence (9.15) is a consequence of the identity

$$\sum_i x_i h \otimes h_i = \sum_i x_i \otimes h_i S(h), \tag{11.5}$$

which can be checked by pairing both sides with $f \otimes z \in H \otimes X$. Indeed,

$$\sum_i \langle x_i h, f \rangle \langle h_i, z \rangle = \langle zh, f \rangle = \langle z, fS(h) \rangle = \sum_i \langle x_i, f \rangle \langle h_i S(h), z \rangle.$$

A more conceptual proof can be given by noticing that formula (11.3) is the same as the formula for the commutator of H-pseudolinear maps. For $\alpha \in \mathscr{C}(\mathscr{L})$ consider the family ad $\alpha(x) \in \mathrm{Hom}(\mathscr{L}, \mathscr{L})$ indexed by $x \in X$. It is easy to see that it satisfies (10.5). So, if it also satisfies (10.4), it would give an H-pseudolinear map from $\mathscr{L}$ to itself. Although this is not true in general, the argument still works because all infinite series that appear will be convergent. (In other words, we embed $\mathscr{C}(\mathscr{L})$ in a certain completion of gc $\mathscr{L}$.) ∎

In order to have the locality (9.13), one has to impose some restrictions on $\mathscr{L}$. In particular, the condition that $\mathscr{L}$ be a linearly compact topological Lie algebra will often suffice to guarantee locality of $\mathscr{C}(\mathscr{L})$.

In what follows, we explain how to reconstruct an H-pseudoalgebra L which is finitely generated over H from its annihilation Lie algebra $\mathscr{A}(L)$. Recall that $\mathscr{A}(L)$ is a linearly compact topological Lie algebra (Proposition 7.4). In many of the proofs we never exploit the algebra structure on

$\mathscr{A}(L)$, so the corresponding statements hold for finite H-modules in general.

We let $\hat{L} = \mathscr{C}\mathscr{A}(L) := \mathrm{Hom}_H^{\mathrm{cont}}(X, X \otimes_H L)$. There is an obvious map

$$\Phi : L \to \hat{L}, \qquad a \mapsto \alpha(x) = x \otimes_H a. \tag{11.6}$$

It is clear by definitions that Φ is a homomorphism of H-pseudoalgebras (or H-modules if L is only an H-module).

11.2. *The Case of Free Modules*

Let L be a Lie H-pseudoalgebra which is free as an H-module: $L = H \otimes L_0$ with the trivial action of H on L_0. Then $\mathscr{L} = \mathscr{A}(L) := X \otimes_H (H \otimes L_0) \simeq X \otimes L_0$ as an H-module.

PROPOSITION 11.2. *When L is a Lie H-pseudoalgebra that is a free H-module, the map Φ defined by* (11.6) *is an isomorphism of Lie H-pseudoalgebras.*

Proof. To construct the inverse of Φ, identify $\mathscr{L}$ with $X \otimes L_0$ and consider

$$\Psi : \hat{L} \to L, \qquad \alpha \mapsto \sum_i S(h_i) \otimes (\varepsilon \otimes \mathrm{id})\, \alpha(x_i).$$

Here, as before, $\{h_i\}$, $\{x_i\}$ are dual bases in H and X, and $\varepsilon(x) = \langle 1, x \rangle$ for $x \in X$. This is well defined, i.e., the sum is finite, because $\alpha(x_i) \in \mathrm{F}_0 X \otimes L_0$ for all but a finite number of x_i and $\varepsilon(\mathrm{F}_0 X) = 0$. Using identity (11.5), it is easy to see that Ψ is H-linear. Next, we have for $a \in L_0$

$$\Psi\Phi(1 \otimes a) = \sum_i S(h_i) \otimes \langle 1, x_i \rangle\, a = S(1) \otimes a = 1 \otimes a,$$

showing that $\Psi\Phi = \mathrm{id}$. In particular, Ψ is surjective.

Assume that $\Psi(\alpha) = 0$ for some $\alpha \in \hat{L}$. This means that $(\langle 1, \cdot \rangle \otimes \mathrm{id})\, \alpha(x) = 0$ for any $x \in X$. But then for any $h \in H$, we have

$$\begin{aligned}(\langle S(h), \cdot \rangle \otimes \mathrm{id})\, \alpha(x) &= (\langle 1, \cdot \rangle \otimes \mathrm{id})((h \otimes 1)\, \alpha(x)) \\ &= (\langle 1, \cdot \rangle \otimes \mathrm{id})(h(\alpha(x))) = (\langle 1, \cdot \rangle \otimes \mathrm{id})\, \alpha(hx) = 0,\end{aligned}$$

which implies $\alpha = 0$. Hence Ψ is injective. ∎

Remark 11.1. If L is only a free H-module, then Φ is an isomorphism of H-modules. Analogous results hold for representations, or for associative H-pseudoalgebras.

11.3. *Reconstruction of a Non-free Module*

Throughout this subsection L will be a (possibly non-free) finitely generated H-module, and H will be the universal enveloping algebra $U(\mathfrak{d})$ of a finite-dimensional Lie algebra $\mathfrak{d}=0$.

The natural map $\Phi\colon L\to\hat{L}$ (see (11.6)) is in general neither injective nor surjective. However, the induced map $\varphi=\mathscr{A}(\Phi)\colon \mathscr{A}(L)\to\mathscr{A}(\hat{L})$ has a left inverse $\psi\colon x\otimes_H \alpha\mapsto\alpha(x)$. This shows that $\mathscr{A}(\Phi)$ is injective, and that ψ is surjective.

We want to figure out to what extent injectivity and surjectivity of Φ fail. First of all let us remark that, by Lemma 7.1, every torsion element $a\in L$ has all zero Fourier coefficients, i.e., it belongs to the kernel of Φ. In fact, we have

PROPOSITION 11.3. *For any finite H-module L, the kernel of $\Phi\colon L\to\hat{L}$ equals the torsion of L.*

Proof. It remains to show that a non-torsion element $a\in L$ does not lie in the kernel of Φ. Consider the map $i\colon L\to F$ constructed in Lemma 2.1. Then $i(a)\neq 0$. The map $\mathscr{A}(i)$ induced by i maps the Fourier coefficient $x\otimes_H a$ of a to the corresponding Fourier coefficient $x\otimes_H i(a)$ of a nonzero element in the free H-module F. Now, it is clear from Proposition 11.2 that $x\otimes_H i(a)\neq 0$ for some $x\in X$, hence $x\otimes_H a$ must be nonzero too. ∎

COROLLARY 11.1. *A finite H-module L is torsion iff $X\otimes_H L=0$.*

COROLLARY 11.2. *Let M, N be finite H-modules, $f\colon M\to N$ be an H-linear map, and assume N to be torsionless. Then $\mathscr{A}(f)=0$ if and only if $f=0$.*

Proof. $\mathscr{A}(f)=0$ means that $X\otimes_H f(M)=0$, hence $f(M)\subset N$ is torsion. ∎

Remark 11.2. By Corollary 10.1, the torsion of a Lie H-pseudoalgebra L is always central, hence the map Φ is injective if L is centerless.

Remark 11.3. If Φ is an isomorphism, then Φ^{-1} induces ψ, i.e., $\psi=\mathscr{A}(\Phi^{-1})$. Corollary 11.2 tells us that if L is torsionless and ψ is induced by some map Ψ, then Φ is an isomorphism and $\Psi=\Phi^{-1}$.

PROPOSITION 11.4. *For any map of finite H-modules $f\colon M\to N$, the following conditions are equivalent*

(1) *$N/f(M)$ is torsion.*

(2) *$\mathscr{A}(f)\colon \mathscr{A}(M) \to \mathscr{A}(N)$ is surjective.*

(3) gw coker $\mathscr{A}(f) < \dim \mathfrak{d}$.

Proof. To show the equivalence of (1) and (2), it is enough to tensor the exact sequence $M \xrightarrow{f} N \to N/f(M) \to 0$ with X, getting the exact sequence $\mathscr{A}(M) \to \mathscr{A}(N) \to X \otimes_H N/f(M) \to 0$, and to apply Corollary 11.1.

Assume that (3) holds but $N/f(M)$ is not torsion. Then it contains a nonzero element a which generates a free H-module. Then $X \otimes_H a \simeq X$ has growth $\dim \mathfrak{d}$, which is a contradiction. If (1) holds, by Corollary 11.1, coker $\mathscr{A}(f) = 0$. ▮

11.4. *Reconstruction of a Lie Pseudoalgebra*

Now let L be a Lie H-pseudoalgebra which is finite as an H-module. Again, $H = U(\mathfrak{d})$ will be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$. Let $\mathscr{L} = \mathscr{A}(L)$ be the annihilation Lie algebra of L, and $\hat{L} = \mathscr{C}(\mathscr{L})$, as before.

By Proposit1ion 11.1, $\hat{L}$ satisfies all the properties of a Lie H-pseudoalgebra except the locality (9.13). An indirect way to establish the locality property for $\hat{L}$ is by embedding it in the bigger (local) Lie pseudoalgebra gc L. In order to map $\hat{L}$ to gc L, we need to assign to each element of $\hat{L}$ a pseudolinear map from L to itself.

This can be done as follows. Recall that $\mathscr{L}$ acts on L by $(x \otimes_H a) \cdot b = [a_x b]$ for $a, b \in L$, $x \in X$ (see Proposition 9.1). Now in terms of x-products the action of $\hat{L}$ on L is given by $\alpha_x b = \alpha(x) \cdot b$. The locality condition $\alpha_x b = 0$ for $x \in \mathrm{F}_n X$, $n \gg 0$ is satisfied because α is continuous and L is a discrete topological $\mathscr{L}$-module (see Proposition 9.1). All the other axioms of a Lie pseudoalgebra representation follow easily from definitions.

We now need to find conditions for the above-defined $j\colon \hat{L} \to \mathrm{gc}\, L$ to be injective. Then $\hat{L}$ would embed into gc$L$, which will show locality.

LEMMA 11.2. *If L is torsionless, the kernel of the above-defined $j\colon \hat{L} \to \mathrm{gc}\, L$ consists of all elements α such that $\alpha(X)$ is contained in the center of $\mathscr{L}$.*

Proof. Since L is torsionless, Φ is injective by Proposition 11.3. Hence, for $a, b \in L$, $x \in X$, one has $[a_x b] = 0$ iff $[a_x b]_y = 0$ for all $y \in X$. By (9.23), (9.24), this is equivalent to $[a_x, b_y] = 0$. Hence, for $l \in \mathscr{L}$, $l \cdot b = 0$ for all b iff l lies in the center of $\mathscr{L}$. Now $\alpha \in \hat{L}$ is in the kernel of j iff $\alpha(x) \cdot b = 0$ for all x and b, which means that $\alpha(x)$ is central for all x. ▮

LEMMA 11.3. *If L is torsionless and $\mathscr{L} = \mathscr{A}(L)$ has a finite-dimensional center, then $j\colon \hat{L} \to \mathrm{gc}\, L$ is injective.*

Proof. Let $\alpha \in \hat{L}$ be in the kernel of j; then by the previous lemma $\alpha(X)$ is contained in the center of $\mathscr{L}$. The latter is finite dimensional by assumption, so the kernel N of α is of finite codimension in X. This implies that N is open in X, and it contains $\mathrm{F}_i X$ for some i. Let $h \in \mathrm{F}^{i+1}H$ but $h \notin \mathrm{F}^i H$; then by Lemma 6.4, $h\mathrm{F}_i X = \mathrm{F}_{-1} X = X$. Since α is H-linear, N is an H-submodule of X. Then $X = h\mathrm{F}_i X \subset hN \subset N$, therefore $N = X$ and $\alpha = 0$. ∎

PROPOSITION 11.5. *Let L be a Lie H-pseudoalgebra which is finite and torsionless as an H-module. If its annihilation Lie algebra $\mathscr{L} = \mathscr{A}(L)$ has a finite-dimensional center, then $\hat{L} = \mathscr{C}(\mathscr{L})$ is a Lie H-pseudoalgebra containing L as an ideal.*

Proof. The only thing that remains to be checked is the locality property for $\hat{L}$. It follows from that of $\mathrm{gc}\, L$, since in this case $j\colon \hat{L} \to \mathrm{gc}\, L$ is injective. ∎

In the proof of Lemma 11.2 we have shown that, if L is finite and torsionless, the kernel of the action of $\mathscr{A}(L)$ on L is exactly the center of $\mathscr{A}(L)$. This implies the following result which was used in the proof of Theorem 8.2.

LEMMA 11.4. *Let $i\colon L \hookrightarrow L_1$ be an injective map of Lie H-pseudoalgebras and assume that L is finite and torsionless. Then the kernel of the induced map $\mathscr{A}(i)\colon \mathscr{A}(L) \to \mathscr{A}(L_1)$ is contained in the center of $\mathscr{A}(L)$.*

Proof. The kernel of $\mathscr{A}(i)$ acts trivially on L_1 and hence on L. ∎

In Section 13.4 we will need the following lemma.

LEMMA 11.5. *Let L be a subalgebra of* gc *V for some finite H-module V (L may be infinite). Then the map $\Phi\colon L \to \hat{L}$ is injective.*

Proof. Assume that a belongs to $\ker \Phi$; then all $x \otimes_H a = 0$, $x \in X$. This implies that all Fourier coefficients $a_x \in \mathrm{End}\, V$ of $a \in \mathrm{gc}\, V$ are zero, hence $a = 0$. ∎

For any topological Lie H-differential algebra $\mathscr{L}$, we have a natural homomorphism $\psi\colon \mathscr{A}\mathscr{C}(\mathscr{L}) \to \mathscr{L}$, given by $x \otimes_H a \mapsto a(x)$ for $a \in \mathscr{C}(\mathscr{L})$, $x \in X$. The map ψ does not need to be surjective, but we have a good control on injectivity, which can sometime prove useful.

LEMMA 11.6. *The kernel of $\psi\colon \mathscr{A}\mathscr{C}(\mathscr{L}) \to \mathscr{L}$ lies in the center of $\mathscr{A}\mathscr{C}(\mathscr{L})$.*

Proof. Follows easily from (9.29) and (9.30). Suppose that α lies in the kernel of ψ. Since ψ is a homomorphism of Lie H-differential algebras, its kernel is an H-stable ideal of $\mathscr{A}\mathscr{C}(\mathscr{L})$. Then by (9.29), $(\alpha \cdot b)_y \in \ker \psi$ for all $b \in \mathscr{C}(\mathscr{L})$, $y \in X$, because in its right-hand side all elements $h_i \alpha$ lie in $\ker \psi$. This means that $(\alpha \cdot b)(y) = 0$ for all $y \in X$, hence $\alpha \cdot b = 0$ for every $b \in \mathscr{C}(\mathscr{L})$. Now, use this in (9.30) to obtain that α is central. ∎

12. RECONSTRUCTION OF PSEUDOALGEBRAS OF VECTOR FIELDS

In this section, we show that the reconstruction procedure of Section 11, when applied to the primitive Lie algebras of vector fields (or current algebras over them), gives the primitive pseudoalgebras of vector fields defined in Section 8 (or current pseudoalgebras over them).

As before, $\mathfrak{d}$ will be an N-dimensional Lie algebra, and $H = U(\mathfrak{d})$ its universal enveloping algebra. $\mathscr{L}$ will be a Lie algebra provided with an action of $\mathfrak{d}$ and a filtration by subspaces $\mathscr{L} = \mathscr{L}_{-1} \supset \mathscr{L}_0 \supset \cdots$. When $\mathscr{L}$ is a subalgebra of W_N, it will always be considered with the filtration induced by the canonical filtration of W_N.

The Lie algebra $\operatorname{Der} \mathscr{L}$ of derivations of $\mathscr{L}$ has the induced filtration

$$(\operatorname{Der} \mathscr{L})_i := \{ d \in \operatorname{Der} \mathscr{L} \mid d(\mathscr{L}_j) \subset \mathscr{L}_{i+j} \ \forall j \}.$$

The action of $\mathfrak{d}$ is called *transitive* if the composition of the homomorphism $\mathfrak{d} \to \operatorname{Der} \mathscr{L}$ and the projection $\operatorname{Der} \mathscr{L} \to \operatorname{Gr}_{-1}(\operatorname{Der} \mathscr{L}) := \operatorname{Der} \mathscr{L} / (\operatorname{Der} \mathscr{L})_0$ is a linear isomorphism. This is equivalent to the following two conditions: $\mathfrak{d} \hookrightarrow \operatorname{Der} \mathscr{L}$ intersects $(\operatorname{Der} \mathscr{L})_0$ trivially and $\dim \operatorname{Gr}_{-1}(\operatorname{Der} \mathscr{L}) = N$.

LEMMA 12.1. *Let L be a current Lie H-pseudoalgebra over a finite-dimensional simple Lie algebra or over one of the primitive pseudoalgebras of vector fields. Then the action of $\mathfrak{d}$ on its annihilation Lie algebra $\mathscr{L} = \mathscr{A}(L)$ is transitive.*

Proof. By Theorem 8.2, $\mathscr{L} = \mathcal{O}_r \hat{\otimes} \mathscr{L}'$ is a current Lie algebra over $\mathscr{L}'$, where $\mathscr{L}'$ is either a finite-dimensional simple Lie algebra $\mathfrak{g}$ (for $r = N = \dim \mathfrak{d}$), or one of the Lie algebras of vector fields $W_{N'}$, $S_{N'}$, $P_{N'}$ or $K_{N'}$ ($N' = N - r$). In particular, we know that $\dim \operatorname{Gr}_{-1}(\operatorname{Der} \mathscr{L}) = N$. By Lemma 7.3, a sufficiently high power of any nonzero element $a \in \mathfrak{d}$ maps any given open subspace of $\mathscr{L}$ surjectively onto $\mathscr{L}$. This cannot hold if a belongs to $(\operatorname{Der} \mathscr{L})_0$, therefore $\mathfrak{d} \to \operatorname{Der} \mathscr{L}$ is injective and the image of $\mathfrak{d}$ intersects $(\operatorname{Der} \mathscr{L})_0$ trivially. Comparing the dimensions, we get that $\mathfrak{d} \to \operatorname{Gr}_{-1}(\operatorname{Der} \mathscr{L})$ is an isomorphism. ∎

The main results of this section can be summarized by Theorem 12.1 below. Its proof follows from Sections 12.1–12.7.

THEOREM 12.1. *Let $\mathscr{L} = \mathcal{O}_r \hat{\otimes} \mathscr{L}'$ be a current Lie algebra over $\mathscr{L}'$, where $\mathscr{L}'$ is a simple linearly compact Lie algebra of growth $N' = N - r$. Assume that $\mathfrak{d}$ acts transitively on $\mathscr{L}$. Then there is a codimension r subalgebra $\mathfrak{d}'$ of $\mathfrak{d}$, acting transitively on $\mathscr{L}'$, such that the H-pseudoalgebra $\mathscr{C}(\mathscr{L})$ is isomorphic to a current pseudoalgebra over the H'-pseudoalgebra $\mathscr{C}(\mathscr{L}')$, where $H' = U(\mathfrak{d}')$. Moreover, $\mathscr{C}(\mathscr{L}')$ is either a finite-dimensional simple Lie algebra (and $\mathfrak{d}' = 0$) or one of the primitive H'-pseudoalgebras of vector fields $W(\mathfrak{d}')$, $S(\mathfrak{d}', \chi')$, $H(\mathfrak{d}', \chi', \omega')$, or $K(\mathfrak{d}', \theta')$.*

12.1. *Reconstruction from W_N*

Recall that $X = H^*$ can be identified with $\mathcal{O}_N = \mathbf{k}[[t_1, ..., t_N]]$. The action of $\mathfrak{d}$ on X gives an action on $\mathcal{O}_N$ in terms of linear differential operators, i.e., an embedding $\mathfrak{d} \hookrightarrow W_N = \operatorname{Der} \mathcal{O}_N$ which we call the *canonical embedding* of $\mathfrak{d}$ in W_N. Note that this embedding is transitive, i.e., $\mathfrak{d} \subset W_N$ is complementary to $\mathrm{F}_0 W_N$.

A structure of an H-differential algebra on W_N is equivalent to a transitive action of $\mathfrak{d}$ on W_N by derivations. Since $\operatorname{Der} W_N = W_N$, this is the same as a transitive embedding of $\mathfrak{d}$ in W_N. By Proposition 6.2, any two such embeddings are equivalent, i.e., conjugate by an automorphism of W_N. With the canonical action of $\mathfrak{d}$, W_N becomes isomorphic to the annihilation algebra of the Lie pseudoalgebra $W(\mathfrak{d})$ defined in Section 8.1. Since $W(\mathfrak{d})$ is a free H-module, Proposition 11.2 shows that the reconstruction of W_N is $W(\mathfrak{d})$, i.e., $\mathscr{C}(W_N) = W(\mathfrak{d})$.

12.2. *Reconstruction from Subalgebras of W_N*

Let $\mathscr{L}$ be a linearly compact Lie subalgebra of W_N, with the induced filtration and with a transitive action of $\mathfrak{d}$ on it. After an automorphism of W_N, we can assume that the action of $\mathfrak{d}$ is the canonical one. Then $\mathscr{C}(\mathscr{L})$ is a subalgebra of $W(\mathfrak{d}) = \mathscr{C}(W_N)$, because the functor $\mathscr{C}$ is left exact. Below we will be concerned with the case when $\mathscr{L}$ is the subalgebra consisting of vector fields annihilating some differential form.

Let $w \in \Omega^n(\mathfrak{d})$ be a pseudoform, and $I \subset H$ be a right ideal. We denote by $W(\mathfrak{d}, w, I)$ the set of all elements $\alpha \in W(\mathfrak{d}) = H \otimes \mathfrak{d}$ such that

$$\alpha * w \in (H \otimes I) \otimes_H \Omega^n(\mathfrak{d}). \tag{12.1}$$

It is easy to check that $W(\mathfrak{d}, w, I)$ is a subalgebra of $W(\mathfrak{d})$.

LEMMA 12.2. *Let $\omega \in \Omega^n_X$ be a differential form, and $W_N(\omega)$ be the Lie subalgebra of W_N consisting of vector fields annihilating ω. If $\omega = y \otimes_H w$*

for some $y \in X$, $w \in \Omega^n(\mathfrak{d})$, *then* $\mathscr{C}(W_N(\omega))$ *is isomorphic to the Lie pseudoalgebra* $W(\mathfrak{d}, w, I)$ *where* $I = \{h \in H \mid yh = 0\}$.

Proof. As was already remarked, $\mathscr{C}(W_N(\omega))$ is a subalgebra of $W(\mathfrak{d})$. Since $\Omega^n(\mathfrak{d}) = H \otimes \bigwedge^n \mathfrak{d}^*$ is a free H-module, we have $(H \otimes H) \otimes_H \Omega^n(\mathfrak{d}) \simeq H \otimes H \otimes \bigwedge^n \mathfrak{d}^*$. For $\alpha \in W(\mathfrak{d})$, write $\alpha * w = \sum_i (f_i \otimes g_i) \otimes_H w_i$ with $f_i, g_i \in H$ and linearly independent $w_i \in \bigwedge^n \mathfrak{d}^*$. Then for any $x \in X$ we have (cf. (7.2))

$$L_{x \otimes_H \alpha} \omega = \sum_i (xf_i)(yg_i) \otimes_H w_i .$$

This is zero for any x iff $yg_i = 0$ for all i, which means $g_i \in I$. ∎

12.3. *Reconstruction from Current Algebras over* $W_{N'}$

Let now $\mathscr{L} = \mathcal{O}_r \mathbin{\hat{\otimes}} W_{N'}$ be a current algebra over $W_{N'}$, and $\mathfrak{d}$ be an $N = N' + r$ dimensional Lie algebra acting transitively on $\mathscr{L}$. Then, by Proposition 6.4, $\mathfrak{d} \hookrightarrow \operatorname{Der} \mathscr{L} = W_r \otimes 1 + \mathcal{O}_r \mathbin{\hat{\otimes}} W_{N'} \subset W_N$. The Lie algebra $\mathscr{L}$ is described as the subalgebra of W_N consisting of vector fields annihilating the functions $t_{N'+1}, ..., t_N$, hence it is an intersection of algebras of the form $W_N(f)$ $(f \in \Omega^0_X = X)$; see Section 12.2.

After an automorphism of W_N, we can assume that the action of $\mathfrak{d}$ on it is the canonical one. Then $\mathscr{L}$ becomes the intersection of $W_N(f_i)$ $(i = N' + 1, ..., N)$ where $f_i \in X$ is the image of t_i. Now Lemma 12.2 implies that $\mathscr{C}(\mathscr{L}) = W(\mathfrak{d}, 1, I)$ where $1 \in \Omega_0(\mathfrak{d}) = H$ and $I = \{h \in H \mid f_i h = 0 \ (i = N' + 1, ..., N)\}$.

Recall that for $\alpha \in W(\mathfrak{d}) = H \otimes \mathfrak{d}$, its action on $1 \in H$ is given by $\alpha * 1 = -\alpha \otimes_H 1 \equiv -\alpha$. Therefore $\alpha \in W(\mathfrak{d}, 1, I)$ iff α belongs to $(H \otimes \mathfrak{d}) \cap (H \otimes I) = H \otimes \mathfrak{d}'$, where the intersection $\mathfrak{d}' = \mathfrak{d} \cap I$ is a Lie subalgebra of $\mathfrak{d}$. Then $H' = U(\mathfrak{d}')$ is a Hopf subalgebra of H, and $H \otimes \mathfrak{d}' \simeq H \otimes_{H'} (H' \otimes \mathfrak{d}')$ is a current pseudoalgebra over $H' \otimes \mathfrak{d}' = W(\mathfrak{d}')$. We have thus proved the following lemma.

LEMMA 12.3. *The reconstruction of a current Lie algebra over* $W_{N'}$, *provided with a transitive action of a Lie algebra* $\mathfrak{d}$, *is a current Lie* H-*pseudoalgebra over* $W(\mathfrak{d}')$ *where* $\mathfrak{d}'$ *is an* N'-*dimensional Lie subalgebra of* $\mathfrak{d}$.

This result is a special case of Lemma 12.5 below.

12.4. *Solving Compatible Systems of Linear Differential Equations*

Let A be any associative $\mathbf{k}$-algebra, and let $\mathcal{O}_r = \mathbf{k}[[t_1, ..., t_r]]$, $W_r = \operatorname{Der} \mathcal{O}_r$, as before. For fixed $n \geqslant 0$, let $f_i(t) \in A[[t_1, ..., t_r]]$ $(i = 1, ..., r + n)$

be formal power series with coefficients in A, where $t=(t_1, ..., t_r)$. Note that W_r acts on $A[[t_1, ..., t_r]]$ by derivations.

Given $r+n$ linear differential operators $D_1, ..., D_{r+n} \in W_r$, consider the following system of differential equations for an unknown $y(t) \in A[[t_1, ..., t_r]]$:

$$D_i(y(t)) = y(t) f_i(t), \qquad i=1, ..., r+n. \tag{12.2}$$

We assume that the operators D_i satisfy

$$[D_i, D_j] = \sum_k c_{ij}^k(t) D_k \qquad \text{with} \quad c_{ij}^k(t) \in \mathcal{O}_r; \tag{12.3}$$

in other words, the space of all operators of the form $\sum_i p_i(t) D_i$ with $p_i(t) \in \mathcal{O}_r$ is a Lie algebra.

Suppose we have found a solution to the system (12.2). Combining Eqs. (12.2) and (12.3), we get

$$\begin{aligned}[D_i, D_j](y) &= D_i D_j(y) - D_j D_i(y) = D_i(yf_j) - D_j(yf_i) \\ &= yf_i f_j + yD_i(f_j) - yf_j f_i - yD_j(f_i),\end{aligned}$$

and

$$[D_i, D_j](y) = \sum_k c_{ij}^k D_k(y) = \sum_k c_{ij}^k y f_k.$$

Hence the right-hand sides of these two equations must be equal. The system (12.2) is called *compatible* if

$$[f_i(t), f_j(t)] + D_i(f_j(t)) - D_j(f_i(t)) = \sum_k c_{ij}^k(t) f_k(t) \qquad \text{for all} \quad i, j. \tag{12.4}$$

Thus, when $y(t)$ is not a divisor of zero in $A[[t_1, ..., t_r]]$ the compatibility of the system is a necessary condition for having a solution. The compatibility (12.4) is equivalent to saying that $\sum_i p_i(t) D_i \mapsto \sum_i p_i(t)(D_i + f_i(t))$ is a homomorphism of Lie algebras.

We will be interested in solving a more general system of equations than (12.2). Before formulating it, let us note that the above remarks have obvious analogues for systems of the form

$$D_i(z(t)) = -h_i(t) z(t), \qquad i=1, ..., r+n \tag{12.5}$$

with $z(t), h_i(t) \in A[[t_1, ..., t_r]]$. The compatibility of (12.5) is equivalent to (12.4) with f_i replaced by h_i.

Now consider the system

$$D_i(g(t)) = g(t) f_i(t) - h_i(t) g(t), \qquad i=1, ..., r+n \tag{12.6}$$

for an unknown $g(t) \in A[[t_1, ..., t_r]]$. We will show it has a solution, provided that both (12.2) and (12.5) are compatible and some initial conditions at $t=0$ are satisfied. (The compatibility of (12.2) and (12.5) implies the compatibility of (12.6).)

PROPOSITION 12.1. *In the above notation, let the operators* $D_i \in W_r$ *satisfy* (12.3) *and*

$$D_i\,|_{t=0} = \begin{cases} \partial_{t_i}, & 1 \leqslant i \leqslant r, \\ 0, & r+1 \leqslant i \leqslant r+n. \end{cases} \tag{12.7}$$

Assume that the systems (12.2) *and* (12.5) *are compatible* (*cf.* (12.4)), *and that*

$$f_i(0) = h_i(0), \qquad r+1 \leqslant i \leqslant r+n. \tag{12.8}$$

Then the system (12.6) *has a unique solution* $g(t) \in A[[t_1, ..., t_r]]$ *for any given initial condition* $g(0) \in A$ *which commutes with* $f_i(0)$ $(r+1 \leqslant i \leqslant r+n)$.

Proof. For $r=0$, both sides of (12.6) are trivial. For $r \geqslant 1$, we will proceed by induction on r.

First of all, note that the compatibility conditions and solvability of the systems (12.2), (12.5), or (12.6) do not change when we apply an automorphism of $\mathcal{O}_r$. The same is true when we make an elementary transformation: multiply one equation by a function (an element of $\mathcal{O}_r$) and add it to another equation. For example, we can replace all D_i $(i \neq r)$ by $D_i - p_i(t)\, D_r$, and correspondingly $f_i(t)$ by $f_i(t) - p_i(t) f_r(t)$ and $h_i(t)$ by $h_i(t) - p_i(t)\, h_r(t)$, as long as we do not violate (12.7), (12.8).

Any vector field $D_r \in W_r$ satisfying $D_r|_{t=0} = \partial_{t_r}$ can be brought to ∂_{t_r} after an automorphism of $\mathcal{O}_r$, so we will assume that $D_r = \partial_{t_r}$. Replacing D_i $(i \neq r)$ by $D_i - D_i(t_r)\, D_r$, we can assume in addition that $D_i(t_r) = 0$ for $i \neq r$.

Now it makes sense to put $t_r = 0$ in the equations with $i \neq r$ in (12.6). Let us denote $\bar{D}_i = D_i\,|_{t_r=0}$, $\bar{f}_i(\bar{t}) = f_i(t_1, ..., t_{r-1}, 0)$, $\bar{h}_i(\bar{t}) = h_i(t_1, ..., t_{r-1}, 0)$, $\bar{t} = (t_1, ..., t_{r-1})$. Consider the reduced system

$$\bar{D}_i(\bar{g}(\bar{t})) = \bar{g}(\bar{t})\, \bar{f}_i(\bar{t}) - \bar{h}_i(\bar{t})\, \bar{g}(\bar{t}), \qquad i = 1, ..., r-1, r+1, ..., r+n \tag{12.9}$$

for an unknown $\bar{g}(\bar{t}) \in A[[t_1, ..., t_{r-1}]]$. Note that, since $D_i(t_r) = \delta_{ir}$, we have $[D_i, D_j](t_r) = 0$ for any i, j, hence $[D_i, D_j]$ does not contain D_r. In particular, putting $t_r = 0$ we see that the operators $\bar{D}_i$ satisfy (12.3). The other assumptions of the proposition are also easy to check, so by induction the system (12.9) has a solution $\bar{g}(\bar{t})$.

The equation

$$\partial_{t_r} g(t) = g(t)\, f_r(t) - h_r(t)\, g(t) \tag{12.10}$$

has a unique solution $g(t)$ satisfying the initial condition

$$g(t_1, ..., t_{r-1}, 0) = \bar{g}(\bar{t}). \tag{12.11}$$

We claim that this $g(t)$ is then a solution of the system (12.6). Indeed, it satisfies (12.6) for $t_r = 0$. Next, we compute for $i \neq r$ (using (12.9), (12.10), and the compatibility of (12.2), (12.5)):

$$\begin{aligned}
D_r D_i(g)|_{t_r=0} &= [D_r, D_i](g)|_{t_r=0} + D_i D_r(g)|_{t_r=0} \\
&= \sum_j \bar{c}_{ri}^{j} \bar{D}_j \bar{g} + \bar{D}_i (\bar{g}\bar{f}_r - \bar{h}_r \bar{g}) \\
&= \sum_j \bar{c}_{ri}^{j} (\bar{g}\bar{f}_j - \bar{h}_j \bar{g}) + (\bar{g}\bar{f}_i - \bar{h}_i \bar{g})\bar{f}_r + \bar{g}\bar{D}_i(\bar{f}_r) \\
&\quad - \bar{D}_i(\bar{h}_r)\,\bar{g} - \bar{h}_r(\bar{g}\bar{f}_i - \bar{h}_i \bar{g}) \\
&= g(D_r(f_i) + f_r f_i)|_{t_r=0} \\
&\quad - (D_r(h_i) - h_i h_r)\, g|_{t_r=0} - \bar{h}_i \bar{g}\bar{f}_r - \bar{h}_r \bar{g}\bar{f}_i \\
&= D_r(g f_i - h_i g)|_{t_r=0}.
\end{aligned}$$

This shows that $\partial_{t_r}(D_i(g) - g f_i + h_i g)|_{t_r=0} = 0$. We can apply the same argument with $[D_r, D_i]$ instead of D_i, and so on, to show that all derivatives with respect to t_r vanish at $t_r = 0$. ∎

Remark 12.1. Any solution $g(t)$ of the system (12.6), such that $g(0)$ is invertible in A, is invertible in $A[[t_1, ..., t_r]]$. Its inverse $g(t)^{-1}$ satisfies (12.6) with f_i replaced by $-h_i$.

12.5. *Reconstruction from a Current Lie Algebra*

Let $\mathscr{L}'$ be a simple linearly compact Lie algebra, and let $\mathscr{L} = \mathcal{O}_r \hat{\otimes} \mathscr{L}'$ be a current algebra over $\mathscr{L}'$. The filtration by subspaces $\mathscr{L}' = \mathscr{L}'_{-1} \supset \mathscr{L}'_0 \supset \cdots$ and the canonical filtration of $\mathcal{O}_r$ give rise to the product filtration of $\mathscr{L}$. Assume that $\mathfrak{d}$ acts on $\mathscr{L}$ transitively by derivations. By Proposition 6.4, we have $\operatorname{Der} \mathscr{L} = W_r \otimes 1 + \mathcal{O}_r \hat{\otimes} \operatorname{Der} \mathscr{L}'$.

Denote by j the embedding $\mathfrak{d} \hookrightarrow \operatorname{Der} \mathscr{L}$, and by p the projection $\operatorname{Der} \mathscr{L} \to W_r$. The preimage $\mathfrak{d}' := (pj)^{-1}(\mathrm{F}_0 W_r)$ is a Lie subalgebra of $\mathfrak{d}$ of codimension r. We have $\mathfrak{d}' \hookrightarrow \mathrm{F}_0 W_r \otimes 1 + \mathcal{O}_r \hat{\otimes} \operatorname{Der} \mathscr{L}'$. The latter contains $\mathrm{F}_0 W_r \otimes 1 + \mathrm{F}_0 \mathcal{O}_r \hat{\otimes} \operatorname{Der} \mathscr{L}'$ as an ideal, hence we get a Lie algebra homomorphism $j' \colon \mathfrak{d}' \to \operatorname{Der} \mathscr{L}'$. It leads to a transitive action of $\mathfrak{d}'$ on $\mathscr{L}'$, because the action of $\mathfrak{d}$ on $\mathscr{L}$ is transitive.

LEMMA 12.4. *Any two transitive embeddings* $j\colon \mathfrak{d} \hookrightarrow \operatorname{Der} \mathcal{L}$, *that induce the same subalgebra* $\mathfrak{d}'$ *and the same* $j'\colon \mathfrak{d}' \to \operatorname{Der} \mathcal{L}'$, *are equivalent up to an automorphism of* $\operatorname{Der} \mathcal{L}$.

Proof. Let us choose a basis $\{\partial_i\}$ of $\mathfrak{d}$ and write $j(\partial_i) = D_i + f_i(t)$ $(i = 1, ..., N = r + n = \dim \mathfrak{d})$ where $D_i \in W_r$ and $f_i(t) \in \mathcal{O}_r \mathbin{\hat{\otimes}} \operatorname{Der} \mathcal{L}'$, $t = (t_1, ..., t_r)$. Note that $D_i = (pj)(\partial_i)$ and $pj\colon \mathfrak{d} \to W_r$ is a Lie algebra homomorphism. We can choose the basis $\{\partial_i\}$ in such a way that $D_i|_{t=0} = \partial_{t_i}$ for $1 \leqslant i \leqslant r$, and $D_i|_{t=0} = 0$ for $r+1 \leqslant i \leqslant r+n$. Then $\{\partial_i\}_{i=r+1,...,r+n}$ is a basis of $\mathfrak{d}'$. Moreover, note that $j'\colon \mathfrak{d}' \to \operatorname{Der} \mathcal{L}'$ is given by $j'(\partial_i) = f_i(0)$.

Let $\tilde{\jmath}$ be another transitive embedding of $\mathfrak{d}$ into $\operatorname{Der} \mathcal{L}$. Since, by Proposition 6.2, the homomorphism pj is uniquely determined by the choice of $\mathfrak{d}'$, we can assume that $(p\tilde{\jmath})(\partial_i) = D_i$. Then $\tilde{\jmath}(\partial_i) = D_i + h_i(t)$ for some $h_i(t) \in \mathcal{O}_r \mathbin{\hat{\otimes}} \operatorname{Der} \mathcal{L}'$. By assumption, $j' = \tilde{\jmath}'$, hence $f_i(0) = h_i(0)$ for $r+1 \leqslant i \leqslant r+n$.

Now we want to find an automorphism $g(t) \in \mathcal{O}_r \mathbin{\hat{\otimes}} \operatorname{Aut} \mathcal{L}'$ such that $g(0) = \mathrm{id}$ and $g(t) \circ (D_i + f_i(t)) = (D_i + h_i(t)) \circ g(t)$. This equation is equivalent to (12.6), and it is easy to see that all conditions of Proposition 12.1 are satisfied: for example, the system (12.2) is compatible because j is a homomorphism. This completes the proof. ∎

Now given the embedding $j'\colon \mathfrak{d}' \to \operatorname{Der} \mathcal{L}'$ we can consider the reconstruction $L' := \mathcal{C}(\mathcal{L}')$ which is a Lie H'-pseudoalgebra, where $H' = U(\mathfrak{d}')$. Given L' we can take the current H-pseudoalgebra $L := \operatorname{Cur} L' = H \otimes_{H'} L'$. Since its annihilation Lie algebra $\mathcal{A}(L)$ is isomorphic to $\mathcal{L}$, we get an embedding $\tilde{\jmath}\colon \mathfrak{d} \hookrightarrow \operatorname{Der} \mathcal{L}$. It induces the same embedding j' as our initial j, so by the previous lemma j and $\tilde{\jmath}$ are equivalent. But then the reconstruction $\mathcal{C}(\mathcal{L})$ of $\mathcal{L}$ provided with j is isomorphic to the reconstruction of $\mathcal{L}$ provided with $\tilde{\jmath}$, which is L. This can be summarized as follows.

LEMMA 12.5. *The reconstruction* $\mathcal{C}(\mathcal{L})$ *of a current Lie algebra* $\mathcal{L} = \mathcal{O}_r \mathbin{\hat{\otimes}} \mathcal{L}'$ *over a simple linearly compact Lie algebra* $\mathcal{L}'$, *provided with a transitive action of a Lie algebra* $\mathfrak{d}$, *is a current Lie* H-*pseudoalgebra over the* H'-*pseudoalgebra* $\mathcal{C}(\mathcal{L}')$, *where* $H' = U(\mathfrak{d}')$ *and* $\mathfrak{d}'$ *is a Lie subalgebra of* $\mathfrak{d}$ *of codimension* r.

12.6. *Reconstruction from* S_N

Now consider S_N with a transitive action of $\mathfrak{d}$ on it. Since $\operatorname{Der} S_N = CS_N \subset W_N$, we have $\mathfrak{d} \hookrightarrow W_N$. After an automorphism of W_N, we can assume $\mathfrak{d} \hookrightarrow W_N$ is the canonical embedding, while S_N becomes $W_N(\omega)$ $(\equiv S_N(\omega))$ where $\omega \in \Omega_X^N$ is a volume form. We can write $\omega = y \otimes_H v$ with $y \in X$ and $v \in \bigwedge^N \mathfrak{d}^*$. Then, by Lemma 12.2, the reconstruction of $W_N(\omega)$ is $W(\mathfrak{d}, v, I)$ where $I=\{h\in H \mid yh=0\}$ is as before.

The action of $W(\mathfrak{d})$ on v is given by (8.8). In the notation of Section 8.4, we have for $\alpha\in W(\mathfrak{d})$

$$\alpha * v = -(\mathrm{div}^{\mathrm{tr\,ad}}(\alpha)\otimes 1+\alpha)\otimes_H v.$$

This shows that $\alpha\in W(\mathfrak{d}, v, I)$ iff $\mathrm{div}^{\mathrm{tr\,ad}}(\alpha)\otimes 1+\alpha\in H\otimes I$.

Note that, since $\omega\neq 0$, we have $I\cap\mathbf{k}=0$. The intersection $I\cap(\mathfrak{d}+\mathbf{k})$ is a Lie algebra. The projection $\pi\colon(\mathfrak{d}+\mathbf{k})\to\mathfrak{d}$ is a Lie algebra homomorphism, which maps $I\cap(\mathfrak{d}+\mathbf{k})$ isomorphically onto a subalgebra $\mathfrak{d}'$ of $\mathfrak{d}$. The inverse isomorphism $\mathfrak{d}'\to I\cap(\mathfrak{d}+\mathbf{k})$ is given by $a\mapsto a+\chi(a)$ for some linear functional $\chi\colon\mathfrak{d}'\to\mathbf{k}$ which vanishes on $[\mathfrak{d}',\mathfrak{d}']$. Conversely, any such χ gives rise to an isomorphism as above.

For $\beta\in H\otimes(\mathfrak{d}+\mathbf{k})$, the equation $\beta\in H\otimes I$ is equivalent to the following two conditions: $(\mathrm{id}\otimes\pi)(\beta)\in H\otimes\mathfrak{d}'$ and $(\mathrm{id}\otimes\pi+\mathrm{id}\otimes\chi\pi)(\beta)=\beta$. Applying this for $\beta=\mathrm{div}^{\mathrm{tr\,ad}}(\alpha)\otimes 1+\alpha$, we get $(\mathrm{id}\otimes\pi)(\beta)=\alpha\in H\otimes\mathfrak{d}'$ and $(\mathrm{id}\otimes\pi+\mathrm{id}\otimes\chi\pi)(\beta)=\alpha+(\mathrm{id}\otimes\chi)(\alpha)=\beta$. The latter equation is equivalent to $(\mathrm{id}\otimes\chi)(\alpha)=\mathrm{div}^{\mathrm{tr\,ad}}(\alpha)\otimes 1$, i.e. to $\mathrm{div}^{\mathrm{tr\,ad}-\chi}(\alpha)=0$. We have proved

LEMMA 12.6. *The reconstruction of the Lie algebra S_N, provided with a transitive action of a Lie algebra $\mathfrak{d}$, is a current Lie H-pseudoalgebra over $S(\mathfrak{d}',\chi')$ where $\mathfrak{d}'$ is a Lie subalgebra of $\mathfrak{d}$ and χ' is a linear functional $\mathfrak{d}'\to\mathbf{k}$ which vanishes on $[\mathfrak{d}',\mathfrak{d}']$.*

In fact, one can show that in this case $\mathfrak{d}'=\mathfrak{d}$, i.e., $\dim I\cap(\mathfrak{d}+\mathbf{k})=N$, but the above statement is sufficient for our purposes.

12.7. *Reconstruction from K_N and H_N*

Now let $\mathscr{L}$ be one of the Lie algebras K_N or P_N, together with a transitive action of $\mathfrak{d}$ on it. We know from Section 6 that as a topological vector space $\mathscr{L}$ is homeomorphic to X. Since, by Proposition 6.2, all transitive actions of $\mathfrak{d}$ on X are equivalent, $\mathscr{L}$ is isomorphic to the "canonical" H-module X, i.e., we may assume that the embedding $\mathfrak{d}\hookrightarrow W_N=\mathrm{Der}\,X$ is the canonical one (Section 12.1). Then, by Proposition 11.2, the reconstruction of $\mathscr{L}$ is isomorphic to H as an H-module. In other words, $\mathscr{C}(\mathscr{L})$ is a free H-module of rank one.

LEMMA 12.7. *The reconstruction of the Lie algebras K_N and H_N, provided with a transitive action of a Lie algebra $\mathfrak{d}$, is a free H-module of rank one.*

Proof. It is enough to show that the reconstruction functor $\mathscr{C}$ gives the same result on the topological H-modules $X=P_N$ and $X/\mathbf{k}=H_N$. In order to do so, we must show that every H-linear continuous homomorphism of X to $X/\mathbf{k}$ can be obtained from a unique H-linear continuous homomorphism of X to itself by composing with the canonical projection $X \to X/\mathbf{k}$.

Since $X/\mathbf{k}$ is linearly compact, by Remark 6.1 there is a bijection between $\mathrm{Hom}_H^{\mathrm{cont}}(X, X/\mathbf{k})$ and $\mathrm{Hom}_H((X/\mathbf{k})^*, H)$. The space $(X/\mathbf{k})^*$ is nothing but the augmentation ideal $H_+ = \ker \varepsilon \subset H$. Therefore we are reduced to show that every H-linear map $\phi\colon H_+ \to H$ is a restriction of a unique H-linear map $H \to H$.

An H-linear $\phi\colon H_+ \to H$ is determined by its value on $\mathfrak{d} \subset H_+$. If $a, b \in \mathfrak{d}$, then $ab - ba = [a, b]$, hence $a\phi(b) - b\phi(a) = \phi([a, b])$. Let d be the maximal degree of $\phi(a)$ for $a \in \mathfrak{d}$. Then $a\phi(b) = b\phi(a)$ modulo $\mathrm{F}^d H$. This means that there exists some $\alpha \in \mathrm{F}^{d-1} H$ such that for every $a \in \mathfrak{d}$, $\phi(a) = a\alpha$ modulo $\mathrm{F}^{d-1} H$. Then the difference between ϕ and right multiplication by α is still H-linear, and its maximal degree on elements from $\mathfrak{d}$ is strictly less than d. The proof now follows by induction. ∎

All Lie pseudoalgebras that are free H-modules of rank one are classified in Theorem 8.1: they are isomorphic to current pseudoalgebras over $K(\mathfrak{d}', \theta')$ or $H(\mathfrak{d}', \chi', \omega')$.

13. STRUCTURE THEORY OF LIE PSEUDOALGEBRAS

13.1. *Structural Correspondence between a Lie Pseudoalgebra and Its Annihilation Algebra*

Recall that a Lie H-pseudoalgebra L is called *finite* if it is finitely generated as an H-module. If H is Noetherian (e.g., $H = U(\mathfrak{d})$ for a finite-dimensional Lie algebra $\mathfrak{d}$) and L is finite, then L is a Noetherian H-module, i.e., every increasing sequence of H-submodules of L stabilizes.

For any two subspaces A and B of L, let

$$[A, B] = \mathrm{span}_{\mathbf{k}} \{[a_x b] \mid a \in A, b \in B, x \in X\}. \tag{13.1}$$

Define the *derived series* of L by $L^{(0)} = L$, $L^{(1)} = [L, L]$, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$. A Lie pseudoalgebra L is called *solvable* if $L^{(n)} = 0$ for some n. Similarly, define the *central series* of L by $L^0 = L$, $L^1 = [L, L]$, $L^{n+1} = [L^n, L]$. The Lie pseudoalgebra L is called *nilpotent* if $L^n = 0$ for some n. As usual, L is called *abelian* if $[L, L] = 0$, i.e., if $[a * b] = 0$ for all $a, b \in L$.

A Lie pseudoalgebra L is called *simple* if it contains no nontrivial ideals and is not abelian. Note that $[L, L]$ is an ideal of L, so in particular, $[L, L] = L$ if L is simple. L is called *semisimple* if it contains no nonzero abelian ideals.

We define the *radical* of L, $\mathrm{Rad}\, L$, to be its maximal solvable ideal. When H is Noetherian and L is finite, $\mathrm{Rad}\, L$ exists because of the Noetherianity of L and part (ii) of the next lemma.

LEMMA 13.1. (i) *If S is a solvable ideal in L and L/S is solvable, then L is solvable.*

(ii) *If S_1, S_2 are solvable ideals in L, then their sum $S_1 + S_2$ is a solvable ideal.*

(iii) *$L/\mathrm{Rad}\, L$ is semisimple. L is semisimple iff $\mathrm{Rad}\, L = 0$.*

Proof. (i) is standard.

(ii) follows from (i) and the fact that $(S_1 + S_2)/S_1 \simeq S_2/(S_1 \cap S_2)$.

(iii) If $L/\mathrm{Rad}\, L$ has an abelian ideal I, then the preimage of I under the natural projection $L \to L/\mathrm{Rad}\, L$ must be solvable and strictly bigger than $\mathrm{Rad}\, L$, which is a contradiction. ∎

It is easy to see, using (9.23), (9.24), that for any two subspaces $A, B \subset L$, we have

$$[X \otimes_H A, X \otimes_H B] = X \otimes_H [A, B] \tag{13.2}$$

as subspaces of $\mathscr{A}(L) = X \otimes_H L$. In particular, if I is an ideal of L, then $X \otimes_H I$ is an ideal of $\mathscr{A}(L)$. We will call an ideal of $\mathscr{A}(L)$ *regular* if it is of the form $X \otimes_H I$ for some ideal I of L.

LEMMA 13.2. *Let L be a Lie H-pseudoalgebra and $I \subset L$ be an ideal. Then*

(i) *$X \otimes_H I = 0$ only if I is central.*

(ii) *$X \otimes_H I = \mathscr{A}(L)$ only if $[L, L] \subset I$.*

Proof. (i) has already been proved, when L is finite, in Corollary 11.1 and Remark 11.2. In the general case, it can be deduced from Proposition 9.1. Let $a \in I$, then $a_x \equiv x \otimes_H a = 0$ for any $x \in X$. Hence the action of a_x on L is trivial, and by (9.22), $[a * b] = 0$ for any $b \in L$.

In order to prove (ii), notice that $X \otimes_H L/I = 0$. Then build a Lie H-pseudoalgebra structure on $\tilde{L} = L \oplus L/I$ by letting L act on the abelian ideal L/I via the adjoint action. Then by part (i), L/I is central in $\tilde{L}$, hence L acts trivially on L/I. This means $[L, L] \subset I$. ∎

Using this lemma and (13.2), it is easy to prove the next two results.

PROPOSITION 13.1. *A Lie pseudoalgebra L is solvable (respectively nilpotent) if and only if its annihilation Lie algebra $\mathscr{A}(L)$ is.*

PROPOSITION 13.2. *Let L be a centerless Lie H-pseudoalgebra which is equal to its derived subalgebra $[L, L]$. Then L is simple if either of the following conditions holds*

(i) *$\mathscr{A}(L)$ has no nontrivial H-invariant ideals.*

(ii) *L is finite or free, and $\mathscr{A}(L)$ has no non-central H-invariant ideals.*

Proof. (i) is immediate from Lemma 13.2.

Assume that (ii) holds but L is not simple. Then $\mathscr{A}(L)$ has a nontrivial central regular ideal. If $a_x \equiv x \otimes_H a$ is central in $\mathscr{A}(L)$ for every $x \in X$, then by (9.24), $[a_x b]_y = 0$ for every $b \in L$, $x, y \in X$. When L is either finite or free, $l_y = 0$ for all $y \in X$ if and only if $l = 0$ (cf. Corollary 11.1). Therefore $[a_x b] = 0$ for all $b \in L$, $x \in X$, and by (9.22) we get $[a * b] = 0$ for any $b \in L$. Hence $a = 0$. ∎

As an immediate consequence we obtain

COROLLARY 13.1. *Let L be a current Lie H-pseudoalgebra over a finite-dimensional simple Lie algebra or over one of the primitive pseudoalgebras of vector fields. Then L is simple.*

Proof. It is easy to check that L satisfies the assumptions of Proposition 13.2 (see Theorem 8.2). ∎

The following proposition will play an important role in the classification of finite simple Lie pseudoalgebras.

PROPOSITION 13.3. *For any Lie H-pseudoalgebra L, any non-central H-invariant ideal J of $\mathscr{A}(L)$ contains a nonzero regular ideal.*

Proof. Let $\alpha \in J$ be non-central. Assume that $X \otimes_H \alpha \cdot l = 0$ for all $l \in L$. Note that by Proposition 9.1, we have $h(\alpha \cdot l) = (h_{(1)}\alpha) \cdot (h_{(2)} l)$ for $h \in H$. This implies $(h\alpha) \cdot l = h_{(1)}(\alpha \cdot (h_{(-2)} l))$, which gives $X \otimes_H (h\alpha) \cdot l = 0$ for any $h \in H$, $l \in L$. Then we can use (9.30) to show that α is central in $\mathscr{A}(L)$, which is a contradiction.

Therefore, there is some $l \in L$ such that $\alpha \cdot l = a$ has a nonzero Fourier coefficient, i.e., $X \otimes_H a \neq 0$. Since $a_y = (\alpha \cdot l)_y = \sum_i [h_i \alpha, l_{yx_i}]$, and J is H-stable, we see that all Fourier coefficients of a lie in J. Then, due to (9.24), all elements in the ideal (a) of L generated by a have all of their Fourier coefficients in J, i.e., $0 \neq X \otimes_H (a) \subset J$. ∎

3.2. *Annihilation Algebras of Finite Simple Lie $U(\mathfrak{d})$-Pseudoalgebras*

We will now approach the problem of classification of all finite simple Lie H-pseudoalgebras. In view of Kostant's Theorem 2.1 and the results of Section 5, we will first restrict ourselves to the case when H is the universal enveloping algebra of a Lie algebra $\mathfrak{d}$. Moreover, we will assume that $\mathfrak{d}$ is finite dimensional; in this case $H = U(\mathfrak{d})$ is filtered by finite-dimensional subspaces. The classification is done in two steps: the first one (done in this subsection) is classifying all Lie algebras that can arise as $\mathscr{A}(L)$ for some finite simple Lie H-pseudoalgebra L, the second step (done in the next subsection) involves a reconstruction of L from its annihilation Lie algebra $\mathscr{A}(L)$ and the action of H on it.

THEOREM 13.1. *If L is a finite simple Lie $H = U(\mathfrak{d})$-pseudoalgebra, then its annihilation Lie algebra $\mathscr{A}(L)$ is isomorphic (as a topological Lie algebra) to an irreducible central extension of a current Lie algebra $\mathcal{O}_r \hat{\otimes} \mathfrak{s}$ where $\mathfrak{s}$ is a simple linearly compact Lie algebra of growth* $\operatorname{gw} \mathfrak{s} = \dim \mathfrak{d} - r$ *(see Remark 7.3 for the definition of* $\operatorname{gw} \mathfrak{s}$).

Proof. First of all, we observe that $\mathscr{L} = \mathscr{A}(L)$ is a linearly compact Lie algebra with respect to the topology defined in Section 7.4, see Proposition 7.4(ii). Consider the *extended annihilation algebra* $\mathscr{L}^e = \mathfrak{d} \ltimes \mathscr{L}$, obtained by letting $\mathfrak{d}$ act on $\mathscr{L} = \mathscr{A}(L)$ according to its $H = U(\mathfrak{d})$-module structure.

LEMMA 13.3. *$\mathscr{L}^e = \mathfrak{d} \ltimes \mathscr{L}$ is a linearly compact Lie algebra possessing a fundamental subalgebra, i.e., an open subalgebra containing no ideals of $\mathscr{L}^e$.*

Proof. Indeed, if L_0 is a finite-dimensional subspace of L generating it over H, then because of (7.14), $\mathscr{L}_i = \mathrm{F}_i X \otimes_H L_0$ is a subalgebra of $\mathscr{L}$ for $i \geqslant s$. None of $\mathscr{L}_i$ contains ideals of $\mathfrak{d} \ltimes \mathscr{L}$, since every such ideal is stable under the action of H and $H \cdot \mathrm{F}_i X = X$, which implies $H \cdot \mathscr{L}_i = \mathscr{L}$. ∎

The center Z of $\mathscr{L}$ is an H-stable closed ideal. The quotient $\mathscr{L}^e/Z = \mathfrak{d} \ltimes (\mathscr{L}/Z)$ is a linearly compact Lie algebra possessing a fundamental subalgebra $\mathscr{L}_s/(Z \cap \mathscr{L}_s)$. Theorem 13.1 will be deduced from Proposition 6.3 applied for $\bar{\mathscr{L}}^e := \mathscr{L}^e/Z$.

By Proposition 13.3, any nonzero H-stable ideal of $\bar{\mathscr{L}} := \mathscr{L}/Z$ contains the image of a nonzero regular ideal of $\mathscr{L}$. Since L is simple, this means that the only nonzero H-stable ideal of $\bar{\mathscr{L}}$ is the whole $\bar{\mathscr{L}}$. Then every nonzero ideal of $\bar{\mathscr{L}}^e$ contained in $\bar{\mathscr{L}}$ must equal $\bar{\mathscr{L}}$. Hence $\bar{\mathscr{L}}$ is a minimal closed ideal of a linearly compact Lie algebra satisfying the assumptions of Proposition 6.3(i), and is therefore of the form stated in part (ii) of this proposition.

Therefore, $\mathscr{L}$ is a central extension of a current Lie algebra over a simple linearly compact Lie algebra. Moreover, $\mathscr{L}$ equals its derived subalgebra due to (13.2). Hence it is an irreducible central extension.

Consider the canonical filtration $\mathrm{F}_n(\mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s}) := \sum_i \mathrm{F}_{n-i}\,\mathcal{O}_r \mathbin{\hat{\otimes}} \mathrm{F}_i\,\mathfrak{s}$, where $\mathrm{F}_i\,\mathfrak{s}$ is the canonical filtration of $\mathfrak{s}$ defined in Section 6 (if $\dim \mathfrak{s} < \infty$ we put $\mathrm{F}_i\,\mathfrak{s} = 0$ for $i \geqslant 0$). Then the growth of $\mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s}$ (with respect to this filtration) equals $\operatorname{gw} \mathcal{O}_r + \operatorname{gw} \mathfrak{s} = r + \operatorname{gw} \mathfrak{s}$. It is clear from Proposition 6.4 that any irreducible central extension of $\mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s}$ has the same growth. On the other hand, with respect to the filtration defined by (7.11), the growth of $\mathscr{L}$ is equal to $N = \dim \mathfrak{d}$ (see Proposition 7.5). We have to show that the two different filtrations give the same growth.

Recall that by Lemma 7.3, a sufficiently high power of any nonzero element $a \in \mathfrak{d}$ maps any given open subspace of $\mathscr{L}$ surjectively onto $\mathscr{L}$. Then the same argument as in the proof of Lemma 12.1 shows that $\mathfrak{d} \hookrightarrow \operatorname{Der} \mathscr{L}$ intersects $\mathrm{F}_0(\operatorname{Der} \mathscr{L})$ trivially, where $\mathrm{F}_0(\operatorname{Der} \mathscr{L})$ is induced by the canonical filtration on $\mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s}$. This implies $N \leqslant r + \operatorname{gw} \mathfrak{s}$.

To show the inverse inequality, note that since $\mathrm{F}_0(\mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s})$ is open in $\bar{\mathscr{L}} = \mathscr{L}/Z \simeq \mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s}$, it contains some $\bar{\mathscr{L}}_m := \mathscr{L}_m/(Z \cap \mathscr{L}_m)$. We have $[\bar{\mathscr{L}}_i, \mathfrak{d}] \subset \bar{\mathscr{L}}_{i-1}$ due to (7.13), and hence

$$\bar{\mathscr{L}}_{i+1} = \{a \in \bar{\mathscr{L}}_i \mid [a, \bar{\mathscr{L}}^e] \subset \bar{\mathscr{L}}_i\} \tag{13.3}$$

due to (6.1). Now (7.13) and (7.14) imply $[\bar{\mathscr{L}}_i, \bar{\mathscr{L}}^e] \subset \bar{\mathscr{L}}_{i-s-1}$, which together with (13.3) leads to $\bar{\mathscr{L}}_{m+n(s+1)} \subset \mathrm{F}_n(\mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s})$ for all $n \geqslant 0$. This implies $N \geqslant r + \operatorname{gw} \mathfrak{s}$.

This completes the proof of Theorem 13.1. ∎

In fact, the above arguments can be used to prove a stronger statement than Theorem 13.1.

COROLLARY 13.2. *Let L be a finite Lie H-pseudoalgebra and M be a minimal nonabelian ideal of L. Then the annihilation algebra of M is one of the Lie algebras described in Theorem* 13.1.

Proof. The only place in the proof of Theorem 13.1 where we used the simplicity of L was where we deduced that any nonzero regular ideal of $\mathscr{A}(L)$ must equal the whole $\mathscr{A}(L)$. This argument is modified as follows. Let $J = X \otimes_H I$ be a nonabelian regular ideal of $\mathscr{A}(L)$ contained in $\mathscr{A}(M)$. Then the minimality of M implies that $I = M$ and $J = \mathscr{A}(M)$. The proof is concluded again by applying Proposition 6.3. ∎

13.3. *Classification of Finite Simple Lie $U(\mathfrak{d})$-Pseudoalgebras*

We will call a *pseudoalgebra of vector fields* any subalgebra of the Lie pseudoalgebra $W(\mathfrak{d})$. As in Section 8, a pseudoalgebra of vector fields is called *primitive* if it is one of the following: $W(\mathfrak{d})$, $S(\mathfrak{d}, \chi)$, $H(\mathfrak{d}, \chi, \omega)$ or $K(\mathfrak{d}, \theta)$ (then its annihilation algebra $\mathscr{A}(L)$ is isomorphic to one of the Lie algebras W_N, S_N, P_N, or K_N).

The following is the main theorem of this section.

THEOREM 13.2. *Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$. Then any finite simple Lie H-pseudoalgebra L is isomorphic to a current pseudoalgebra over a finite-dimensional simple Lie algebra or over one of the primitive pseudoalgebras of vector fields.*

Explicitly, $L \simeq \operatorname{Cur}_{H'}^{H} L'$, *where* $H' = U(\mathfrak{d}')$, $\mathfrak{d}'$ *is a subalgebra of* $\mathfrak{d}$, *and* L' *is one of the following*

(a) L' *is a finite-dimensional simple Lie algebra and* $\mathfrak{d}' = 0$;

(b) $L' = W(\mathfrak{d}')$, $\mathfrak{d}'$ *is arbitrary*;

(c) $L' = S(\mathfrak{d}', \chi')$, *where* $\mathfrak{d}'$ *is arbitrary and* $\chi' \in (\mathfrak{d}')^*$ *is such that* $\chi'([\mathfrak{d}', \mathfrak{d}']) = 0$;

(d) $L' = H(\mathfrak{d}', \chi', \omega')$, *where* $N' = \dim \mathfrak{d}'$ *is even,* χ' *is as in* (c), *and* $\omega' \in \bigwedge^2 (\mathfrak{d}')^*$ *is such that* $(\omega')^{N'/2} \neq 0$ *and* $\mathrm{d}\omega' + \chi' \wedge \omega' = 0$;

(e) $L' = K(\mathfrak{d}', \theta')$, *where* $N' = \dim \mathfrak{d}'$ *is odd and* $\theta' \in (\mathfrak{d}')^*$ *is such that* $\theta' \wedge (\mathrm{d}\theta')^{(N'-1)/2} \neq 0$.

Proof. By Theorem 13.1, the annihilation algebra $\mathscr{L}$ of L is an irreducible central extension of a current Lie algebra $\bar{\mathscr{L}} = \mathcal{O}_r \mathbin{\hat{\otimes}} \bar{\mathfrak{s}}$, where $\bar{\mathfrak{s}}$ is a simple linearly compact Lie algebra of growth $N' = N - r$. We have surjective maps

$$\mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s} \to \mathscr{L} \to \mathcal{O}_r \mathbin{\hat{\otimes}} \bar{\mathfrak{s}}, \tag{13.3}$$

where $\mathfrak{s}$ is the universal central extension of $\bar{\mathfrak{s}}$. By Theorem 6.1, $\bar{\mathfrak{s}}$ is either finite dimensional (when $N' = 0$) or one of the Lie algebras $W_{N'}$, $S_{N'}$, $H_{N'}$ or $K_{N'}$. By Proposition 6.4, we have $\mathfrak{s} = \bar{\mathfrak{s}}$ in all cases, except $\bar{\mathfrak{s}} = H_{N'}$ in which case the center of $\mathfrak{s} = P_{N'}$ is 1-dimensional.

Note that $\operatorname{Der} \mathfrak{s} = \operatorname{Der} \bar{\mathfrak{s}}$, and therefore, by Proposition 6.4, we have $\operatorname{Der}(\mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s}) = \operatorname{Der}(\mathcal{O}_r \mathbin{\hat{\otimes}} \bar{\mathfrak{s}})$. This implies $\operatorname{Der}(\mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s}) = \operatorname{Der} \mathscr{L} = \operatorname{Der}(\mathcal{O}_r \mathbin{\hat{\otimes}} \bar{\mathfrak{s}})$. Then the action of $\mathfrak{d}$ on $\mathscr{L}$ induces actions on $\mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s}$ and $\mathcal{O}_r \mathbin{\hat{\otimes}} \bar{\mathfrak{s}}$. The argument from the proof of Lemma 12.1 shows that these actions are transitive.

Now, let us apply the reconstruction functor $\mathscr{C}$ to the maps in (13.3). By Theorem 12.1, $\mathscr{C}(\mathcal{O}_r \mathbin{\hat{\otimes}} \mathfrak{s}) \simeq \operatorname{Cur}_{H'}^{H} \mathscr{C}(\mathfrak{s})$, and $S := \mathscr{C}(\mathfrak{s})$ is one of the Lie pseudoalgebras described in (a)–(e) above. Moreover, by Lemma 12.7, we have $\mathscr{C}(\bar{\mathfrak{s}}) \simeq \mathscr{C}(\mathfrak{s}) = S$, and hence $\mathscr{C}(\mathcal{O}_r \mathbin{\hat{\otimes}} \bar{\mathfrak{s}}) \simeq \operatorname{Cur}_{H'}^{H} S$. We therefore obtain H-linear maps $\operatorname{Cur}_{H'}^{H} S \to \hat{L} \to \operatorname{Cur}_{H'}^{H} S$ whose composition is the identity. Hence $\hat{L} := \mathscr{C}(\mathscr{L})$ is isomorphic to $\operatorname{Cur}_{H'}^{H} S$, which is a simple Lie pseudoalgebra (Corollary 13.1).

The homomorphism $\Phi\colon L \to \hat{L}$ given by (11.6) is injective because L is centerless (Remark 11.2). The action of $\hat{L}$ on L built in Section 11.4 shows that the image of Φ is an ideal of $\hat{L}$. Since $\hat{L}$ is simple, it follows that Φ is an isomorphism. ∎

Corollary 13.2 and the above proof imply the following result.

COROLLARY 13.3. *Let L be a finite Lie pseudoalgebra and M be a minimal nonabelian ideal of L. Then M is a simple Lie pseudoalgebra.*

LEMMA 13.4. *If L is a centerless Lie pseudoalgebra, then any nonzero finite ideal of L contains a nonzero minimal ideal.*

Proof. By Zorn's Lemma, it is enough to show that $\bigcap I_\alpha \neq 0$ for any collection of finite ideals $\{I_\alpha\}_{\alpha\in A}$ of L such that $I_\alpha \subset I_\beta$ for $\alpha<\beta$, where A is a totally ordered index set. Assume that $\bigcap I_\alpha = 0$. Then there is a chain of ideals $\{I_\alpha\}_{\alpha\in A'}$ $(A'\subset A)$ of the same rank whose intersection is zero. Fix some $\alpha_0 \in A'$. Then for any $\beta\in A'$, $\beta<\alpha_0$, the module I_{α_0}/I_β is torsion, so by Corollary 10.1, L acts trivially on it. This implies $[L, I_{\alpha_0}]\subset I_\beta$ for each such β, hence I_{α_0} is central. ∎

13.4. *Derivations of Finite Simple Lie $U(\mathfrak{d})$-Pseudoalgebras*

We will determine all derivations of a finite simple Lie $H=U(\mathfrak{d})$-pseudoalgebra L (see Definition 10.2).

First let us consider the case when $L=\operatorname{Cur}\mathfrak{g} := H\otimes\mathfrak{g}$ is a current pseudoalgebra over a finite-dimensional Lie algebra $\mathfrak{g}$. The Lie pseudoalgebra $W(\mathfrak{d})$ acts on L by just acting on the first factor in $H\otimes\mathfrak{g}$ (cf. (8.4)):

$$(f\otimes a) * (g\otimes b) = -(f\otimes ga)\otimes_H (1\otimes b), \qquad f, g\in H,\ a\in\mathfrak{d},\ b\in\mathfrak{g}. \tag{13.4}$$

We also have an embedding $\operatorname{Cur}\operatorname{Der}\mathfrak{g}\subset\operatorname{Der} L$. The image of $\operatorname{Cur}\operatorname{Der}\mathfrak{g}$ in $\operatorname{Der} L$ is normalized by that of $W(\mathfrak{d})$, and the two form a semidirect sum $W(\mathfrak{d})\ltimes\operatorname{Cur}\operatorname{Der}\mathfrak{g}$ which as an H-module is isomorphic to $H\otimes(\mathfrak{d}\oplus\operatorname{Der}\mathfrak{g})$.

PROPOSITION 13.4. *For any simple finite-dimensional Lie algebra $\mathfrak{g}$, we have* $\operatorname{Der}\operatorname{Cur}\mathfrak{g} = W(\mathfrak{d})\ltimes\operatorname{Cur}\mathfrak{g}$.

Proof. By Lemma 10.2(iii), the annihilation algebra $\mathscr{A}(\operatorname{Der}\operatorname{Cur}\mathfrak{g})\subset\operatorname{Der}\mathscr{A}(\operatorname{Cur}\mathfrak{g}) = \operatorname{Der}(X\otimes\mathfrak{g})$. By Proposition 6.4(ii), the latter is isomorphic to $W_N\otimes 1+\mathcal{O}_N\otimes\mathfrak{g}$, since $X\simeq\mathcal{O}_N$. Then $\mathscr{C}\mathscr{A}(\operatorname{Der}\operatorname{Cur}\mathfrak{g})\subset\mathscr{C}(\operatorname{Der}(X\otimes\mathfrak{g})) = W(\mathfrak{d})+\operatorname{Cur}\mathfrak{g}$ (see Theorem 12.1). Now by Lemma 11.5, $\operatorname{Der}\operatorname{Cur}\mathfrak{g}\subset\mathscr{C}\mathscr{A}(\operatorname{Der}\operatorname{Cur}\mathfrak{g})\subset W(\mathfrak{d})+\operatorname{Cur}\mathfrak{g}$. ∎

A similar argument as in the proof of the proposition shows that $\operatorname{Der} L = L$ when L is one of the primitive pseudoalgebras of vector fields $W(\mathfrak{d})$, $S(\mathfrak{d},\chi)$, $H(\mathfrak{d},\chi,\omega)$ or $K(\mathfrak{d},\theta)$. In fact, the same holds when L is a current pseudoalgebra over one of them.

PROPOSITION 13.5. *Let L be a simple pseudoalgebra of vector fields* (*i.e., L is a current pseudoalgebra over one of the primitive ones*). *Then* $\operatorname{Der} L = L$.

Proof. Let L be a current pseudoalgebra over L', and $L' \subset W(\mathfrak{d}')$ be one of the primitive pseudoalgebras of vector fields, where $\mathfrak{d}'$ is a Lie subalgebra of $\mathfrak{d}$. Then, by Theorem 8.2, the annihilation algebra $\mathscr{L} = \mathscr{A}(L)$ is a current Lie algebra over $\mathscr{L}' = \mathscr{A}(L')$: $\mathscr{L} = \mathcal{O}_r \mathbin{\hat{\otimes}} \mathscr{L}'$, and $\mathscr{L}'$ is isomorphic to $W_{N'}$, $S_{N'}$, $P_{N'}$, or $K_{N'}$, where $N' = \dim \mathfrak{d}' = N - r$, $N = \dim \mathfrak{d}$.

As in the proof of Proposition 13.4, we have $\operatorname{Der} L \subset \mathscr{C}\mathscr{A}(\operatorname{Der} L) \subset \mathscr{C}(\operatorname{Der} \mathscr{L})$. By Proposition 6.4, we have $\operatorname{Der} \mathscr{L} = W_r \otimes 1 + \mathcal{O}_r \mathbin{\hat{\otimes}} \operatorname{Der} \mathscr{L}'$, and $\operatorname{Der} \mathscr{L}' =$ $W_{N'}$, $CS_{N'}$, $CH_{N'}$, or $K_{N'}$ is a Lie subalgebra of $W_{N'}$. In particular, we see that $\operatorname{Der} \mathscr{L} \subset W_N$, and hence $\mathscr{C}(\operatorname{Der} \mathscr{L})$ is a subalgebra of $W(\mathfrak{d})$.

So, we have: $\operatorname{Der} L \subset W(\mathfrak{d})$, $L = \operatorname{Cur} L' \subset \operatorname{Cur} W(\mathfrak{d}') \simeq H \otimes \mathfrak{d}' \subset W(\mathfrak{d})$. Take any two nonzero elements $a \in W(\mathfrak{d})$, $b \in H \otimes \mathfrak{d}'$. Then we claim that $[a * b] \in H \otimes \mathfrak{d}'$ implies $a \in H \otimes \mathfrak{d}'$. This follows easily from the definition (8.3), using that $\mathfrak{d}'$ is a subalgebra of $\mathfrak{d}$ (see the proof of Proposition 13.6 below for a similar argument).

Therefore $\operatorname{Der} L \subset \operatorname{Cur} W(\mathfrak{d}')$. For $a \in \operatorname{Cur} W(\mathfrak{d}')$, we can write $a = \sum h_i \otimes_{H'} a_i$ for some $a_i \in W(\mathfrak{d}')$ and $h_i \in H$ such that the classes $h_i H'$ are linearly independent in H/H'. Then if $a \in \operatorname{Der} L$, it is easy to see that all a_i must belong to $\operatorname{Der} L'$. Hence $\operatorname{Der} L = \operatorname{Cur} \operatorname{Der} L'$. But $\operatorname{Der} L' \subset \mathscr{C}(\operatorname{Der} \mathscr{L}') = L'$, so $\operatorname{Der} L = \operatorname{Cur} L' = L$. ∎

13.5. *Finite Semisimple Lie $U(\mathfrak{d})$-Pseudoalgebras*

Recall that a Lie H-pseudoalgebra L is called *semisimple* if it contains no nonzero abelian ideals. Let $H = U(\mathfrak{d})$, for a finite-dimensional Lie algebra $\mathfrak{d}$.

If $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, then by Proposition 13.4, we have $\operatorname{Der} \operatorname{Cur} \mathfrak{g} = W(\mathfrak{d}) \ltimes \operatorname{Cur} \mathfrak{g}$. It is easy to see that for any subalgebra A of the Lie pseudoalgebra $W(\mathfrak{d})$, the Lie pseudoalgebra $A \ltimes \operatorname{Cur} \mathfrak{g}$ is semisimple. Indeed, assume that $I \subset A \ltimes \operatorname{Cur} \mathfrak{g}$ is an abelian ideal. Then $I \cap \operatorname{Cur} \mathfrak{g}$ is an abelian ideal in $\operatorname{Cur} \mathfrak{g}$, hence $I \cap \operatorname{Cur} \mathfrak{g} = 0$. But this is impossible unless $I = 0$ because the pseudobracket of any element from $(W(\mathfrak{d}) + \operatorname{Cur} \mathfrak{g}) \backslash \operatorname{Cur} \mathfrak{g}$ with elements from $\operatorname{Cur} \mathfrak{g}$ gives nonzero elements from $\operatorname{Cur} \mathfrak{g}$ (see (13.4)). Note that this argument implies that any nonzero ideal of $A \ltimes \operatorname{Cur} \mathfrak{g}$ contains $\operatorname{Cur} \mathfrak{g}$.

Now we can classify all finite semisimple Lie $U(\mathfrak{d})$-pseudoalgebras.

THEOREM 13.3. *Any finite semisimple Lie $U(\mathfrak{d})$-pseudoalgebra L is a direct sum of finite simple Lie pseudoalgebras* (*described by Theorem* 13.2) *and of pseudoalgebras of the form $A \ltimes \operatorname{Cur} \mathfrak{g}$, where A is a subalgebra of $W(\mathfrak{d})$ and $\mathfrak{g}$ is a simple finite-dimensional Lie algebra.*

Proof. Consider the set $\{M_i\}$ of all minimal nonzero ideals of L. This set is nonempty by Lemma 13.4, and finite because L is a Noetherian

H-module. The adjoint action of L on M_i gives a homomorphism of Lie pseudoalgebras $L \to \operatorname{Der} M_i$, cf. Lemma 10.2(ii).

We claim that the direct sum of these homomorphisms is an injective map. Indeed, let $N \subset L$ be the set of all elements that act trivially on all M_i. This set is an ideal of L. If it is nonzero it must contain some minimal ideal M_i. But then this M_i is abelian, which contradicts the semisimplicity of L.

Therefore we have embeddings $\bigoplus M_i \subset L \subset \bigoplus \operatorname{Der} M_i$. By Corollary 13.3 all M_i are simple Lie pseudoalgebras. If M_i is not a current pseudoalgebra over a finite-dimensional Lie algebra, then by Proposition 13.5, $\operatorname{Der} M_i = M_i$. For $M_i = \operatorname{Cur} \mathfrak{g}$, we have $\operatorname{Der} \operatorname{Cur} \mathfrak{g} = W(\mathfrak{d}) \ltimes \operatorname{Cur} \operatorname{Der} \mathfrak{g}$. Any subalgebra of $W(\mathfrak{d}) \ltimes \operatorname{Cur} \mathfrak{g}$ containing $\operatorname{Cur} \mathfrak{g}$ is of the form $A \ltimes \operatorname{Cur} \mathfrak{g}$, where A is a subalgebra of $W(\mathfrak{d})$. ∎

Recall that a pseudoalgebra of vector fields is any subalgebra of the Lie pseudoalgebra $W(\mathfrak{d})$.

PROPOSITION 13.6. *For any two nonzero elements $a, b \in W(\mathfrak{d})$, we have $[a * b] \neq 0$. In particular, $W(\mathfrak{d})$ does not contain nonzero abelian elements, i.e., elements a such that $[a * a] = 0$.*

Proof. Let us write

$$a = \sum_i h_i \otimes \partial_i, \qquad b = \sum_j k_j \otimes \partial_j,$$

where $h_i, k_j \in H$ and $\{\partial_i\}$ is a basis of $\mathfrak{d}$. Denote by m (respectively n) the maximal degree of the h_i (respectively k_j). We have (cf. (8.3))

$$\begin{aligned}[][a * b] = &\sum_{i,j} (h_i \otimes k_j) \otimes_H (1 \otimes [\partial_i, \partial_j]) \\ &- \sum_{i,j} (h_i \otimes k_j \partial_i) \otimes_H (1 \otimes \partial_j) + \sum_{i,j} (h_i \partial_j \otimes k_j) \otimes_H (1 \otimes \partial_i).\end{aligned}$$

Assume that $[a * b] = 0$. Notice that only the third summand contains coefficients from $H \otimes H$ of degree $(m+1, n)$, hence it must be zero modulo $\mathrm{F}^m H \otimes \mathrm{F}^n H$. Since the ∂_i are linearly independent, the same is true for each term $\sum_j h_i \partial_j \otimes k_j$. If we choose i such that h_i is of degree exactly m, we get a contradiction. ∎

COROLLARY 13.4. *A finite Lie $U(\mathfrak{d})$-pseudoalgebra L contains no nonzero abelian elements iff it is a direct sum of pseudoalgebras of vector fields.*

Proof. Assume that L is not a direct sum of pseudoalgebras of vector fields. If L is not semisimple, then $\operatorname{Rad} L$ contains nonzero abelian elements. If L is semisimple, Theorem 13.3 implies that L contains a subalgebra of the form $A \ltimes \operatorname{Cur} \mathfrak{g}$ with $\mathfrak{g} \neq 0$, and therefore contains nonzero abelian elements (for example, $1 \otimes a$ for any $a \in \mathfrak{g}$).

The converse statement follows from Proposition 13.6. ∎

THEOREM 13.4. *Any pseudoalgebra of vector fields is simple.*

Proof. By Proposition 13.6, a pseudoalgebra L of vector fields does not contain nonzero abelian elements, and hence is semisimple. Then, by Theorem 13.3, L is a direct sum of finite simple Lie pseudoalgebras and of pseudoalgebras of the form $A \ltimes \operatorname{Cur} \mathfrak{g}$. In fact, there is only one summand, as $[a * b] \neq 0$ for any two nonzero elements $a, b \in W(\mathfrak{d})$. Furthermore, L cannot be of the form $A \ltimes \operatorname{Cur} \mathfrak{g}$ with $\mathfrak{g} \neq 0$, because $\operatorname{Cur} \mathfrak{g}$ contains nonzero abelian elements. ∎

COROLLARY 13.5. *Any finite semisimple Lie $U(\mathfrak{d})$-pseudoalgebra L is a direct sum of pseudoalgebras of the form $A \ltimes \operatorname{Cur} \mathfrak{g}$, where A is either 0 or one of the simple pseudoalgebras of vector fields* (*described by Theorem* 13.2), *and $\mathfrak{g}$ is either 0 or a simple finite-dimensional Lie algebra.*

We can also describe all ideals of a finite semisimple Lie pseudoalgebra L. By the above corollary, it is enough to consider the case $L = A \ltimes \operatorname{Cur} \mathfrak{g}$ with $A \neq 0$, $\mathfrak{g} \neq 0$.

PROPOSITION 13.7. *Let $L = A \ltimes \operatorname{Cur} \mathfrak{g}$ where A is a pseudoalgebra of vector fields and $\mathfrak{g}$ is a simple finite-dimensional Lie algebra. Then the only nonzero proper ideal of L is* $\operatorname{Cur} \mathfrak{g}$.

Proof. We have already noticed (see the paragraph before Theorem 13.3) that any nonzero ideal I of L contains $\operatorname{Cur} \mathfrak{g}$. Then $I/\operatorname{Cur} \mathfrak{g}$ is an ideal of $L/\operatorname{Cur} \mathfrak{g} \simeq A$, but A is simple by Theorem 13.4. ∎

13.6. *Homomorphisms between Finite Simple Lie $U(\mathfrak{d})$-Pseudoalgebras*

In this subsection, $H = U(\mathfrak{d})$ is again the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$.

THEOREM 13.5. *For any finite-dimensional Lie algebra $\mathfrak{g}$ and any pseudoalgebra of vector fields L, there are no nontrivial homomorphisms between L and* $\operatorname{Cur} \mathfrak{g}$.

Proof. Any homomorphism $\operatorname{Cur} \mathfrak{g} \to L$ leads to abelian elements in L, and therefore is zero (see Proposition 13.6).

Let f be a homomorphism from L to $\operatorname{Cur} \mathfrak{g}$. Then f induces a homomorphism of Lie algebras $\mathscr{A}(f)\colon \mathscr{A}(L) \to \mathscr{A}(\operatorname{Cur} \mathfrak{g})$. By Theorem 13.4, L is simple, so $L = \operatorname{Cur}_{H'}^{H} L'$ where L' is a primitive H'-pseudoalgebra of vector fields ($H' = U(\mathfrak{d}')$ and $\mathfrak{d}'$ is a subalgebra of $\mathfrak{d}$). By Theorem 8.2, the annihilation algebra $\mathscr{L} = \mathscr{A}(L)$ is isomorphic to a current Lie algebra $\mathcal{O}_r \hat{\otimes} \mathscr{L}'$ over $\mathscr{L}' = \mathscr{A}(L')$. Moreover, the quotient of $\mathscr{L}'$ by its center is infinite dimensional and simple. On the other hand, the annihilation algebra $\mathscr{A}(\mathrm{Cur}\,\mathfrak{g}) \simeq X \otimes \mathfrak{g}$ is a current Lie algebra over $\mathfrak{g}$, which is a projective limit of finite-dimensional Lie algebras $(X/\mathrm{F}_n X) \otimes \mathfrak{g}$. Hence the adjoint action of $\mathscr{L}' \equiv 1 \otimes \mathscr{L}'$ on $\mathscr{L}$ maps trivially to each of them via $\mathscr{A}(f)$. But since $[\mathscr{L}', \mathscr{L}] = \mathscr{L}$, this implies that each $\mathscr{L} \to (X/\mathrm{F}_n X) \otimes \mathfrak{g}$ is trivial. Therefore $\mathscr{A}(f) = 0$, and by Corollary 11.2, we get $f = 0$. ∎

THEOREM 13.6. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be finite-dimensional simple Lie algebras. Then any isomorphism f*: Cur $\mathfrak{g} \xrightarrow{\sim}$ Cur $\mathfrak{h}$ *maps* $1 \otimes \mathfrak{g}$ *onto* $1 \otimes \mathfrak{h}$, *and thus is induced by some isomorphism of Lie algebras $\mathfrak{g} \xrightarrow{\sim} \mathfrak{h}$. In particular,* Aut Cur $\mathfrak{g} \simeq$ Aut $\mathfrak{g}$.

Recall that $\mathscr{A}(\mathrm{Cur}\,\mathfrak{g}) \simeq X \otimes \mathfrak{g}$ is a current Lie algebra. In the proof of the theorem we are going to use the following lemma.

LEMMA 13.5. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and R be a commutative associative algebra. Then all ideals of $R \otimes \mathfrak{g}$ are of the form $I \otimes \mathfrak{g}$ where I is an ideal of R.*

Proof. As a $\mathfrak{g}$-module, $R \otimes \mathfrak{g}$ is isomorphic to a direct sum of several copies of $\mathfrak{g}$. Any ideal J of $R \otimes \mathfrak{g}$ is in particular a $\mathfrak{g}$-module, hence it is spanned over $\mathbf{k}$ by elements of the form $r \otimes a \in J$ where $r \in R$ and a is a root vector in $\mathfrak{g}$. If $r \neq 0$ is such that $r \otimes a \in J$ for some nonzero $a \in \mathfrak{g}$, then $r \otimes \mathfrak{g} \subset J$, since $\{a \in \mathfrak{g} \mid r \otimes a \in J\}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}$ is simple. Setting $I = \{r \in R \mid r \otimes \mathfrak{g} \subset J\}$, we see that I is an ideal of R and $J = I \otimes \mathfrak{g}$. ∎

Proof of Theorem 13.6. Define a map $\rho: \mathscr{A}(\mathrm{Cur}\,\mathfrak{h}) \to \mathfrak{h}$ by the formula

$$\rho(x \otimes_H (1 \otimes a)) = \langle x, 1 \rangle\, a, \qquad x \in X,\ a \in \mathfrak{h}. \tag{13.5}$$

Then for $a = \sum_i h_i \otimes a_i \in \mathrm{Cur}\,\mathfrak{h} = H \otimes \mathfrak{h}$, we have

$$\rho(x \otimes_H a) = \sum_i \langle S(x), h_i \rangle\, a_i. \tag{13.6}$$

It is easy to see that ρ is a surjective Lie algebra homomorphism.

Any isomorphism $f: \mathrm{Cur}\,\mathfrak{g} \xrightarrow{\sim} \mathrm{Cur}\,\mathfrak{h}$ induces an isomorphism of Lie algebras $\varphi = \mathscr{A}(f): \mathscr{A}(\mathrm{Cur}\,\mathfrak{g}) \xrightarrow{\sim} \mathscr{A}(\mathrm{Cur}\,\mathfrak{h})$. By Lemma 13.5, $\ker \rho\varphi = I \otimes \mathfrak{g}$ for some proper ideal I of X. Recall that X is isomorphic as a topological algebra to $\mathcal{O}_N = \mathbf{k}[[t_1, ..., t_N]]$ ($N = \dim \mathfrak{d}$), and $\mathcal{O}_N$ has a unique maximal ideal $M_0 = (t_1, ..., t_N)$. Noting that M_0 corresponds to $\mathrm{F}_0 X := \{x \in X \mid \langle x, 1 \rangle = 0\}$ via the isomorphism $X \simeq \mathcal{O}_N$, we deduce that $I \subset \mathrm{F}_0 X$. If $I \neq \mathrm{F}_0 X$, then $(\mathrm{F}_0 X/I) \otimes \mathfrak{g}$ is a nontrivial ideal of $(X/I) \otimes \mathfrak{g} \simeq \mathfrak{h}$, which is impossible because $\mathfrak{h}$ is simple. It follows that $\ker \rho\varphi = \mathrm{F}_0 X \otimes \mathfrak{g}$.

Now fix $a \in \mathfrak{g}$ and write $f(a) = \sum_i h_i \otimes a_i$ for some $h_i \in H$ and linearly independent $a_i \in \mathfrak{h}$. Assume that, say, $h_1 \notin \mathbf{k} = \mathrm{F}^0 H$. Then we can find $x \in \mathrm{F}_0 X$ such that $\langle S(x), h_1 \rangle \neq 0$. Then, by (13.6), the element $x \otimes a \in \mathrm{F}_0 X \otimes \mathfrak{g}$ is mapped by $\rho\varphi$ to $\sum_i \langle S(x), h_i \rangle \otimes a_i \neq 0$, which is a contradiction. This shows that $f(a) \in 1 \otimes \mathfrak{h}$, completing the proof. ▮

We turn now to the description of subalgebras of $W(\mathfrak{d})$. Recall that the Lie pseudoalgebra $W(\mathfrak{d})$ acts on $H = U(\mathfrak{d})$ by (8.4). Hence any homomorphism of Lie pseudoalgebras $L \to W(\mathfrak{d})$ gives rise to a structure of an L-module on H.

Let us first consider the case when L is a free H-module of rank one: $L = He$ with a pseudobracket $[e * e] = \alpha \otimes_H e$, $\alpha \in H \otimes H$. Let $M = Hm$ be an L-module, with action $e * m = \beta \otimes_H m$, $\beta \in H \otimes H$. We already know (Lemma 4.1) that α must be of the form $\alpha = r + s \otimes 1 - 1 \otimes s$ where $r \in \mathfrak{d} \wedge \mathfrak{d}$, $s \in \mathfrak{d}$. Moreover r and s satisfy Eqs. (4.3) and (4.4). Furthermore, β defines a representation of L if and only if it satisfies the following equation in $H \otimes H \otimes H$ (cf. Proposition 4.1)

$$(1 \otimes \beta)(\mathrm{id} \otimes \Delta)(\beta) - (\sigma \otimes \mathrm{id})((1 \otimes \beta)(\mathrm{id} \otimes \Delta)(\beta)) = (\alpha \otimes 1)(\Delta \otimes \mathrm{id})(\beta). \tag{13.7}$$

PROPOSITION 13.8. *If $L = He$ is a Lie pseudoalgebra with $[e * e] = \alpha \otimes_H e$, $\alpha = r + s \otimes 1 - 1 \otimes s$, then the only nonzero homomorphism $L \to W(\mathfrak{d})$ is given by $e \mapsto -r + 1 \otimes s$.*

Proof. The statement of the proposition is equivalent to saying that all solutions β of (13.7) with $\beta \in H \otimes \mathfrak{d}$ are either trivial or of the form $\beta = r - 1 \otimes s$. It is easy to check that the latter is indeed a solution (cf. Lemma 8.3).

Let us choose a basis $\{\partial_i\}$ of $\mathfrak{d}$, and write $\beta = \sum h^i \otimes \partial_i$ and $r = \sum_{i,j} r^{ij} \partial_i \otimes \partial_j$ for some $h^i \in H$, $r^{ij} \in \mathbf{k}$. We will assume throughout the proof that $\beta \neq 0$, and denote by d the maximal degree of the h^i. Substituting the above expressions for α and β in (13.7), we get

$$\sum_{i,j} h^j \otimes h^i \otimes [\partial_i, \partial_j] + \sum_{i,j} (h^j \otimes h^i \partial_j - h^i \partial_j \otimes h^j) \otimes \partial_i$$
$$= \sum_{i,j,k} r^{ij} \partial_i h^k_{(1)} \otimes \partial_j h^k_{(2)} \otimes \partial_k + \sum_k (s h^k_{(1)} \otimes h^k_{(2)} - h^k_{(1)} \otimes s h^k_{(2)}) \otimes \partial_k.$$

If $d > 1$, expressing all h^k in the Poincaré–Birkhoff–Witt basis relative to the basis $\{\partial_i\}$, we see that $H \otimes H$-coefficients of degree $2d + 1$ in the second summation in the left-hand side cannot cancel with terms from other summations, which only contribute lower degree terms. Therefore

$$\sum_{i,j} (h^j \otimes h^i \partial_j - h^i \partial_j \otimes h^j) \otimes \partial_i = 0 \qquad \mathrm{mod}\ \mathrm{F}^{2d}(H \otimes H) \otimes \mathfrak{d}.$$

This implies that $\sum_j h^j \otimes h^i \partial_j = \sum_j h^i \partial_j \otimes h^j \mod \mathrm{F}^{2d}(H \otimes H)$ for every i, which gives a contradiction, since we can choose h^i to have degree exactly d. So $d \leqslant 1$, and we can write $\beta = \sum_{i,j} \beta^{ij} \partial_i \otimes \partial_j + 1 \otimes t$ with $\beta^{ij} \in \mathbf{k}$, $t \in \mathfrak{d}$.

Substituting this into (13.7) and comparing degree four terms we get $\beta^{ij} \beta^{kl} = r^{ij} \beta^{kl}$ for all i, j, k, l. Since $\beta \neq 0$ we conclude that $\beta^{ij} = r^{ij}$ for all i, j. We are only left with showing that $t = -s$. Substitute $\alpha = r + s_1 - s_2$ and $\beta = r + t_2$ into (13.7), and then use (4.4) to obtain

$$\begin{aligned} & r_{12}(s_3 + t_3) + [t_3, r_{13} - r_{23}] + r_{23} t_2 - r_{13} t_1 - s_1 r_{13} + s_2 r_{23} \\ & \quad = t_3(t_1 - t_2 + s_1 - s_2). \end{aligned} \tag{13.8}$$

Notice that $r_{12}(s_3 + t_3)$ is the only term lying in $\mathfrak{d} \otimes \mathfrak{d} \otimes \mathfrak{d}$ and everything else belongs to $H \otimes H \otimes \mathbf{k} + H \otimes \mathbf{k} \otimes H + \mathbf{k} \otimes H \otimes H$. Hence $r_{12}(s_3 + t_3) = 0$, which is only possible if $r = 0$ or $s + t = 0$. In the latter case we are done. In the former, the left-hand side of (13.8) becomes zero, and $t \neq 0$ since $\beta \neq 0$. Thus $t_1 + s_1 - t_2 - s_2 = 0$ and $t + s = 0$. ∎

Proposition 13.8 shows that nonabelian Lie pseudoalgebras that are free of rank one over H embed uniquely in $W(\mathfrak{d})$. We will show that the other simple pseudoalgebras of vector fields are spanned as Lie pseudoalgebras by subalgebras of rank one, and therefore also embed uniquely in $W(\mathfrak{d})$. Recall that any pseudoalgebra of vector fields is in fact simple (Theorem 13.4).

THEOREM 13.7. (i) *For any subalgebra L of $W(\mathfrak{d})$, there is a unique nonzero homomorphism $L \to W(\mathfrak{d})$.*

(ii) *There is at most one nonzero homomorphism between any two pseudoalgebras of vector fields.*

Proof. Part (ii) is an immediate consequence of (i) and Theorem 13.4.

By the above remarks, it remains to prove (i) in the cases when L is a current pseudoalgebra over either $W(\mathfrak{d}')$ or $S(\mathfrak{d}', \chi')$, where $\mathfrak{d}'$ is a subalgebra of $\mathfrak{d}$.

In the former case ($L = \mathrm{Cur}\, W(\mathfrak{d}') := H \otimes_{H'} (H' \otimes \mathfrak{d}')$, $H' = U(\mathfrak{d}')$), note that L is spanned over $H = U(\mathfrak{d})$ by elements $\tilde{a} = 1 \otimes_{H'} (1 \otimes a)$ for $a \in \mathfrak{d}'$. Then $[\tilde{a} * \tilde{a}] = (a \otimes 1 - 1 \otimes a) \otimes_H \tilde{a}$, and by Proposition 13.8 we know that the only nonzero homomorphism of the Lie H-pseudoalgebra

$H\tilde{a}$ to $W(\mathfrak{d})$ maps $\tilde{a}$ to $1 \otimes a$. Hence any embedding of L in $W(\mathfrak{d})$ maps each $\tilde{a}$ to the corresponding element $1 \otimes a$ of $W(\mathfrak{d})$.

Now let L be a current pseudoalgebra over $S(\mathfrak{d}', \chi')$. We will give the proof in the case when $L = S(\mathfrak{d}, \chi)$, the case of currents being completely analogous. We are going to make use of the following lemma.

Lemma 13.6. *If $\mathfrak{d}$ is a finite-dimensional Lie algebra of dimension $N > 1$, then there exist 2-dimensional subalgebras $\mathfrak{d}_i$ $(i = 1, ..., N-1)$ such that* $\dim \sum_{i=1}^{r} \mathfrak{d}_i = r + 1$ *for every* $r = 1, ..., N-1$.

Proof. If $\mathfrak{d}$ has a semisimple element h, we complement it to a basis of $\operatorname{ad} h$ eigenvectors $\{h, h_1, ..., h_{N-1}\}$. The subalgebras $\mathfrak{d}_i = \mathbf{k}h + \mathbf{k}h_i$ then satisfy the statement of the lemma.

If $\mathfrak{d}$ has no semisimple elements, then from Levi's theorem we know that $\mathfrak{d}$ must be solvable. In this case it has a 1-dimensional ideal $\mathbf{k}h$. Complementing h to a basis $\{h, h_1, ..., h_{N-1}\}$, we conclude as before. ∎

Now consider a 2-dimensional subalgebra of $\mathfrak{d}$ with basis $\{a, b\}$. Then the element $e_{ab} \in S(\mathfrak{d}, \chi)$ from Proposition 8.1 depends on the choice of basis only up to multiplication by a nonzero element of $\mathbf{k}$. Moreover, the H-span of this element is a (free) rank one subalgebra of $S(\mathfrak{d}, \chi)$, as can be easily checked (cf. Remark 8.3).

Let S_i be the rank one subalgebras of $S(\mathfrak{d}, \chi)$ associated as above with the 2-dimensional subalgebras $\mathfrak{d}_i$ of $\mathfrak{d}$ constructed in Lemma 13.6. Then by comparing second tensor factors, we see that $S_{r+1} \cap \sum_{i=1}^{r} S_i = 0$ for each $r = 1, ..., N-2$. Therefore the sum of all S_i is a free H-submodule F of $S(\mathfrak{d}, \chi)$ of rank $N-1$. Since the rank of $S(\mathfrak{d}, \chi)$ is also $N-1$, we see that $S(\mathfrak{d}, \chi)/F$ is a torsion H-module.

Denote by S the subalgebra of $S(\mathfrak{d}, \chi)$ generated by F. Since $S(\mathfrak{d}, \chi)/S$ is a torsion H-module, we conclude, by Corollary 10.1, that S is an ideal of $S(\mathfrak{d}, \chi)$. Hence $S(\mathfrak{d}, \chi) = S$ by simplicity of $S(\mathfrak{d}, \chi)$. Now by Proposition 13.8, each subalgebra S_i embeds uniquely in $W(\mathfrak{d})$. Hence $S = S(\mathfrak{d}, \chi)$ embeds uniquely in $W(\mathfrak{d})$.

This completes the proof of Theorem 13.7. ∎

Combining a number of previous results, we get an explicit description of all subalgebras of $W(\mathfrak{d})$, and of all isomorphisms between the simple Lie pseudoalgebras listed in Theorem 13.2.

Corollary 13.6. *A complete list of all subalgebras L of $W(\mathfrak{d}) = H \otimes \mathfrak{d}$ is*

(a) $L = H \otimes \mathfrak{d}' \simeq \operatorname{Cur}_{H'}^{H} W(\mathfrak{d}')$, *where $\mathfrak{d}'$ is any subalgebra of $\mathfrak{d}$ and* $H' = U(\mathfrak{d}')$;

(b) $L = \{\sum_i h_i \otimes a_i \in H \otimes \mathfrak{d}' \mid \sum_i h_i(a_i + \chi'(a_i)) = 0\} \simeq \mathrm{Cur}_{H'}^{H}\, S(\mathfrak{d}', \chi')$, *where* $\mathfrak{d}'$ *is any subalgebra of* $\mathfrak{d}$ *and* $\chi' \in (\mathfrak{d}')^*$ *is such that* $\chi'([\mathfrak{d}', \mathfrak{d}']) = 0$;

(c) $L = \{(h \otimes 1)(r - 1 \otimes s) \mid h \in H\}$, *where* $r \in \mathfrak{d} \wedge \mathfrak{d}$ *and* $s \in \mathfrak{d}$ *satisfy* (4.3), (4.4). *In this case, L is isomorphic to a current pseudoalgebra over* $H(\mathfrak{d}', \chi', \omega')$ *or* $K(\mathfrak{d}', \theta')$ (*see Sections* 8.6 *and* 8.7).

COROLLARY 13.7. *All nontrivial isomorphisms among the simple Lie* $U(\mathfrak{d})$*-pseudoalgebras listed in Theorem* 13.2 *are the following* ($H = U(\mathfrak{d})$, $H' = U(\mathfrak{d}')$)

(i) $\mathrm{Cur}\, \mathfrak{g}' \simeq \mathrm{Cur}\, \mathfrak{g}''$ *when* $\mathfrak{g}'$ *and* $\mathfrak{g}''$ *are isomorphic Lie algebras.*

(ii) $\mathrm{Cur}_{H'}^{H}\, H(\mathfrak{d}', \chi', \omega') \simeq \mathrm{Cur}_{H'}^{H}\, H(\mathfrak{d}', \chi', \omega'')$ *when* $\omega'' = c\omega'$ *for some nonzero* $c \in \mathbf{k}$.

(iii) $\mathrm{Cur}_{H'}^{H}\, K(\mathfrak{d}', \theta') \simeq \mathrm{Cur}_{H'}^{H}\, K(\mathfrak{d}', \theta'')$ *when* $\theta'' = c\theta'$ *for some nonzero* $c \in \mathbf{k}$.

(iv) $\mathrm{Cur}_{H'}^{H}\, W(\mathfrak{d}') \simeq \mathrm{Cur}_{H'}^{H}\, K(\mathfrak{d}', \theta')$ *when* $\dim \mathfrak{d}' = 1$.

(v) $\mathrm{Cur}_{H'}^{H}\, H(\mathfrak{d}', \chi', \omega) \simeq \mathrm{Cur}_{H'}^{H}\, S(\mathfrak{d}', \chi'')$ *when* $\dim \mathfrak{d}' = 2$ *and* $\chi'' = -\chi' + \mathrm{tr\,ad}$.

13.7. *Finite Simple and Semisimple Lie* $(U(\mathfrak{d}) \sharp \mathbf{k}[\Gamma])$*-Pseudoalgebras*

Let, as before, $H = U(\mathfrak{d})$ be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$. Let Γ be a (not necessarily finite) group acting on $\mathfrak{d}$ by automorphisms. The action of Γ on $\mathfrak{d}$ can be extended to an action on H which we denote by $g \cdot f$ for $g \in \Gamma$, $f \in H$. Recall that the smash product $\tilde{H} = H \sharp \mathbf{k}[\Gamma]$ is a Hopf algebra, with the product determined by $g \cdot f = gfg^{-1}$, and coproduct $\Delta(fg) = \Delta(f)\,\Delta(g)$ ($g \in \Gamma$, $f \in H$).

A left $\tilde{H}$-module L is the same as an H-module together with an action of Γ on it which is compatible with that of H. An $\tilde{H}$-module L will be called *finite* if it is finite as an H-module.

Let $\tilde{L}$ be a Lie $\tilde{H}$-pseudoalgebra with a pseudobracket denoted as $[a \,\tilde{*}\, b]$. By Corollary 5.1, $\tilde{L}$ is also a Lie H-pseudoalgebra, which we denote as L with a pseudobracket $[a * b]$. L is equipped with an action of Γ, and $[a * b]$ is Γ-equivariant; see (5.5). As an $\tilde{H}$-module, $L = \tilde{L}$. The relationship between the two pseudobrackets is given by (5.7).

Then the following statements are easy to check.

LEMMA 13.7. (i) $[a \,\tilde{*}\, b] = 0$ *iff* $[ga * b] = 0$ *for all* $g \in \Gamma$.

(ii) $I \subset L = \tilde{L}$ *is an ideal of the Lie* $\tilde{H}$*-pseudoalgebra* $\tilde{L}$ *iff it is a* Γ*-invariant ideal of the Lie* H*-pseudoalgebra* L.

(iii) *If* I *is as in* (ii), *then its derived pseudoalgebra* $[I, I]$ *is the same with respect to both pseudobrackets* $[a \,\tilde{*}\, b]$ *and* $[a * b]$.

(iv) $\mathrm{Rad}\, \tilde{L} = \mathrm{Rad}\, L$.

Proof. (i) If $[a \tilde{*} b] = 0$ then all its coefficients in front of $(gH \otimes \mathbf{k}) \otimes_{\tilde{H}} L$ are zero for different $g \in \Gamma$. Since $[a * b] \in (H \otimes H) \otimes_H L$, it follows that all $[ga * b] = 0$.

Parts (ii) and (iii) are clear by (5.7).

Part (iv) follows from (i)–(iii) and the fact that $\operatorname{Rad} L$ is Γ-invariant. ($\operatorname{Rad} L$ is Γ-invariant because $[a * b]$ is Γ-equivariant, see (5.5).) ∎

PROPOSITION 13.9. *The Lie $\tilde{H}$-pseudoalgebra $\tilde{L}$ is solvable (respectively semisimple) if and only if the Lie H-pseudoalgebra L is.*

Proof. Follows from Lemmas 13.7(iv) and 13.1(iii). ∎

PROPOSITION 13.10. *The Lie $\tilde{H}$-pseudoalgebra $\tilde{L}$ is finite and simple if and only if the Lie H-pseudoalgebra L is a finite direct sum of isomorphic finite simple Lie H-pseudoalgebras and Γ acts on them transitively.*

Proof. By Lemma 13.7, $\tilde{L}$ is simple iff L is not abelian and has no non-trivial Γ-invariant ideals. In particular, L is semisimple. Using Theorem 13.3 and the fact that $\mathbf{k}[\Gamma] I$ is an ideal of L if I is an ideal, we see that L is a direct sum of isomorphic finite simple Lie H-pseudoalgebras. ∎

13.8. *Examples of Infinite Simple Subalgebras of* gc_n

In this subsection, H is an arbitrary cocommutative Hopf algebra. Let us define a map $\omega \colon H \otimes H \to H \otimes H$ by the formula

$$\omega(f \otimes a) = f a_{(-1)} \otimes a_{(-2)} = (f \otimes 1)\, \Delta(S(a)). \tag{13.9}$$

It is easy to check that $\omega^2 = \mathrm{id}$; this also follows from the identities $\omega = \mathscr{F}^{-1}(\mathrm{id} \otimes S) = (\mathrm{id} \otimes S)\, \mathscr{F}$ where $\mathscr{F}$ is the Fourier transform defined by (2.33).

LEMMA 13.8. *The above ω is an anti-involution of* $\mathrm{Cend}_1 = H \otimes H$, *i.e., it is an H-linear map satisfying* $\omega^2 = \mathrm{id}$ *and*

$$\omega(a) * \omega(b) = (\sigma \otimes_H \omega)\,(b * a), \qquad a, b \in \mathrm{Cend}_1, \tag{13.10}$$

where, as before, $\sigma \colon H \otimes H \to H \otimes H$ is the permutation of the factors.

Proof. It only remains to check (13.10), which is straightforward. ∎

When $H = U(\mathfrak{d})$, the annihilation algebra $\mathscr{A}(\mathrm{Cend}_1)$ is isomorphic to the associative algebra of all differential operators on X, and ω induces its standard anti-involution $*$ (formal adjoint).

Let $\gamma\colon \mathrm{End}(\mathbf{k}^n) \to \mathrm{End}(\mathbf{k}^n)$ be an anti-involution, i.e., $\gamma^2 = \mathrm{id}$ and $\gamma(A)\,\gamma(B) = \gamma(BA)$. Then we can define an anti-involution ω of $\mathrm{Cend}_n = H \otimes H \otimes \mathrm{End}(\mathbf{k}^n)$ by the formula (cf. (13.9))

$$\omega(f \otimes a \otimes A) = f a_{(-1)} \otimes a_{(-2)} \otimes \gamma(A). \tag{13.11}$$

Let $\mathrm{gc}_n(\omega)$ be the set of all $a \in \mathrm{Cend}_n$ such that $\omega(a) = -a$. This is a subalgebra of the Lie pseudoalgebra gc_n. Indeed, it is an H-submodule because ω is H-linear. If $\omega(a) = -a$, $\omega(b) = -b$, then

$$\begin{aligned}(\mathrm{id} \otimes_H \omega)\,[a * b] &= (\mathrm{id} \otimes_H \omega)\,(a * b - (\sigma \otimes_H \mathrm{id})\,(b * a)) \\ &= (\sigma \otimes_H \mathrm{id})\,\omega(b) * \omega(a) - \omega(a) * \omega(b) = -[a * b].\end{aligned}$$

Two important examples of Lie pseudoalgebras $\mathrm{gc}_n(\omega)$ are obtained when $\mathbf{k}^n$ is endowed with a symmetric or skew-symmetric nondegenerate bilinear form, and $\gamma(A)$ is the adjoint of A with respect to this form. In these cases, we denote $\mathrm{gc}_n(\omega)$ by oc_n and spc_n, respectively.

Proposition 13.11. *Let $H = U(\mathfrak{d})$, $\mathfrak{d} \neq 0$. Then* oc_n *and* spc_n *are infinite subalgebras of* gc_n *that act irreducibly on $H \otimes \mathbf{k}^n$. We have*: $\mathrm{oc}_n \cap \mathrm{Cur}\,\mathfrak{gl}_n = \mathrm{Cur}\,\mathfrak{o}_n$ *and* $\mathrm{spc}_n \cap \mathrm{Cur}\,\mathfrak{gl}_n = \mathrm{Cur}\,\mathfrak{sp}_n$.

Proof. The second statement is obvious by the definitions. Since $\mathfrak{o}_n$ $(n \geqslant 3)$ and $\mathfrak{sp}_n$ $(n \geqslant 2)$ act irreducibly on $\mathbf{k}^n$, we only have to check that the action of oc_n on $H \otimes \mathbf{k}^n$ is irreducible for $n = 1, 2$. Using diagonal matrices, we see that it suffices to check that oc_1 acts irreducibly on H.

Recall that this action is given by (see (10.12)

$$\alpha * h = (1 \otimes h)\,\alpha \otimes_H 1 \qquad \text{for} \quad \alpha \in \mathrm{gc}_1 = H \otimes H,\ h \in H.$$

For $a \in \mathfrak{d}$, let $\alpha = 1 \otimes a - \omega(1 \otimes a) = 2 \otimes a + a \otimes 1 \in \mathrm{oc}_1$. Then $\alpha * h = (1 \otimes ha) \otimes_H 1 + (1 \otimes h) \otimes_H a$. If $M \subset H$ is an oc_1-submodule, and $h \in M$, $h \neq 0$, then the previous formula implies $1 \in M$. Therefore $M = H$. ∎

Remark 13.1. It follows from Theorem 14.2 below that in the case $H = U(\mathfrak{d})$, $\mathfrak{d} \neq 0$, the Lie pseudoalgebras gc_n, oc_n, and spc_n are semisimple. In fact, one can show that in this case they are simple. In the case $H = \mathbf{k}[\Gamma]$ with a finite group Γ, the Lie pseudoalgebra gc_n has a center that is a free H-module of rank 1, the quotient by which is simple.

If I is a left or right ideal of the associative pseudoalgebra Cend_n and L is a subalgebra of the Lie pseudoalgebra gc_n, then their intersection $I \cap L$ is again a subalgebra of gc_n. All ideals of Cend_n are described in the next proposition.

PROPOSITION 13.12. (i) *Any left ideal of the associative pseudoalgebra* Cend_n *is a sum of ideals of the form* $H \otimes R \otimes E$ *where* $R \subset H$ *is a right ideal and* $E \subset \mathrm{End}(\mathbf{k}^n)$ *is a left ideal.*

(ii) *Any right ideal of* Cend_n *is of the form* $\omega(I)$ *for a unique left ideal* I.

(iii) Cend_n *has no two-sided ideals, i.e., it is a simple associative pseudoalgebra.*

Proof. Let $I \subset \mathrm{Cend}_n$ be a left ideal, $\alpha = 1 \otimes a \otimes A \in \mathrm{Cend}_n$, and $\beta = \sum_i g_i \otimes b_i \otimes B_i \in I$ with linearly independent g_i. Then

$$\alpha * \beta = \sum_i (1 \otimes g_i a_{(1)}) \otimes_H (1 \otimes b_i a_{(2)} \otimes AB_i) \in (H \otimes H) \otimes_H I.$$

Taking $a = 1$, we see that all $1 \otimes b_i \otimes AB_i \in I$. In particular, $1 \otimes b_i \otimes B_i \in I$, and hence each element from I is an H-linear combination of elements of the form $\beta = 1 \otimes b \otimes B$. For such β, we have $\alpha * \beta = (1 \otimes a_{(1)}) \otimes_H (1 \otimes ba_{(2)} \otimes AB)$. For $a \in \mathfrak{d}$, $A = \mathrm{Id}$, we get that $1 \otimes ba \otimes B \in I$. This proves the first part of the proposition.

Part (ii) is obvious, and part (iii) follows easily from (i) and (ii). ∎

14. REPRESENTATION THEORY OF LIE PSEUDOALGEBRAS

14.1. *Conformal Version of the Lie Lemma*

Let L be a Lie H-pseudoalgebra and V be an L-module. In this subsection, $H = U(\mathfrak{d})$ will be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$. In particular, H is a Noetherian ring with no divisors of zero.

Let $I \subset L$ be an ideal and $\varphi \in \mathrm{Hom}_H(I, H)$ be such that

$$V_\varphi := \{ v \in V \mid a * v = (\varphi(a) \otimes 1) \otimes_H v \;\; \forall a \in I \} \tag{14.1}$$

is nonzero. We will call the elements of V_φ *eigenvectors* for I with an *eigenvalue* φ. Note that every $v \in V_\varphi$ is an eigenvector for the action of $X \otimes_H I \subset \mathscr{A}(L)$ on V. By abuse of notation, we will also write $a_x v = \varphi(a_x) v$ for $a \in I$, $x \in X$, $v \in V_\varphi$, where $\varphi(a_x) = \langle S(x), \varphi(a) \rangle$, cf. (9.6).

Clearly, if $\varphi = 0$, then V_0 is an L-submodule of V.

LEMMA 14.1. *If* $\varphi \neq 0$, *then* HV_φ *is a free* H-*module, isomorphic to* $H \otimes V_\varphi$ *with* H *acting on the first tensor factor.*

Proof. Assume that

$$\sum_i f_i v_i = 0 \tag{14.2}$$

for some $f_i \in H$, $v_i \in V_\varphi$. Let (14.2) be a relation of this form with $f_i \in \mathrm{F}^{n_i}H$ so that $\sum_i n_i$ is minimal. We call $\sum_i n_i$ the degree of the relation (14.2). Assume that v_i's are linearly independent, so that the degree of (14.2) is positive.

We can find $a \in I$, $x \in X$ such that $\varphi(a_x) \neq 0$. Applying a_x to (14.2) and using (9.16), we obtain

$$\sum_i f_{i(2)}\, \varphi(a_{f_{i(-1)}x})\, v_i = 0.$$

Subtracting this from (14.2), we get a relation of lower degree than (14.2), because $\Delta(f) \in 1 \otimes f + \sum_{j=1}^{n} \mathrm{F}^j H \otimes \mathrm{F}^{n-j} H$ for $f \in \mathrm{F}^n H$. ∎

The following result is an analogue of Lie's Lemma.

PROPOSITION 14.1. *If V is finite as an H-module, then*

$$L * V_\varphi \subset (H \otimes \mathbf{k}) \otimes_H (\mathfrak{d} V_\varphi + V_\varphi). \tag{14.3}$$

In other words, for every $\beta \in \mathscr{A}(L)$, there exist $\partial_\beta \in \mathfrak{d}$ and $A_\beta \in \operatorname{End} V_\varphi$ such that

$$\beta v = (\partial_\beta + A_\beta)\, v \qquad \text{for any} \quad v \in V_\varphi. \tag{14.4}$$

In particular, HV_φ is an L-submodule of V.

Proof. Fix nonzero elements $w \in V_\varphi$, $\beta \in \mathscr{A}(L)$, and let $w_n = \beta^n w$. Let W_n be the linear span of $w_0, \ldots, w_n$; we set $W_n = 0$ for $n < 0$. For $a \in I$, $x \in X$, we have $a_x w = \varphi(a_x)\, w$, and by induction,

$$a_x w_n \in \varphi(a_x)\, w_n + n\varphi([a_x, \beta])\, w_{n-1} + W_{n-2}. \tag{14.5}$$

In particular, all HW_n are I-modules.

Since V is a Noetherian H-module, there exists $N \geqslant 0$ such that $HW_{N-1} \neq HW_N = HW_{N+1}$. In particular,

$$w_{N+1} \in (N+1)\, h w_N + HW_{N-1} \tag{14.6}$$

for some $h \in H$.

Writing (14.5) for $n=N+1$ and using (14.6), we get

$$a_x w_{N+1} \in \varphi(a_x)(N+1)\, h w_N + (N+1)\, \varphi([a_x, \beta])\, w_N + H W_{N-1}. \tag{14.7}$$

On the other hand, applying a_x to both sides of (14.6) and using the H-sesqui-linearity gives

$$a_x w_{N+1} \in \varphi(a_{h_{(-1)} x})(N+1)\, h_{(2)} w_N + H W_{N-1}. \tag{14.8}$$

Subtracting (14.8) from (14.7) gives

$$f w_N \in H W_{N-1} \qquad \text{for} \quad f = \varphi(a_x)\, h + \varphi([a_x, \beta]) - \varphi(a_{h_{(-1)} x})\, h_{(2)}. \tag{14.9}$$

If $f \neq 0$, then the module HW_N / HW_{N-1} is torsion, hence I acts on it as zero by Corollary 10.1. This gives $a_x w_N \in HW_{N-1}$ for all $a \in I$, $x \in X$. Then (14.5) implies $\varphi(a_x)\, w_N \in HW_{N-1}$. Since $HW_{N-1} \neq HW_N$, it follows that $\varphi = 0$, which contradicts the assumption $f \neq 0$.

Therefore $f = 0$. This is possible only when $h \in \mathrm{F}^1 H = \mathfrak{d} + \mathbf{k}$. Then for any $v \in V_\varphi$, one has

$$0 = fv = h(a_x v) + [a_x, \beta]\, v - h_{(2)}(a_{h_{(-1)} x} v) = [a_x, \beta - h]\, v.$$

This implies that $(\beta - h)\, v \in V_\varphi$, proving (14.4). ∎

14.2. *Conformal Version of the Lie Theorem*

THEOREM 14.1. *Let $H = U(\mathfrak{d}) \sharp \mathbf{k}[\Gamma]$ with $\dim \mathfrak{d} < \infty$. Let L be a solvable Lie H-pseudoalgebra and V be an L-module which is finite over $U(\mathfrak{d})$. Then there exists an eigenvector for the action of L on V, i.e., $v \in V \backslash \{0\}$ and $\varphi \in \mathrm{Hom}_H(L, H)$ such that $a * v = (\varphi(a) \otimes 1) \otimes_H v$ for all $a \in L$.*

Proof. Using Corollary 5.1 and Proposition 13.9, we can assume that $H = U(\mathfrak{d})$. The proof will be by induction on the length of the derived series of L.

First consider the case when L is abelian. By a Zorn's Lemma argument, it is enough to find an eigenvector when $L = Ha$ is abelian generated by one element a. We may assume that $\ker a = 0$; then by Lemma 10.3 all $\ker_n a$ are finite dimensional. Let n be such that $\ker_n a \neq 0$. Then the statement follows from the usual Lie Theorem applied to the $\mathscr{A}(L)$-module $\ker_n a$.

Now let L be nonabelian, $I = [L, L] \neq 0$. By the inductive assumption, I has a space of eigenvectors $V_\varphi \neq 0$. If $\varphi = 0$, then V_0 is an L-submodule of V on which I acts as zero. The abelian H-pseudoalgebra L/I has an eigenvector in V_0, which is also an eigenvector for L.

Now assume that $\varphi \neq 0$. By Proposition 14.1, we have for $\alpha, \beta \in \mathscr{A}(L)$, $v \in V_\varphi$,

$$\alpha v = (\partial_\alpha + A_\alpha)\, v,$$
$$\beta v = (\partial_\beta + A_\beta)\, v,$$
$$[\alpha, \beta] v = \varphi([\alpha, \beta])\, v.$$

On the other hand, we can compute

$$\begin{aligned} \alpha\beta v &= \alpha(\partial_\beta + A_\beta)\, v = \partial_\beta(\alpha v) - (\partial_\beta \alpha)\, v + \alpha(A_\beta v) \\ &= \partial_\beta(\partial_\alpha + A_\alpha)\, v - (\partial_{\partial_\beta \alpha} + A_{\partial_\beta \alpha})\, v + (\partial_\alpha + A_\alpha)\, A_\beta v \\ &= \partial_\beta \partial_\alpha v - \partial_{\partial_\beta \alpha} v + \partial_\beta A_\alpha v + \partial_\alpha A_\beta v - A_{\partial_\beta \alpha} v + A_\alpha A_\beta v. \end{aligned}$$

It follows that

$$[\partial_\alpha, \partial_\beta] = \partial_{\partial_\alpha \beta} - \partial_{\partial_\beta \alpha}.$$

Assume that $\partial_{a_x} \neq 0$ for some $a \in L$, $x \in X$, and write $\partial_x = \partial_{a_x}$ for short. For $\alpha = a_x$, $\beta = a_y$, the above equation becomes

$$[\partial_x, \partial_y] = \partial_{\partial_x y} - \partial_{\partial_y x}$$

(recall that $h a_x = a_{hx}$ for $h \in H$). Note that $\partial_y = 0$ if $y \in \mathrm{F}_n X$ for sufficiently large n. Take the minimal such n, and let $x \in \mathrm{F}_{n-1} X$ be such that $\partial_x \neq 0$. By Lemma 6.4, there exists $y \in \mathrm{F}_n X$ such that $x = \partial_x y$. Then $\partial_y = 0$ and $\partial_x = \partial_{\partial_x y} - \partial_{\partial_y x} = [\partial_x, \partial_y] = 0$, which is a contradiction.

It follows that all $\partial_{a_x} = 0$, hence L preserves V_φ. By Lemma 14.1, $\dim V_\varphi < \infty$, and therefore L has an eigenvector by the usual Lie Theorem for $\mathscr{A}(L)$. ▮

COROLLARY 14.1. *Let L be a solvable Lie H-pseudoalgebra and V be a finite L-module (i.e., finite over* $U(\mathfrak{d})$*). Then V has a filtration by L-submodules* $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ *such that for any i the L-module* V_{i+1}/V_i *is generated over H by eigenvectors of some given eigenvalue* $\varphi_i \in \mathrm{Hom}_H(L, H)$.

14.3. *Conformal Version of the Cartan–Jacobson Theorem*

THEOREM 14.2. *Let* $H = U(\mathfrak{d})$ *be the universal enveloping algebra of a finite-dimensional Lie algebra* $\mathfrak{d}$*. Let L be a Lie H-pseudoalgebra acting faithfully and irreducibly on the finite H-module V. Then one of the following two possibilities holds*

(i) *L is semisimple, either finite or infinite.*

(ii) *L is finite,* $\operatorname{Rad} L$ *is abelian and of rank one as an H-module. In this case, there is a subspace $\bar{V}$ of V such that $V \simeq H \otimes \bar{V}$ is a free H-module and L can be identified with* $(A \ltimes \operatorname{Cur} \mathfrak{g}) \ltimes (R \otimes \mathrm{id}_{\bar{V}}) \subset \mathrm{gc}\, V$, *where A is a subalgebra of $W(\mathfrak{d})$, $\mathfrak{g}$ is zero or a semisimple subalgebra of* $\mathfrak{sl}\,\bar{V}$, *and R is a nonzero left ideal of H.*

Proof. Assume that L is not semisimple; i.e., it has a nonzero abelian ideal I. Then, by Theorem 14.1, I has an eigenvector in V. If $\bar{V} = V_\varphi$ is the corresponding eigenspace in V, then, by Proposition 14.1, $H\bar{V}$ is an L-submodule of V. The irreducibility of V implies that $V = H\bar{V}$. Now, by Lemma 14.1, $V \simeq H \otimes \bar{V}$ is a free H-module, since $\varphi \neq 0$ by the faithfulness of V.

Proposition 14.1 and the faithfulness of V also show that L embeds into $W(\mathfrak{d}) \ltimes \operatorname{Cur} \mathfrak{gl}\,\bar{V} \subset \mathrm{gc}\, V$. In particular, L is finite. Then $\operatorname{Rad} L$ exists, and we can assume that $\bar{V}$ is an eigenspace for $\operatorname{Rad} L$. For each element $a \in \operatorname{Rad} L$ and $v \in \bar{V}$ we have $a * v = (\varphi(a) \otimes 1) \otimes_H v$, which means that $\operatorname{Rad} L$ is identified with $R \otimes \mathrm{id}_{\bar{V}} \subset \operatorname{Cur} \mathfrak{gl}\,\bar{V}$ for $R = \varphi(\operatorname{Rad} L)$. Note that R is of rank one, because $R \neq 0$ and H has no zero divisors.

Let $L_1 = L \cap (W(\mathfrak{d}) \ltimes \operatorname{Cur} \mathfrak{sl}\,\bar{V})$. Notice that L_1 is a subalgebra of $L_2 := L/\operatorname{Rad} L$ and $L_1 + \operatorname{Rad} L$ is a semidirect sum, because $W(\mathfrak{d}) \ltimes \operatorname{Cur} \mathfrak{gl}\,\bar{V} = (W(\mathfrak{d}) \ltimes \operatorname{Cur} \mathfrak{sl}\,\bar{V}) \ltimes (H \otimes \mathrm{id}_{\bar{V}})$. Since $\operatorname{Rad} L = R \otimes \mathrm{id}_{\bar{V}}$, this also implies that RL is contained in $L_1 + \operatorname{Rad} L$. Then RL_2 embeds in L_1, i.e., $R(L_2/L_1) = 0$. Hence L_2/L_1 is torsion, and by Corollary 10.1, L_1 is an ideal of L_2. But L_2 is semisimple, and by Proposition 13.7 a finite semisimple Lie pseudoalgebra does not have proper ideals of the same rank. Therefore, $L_1 = L_2$ and $L = L_1 \ltimes \operatorname{Rad} L$ is a semidirect sum of pseudoalgebras.

Finally, to show that L_1 is of the form $A \ltimes \operatorname{Cur} \mathfrak{g}$, notice that $L_1 \cap \operatorname{Cur} \mathfrak{sl}\,\bar{V}$ is an ideal of L_1, since $\operatorname{Cur} \mathfrak{sl}\,\bar{V}$ is an ideal of $W(\mathfrak{d}) \ltimes \operatorname{Cur} \mathfrak{sl}\,\bar{V}$. This ideal is generated over H by abelian elements, so by Propositions 13.6 and 13.7 if it is nonzero it is of the form $\operatorname{Cur} \mathfrak{g}$ for some semisimple subalgebra $\mathfrak{g}$ of $\mathfrak{sl}\,\bar{V}$. Hence $\operatorname{Cur} \mathfrak{g} \subset L_1 \subset W(\mathfrak{d}) \ltimes \operatorname{Cur} \mathfrak{g}$. But any subalgebra of $W(\mathfrak{d}) \ltimes \operatorname{Cur} \mathfrak{g}$ containing $\operatorname{Cur} \mathfrak{g}$ is equal to $A \ltimes \operatorname{Cur} \mathfrak{g}$ for some subalgebra A of $W(\mathfrak{d})$. This completes the proof. ∎

14.4. *Conformal Version of Engel's Theorem*

As an application of the results of Section 14.2, we can prove a conformal analogue of Engel's Theorem.

THEOREM 14.3. *Let $H = U(\mathfrak{d}) \sharp \mathbf{k}[\Gamma]$ with* $\dim \mathfrak{d} < \infty$, *and let L be a finite Lie H-pseudoalgebra (i.e., finite over $U(\mathfrak{d})$). Assume that the action of any element $\alpha \in \mathscr{A}(L)$ on L is nilpotent. Then L is a nilpotent Lie pseudoalgebra.*

Proof. First of all, note that the property that any element α of $\mathscr{A}(L)$ acts nilpotently on L remains valid when we replace L by any quotient of L by an ideal. In particular, $L/\mathrm{Rad}\, L$ will have that property. However, $L/\mathrm{Rad}\, L$ is semisimple, and from the classification of finite semisimple Lie pseudoalgebras we see that this is impossible, unless $L/\mathrm{Rad}\, L = 0$.

Therefore L is solvable. The nilpotence of all $\alpha \in \mathscr{A}(L)$ imply that all eigenvalues for L are zero. Now Corollary 14.1 implies that L is a nilpotent Lie pseudoalgebra. ∎

14.5. *Generalized Weight Space Decomposition for Nilpotent Lie Pseudoalgebras*

Let L be a (not necessarily finite) Lie H-pseudoalgebra, and V be a finite L-module, where $H = U(\mathfrak{d})$ for a finite-dimensional Lie algebra $\mathfrak{d}$.

Recall that for any $\varphi \in \mathrm{Hom}_H(L, H)$, the *eigenspace* V_φ of V is defined by

$$V_\varphi = \{ v \in V \mid a * v = (\varphi(a) \otimes 1) \otimes_H v \ \ \forall a \in L \}. \tag{14.10}$$

Let $V^\varphi_{-1} = 0$ and set inductively

$$V^\varphi_{i+1} = H\{ v \in V \mid a * v - (\varphi(a) \otimes 1) \otimes_H v \in (H \otimes H) \otimes_H V^\varphi_i \ \forall a \in L \}. \tag{14.11}$$

Then $V^\varphi_0 = HV_\varphi$ and $V^\varphi_{i+1}/V^\varphi_i = H(V/V^\varphi_i)_\varphi$. The V^φ_i form an increasing sequence of H-submodules of V which stabilizes (because of Noetherianity) to some H-submodule of V denoted V^φ. If $V^\varphi_{n-1} \neq V^\varphi_n = V^\varphi$, then we set the *depth* of V^φ to be n. We call V^φ the *generalized weight submodule* of V relative to the *weight* φ.

When L is nilpotent, it is solvable, and, by Corollary 14.1, any finite L-module V has a filtration by L-submodules so that the successive quotients are generalized weight modules.

The main result of this subsection is the following theorem.

THEOREM 14.4. *Let L be a nilpotent Lie H-pseudoalgebra and V be a finite L-module. Then V decomposes as a direct sum of generalized weight modules.*

Proof. In order to prove the statement, it is enough to show that all L-module extensions between generalized weight modules relative to distinct weights are trivial.

The strategy is to consider first the case when $L = \langle T \rangle$ is the Lie pseudoalgebra generated by one element $T \in \mathrm{gc}\, V$. Then in the general case, we show that the generalized weight spaces V^φ relative to some element $T \in L$ are L-invariant.

LEMMA 14.2. *Let V be a finite H-module, $T \in \mathrm{gc}\, V$, and $L = \langle T \rangle$ be a nilpotent Lie pseudoalgebra. If V contains a T-generalized weight module V^φ and $V/V^\varphi = W = W^\psi$ with $\psi \neq \varphi$, then $V \simeq V^\varphi \oplus W$ as L-modules.*

Proof. Since $W^\psi_{i+1}/W^\psi_i = H(W/W^\psi_i)_\psi$ for any i, it suffices to prove the statement when $W = W^\psi_0 = HW_\psi$.

Let us first consider the case when $W = H\bar{v}$ is a cyclic H-module. In order to prove that the extension is trivial, we need to find a lifting $v \in V$ of $\bar{v}$ such that $T * v = (\psi \otimes 1) \otimes_H v$ and to show that $Hv + V^\varphi$ is a direct sum of H-modules (here and below, we write just ψ instead of $\psi(T)$). We will prove this by induction on the depth of V^φ, the basis of induction being trivial.

Let thus the statement be true for all T-generalized weight modules of depth $\leqslant n$ and consider a module V^φ of depth $n+1$. Fix an arbitrary lifting $v \in V$ of $\bar{v}$; then

$$T * v = (\psi \otimes 1) \otimes_H v \qquad \mathrm{mod}\ (H \otimes H) \otimes_H V^\varphi. \tag{14.12}$$

Set

$$T_1 = T, \qquad T_{m+1} = [T_m * T] \in H^{\otimes(m+1)} \otimes_H L \qquad \text{for} \quad m \geqslant 1.$$

Then we claim that for $m > 1$, $T_{m+1} * v \in H^{\otimes(m+2)} \otimes_H V^\varphi_n$ implies $T_m * v \in H^{\otimes(m+1)} \otimes_H V^\varphi_n$. We are going to show this first in the case when $\varphi \neq 0$, the proof for $\varphi = 0$ only requiring minor changes.

So, let $\varphi \neq 0$. Then $V^\varphi/V^\varphi_n = V^\varphi_{n+1}/V^\varphi_n = H(V^\varphi/V^\varphi_n)_\varphi$ is a free H-module, because it is generated by its φ-eigenspace and we can apply Lemma 14.1. We pick some H-basis $\{w^j\}$ for V^φ modulo V^φ_n. If $\{h^i\}$ is some $\mathbf{k}$-basis of H compatible with its filtration, we write

$$T_m * v = \sum_{i,j} (\alpha^i_j \otimes h^i) \otimes_H w^j \qquad \mathrm{mod}\ H^{\otimes(m+1)} \otimes_H V^\varphi_n, \tag{14.13}$$

where $\alpha^i_j \in H^{\otimes m}$.

Notice that for $m > 1$, T_m belongs to $H^{\otimes m} \otimes_H [L, L]$ where $[L, L]$ is the derived algebra of L, hence all weights are zero on it. This means that

$$T_m * V^\varphi \subset H^{\otimes(m+1)} \otimes_H V^\varphi_n \qquad \text{for} \quad m > 1. \tag{14.14}$$

We have

$$T_{m+1} * v = [T_m * T] * v = T_m * (T * v) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\, T * (T_m * v).$$

We compute the right-hand side, using (3.16), (3.19), and (14.12)–(14.14), and obtain

$$T_{m+1} * v = \sum_{i,j} (\alpha^i_j \otimes \psi h^i_{(1)} \otimes h^i_{(2)} - \alpha^i_j \otimes \varphi \otimes h^i) \otimes_H w^j$$
$$\mod H^{\otimes (m+2)} \otimes_H V^\varphi_n.$$

Now the assumption $T_{m+1} * v \in H^{\otimes (m+2)} \otimes_H V^\varphi_n$ implies that coefficients of all w^j must be zero. Let us choose the highest degree d for which there is some h^i of degree d such that $\alpha^i_j \neq 0$ for some j. Then we get $\alpha^i_j \otimes (\psi - \varphi) = 0$ for all j and all h^i of degree d, hence $\alpha^i_j = 0$, giving a contradiction. This proves that all $\alpha^i_j = 0$, and therefore $T_m * v \in H^{\otimes (m+1)} \otimes_H V^\varphi_n$.

Now, because of nilpotence of L, $T_N = 0$ for $N \gg 0$, and obviously $0 * v \in H^{\otimes (N+1)} \otimes_H V^\varphi_n$. Thus we can pull the statement back to $m = 2$ to obtain that $[T * T]$ maps any lifting v of $\bar{v}$ inside $H^{\otimes 3} \otimes_H V^\varphi_n$.

Now we can choose the lifting v of $\bar{v}$ so that $T * v - (\psi \otimes 1) \otimes_H v \in H^{\otimes 2} \otimes_H V^\varphi_n$. Indeed, performing the same computation as above, using instead of (14.12)

$$T * v = (\psi \otimes 1) \otimes_H v + \sum_{i,j} (\alpha^i_j \otimes h^i) \otimes_H w^j \qquad \mod H^{\otimes 2} \otimes_H V^\varphi_n$$

for some $\alpha^i_j \in H$, we get $\alpha^i_j \otimes (\varphi - \psi) - (\varphi - \psi) \otimes \alpha^i_j = 0$. This shows that $\alpha^i_j = c^i_j(\varphi - \psi)$ for some choice of $c^i_j \in \mathbf{k}$. Now choose v to be the lifting of $\bar{v}$ minimizing the top degree d of h^i such that some α^i_j is nonzero. Then if we replace v by $v' = v + \sum c^i_j h^i w^j$, all coefficients α^i_j in degree d vanish, against minimality of v. This contradiction shows that the lifting v can be chosen in such a way that $\alpha^i_j = 0$ for all i, j, and $T * v = (\psi \otimes 1) \otimes_H v$ modulo $H^{\otimes 2} \otimes_H V^\varphi_n$.

This shows that $Hv + V^\varphi_n$ is indeed a submodule of V, and it satisfies the hypotheses of our claim. Moreover, V^φ_n is of depth n and we can apply the inductive assumption to show that $Hv + V^\varphi_n$ decomposes as a direct sum of L-submodules. This means that we can find a lifting $\tilde{v}$ of $v + V^\varphi_n$ for which $T * \tilde{v} = (\psi \otimes 1) \otimes_H \tilde{v}$ holds exactly.

We have found a lifting $\tilde{v}$ of $\bar{v}$ proving that $V = H\tilde{v} + V^\varphi$. We are left with showing that this is a direct sum of H-modules. This is clear if $\psi \neq 0$ since in this case $H\tilde{v}$ is free, hence projective. If instead $\psi = 0$, assume the sum not to be free. This means that some multiple $h\tilde{v}$ of $\tilde{v}$ lies in V^φ. Since $\tilde{v}$ is killed by T, so is $h\tilde{v}$, showing $h\tilde{v} = 0$ as no other vector in a generalized weight module of nonzero weight φ is killed by T. This concludes the proof in case $\varphi \neq 0$.

If $\varphi = 0$, then we choose a $\mathbf{k}$-basis of V^{φ} modulo V^{φ}_n, and use in (14.13) coefficients of the form $\alpha_j \otimes 1$. The rest of the proof is the same.

Finally, consider the general case of a non-cyclic H-module W. We distinguish two cases. If $\psi \neq 0$, then $W = HW_{\psi}$ is free by Lemma 14.1, and it decomposes as a direct sum of cyclic modules to which we can apply the above argument independently. If $\psi = 0$, then we choose generators $\bar{v}^i$ of W over H, lift them to elements v^i of V in such a way that each of them is mapped by T to zero, and then argue that if $\sum h_i \bar{v}^i = 0$ then $\sum h_i v^i$ is an element of V^{φ} killed by T, hence is zero. Therefore the extension of H-modules splits, and so does that of L-modules, by the above computation. ∎

Now let L be any nilpotent Lie H-pseudoalgebra, V be a finite L-module, and $T \in L$, $T \neq 0$.

Lemma 14.3. *Every T-generalized weight submodule of V is stabilized by the action of L.*

Proof. We set

$$L_{(-1)} = 0, \qquad L_{(i+1)} = \{a \in L \mid [T * a] \in (H \otimes H) \otimes_H L_{(i)}\} \tag{14.15}$$

and

$$V_{(-1)} = 0, \quad V_{(i+1)} = \{v \in V \mid T * v - (\varphi(T) \otimes 1) \otimes_H v \in (H \otimes H) \otimes_H V_{(i)}\}. \tag{14.16}$$

Then the $L_{(i)}$ are H-submodules of L whose union is L (since L is nilpotent), and the $V_{(i)}$ are vector subspaces of V whose H-span is the T-generalized weight space V^{φ} (because $V^{\varphi}_i = HV_{(i)}$ for all i).

It is easy to show by induction on $n = i + j$ that

$$L_{(i)} * V_{(j)} \subset (H \otimes H) \otimes_H V_{(i+j)}. \tag{14.17}$$

Indeed, the basis of induction (say $n = -1$) is trivial, and the inductive step follows from (14.15), (14.16) and the identity $[T * a] * v = T * (a * v) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\, a * (T * v)$. Equation (14.17) implies that $L * V^{\varphi} \subset (H \otimes H) \otimes_H V^{\varphi}$, as desired. ∎

We are now able to complete the proof of Theorem 14.4. Let $V = \bigoplus V_i$ be finest among all decompositions into direct sum of L-submodules of V such that all of the H-torsion of V is contained in one of the V_i. Note that such a finest decomposition always exists, because any decomposition defines a partition of rank V into non-negative integers, and finer decompositions define finer partitions.

We claim that each V_i is a generalized weight module for L. Otherwise, there must be some element $T \in L$ for which some of the V_i is not a T-generalized weight module. But if so, then V_i decomposes into a direct sum of its T-generalized weight submodules, and all torsion elements lie in the T-eigenspace of eigenvalue 0. Since all T-generalized weight submodules are L-invariant, we obtain a contradiction. Therefore V is a direct sum of its generalized weight submodules. ∎

14.6. *Representations of a Lie Pseudoalgebra and of Its Annihilation Algebra*

Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$, and L be a finite Lie H-pseudoalgebra.

Recall that the annihilation algebra $\mathscr{L} = \mathscr{A}(L)$ of L possesses a filtration by subspaces $\mathscr{L} = \mathscr{L}_{-1} \supset \mathscr{L}_0 \supset \cdots$ satisfying (7.14),

$$[\mathscr{L}_i, \mathscr{L}_j] \subset \mathscr{L}_{i+j-s} \qquad \text{for all } i, j \text{ and some fixed } s,$$

that make $\mathscr{L}$ a linearly compact Lie algebra (Proposition 7.4). Moreover, $\mathscr{L}$ is an H-differential algebra, i.e., $\mathfrak{d}$ acts on it by derivations. Let $\mathscr{L}^e := \mathfrak{d} \ltimes \mathscr{L}$ be the extended annihilation algebra. Letting $\mathscr{L}^e_n = \mathscr{L}_n$ for all n makes $\mathscr{L}^e$ a topological Lie algebra as well.

An $\mathscr{L}^e$-module (or $\mathscr{L}$-module) V is called *conformal* if any $v \in V$ is killed by some $\mathscr{L}_n$; in other words, if V is a topological $\mathscr{L}^e$-module when endowed with the discrete topology. Now Proposition 9.1 can be reformulated as follows.

Proposition 14.2. *Any module V over the Lie pseudoalgebra L has a natural structure of a conformal $\mathscr{L}^e$-module, and vice versa. Moreover, V is irreducible as an L-module iff it is irreducible as an $\mathscr{L}^e$-module.*

Together with the next two lemmas, this proposition is an important tool in the study of representation theory of Lie pseudoalgebras.

Lemma 14.4. *Let L be a finite Lie pseudoalgebra and V be a finite L-module. For $n \geqslant -1$, let*

$$\ker_n V = \{v \in V \mid \mathscr{L}_n v = 0\},$$

so that, for example, $\ker_{-1} V = \ker V$ *and* $V = \bigcup_n \ker_n V$. *Then all vector spaces* $\ker_n V/\ker V$ *are finite dimensional.*

Proof. The proof is an application of Lemma 10.3, using the following fact: If A is a vector space and $A_i \supset B_i$ $(i = 1, ..., k)$ are subspaces of A

such that all A_i/B_i are finite dimensional, then $\bigcap A_i/\bigcap B_i$ is finite dimensional. It is enough to show this for $k=2$, in which case it follows from the isomorphism

$$\frac{(A_1 \cap A_2)/(B_1 \cap B_2)}{(A_1 \cap B_2)/(B_1 \cap B_2)} \simeq \frac{A_1 \cap A_2}{A_1 \cap B_2}. \quad \blacksquare$$

Note that $[\mathscr{L}_s, \mathscr{L}_n] \subset \mathscr{L}_n$ for any n, and in particular $\mathscr{L}_s$ is a Lie algebra.

LEMMA 14.5. *Let L be a finite Lie pseudoalgebra and V be a finite L-module such that* $\ker V = 0$. *Then V is locally finite as an $\mathscr{L}_s$-module, i.e., any vector $v \in V$ is contained in a finite-dimensional subspace invariant under $\mathscr{L}_s$.*

Proof. Any $v \in V$ is contained in some $\ker_n V$, which is finite dimensional by Lemma 14.4, and $\mathscr{L}_s$-invariant because $[\mathscr{L}_s, \mathscr{L}_n] \subset \mathscr{L}_n$. $\blacksquare$

Let V be a finite irreducible L-module. Then $\ker V = 0$. Take some n such that $\ker_n V \neq 0$. This space is finite dimensional and $\mathscr{L}_s$-invariant; let U be an irreducible $\mathscr{L}_s$-submodule of $\ker_n V$. The $\mathscr{L}^e$-submodule of V generated by U is a factor of the induced module $\operatorname{Ind}_{\mathscr{L}_s}^{\mathscr{L}^e} U$. Thefore, V is a factor module of $\operatorname{Ind}_{\mathscr{L}_s}^{\mathscr{L}^e} U$.

In many cases $\mathfrak{d}$ acts on $\mathscr{L}$ by inner derivations so that we have an injective homomorphism $\mathfrak{d} \hookrightarrow \mathscr{L}$. In this case, $\mathscr{L}^e$ is isomorphic to the direct sum of $\mathfrak{d}$ and $\mathscr{L}$, and we have $\operatorname{Ind}_{\mathscr{L}_s}^{\mathscr{L}^e} U \simeq H \otimes \operatorname{Ind}_{\mathscr{L}_s}^{\mathscr{L}} U$.

The above results, combined with the results of [Ru1, Ru2] and [Ko], will allow us to classify all finite irreducible representations of all finite semisimple Lie pseudoalgebras (work in progress).

15. COHOMOLOGY OF LIE PSEUDOALGEBRAS

15.1. *The Complexes $C^\bullet(L, M)$ and $\tilde{C}^\bullet(L, M)$*

Recall that in Section 3 we defined cohomology of a Lie algebra in any pseudotensor category (Definition 3.4). Now we will spell out this definition for the case of Lie H-pseudoalgebras, i.e., for the pseudotensor category $\mathscr{M}^*(H)$ (see (3.4)). As before, H is a cocommutative Hopf algebra. Let L be a Lie H-pseudoalgebra and M be an L-module.

By definition, $C^n(L, M)$, $n \geqslant 1$, consists of all

$$\gamma \in \operatorname{Lin}(\{\underbrace{L, ..., L}_{n}\}, M) := \operatorname{Hom}_{H^{\otimes n}}(L^{\otimes n}, H^{\otimes n} \otimes_H M) \qquad (15.1)$$

that are skew-symmetric (see Fig. 4). Explicitly, γ has the following defining properties (cf. (3.23), (3.24)):

H-polylinearity,

$$\gamma(h_1 a_1 \otimes \cdots \otimes h_n a_n) = ((h_1 \otimes \cdots \otimes h_n) \otimes_H 1)\, \gamma(a_1 \otimes \cdots \otimes a_n) \tag{15.2}$$

for $h_i \in H$, $a_i \in L$.

Skew-symmetry,

$$\begin{aligned}\gamma(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_n)&\\ = -(\sigma_{i,\,i+1} \otimes_H \mathrm{id})\, \gamma(a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n),&\end{aligned} \tag{15.3}$$

where $\sigma_{i,\,i+1}\colon H^{\otimes n} \to H^{\otimes n}$ is the transposition of the ith and $(i+1)$st factors.

For $n=0$, we put $C^0(L, M) = \mathbf{k} \otimes_H M \simeq M/H_+ M$, where $H_+ = \{h \in H \mid \varepsilon(h) = 0\}$ is the augmentation ideal. The differential $d\colon C^0(L, M) = \mathbf{k} \otimes_H M \to C^1(L, M) = \mathrm{Hom}_H(L, M)$ is given by

$$\begin{aligned}(d(1 \otimes_H m))(a) = \sum_i (\mathrm{id} \otimes \varepsilon)(h_i)\, m_i \in M&\\ \text{if} \quad a * m = \sum_i h_i \otimes_H m_i \in H^{\otimes 2} \otimes_H M&\end{aligned} \tag{15.4}$$

for $a \in L$, $m \in M$.

For $n \geqslant 1$, the differential $d\colon C^n(L, M) \to C^{n+1}(L, M)$ is given by Fig. 5. Explicitly,

$$\begin{aligned}(d\gamma)(a_1 \otimes \cdots \otimes a_{n+1})&\\ = \sum_{1 \leqslant i \leqslant n+1} (-1)^{i+1} (\sigma_{1 \to i} \otimes_H \mathrm{id})\, a_i * \gamma(a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes a_{n+1})&\\ + \sum_{1 \leqslant i < j \leqslant n+1} (-1)^{i+j} (\sigma_{1 \to i,\, 2 \to j} \otimes_H \mathrm{id})&\\ \times \gamma([a_i * a_j] \otimes a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes \hat{a}_j \otimes \cdots \otimes a_{n+1}),&\end{aligned} \tag{15.5}$$

where $\sigma_{1 \to i}$ is the permutation $h_i \otimes h_1 \otimes \cdots \otimes h_{i-1} \otimes h_{i+1} \otimes \cdots \otimes h_{n+1} \mapsto h_1 \otimes \cdots \otimes h_{n+1}$, and $\sigma_{1 \to i,\, 2 \to j}$ is the permutation $h_i \otimes h_j \otimes h_1 \otimes \cdots \otimes h_{i-1} \otimes h_{i+1} \otimes \cdots \otimes h_{j-1} \otimes h_{j+1} \otimes \cdots \otimes h_{n+1} \mapsto h_1 \otimes \cdots \otimes h_{n+1}$.

In (15.5) we also use the following conventions. If $a * b = \sum_i f_i \otimes_H c_i \in H^{\otimes 2} \otimes_H M$ for $a \in L$, $b \in M$, then for any $f \in H^{\otimes n}$ we set

$$a * (f \otimes_H b) = \sum_i (1 \otimes f)(\mathrm{id} \otimes \Delta^{(n-1)})(f_i) \otimes_H c_i \in H^{\otimes (n+1)} \otimes_H M,$$

where $\Delta^{(n-1)} = (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta) \cdots (\mathrm{id} \otimes \Delta)\,\Delta\colon H \to H^{\otimes n}$ is the iterated comultiplication ($\Delta^{(0)} := \mathrm{id}$). Similarly, if $\gamma(a_1 \otimes \cdots \otimes a_n) = \sum_i g_i \otimes_H v_i \in H^{\otimes n} \otimes_H M$, then for $g \in H^{\otimes 2}$ we set

$$\gamma((g \otimes_H a_1) \otimes a_2 \otimes \cdots \otimes a_n)$$
$$= \sum_i (g \otimes 1^{\otimes(n-1)})(\Delta \otimes \mathrm{id}^{\otimes(n-1)})(g_i) \otimes_H v_i \in H^{\otimes(n+1)} \otimes_H M.$$

These conventions reflect the compositions of polylinear maps in $\mathcal{M}^*(H)$, see (3.8). Note that (15.5) holds also for $n=0$ if we define $\Delta^{(-1)} := \varepsilon$.

The fact that $d^2=0$ is most easily checked using Fig. 5 and the same argument as in the usual Lie algebra case. The cohomology of the resulting complex $C^\bullet(L, M)$ is called the *cohomology of L with coefficients in M* and is denoted by $\mathrm{H}^\bullet(L, M)$.

One can also modify the above definition by replacing everywhere $\otimes_H$ by $\otimes$. Let $\tilde{C}^n(L, M)$ consist of all skew-symmetric $\gamma \in \mathrm{Hom}_{H^{\otimes n}}(L^{\otimes n}, H^{\otimes n} \otimes M)$, cf. (15.2), (15.3). Then we can define a differential $d\colon \tilde{C}^n(L, M) \to \tilde{C}^{n+1}(L, M)$ by (15.5) with $\otimes_H$ replaced everywhere by $\otimes$; then again $d^2=0$. (In fact, one can define a pseudotensor category $\tilde{\mathcal{M}}^*(H)$ by replacing $\otimes_H$ by $\otimes$ everywhere in the definition of $\mathcal{M}^*(H)$.) The corresponding cohomology $\tilde{\mathrm{H}}^\bullet(L, M)$ will be called the *basic cohomology* of L with coefficients in M. In contrast, $\mathrm{H}^\bullet(L, M)$ is sometimes called the *reduced* cohomology (cf. [BKV]).

15.2. *Extensions and Deformations*

We will show that the cohomology theory of Lie pseudoalgebras defined in Section 15.1 describes extensions and deformations, just as any cohomology theory. This result is a straightforward generalization of Theorem 3.1 from [BKV].

THEOREM 5.1. (i) *The isomorphism classes of H-split extensions*

$$0 \to M \to E \to N \to 0$$

of finite modules over a Lie H-pseudoalgebra L are in one-to-one correspondence with elements of $\mathrm{H}^1(L, \mathrm{Chom}(N, M))$.

(ii) *Let C be an L-module, considered as a Lie H-pseudoalgebra with respect to the zero pseudobracket. Then the equivalence classes of H-split "abelian" extensions*

$$0 \to C \to \hat{L} \to L \to 0$$

of the Lie H-pseudoalgebra L correspond bijectively to $\mathrm{H}^2(L, C)$.

(iii) *The equivalence classes of first-order deformations of a Lie H-pseudoalgebra L* (*leaving the H-action intact*) *correspond bijectively to* $\mathrm{H}^2(L, L)$.

Proof. (i) Let

$$0 \longrightarrow M \xrightarrow{i} E \xrightarrow{p} N \longrightarrow 0$$

be an extension of L-modules, which is split over H. Choose a splitting $E = M \oplus N = \{m + n \mid m \in M,\ n \in N\}$ as H-modules. The fact that i and p are homomorphisms of L-modules implies ($a \in L$, $m \in M$, $n \in N$)

$$a *_E m = a *_M m, \tag{15.6}$$

$$a *_E n - a *_N n =: \gamma(a)(n) \in H^{\otimes 2} \otimes_H M. \tag{15.7}$$

It is clear that $\gamma(a) \in \mathrm{Chom}(N, M)$ and $\gamma\colon L \to \mathrm{Chom}(N, M)$ is H-linear; in other words, $\gamma \in C^1(L, \mathrm{Chom}(N, M)) = \mathrm{Hom}_H(L, \mathrm{Chom}(N, M))$.

For $a, b \in L$, $n \in N$, we have (cf. (3.26))

$$[a * b] *_E n = a *_E (b *_E n) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\,(b *_E (a *_E n)),$$

$$[a * b] *_N n = a *_N (b *_N n) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\,(b *_N (a *_N n)).$$

Subtracting these two equations and using (15.6), (15.7), we get

$$\begin{aligned}\gamma([a * b])(n) = a *_M \gamma(b)(n) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\,\gamma(b)(a *_N n)\\ - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\, b *_M \gamma(a)(n) + \gamma(a)(b *_N n)\\ = ((a * \gamma)(b))(n) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})((b * \gamma)(a))(n)\end{aligned}$$

(recall that the action of L on $\mathrm{Chom}(N, M)$ was defined in Remark 10.2). The last equation means that $d\gamma = 0$.

If we choose another splitting of H-modules $E = M \oplus' N = \{m +' n \mid m \in M,\ n \in N\}$, then it will differ by an element φ of $\mathrm{Hom}_H(N, M)$: $m + n = (m + \varphi(n)) +' n$. Then the corresponding

$$\gamma(a)(n) = a *_M \varphi(n) - (\mathrm{id}_{H \otimes H} \otimes_H \varphi)\,(a *_N n) + \gamma'(a)(n).$$

Since $\mathrm{Hom}_H(N, M) \simeq \mathbf{k} \otimes_H \mathrm{Chom}(N, M) = C^0(L, \mathrm{Chom}(N, M))$ (see Remark 10.1), we get $\gamma(a) = a * \varphi + \gamma'(a)$, i.e., $\gamma = d\varphi + \gamma'$.

Conversely, given an element of $\mathrm{H}^1(L, \mathrm{Chom}(N, M))$, we can choose a representative $\gamma \in C^1(L, \mathrm{Chom}(N, M))$ and define an action $*_E$ of L on $E = M \oplus N$ by (15.6), (15.7), which will depend only on the cohomology class of γ. This proves (i).

The proof of (ii) is similar. Write $\hat{L} = L \oplus C = \{a + c \mid a \in L,\, c \in C\}$ as H-modules. Denoting the pseudobracket of $\hat{L}$ by $[a \,\hat{*}\, b]$, we have for $a, b \in L$, $c, c_1 \in C$:

$$[a \,\hat{*}\, c] = a * c,$$

$$[c \,\hat{*}\, c_1] = 0,$$

$$[a \,\hat{*}\, b] - [a * b] =: \gamma(a \otimes b) \in H^{\otimes 2} \otimes_H C.$$

It is clear that $\gamma \in C^2(L, C)$, and the Jacobi identity for $\hat{L}$ implies $d\gamma = 0$.

(iii) A first-order deformation of L is the structure of a Lie H-pseudoalgebra on $\hat{L} = L[\epsilon]/(\epsilon^2) = L \oplus L\epsilon$, where H acts trivially on ϵ, such that the map $\hat{L} \to L$ given by putting $\epsilon = 0$ is a homomorphism of Lie pseudoalgebras. This means that

$$0 \to L\epsilon \to \hat{L} \to L \to 0$$

is an abelian extension of Lie pseudoalgebras, so (iii) follows from (ii). ∎

15.3. *Relation to Gelfand–Fuchs Cohomology*

Let again L be a Lie H-pseudoalgebra and $\mathscr{L} = \mathscr{A}(L) := X \otimes_H L$ be its annihilation Lie algebra. Recall that (by Proposition 9.1) any L-module M has a natural structure of an $\mathscr{L}$-module, given by $(x \otimes_H a) \cdot m = a_x m$ ($a \in L$, $x \in X$, $m \in M$), where $a_x m$ is the x-product defined by (cf. (9.6))

$$a_x m = \sum_i \langle S(x), g_i \rangle \, v_i \qquad \text{if} \quad a * m = \sum_i (g_i \otimes 1) \otimes_H v_i \in H^{\otimes 2} \otimes_H M.$$

Similarly, for $\gamma \in \tilde{C}^n(L, M)$ and $x_1, ..., x_n \in X$, we define

$$\gamma_{x_1, ..., x_n}(a_1 \otimes \cdots \otimes a_n) = \sum_i \langle S(x_1), g_{i,1} \rangle \cdots \langle S(x_n), g_{i,n} \rangle \, v_i$$

if

$$\gamma(a_1 \otimes \cdots \otimes a_n) = \sum_i (g_{i,1} \otimes \cdots \otimes g_{i,n}) \otimes v_i \in H^{\otimes n} \otimes M.$$

The H-polylinearity (15.2) of γ implies that the map $\mathscr{A}\gamma\colon \mathscr{L}^{\otimes n} \to M$, given by

$$(\mathscr{A}\gamma)((x_1 \otimes_H a_1) \otimes \cdots \otimes (x_n \otimes_H a_n)) := \gamma_{x_1, ..., x_n}(a_1 \otimes \cdots \otimes a_n),$$

is well defined. Moreover, $\mathscr{A}\gamma$ is skew-symmetric (i.e., it is a map from $\bigwedge^n \mathscr{L}$ to M) because of skew-symmetry (15.3) of γ.

Therefore, we can consider $\mathscr{A}\gamma$ as an n-cochain for the Lie algebra $\mathscr{L}$ with coefficients in M. It is not difficult to check that the map $\mathscr{A}: \tilde{C}^n(L, M) \to C^n(\mathscr{L}, M)$ commutes with the differentials (this also follows from the results of Section 7.2). The following result is proved in the same way as Proposition 9.1.

Proposition 15.1. *The above map $\mathscr{A}: \tilde{C}^\bullet(L, M) \to C^\bullet(\mathscr{L}, M)$ is an isomorphism from the complex $\tilde{C}^\bullet(L, M)$ to the subcomplex $C^\bullet_{\mathrm{GF}}(\mathscr{L}, M)$ of $C^\bullet(\mathscr{L}, M)$ consisting of local cochains, i.e., cochains $\mathscr{A}\gamma$ satisfying*

$$(\mathscr{A}\gamma)((x_1 \otimes_H a_1) \otimes \cdots \otimes (x_n \otimes_H a_n)) = 0, \tag{15.8}$$

for any fixed $x_2, ..., x_n$ and $a_1, ..., a_n$, and $x_1 \in \mathrm{F}_k X$ for $k \gg 0$.

Note that the locality condition (15.8) means that $\mathscr{A}\gamma$ is continuous when M is endowed with the discrete topology and $\mathscr{L}$ with the topology defined in Section 7.4. Therefore we have

Corollary 15.1. *The basic cohomology $\tilde{\mathrm{H}}^\bullet(L, M)$ of a Lie pseudoalgebra L is isomorphic to the Gelfand–Fuchs cohomology $\mathrm{H}^\bullet_{\mathrm{GF}}(\mathscr{L}, M)$ of its annihilation Lie algebra $\mathscr{L}$.*

Recall that H acts on $\mathscr{L} = X \otimes_H L$ via its left action on X: $h(x \otimes_H a) = hx \otimes_H a$ $(h \in H, x \in X, a \in L)$. Using the comultiplication $\Delta^{(n-1)}(h) = \sum h_{(1)} \otimes \cdots \otimes h_{(n)}$, we also get an action of H on $\mathscr{L}^{\otimes n}$. It follows from (2.18), (2.25) that for $h \in H$, $\alpha \in \mathscr{L}^{\otimes n}$, $\gamma \in \tilde{C}^n(L, M)$, one has

$$(\mathscr{A}\gamma)(h\alpha) = (\mathscr{A}(\gamma \cdot h))(\alpha),$$

where $\gamma \cdot h \in \tilde{C}^n(L, M)$ is defined by

$$(\gamma \cdot h)(a_1 \otimes \cdots \otimes a_n) = \sum_i g_i \Delta^{(n-1)}(h) \otimes v_i$$

if

$$\gamma(a_1 \otimes \cdots \otimes a_n) = \sum_i g_i \otimes v_i \in H^{\otimes n} \otimes M.$$

Considering $C^n(L, M)$ instead of $\tilde{C}^n(L, M)$ amounts to replacing $\otimes$ by $\otimes_H$, i.e., to factoring by the relations

$$(\gamma \cdot h)(a_1 \otimes \cdots \otimes a_n) - (1^{\otimes n} \otimes h)\, \gamma(a_1 \otimes \cdots \otimes a_n), \qquad h \in H.$$

In terms of $\mathscr{A}\gamma$, this corresponds to factoring by

$$h((\mathscr{A}\gamma)(\alpha)) - (\mathscr{A}\gamma)(h\alpha) = (h \circ (\mathscr{A}\gamma) - (\mathscr{A}\gamma) \circ h)(\alpha).$$

This implies the next result.

PROPOSITION 15.2. *The isomorphism* $\mathscr{A}\colon \tilde{C}^\bullet(L, M) \xrightarrow{\sim} C^\bullet_{\mathrm{GF}}(\mathscr{L}, M)$ *induces an isomorphism from* $C^\bullet(L, M)$ *to the quotient complex of* $C^\bullet_{\mathrm{GF}}(\mathscr{L}, M)$ *by the subcomplex* $\{h \circ c - c \circ h \mid c \in C^\bullet_{\mathrm{GF}}(\mathscr{L}, M),\ h \in H\}$.

When $H = U(\mathfrak{d})$, we can define an action of H on $C^\bullet_{\mathrm{GF}} \equiv C^\bullet_{\mathrm{GF}}(\mathscr{L}, M)$ by $h \cdot c := h \circ c + c \circ S(h)$. This action commutes with the differential d, and $\mathscr{A}$ induces an isomorphism from $C^\bullet(L, M)$ to the quotient complex $C^\bullet_{\mathrm{GF}}/H \cdot C^\bullet_{\mathrm{GF}}$. The Lie algebra $\mathfrak{d}$ acts on $C^\bullet_{\mathrm{GF}}$, and clearly $C^\bullet_{\mathrm{GF}}/H \cdot C^\bullet_{\mathrm{GF}} = C^\bullet_{\mathrm{GF}}/\mathfrak{d} \cdot C^\bullet_{\mathrm{GF}}$. We have an exact sequence of complexes

$$0 \to \mathfrak{d} \cdot C^\bullet_{\mathrm{GF}} \to C^\bullet_{\mathrm{GF}} \to C^\bullet_{\mathrm{GF}}/\mathfrak{d} \cdot C^\bullet_{\mathrm{GF}} \to 0,$$

which gives a long exact sequence for cohomology

$$\begin{aligned} \cdots \to \mathrm{H}^i(\mathfrak{d} \cdot C^\bullet_{\mathrm{GF}}) \to \mathrm{H}^i(C^\bullet_{\mathrm{GF}}) \to \mathrm{H}^i(C^\bullet_{\mathrm{GF}}/\mathfrak{d} \cdot C^\bullet_{\mathrm{GF}}) \\ \to \mathrm{H}^{i+1}(\mathfrak{d} \cdot C^\bullet_{\mathrm{GF}}) \to \mathrm{H}^{i+1}(C^\bullet_{\mathrm{GF}}) \to \cdots . \end{aligned} \tag{15.9}$$

Remark 15.1. [BKV]. If $\dim \mathfrak{d} = 1$, then $\mathfrak{d}$ acts freely on C^i_{GF} for $i > 0$, and we have $\mathrm{H}^i(\mathfrak{d} \cdot C^\bullet_{\mathrm{GF}}) \simeq \mathrm{H}^i(C^\bullet_{\mathrm{GF}})$ for $i > 0$. When M is a free H-module, this is also true for $i = 0$.

PROPOSITION 15.3. *Assume that* $\mathfrak{d}$ *acts on* $\mathscr{L}$ *by inner derivations and that the action of* $\mathfrak{d}$ *on* M *coincides with that of its image in* $\mathscr{L}$. *Then for any* $i \geqslant 0$, *we have isomorphisms*

$$\mathrm{H}^i(L, M) \simeq \mathrm{H}^i_{\mathrm{GF}}(\mathscr{L}, M) \oplus \mathrm{H}^{i+1}(\mathfrak{d} \cdot C^\bullet_{\mathrm{GF}}). \tag{15.10}$$

If, in addition, $\dim \mathfrak{d} = 1$, *then we have for* $i \geqslant 0$

$$\mathrm{H}^i(L, M) \simeq \mathrm{H}^i_{\mathrm{GF}}(\mathscr{L}, M) \oplus \mathrm{H}^{i+1}_{\mathrm{GF}}(\mathscr{L}, M). \tag{15.11}$$

Proof. Since the adjoint action of $\mathscr{L}$ on $\mathrm{H}^\bullet_{\mathrm{GF}}(\mathscr{L}, M)$ is trivial, we obtain that $\mathrm{H}^i(\mathfrak{d} \cdot C^\bullet_{\mathrm{GF}})$ maps to zero in the exact sequence (15.9). Therefore we have exact sequences

$$0 \to \mathrm{H}^i(C^\bullet_{\mathrm{GF}}) \to \mathrm{H}^i(C^\bullet_{\mathrm{GF}}/\mathfrak{d} \cdot C^\bullet_{\mathrm{GF}}) \to \mathrm{H}^{i+1}(\mathfrak{d} \cdot C^\bullet_{\mathrm{GF}}) \to 0,$$

which lead to isomorphisms (15.10). Formula (15.11) follows from Remark 15.1.

Note that in general we have

$$\dim \mathrm{H}^i(L, M) \leqslant \dim \mathrm{H}^i_{\mathrm{GF}}(\mathscr{L}, M) + \dim \mathrm{H}^{i+1}(\mathfrak{d} \cdot C^{\bullet}_{\mathrm{GF}}).$$

The above results provide a tool for computing the cohomology of Lie pseudoalgebras, by making use of the known results on Gelfand–Fuchs cohomology of Lie algebras of vector fields [Fu].

15.4. *Central Extensions of Finite Simple Lie Pseudoalgebras*

In this section we determine by a direct computation all nontrivial central extensions of a finite simple Lie pseudoalgebra L with trivial coefficients (see Theorem 15.2 below).

Such a central extension of L is isomorphic as an H-module to $\hat{L} = L \oplus \mathbf{k}1$, where the action of H on 1 is given by $h \cdot 1 = \varepsilon(h)\, 1$. The pseudobracket is then

$$[a \mathbin{\hat{*}} b] = [a * b] + \gamma(a, b) \otimes_H 1, \qquad a, b \in L, \tag{15.12}$$

where $\gamma(a, b) \in H \otimes H$.

Notice that a tensor product $(h^1 \otimes \cdots \otimes h^n) \otimes_H 1 \in H^{\otimes n} \otimes_H \mathbf{k}$ can always be re-expressed as $(h^1 h^n_{(-1)} \otimes \cdots \otimes h^{n-1} h^n_{(-(n-1))} \otimes 1) \otimes_H 1$, and this coefficient is unique in $H^{\otimes (n-1)} \otimes 1$ (see Lemma 2.3).

Therefore, the above bracket is uniquely determined by the unique $\beta(a, b) \in H$ such that

$$\gamma(a, b) \otimes_H 1 = (\beta(a, b) \otimes 1) \otimes_H 1; \tag{15.13}$$

we will call this map $\beta\colon L \otimes L \to H$ the *cocycle* representing the central extension. Then H-bilinearity and skew-symmetry of the pseudobracket give the following properties of this cocycle,

$$\beta(ha, b) = h\beta(a, b), \qquad \beta(a, hb) = \beta(a, b)\, S(h), \qquad \beta(a, b) = -S(\beta(b, a)), \tag{5.14}$$

for all $a, b \in L$, $h \in H$.

Two central extensions $\hat{L}_i = L \oplus \mathbf{k}1$ $(i = 1, 2)$ are called equivalent if there exists a Lie pseudoalgebra isomorphism $\Phi\colon \hat{L}_1 \to \hat{L}_2$ of the form $\Phi(a \oplus c1) = a \oplus (\phi(a) + c)\, 1$, where $\phi\colon L_1 \to \mathbf{k}$ is an H-linear map. Then the cocycles representing the two equivalent central extensions differ by $\tau_\phi(a, b)$ such that

$$(\tau_\phi(a, b) \otimes 1) \otimes_H 1 = (\mathrm{id}_{H \otimes H} \otimes_H \phi)([a * b]). \tag{15.15}$$

This is called a *trivial cocycle.*

If $L = He$ is an $H = U(\mathfrak{d})$-module which is free on the generator e, such that

$$[e * e] = \alpha \otimes_H e, \qquad \alpha = r + s \otimes 1 - 1 \otimes s, \qquad r \in \mathfrak{d} \wedge \mathfrak{d},\ s \in \mathfrak{d},$$

then a cocycle $\beta(a, b)$ is completely determined by its value $\beta = \beta(e, e) \in H$. Trivial cocycles are of the form

$$\tau = \tau(e, e) = \phi(e)(2s - x), \qquad \phi(e) \in \mathbf{k},$$

where

$$x = \tfrac{1}{2} \sum_{i, j} r^{ij} [\partial_i, \partial_j] \qquad \text{if} \quad r = \sum_{i, j} r^{ij} \partial_i \otimes \partial_j. \tag{15.16}$$

LEMMA 15.1. *Let $L = He$ be a Lie pseudoalgebra as above. Then $\mathrm{H}^2(L, \mathbf{k}) \simeq B/\mathbf{k}(2s - x)$, where B is the space of elements $\beta \in H$ satisfying the following two conditions*

$$\beta = -S(\beta), \tag{15.17}$$

$$\alpha \Delta(\beta) = (\beta \otimes 1 + 1 \otimes \beta)\, \alpha + \beta \otimes (3s - x) - (3s - x) \otimes \beta. \tag{15.18}$$

Moreover, when $r \neq 0$, then $\beta \in \mathfrak{d}$, and (15.18) *becomes equivalent to the following system of equations*

$$[s, \beta] = 0, \tag{15.19}$$

$$[r, \Delta(\beta)] = \beta \otimes (3s - x) - (3s - x) \otimes \beta. \tag{15.20}$$

Proof. Let $\hat{L} = He + \mathbf{k}1$ be a central extension of L with a pseudo-bracket

$$[e \,\hat{*}\, e] = \alpha \otimes_H e + (\beta \otimes 1) \otimes_H 1,$$

where $h \cdot 1 = \varepsilon(h)\, 1$ for $h \in H$.

The skew-symmetry of $[e \,\hat{*}\, e]$ is equivalent to (15.17). The Jacobi identity is equivalent to Jacobi identity for $[e * e]$ together with the following cocycle condition for $\gamma = \beta \otimes 1$ (cf. Proposition 4.1):

$$(\alpha \otimes 1)\,(\Delta \otimes \mathrm{id})(\gamma) \otimes_H 1 = (1 \otimes \alpha)\,(\mathrm{id} \otimes \Delta)(\gamma) \otimes_H 1 \\ - (\sigma \otimes \mathrm{id})((1 \otimes \alpha)(\mathrm{id} \otimes \Delta)(\gamma)) \otimes_H 1. \qquad (15.21)$$

With the usual notation $r_{12} = r \otimes 1$, $s_1 = s \otimes 1 \otimes 1$, etc., we have

$$(\alpha \otimes 1)(\Delta \otimes \mathrm{id})(\gamma) \otimes_H 1 = (\alpha \Delta(\beta) \otimes 1) \otimes_H 1$$

and

$$\begin{aligned}(1 \otimes \alpha)(\mathrm{id} \otimes \Delta)(\gamma) \otimes_H 1 &= (r_{23} + s_2 - s_3)\,\beta_1 \otimes_H 1 \\ &= \beta_1 (r_{23} + s_2 - s_3) \otimes_H 1 \\ &= \beta_1 (-r_{21} - x_2 + s_1 + 2s_2) \otimes_H 1 \\ &= \beta_1 (\alpha_{12} + 3s_2 - x_2) \otimes_H 1.\end{aligned}$$

From here it is easy to see that (15.21) is equivalent to (15.18).

Let now r be nonzero. Rewrite (15.18) in the form

$$\alpha(\Delta(\beta) - \beta \otimes 1 - 1 \otimes \beta) = [\beta \otimes 1 + 1 \otimes \beta, \alpha] + \beta \otimes (3s - x) - (3s - x) \otimes \beta.$$

If $\beta \notin \mathfrak{d} + \mathbf{k}$, then the degree of the left-hand side equals $\deg \beta + 2$ while that of the right-hand side is at most $\deg \beta + 1$, giving a contradiction. So $\beta \in \mathfrak{d} + \mathbf{k}$, and (15.17) shows that $\beta \in \mathfrak{d}$. ∎

PROPOSITION 15.4. *Let $\mathfrak{d}' \subset \mathfrak{d}$ be finite-dimensional Lie algebras, $H = U(\mathfrak{d})$, $H' = U(\mathfrak{d}')$, and let $L = \mathrm{Cur}_{H'}^{H} W(\mathfrak{d}')$.*

(i) *If* $\dim \mathfrak{d}' = 1$, *then* $\mathrm{H}^2(L, \mathbf{k})$ *is* 1-*dimensional.*

(ii) *If* $\mathfrak{d}$ *is abelian and* $\dim \mathfrak{d}' > 1$, *then* $\mathrm{H}^2(L, \mathbf{k}) = 0$.

Proof. (i) The Lie pseudoalgebra $L = He$ is free of rank one, with $e = 1 \otimes s$, $s \in \mathfrak{d}' \backslash \{0\}$, hence we can use Lemma 15.1. In this case $\alpha = s \otimes 1 - 1 \otimes s$, and Eq. (15.18) becomes

$$(s \otimes 1 - 1 \otimes s)\Delta(\beta) = 2(\beta \otimes s - s \otimes \beta) + \beta s \otimes 1 - 1 \otimes \beta s$$

for $\beta \in H$. We choose a basis $\{\partial_i\}$ of $\mathfrak{d}$ such that $\partial_1 = s$, and express β in a Poincaré–Birkhoff–Witt basis as $\beta = \sum_I \beta_I \partial^{(I)}$, $\beta_I \in \mathbf{k}$ (see Example 2.1). Then the above equation becomes

$$\sum_{I, J} \beta_{I+J}(\partial_1 \partial^{(I)} \otimes \partial^{(J)} - \partial^{(I)} \otimes \partial_1 \partial^{(J)})$$
$$= \sum_I 2\beta_I(\partial^{(I)} \otimes \partial_1 - \partial_1 \otimes \partial^{(I)}) + \sum_I \beta_I(\partial^{(I)}\partial_1 \otimes 1 - 1 \otimes \partial^{(I)}\partial_1).$$

Comparing terms of the form $h \otimes \partial_j$ $(j \neq 1)$ we find that β_I is zero unless $I = (i, 0, ..., 0)$ for some i. Hence $\beta = \sum_i \beta_i s^i$, $\beta_i \in \mathbf{k}$. Substituting and comparing coefficients, we obtain that $\beta = \beta_1 s + \beta_3 s^3$. This obviously satisfies (15.17). The trivial cocycles are multiples of $2s$, hence s^3 is the unique central extension up to scalar multiples. This is the well-known Virasoro central extension.

(ii) Choose a basis of $\mathfrak{d}'$ and let β be a cocycle representing a central extension of $L \simeq H \otimes \mathfrak{d}'$. Then for each basis element a, β restricts to a cocycle of $H \otimes a \subset L$, which is a current Lie pseudoalgebra over $W(\mathbf{k}a)$. By part (i) we can then add to β a trivial cocycle as to make $\beta(1 \otimes a, 1 \otimes a) = c_a a^3$, $c_a \in \mathbf{k}$, for every such basis element $a \in \mathfrak{d}'$. Denoting $\beta = \beta(1 \otimes a, 1 \otimes b)$, the Jacobi identity for elements $1 \otimes a, 1 \otimes a, 1 \otimes b$ then gives

$$(a \otimes 1 - 1 \otimes a)\, \Delta(\beta) = c_a(a^3 \otimes b - b \otimes a^3) + (\beta \otimes 1 - 1 \otimes \beta)\, \Delta(a). \tag{15.22}$$

Let a, b be distinct elements in the above basis, which we extend to a basis $\{\partial_i\}$ of $\mathfrak{d}$ with $\partial_1 = a$, $\partial_2 = b$. We substitute the Poincaré–Birkhoff–Witt basis expression $\beta = \sum_I \beta_I \partial^{(I)}$ in (15.22), to get

$$c_a(\partial_1^3 \otimes \partial_2 - \partial_2 \otimes \partial_1^3)$$
$$= \sum_{I, J} \beta_{I+J}(\partial_1 \partial^{(I)} \otimes \partial^{(J)} - \partial^{(I)} \otimes \partial_1 \partial^{(J)})$$
$$- \sum_J \beta_J(\partial_1 \partial^{(J)} \otimes 1 + \partial^{(J)} \otimes \partial_1 - \partial_1 \otimes \partial^{(J)} - 1 \otimes \partial_1 \partial^{(J)}).$$

Comparing coefficients of the form $h \otimes \partial_j$ for $j \neq 1$, we find that β_I can be nonzero only when $I = (2, 1, 0, ..., 0)$, in which case $\beta_I = 2c_a$, and when $I = (i, 0, ..., 0)$ for some i. This means that $\beta = \beta(1 \otimes a, 1 \otimes b) = f(a) + c_a a^2 b$ for some polynomial f.

We can repeat the same argument after switching the roles of a and b, to get: $\beta(1 \otimes b, 1 \otimes a) = g(b) + c_b b^2 a$. Then the skew-symmetry $\beta(1 \otimes a, 1 \otimes b) = -S(\beta(1 \otimes b, 1 \otimes a))$ implies: $f(a) + c_a a^2 b = -g(-b) + c_b a b^2$. This is possible only when $f = 0$, $c_a = 0$. Therefore β is identically zero. ∎

PROPOSITION 15.5. *Let $\mathfrak{d}' \subset \mathfrak{d}$ be abelian finite-dimensional Lie algebras, $H = U(\mathfrak{d})$, $H' = U(\mathfrak{d}')$, and let $L = \mathrm{Cur}_{H'}^{H} H(\mathfrak{d}', \chi, \omega)$. Then $\mathrm{H}^2(L, \mathbf{k})$ is isomorphic to $\mathfrak{d}$ if $\chi = 0$, and is trivial otherwise.*

Proof. L is free of rank one and $r \neq 0$, hence (15.18) becomes $3(\beta \otimes s - s \otimes \beta) = 0$, $\beta \in \mathfrak{d}$. This is satisfied only by multiples of s if $s \neq 0$ and by all elements of $\mathfrak{d}$ otherwise. Since $\mathfrak{d}$ is abelian, then $x = 0$ and trivial cocycles are multiples of s. ∎

PROPOSITION 15.6. *Let $\mathfrak{d}'$ be the Heisenberg Lie algebra of dimension $N = 2n + 1 \geqslant 3$, and $\mathfrak{d} = \mathfrak{d}' \oplus \mathfrak{d}_0$ be the direct sum of $\mathfrak{d}'$ and an abelian Lie algebra $\mathfrak{d}_0$. Let $H = U(\mathfrak{d})$, $H' = U(\mathfrak{d}')$, and $L = \mathrm{Cur}_{H'}^{H} K(\mathfrak{d}', \theta)$. Then $\mathrm{H}^2(L, \mathbf{k}) = 0$.*

Proof. L is free of rank one, and

$$\alpha = \sum_{i=1}^{n} (a_i \otimes b_i - b_i \otimes a_i) - c \otimes 1 + 1 \otimes c,$$

where $\{a_i, b_i, c\}$ is a basis of $\mathfrak{d}'$ with the only nonzero commutation relations $[a_i, b_i] = c$, $1 \leqslant i \leqslant n$ (see Example 8.4).

It is immediate to check that $[r, d \otimes 1 + 1 \otimes d] = c \otimes d - d \otimes c$ for all $d \in \mathfrak{d}'$. Moreover, the element x from (15.16) equals nc. Then, if $\beta = \beta' + \beta_0$ with $\beta' \in \mathfrak{d}'$, $\beta_0 \in \mathfrak{d}_0$, Eq. (15.20) becomes

$$\beta' \otimes c - c \otimes \beta' = (n+3)(\beta \otimes c - c \otimes \beta).$$

All solutions β of this equation are multiples of c. Trivial cocycles are multiples of $2s - x = -(n+2)\, c$, hence all cocycles are trivial. ∎

PROPOSITION 15.7. *Let $\mathfrak{d}' \subset \mathfrak{d}$ be abelian finite-dimensional Lie algebras such that* $\dim \mathfrak{d}' > 2$, *let $H = U(\mathfrak{d})$, $H' = U(\mathfrak{d}')$, and let $L = \mathrm{Cur}_{H'}^{H} S(\mathfrak{d}', 0)$. Then $\mathrm{H}^2(L, \mathbf{k}) = 0$.*

Proof. By Proposition 8.1, L is spanned over H by elements

$$e_{ab} = a \otimes b - b \otimes a, \qquad a, b \in \mathfrak{d}',$$

satisfying the relations $e_{ab} = -e_{ba}$ and

$$a e_{bc} + b e_{ca} + c e_{ab} = 0. \tag{15.23}$$

The pseudobrackets are (see (8.24))

$$\begin{aligned}[] [e_{ab} * e_{cd}] = {} & (a \otimes d) \otimes_H e_{bc} + (b \otimes c) \otimes_H e_{ad} \\ & - (a \otimes c) \otimes_H e_{bd} - (b \otimes d) \otimes_H e_{ac},\end{aligned}$$

and in particular

$$[e_{ab} * e_{ac}] = -(a \otimes c) \otimes_H e_{ab} + (b \otimes a) \otimes_H e_{ac} - (a \otimes a) \otimes_H e_{bc},$$

$$[e_{ab} * e_{ab}] = (b \otimes a - a \otimes b) \otimes_H e_{ab}.$$

Trivial cocycles τ_ϕ are determined by the identity (see (15.15))

$$\begin{aligned}(\tau_\phi(e_{ab}, e_{cd}) \otimes 1) \otimes_H 1 \\ = (a \otimes d) \otimes_H \phi_{bc} + (b \otimes c) \otimes_H \phi_{ad} - (a \otimes c) \otimes_H \phi_{bd} - (b \otimes d) \otimes_H \phi_{ac},\end{aligned}$$

where $\phi_{ab} = \phi(e_{ab}) = -\phi_{ba} \in \mathbf{k}$, which gives

$$\tau_\phi(e_{ab}, e_{cd}) = -ad\phi_{bc} - bc\phi_{ad} + ac\phi_{bd} + bd\phi_{ac}. \tag{15.24}$$

Let β be a cocycle for L representing a central extension. Write $\beta_{ab,\,cd} = \beta(e_{ab}, e_{cd})$ for short. Equations (15.14), (15.23) give the identities

$$\beta_{ab,\,cd} = -\beta_{ba,\,cd} = -\beta_{ab,\,dc} = -S(\beta_{cd,\,ab}), \tag{15.25}$$

$$a\beta_{bc,\,cd} + b\beta_{ca,\,cd} + c\beta_{ab,\,cd} = 0. \tag{15.26}$$

Using this, Jacobi identity for the elements e_{ab}, e_{ab}, e_{ac} gives the following equation for β

$$\begin{aligned}(b \otimes a - a \otimes b)(\Delta(\beta_{ab,\,ac}) - \beta_{ab,\,ac} \otimes 1 - 1 \otimes \beta_{ab,\,ac}) \\ = ab \otimes \beta_{ab,\,ac} - \beta_{ab,\,ac} \otimes ab + \beta_{ab,\,bc} \otimes a^2 - a^2 \otimes \beta_{ab,\,bc} \\ + \beta_{ab,\,ab} \otimes ac - ac \otimes \beta_{ab,\,ab}.\end{aligned} \tag{15.27}$$

This is a homogeneous equation and can be solved degree by degree. If β is homogeneous of degree one, then the left-hand side is zero, and we immediately see that $\beta_{ab,\,ac} = \beta_{ab,\,bc} = \beta_{ab,\,ab} = 0$. Then, by (15.26), $\beta_{ab,\,cd} = 0$.

If β is homogeneous of degree other than one, then $\beta_{ab,\,ab} = 0$, since β restricts to a cocycle of the free rank one Lie pseudoalgebra He_{ab}, which has been shown in Proposition 15.5 to take values in $\mathfrak{d}$. Then Eqs. (15.25), (15.26) give $a\beta_{ab,\,bc} = b\beta_{ab,\,ac}$. Hence if a and b are linearly independent, we can find some $p = p^a_{bc} \in H$ such that

$$\beta_{ab,\,ac} = ap^a_{bc}, \qquad \beta_{ab,\,bc} = bp^a_{bc}. \tag{15.28}$$

We substitute this into (15.27) and after simplification obtain $\Delta(p) = p \otimes 1 + 1 \otimes p$. Therefore, $p \in \mathfrak{d}$, hence the only nonzero solutions to (15.27) occur in degree two.

Now using (15.26) and (15.28), we get

$$\beta_{ab,\,cd} = ap^c_{bd} - bp^c_{ad}\,. \tag{15.29}$$

The skew-symmetry $\beta_{ab,\,cd} = -\beta_{cd,\,ab}$ gives the equations $p^a_{bc} = -p^a_{cb}$ and

$$ap^c_{bd} - bp^c_{ad} = cp^a_{bd} - dp^a_{bc}\,. \tag{15.30}$$

From this we obtain that p^a_{bd} lies in the linear span of a, b, d. Comparing the coefficients in front of ac in (15.30), we see that the coefficient of a in p^a_{bd} is equal to the coefficient of c in p^c_{bd}. Call this coefficient ϕ_{bd}; then $\phi_{bd} = -\phi_{db}$. Then comparison of other coefficients in (15.30) shows that

$$p^a_{bc} = a\phi_{bc} + b\phi_{ca} + c\phi_{ab} \tag{15.31}$$

for all $a, b, c \in \mathfrak{d}'$. Substitute this in (15.29) to obtain that $\beta = \tau_\phi$ is trivial (cf. (15.24)). ∎

Proposition 15.8. *Let $H = U(\mathfrak{d})$, and let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. If $L = \operatorname{Cur}\mathfrak{g}$, then $\mathrm{H}^2(L, \mathbf{k}) \simeq \mathfrak{d}$.*

Proof. Let β be a cocycle for L. We will write $\beta(a, b) = \beta(1 \otimes a, 1 \otimes b)$ for $a, b \in \mathfrak{g}$. Then Jacobi identity leads to the equation

$$\beta(a, [b, c]) \otimes 1 - 1 \otimes \beta(b, [a, c]) = \Delta(\beta([a, b], c)). \tag{15.32}$$

This immediately implies $\beta([a, b], c) \in \mathfrak{d} + \mathbf{k}$. Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ this shows that $\beta(a, b) \in \mathfrak{d} + \mathbf{k}$ for all $a, b \in \mathfrak{g}$.

We can now solve the homogeneous equation (15.32) degree by degree. Solutions of degree zero are cocycles of the Lie algebra $\mathfrak{g}$, hence they are all trivial. Solutions of degree one satisfy $\beta(a, [b, c]) = \beta([a, b], c)$, and skew-symmetry implies $\beta(a, b) = -S(\beta(b, a)) = \beta(b, a)$. Therefore every such β is an invariant symmetric bilinear form on $\mathfrak{g}$ with values in $\mathfrak{d}$. Any such bilinear form can be written as $\beta(a, b) = (a \mid b)\, d$ where $(\cdot \mid \cdot)$ is the Killing form on $\mathfrak{g}$ and d is some element of $\mathfrak{d}$. Such cocycles β are nontrivial, hence inequivalent central extensions are in one-to-one correspondence with elements of $\mathfrak{d}$. ∎

Theorem 15.2. *Let $H = U(\mathfrak{d})$ and L be a simple Lie H-pseudoalgebra. Then L may have a nontrivial central extension (given by (15.12), (15.13)) only if*

(i) $L = \mathrm{Cur}\,\mathfrak{g}$, *in which case* $\mathrm{H}^2(L, \mathbf{k}) \simeq \mathfrak{d}$ *and cocycles* β *are given by* $\beta_d(1 \otimes a, 1 \otimes b) = (a \mid b)\, d$ *for* $a, b \in \mathfrak{g}$, *where* $d \in \mathfrak{d}$ *and* $(\cdot|\cdot)$ *is the Killing form.*

(ii) $L = \mathrm{Cur}_{H'}^{H} W(\mathfrak{d}')$ *with* $\mathbf{k}s = \mathfrak{d}' \subset \mathfrak{d}$, $\dim \mathfrak{d}' = 1$, *in which case* $\mathrm{H}^2(L, \mathbf{k})$ *is of dimension one, generated by the Virasoro cocycle* $\beta(1 \otimes s, 1 \otimes s) = s^3$.

(iii) $L = \mathrm{Cur}_{H'}^{H} H(\mathfrak{d}', \chi', \omega')$ *with* $\mathfrak{d}' \subset \mathfrak{d}$, *in which case* $\mathrm{H}^2(L, \mathbf{k})$ *is isomorphic to the quotient of the space of all solutions* $\beta \in \mathfrak{d}$ *to Eqs.* (15.19), (15.20) *by the subspace* $\mathbf{k}(2s - x)$, *where* $r \in \mathfrak{d}' \wedge \mathfrak{d}'$ *is dual to* ω', $s \in \mathfrak{d}'$ *is such that* $\chi' = \iota_s \omega'$, *and* x *is given by* (15.16).

Proof. (i) and (ii) follow from Propositions 15.8 and 15.4(i), and (iii) from a direct application of Lemma 15.1.

For any other simple pseudoalgebra L, the strategy is to construct a continuous family of pseudoalgebras L_t, indexed by $t \in \mathbf{k}$ endowed with the Zariski topology, that are all isomorphic to L when $t \neq 0$, and whose fiber at $t = 0$ is one of the pseudoalgebras already considered in Propositions 15.4(ii), 15.6, and 15.7. Then, since $\mathrm{H}^2(L_t, \mathbf{k}) = 0$ for $t = 0$, it will follow that $\mathrm{H}^2(L_t, \mathbf{k}) = 0$ whenever t lies in a neighborhood of 0, hence for all $t \in \mathbf{k}$.

In the case of a current pseudoalgebra over a W or S type Lie pseudoalgebra, choose a basis $\{\partial_i\}$ of $\mathfrak{d}$ that contains a basis of $\mathfrak{d}'$, and construct the family $\mathfrak{d}'_t \subset \mathfrak{d}_t$ of Lie algebras indexed by $t \in \mathbf{k}$ generated by elements $\{\partial_i^t\}$ with Lie bracket $[\partial_i^t, \partial_j^t] = t[\partial_i, \partial_j]^t$. Then for $t \neq 0$ we have an isomorphism $\mathfrak{d}_t \ni \partial_i^t \mapsto t\partial_i \in \mathfrak{d}$, whereas $\mathfrak{d}_0$ is an abelian Lie algebra. Then $\{\mathrm{Cur}_{H'_t}^{H_t} W(\mathfrak{d}'_t)\}_{t \in \mathbf{k}}$, where $H_t = U(\mathfrak{d}_t)$, $H'_t = U(\mathfrak{d}'_t)$, is a family of pseudoalgebras all isomorphic to $\mathrm{Cur}_{H'}^{H} W(\mathfrak{d}')$ for $t \neq 0$. The fiber of this family at $t = 0$ has been shown in Proposition 5.14(ii) to have no nontrivial central extensions. In the same way, if we set $\chi_t(\partial_i^t) = t\chi(\partial_i)$, then $\{\mathrm{Cur}_{H'_t}^{H_t} S(\mathfrak{d}'_t, \chi_t)\}_{t \in \mathbf{k}}$ is a family of pseudoalgebras all isomorphic to $\mathrm{Cur}_{H'}^{H} S(\mathfrak{d}', \chi)$ for $t \neq 0$, and the fiber at $t = 0$ is $\mathrm{Cur}_{H'_0}^{H_0} S(\mathfrak{d}'_0, 0)$ where $\mathfrak{d}'_0 \subset \mathfrak{d}_0$ are isomorphic to $\mathfrak{d}' \subset \mathfrak{d}$ as vector spaces but have trivial bracket.

If L is a current pseudoalgebra over $K(\mathfrak{d}', \theta)$, for finite-dimensional Lie algebras $\mathfrak{d}' \subset \mathfrak{d}$, choose a basis $\{a_i, b_i, s\}$ of $\mathfrak{d}'$ as in Lemma 8.4, and complete it with $\{d_1, ..., d_r\}$ to a basis of $\mathfrak{d}$. Then a continuous family $\{\mathfrak{d}_t\}$ of Lie algebras can be constructed for $t \neq 0$ as $\mathfrak{d}_t \simeq \mathfrak{d}$ spanned by $a_i^t = ta_i$, $b_i^t = tb_i$, $s^t = t^2 s$, $d_i^t = t^2 d_i$, and by setting $a_i^0, b_i^0, c^0 = -s^0$ to span a Heisenberg algebra, and all brackets involving d_i^0 to be zero. Define $\theta_t \in (\mathfrak{d}'_t)^*$ by $\theta_t(a_i^t) = \theta_t(b_i^t) = 0$, $\theta_t(s^t) = -1$. Then $\mathrm{Cur}_{H'_0}^{H_0} K(\mathfrak{d}'_0, \theta_0)$ is the limit of the Lie pseudoalgebras $\{\mathrm{Cur}_{H'_t}^{H_t} K(\mathfrak{d}'_t, \theta_t)\}_{t \neq 0}$, which are all isomorphic to $\mathrm{Cur}_{H'}^{H} K(\mathfrak{d}', \theta)$, and the former Lie pseudoalgebra is of the type treated in Proposition 15.6. ∎

16. APPLICATION TO THE CLASSIFICATION OF POISSON BRACKETS IN CALCULUS OF VARIATIONS

In calculus of variations the phase space consists of C^∞ vector functions $u=(u_1(x), ..., u_r(x))$ where $u_i(x)$ are, for example, functions with compact support on a closed N-dimensional manifold. We will consider *linear* local Poisson brackets,

$$\{u_i(x), u_j(y)\} = \sum_\alpha B_{\alpha ij}(y)\, \partial_y^\alpha\, \delta(x-y) \tag{16.1}$$

where the sum runs over a finite set of multi-indices $\alpha=(\alpha_1, ..., \alpha_N) \in \mathbb{Z}_+^N$, the $B_{\alpha ij}$ are linear combinations of the u_k and of their derivatives $u_k^{(\gamma)} := \partial_x^\gamma u_k$, where $\partial_x^\gamma := (\partial/\partial x_1)^{\gamma_1} \cdots (\partial/\partial x_N)^{\gamma_N}$, and $\delta(x-y)$ is the delta-function (defined by $\int f(x)\, \delta(x-y)\, \mathrm{d}x = f(y)$). By Leibniz rule and bilinearity, the Poisson bracket (16.1) extends to arbitrary polynomials P and Q in the u_i and their derivatives. Explicitly

$$\{P(x), Q(y)\} = \sum_{\alpha, \beta, i, j} \frac{\partial P(x)}{\partial u_i^{(\alpha)}} \frac{\partial Q(y)}{\partial u_j^{(\beta)}}\, \partial_x^\alpha\, \partial_y^\beta \{u_i(x), u_j(y)\}. \tag{16.2}$$

This bracket, apart from bilinearity and Leibniz rule, should satisfy skew-commutativity and the Jacobi identity.

The basic quantities in calculus of variations are local functionals (Hamiltonians) $I_P = \int P(x)\, \mathrm{d}x$. Using bilinearity and integration by parts ($\int (\partial P/\partial x_i)\, Q\, \mathrm{d}x = -\int P(\partial Q/\partial x_i)\, \mathrm{d}x$), we get from (16.2) the following well-known formula,

$$\{I_P, I_Q\} = \sum_{i,j} \iint \frac{\delta P(x)}{\delta u_i} \frac{\delta Q(y)}{\delta u_j} \{u_i(x), u_j(y)\}\, \mathrm{d}x\, \mathrm{d}y, \tag{16.3}$$

where

$$\frac{\delta P(x)}{\delta u_i} = \sum_\alpha (-\partial_x)^\alpha \frac{\partial P(x)}{\partial u_i^{(\alpha)}} \tag{16.4}$$

is the variational derivative.

More generally, one usually considers a class of Poisson brackets of the form

$$\{u_i(x), u_j(y)\} = B_{ij}(y, \partial_y^\alpha)\, \delta(x-y), \tag{16.5}$$

where B_{ij} are differential operators in ∂_y^α whose coefficients are polynomials in $u_k^{(\gamma)}(y)$. Then the $r \times r$ matrix $B=(B_{ij})$ is called a Hamiltonian operator, and, integrating by parts, formula (16.3) can be rewritten in its most familiar form

$$\{I_P, I_Q\} = \int \left(B\,\frac{\delta P(x)}{\delta u}\right)\frac{\delta Q(x)}{\delta u}\,\mathrm{d}x. \tag{16.6}$$

Given a Hamiltonian $h = \int P(x)\,\mathrm{d}x$, we have the corresponding Hamiltonian system of evolutionary partial differential equations

$$\dot{u}_i = \{h, u_i\} \equiv \sum_j B_{ij}\,\frac{\delta P}{\delta u_j}, \tag{16.7}$$

so that if another Hamiltonian h_1 is in involution with h, i.e., $\{h, h_1\} = 0$, then h_1 is an integral of motion of (16.7), i.e., $\dot{h}_1 = 0$.

It is shown in [DN1, M] that for $r \geqslant 2$, any Poisson bracket of hydrodynamic type (i.e., linear in the derivatives) under certain non-degeneracy conditions can be transformed into a linear Poisson bracket of hydrodynamic type by a change of the field variables. The latter Poisson brackets have been studied rather extensively (see [Do, DN2] and references there, [GD, M, Z]).

Let $H = \mathbb{C}[\partial/\partial x_1, ..., \partial/\partial x_N]$ be the universal enveloping algebra of the N-dimensional abelian Lie algebra $\mathfrak{d}$. Let $F = \bigoplus_{i=1}^r Hu_i$ be the free H-module of rank r on generators u_i. Consider a linear Poisson bracket (16.1). For any three subspaces A, B, C of F, we will use the notation $\{A, B\} \subset C$ if for any $a \in A$, $b \in B$ all coefficients in front of $\partial_y^\alpha \delta(x - y)$ in $\{a(x), b(y)\}$ belong to C. We call a *linear Poisson algebra* any H-submodule L of F which is closed under the Poisson bracket, i.e., such that $\{L, L\} \subset L$. By an isomorphism of two such algebras we mean a $\mathbb{C}$-linear isomorphism preserving Poisson brackets.

If L is a linear Poisson algebra, we define the *λ-bracket* ($\lambda = (\lambda_1, ..., \lambda_N)$) on L as the Fourier transform of the linear Poisson bracket (16.1):

$$[u_{i\lambda} u_j] = \sum_\alpha \lambda^\alpha B_{\alpha ij}. \tag{16.8}$$

Then we get a Lie conformal algebra in N (commuting) indeterminates (defined in the same way as for the $N = 1$ case in the introduction). Thus, the classification of linear Poisson algebras follows from the classification of Lie $U(\mathfrak{d})$-conformal algebras, where $\mathfrak{d}$ is the N-dimensional abelian Lie algebra.

Recall that the structure of a Lie conformal algebra is equivalent to the structure of a Lie pseudoalgebra (see Section 9). The relationship between the linear Poisson bracket (16.1) and the pseudobracket can be described explicitly as

$$[u_i * u_j] = \sum_k P_{ij}^k(\partial \otimes 1, 1 \otimes \partial) \otimes_H u_k, \tag{16.9}$$

if

$$\{u_i(x), u_j(y)\} = \sum_k P_{ij}^k(\partial_x, \partial_y)(u_k(y)\,\delta(x-y)) \tag{16.10}$$

for some polynomials P_{ij}^k. Note that Eq. (16.1) can always be written in the form (16.10). Indeed, if

$$\{u_i(x), u_j(y)\} = \sum_k Q_{ij}^k(\partial_y, \partial_t)(u_k(t)\,\delta(x-y))|_{t=y} \tag{16.11}$$

for some polynomials Q_{ij}^k, then we have (16.10) with $P_{ij}^k(z, w) = Q_{ij}^k(-z, z+w)$. In this case, the λ-bracket (16.8) is given by

$$[u_{i\lambda} u_j] = \sum_k Q_{ij}^k(\lambda, \partial)\, u_k. \tag{16.12}$$

Remark 16.1. The constant terms of $B_{\alpha ij}$ in (16.1) give a central extension of the linear Poisson algebra corresponding to the $B_{\alpha ij}$ with constant terms removed. In terms of the associated Lie pseudoalgebras this corresponds to a central extension by $\mathbb{C}$ with a trivial action of H. By Theorem 15.1, these central extensions are parameterized by $\mathrm{H}^2(L, \mathbb{C})$.

EXAMPLE 16.1. (cf. [M, DN2, K4, BKV]).

(1) *General Poisson algebra* $W_{r,N}$, where $1 \leqslant r \leqslant N$ $(1 \leqslant i, j \leqslant r)$:

$$\{u_i(x), u_j(y)\} = \frac{\partial u_j(y)}{\partial y_i}\,\delta(x-y) + u_j(y)\,\frac{\partial}{\partial y_i}\,\delta(x-y) + u_i(y)\,\frac{\partial}{\partial y_j}\,\delta(x-y).$$

(2) *Special Poisson algebra* $S_{r,N,\chi}$, where $2 \leqslant r \leqslant N$ and $\chi = (\chi_1, \ldots, \chi_r) \in \mathbb{C}^r$, is the following subalgebra of $W_{r,N}$:

$$\left\{ \sum_{i=1}^r P_i(\partial_x)\, u_i(x) \,\middle|\, \sum_{i=1}^r \left(\frac{\partial}{\partial x_i} + \chi_i\right) P_i(\partial_x) = 0 \right\}.$$

It is generated over $H = \mathbb{C}[\partial/\partial x_1, \ldots, \partial/\partial x_N]$ by elements

$$u_{ij}(x) = \left(\frac{\partial}{\partial x_i} + \chi_i\right) u_j(x) - \left(\frac{\partial}{\partial x_j} + \chi_j\right) u_i(x).$$

(3) *Hamiltonian Poisson algebra* $H_{2s,N}$, $2 \leqslant r = 2s \leqslant N$:

$$\{u(x), u(y)\} = \sum_{i=1}^{s} \left(\frac{\partial u(y)}{\partial y_i} \frac{\partial}{\partial y_{i+s}} \delta(x-y) - \frac{\partial u(y)}{\partial y_{i+s}} \frac{\partial}{\partial y_i} \delta(x-y) \right).$$

We have an inclusion $H_{2s,N} \subset W_{2s,N}$ by letting

$$u(x) = \sum_{i=1}^{s} \left(\frac{\partial u_{i+s}(x)}{\partial x_i} - \frac{\partial u_i(x)}{\partial x_{i+s}} \right).$$

(4) *Current Poisson algebra* $\mathrm{Cur}_N \mathfrak{g}$ associated to a simple r-dimensional Lie algebra $\mathfrak{g}$ with structure constants c_{ij}^k $(1 \leqslant i, j, k \leqslant r)$

$$\{v_i(x), v_j(y)\} = \sum_{k=1}^{r} c_{ij}^k v_k(y)\, \delta(x-y).$$

(5) Semidirect sum of $W_{r,N}$ or one of its subalgebras $S_{r,N,\chi}$, $H_{2s,N}$ with $\mathrm{Cur}_N \mathfrak{g}$ defined by $(1 \leqslant i \leqslant r,\ v(x) \in \mathrm{Cur}_N \mathfrak{g})$

$$\{u_i(x), v(y)\} = \frac{\partial v(y)}{\partial y_i} \delta(x-y) + v(y) \frac{\partial}{\partial y_i} \delta(x-y).$$

A subspace I of a Poisson algebra L is called an *ideal* if it is invariant under taking Poisson brackets with elements of L, i.e., if $\{L, I\} \subset I$. A Poisson algebra L is called *simple* (respectively *semisimple*) if the Poisson bracket is not identically zero and L contains no nonzero H-invariant ideals I such that $I \neq L$ (respectively $\{I, I\} \neq 0$). Note that the Poisson algebras that we consider here are *finite*, i.e., finitely generated as H-modules.

Then Theorems 13.2 and 13.3 and Corollary 13.6 imply:

THEOREM 16.1. (i) *Any simple linear Poisson algebra is isomorphic to one of the Poisson algebras* $W_{r,N}$, $S_{r,N,\chi}$, $H_{2s,N}$, $\mathrm{Cur}_N \mathfrak{g}$.

(ii) *Any semisimple linear Poisson algebra is a direct sum of simple ones and of the semidirect sums described in Example* 16.1(5).

Remark 16.2. It follows from Remark 16.1 and the results of Section 15.4 that all nontrivial central extensions of simple Poisson algebras are described by the following 2-cocycles.

For $\alpha \in \mathbb{C}^r$ let

$$\psi_\alpha(x, y) = \sum_{i=1}^{r} \alpha_i \frac{\partial}{\partial y_i} \delta(x-y).$$

Then all nontrivial 2-cocycles for $H_{r,N}$ are

$$\gamma_\alpha(u(x), u(y)) = \psi_\alpha(x, y).$$

All nontrivial 2-cocycles for $\mathrm{Cur}_N\, \mathfrak{g}$ are

$$\gamma_\alpha(v_i(x), v_j(y)) = b_{ij}\psi_\alpha(x, y),$$

where $b_{ij} = (v_i \mid v_j)$ is the invariant scalar product.

The Poisson algebra $W_{r,N}$ has a nontrivial central extension iff $r = 1$, and in the latter case it is given by the well-known Virasoro cocycle,

$$\gamma(u(x), u(y)) = \left(\frac{\partial}{\partial y}\right)^3 \delta(x - y).$$

The Poisson algebra $S_{r,N,\chi}$ has no nontrivial central extensions if $r > 2$ or $\chi \neq 0$, and $S_{2,N,0} \simeq H_{2,N}$ has nontrivial central extensions described above.

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REFERENCES

[BKV] B. Bakalov, V. G. Kac, and A. A. Voronov, Cohomology of conformal algebras, *Comm. Math. Phys.* **200** (1999), 561–598.

[BD] A. Beilinson and V. Drinfeld, Chiral algebras, preprint.

[BPZ] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nuclear Phys. B* **241**, No. 2 (1984), 333–380.

[B1] R. J. Blattner, A theorem of Cartan and Guillemin, *J. Differential Geom.* **5** (1970), 295–305.

[B2] R. J. Blattner, Induced and produced representations of Lie algebras, *Trans. Amer. Math. Soc.* **144** (1969), 457–474.

[Bo1] R. E. Borcherds, Vertex algebras, Kac–Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. U.S.A.* **83** (1986), 3068–3071.

[Bo2] R. E. Borcherds, Vertex algebras, *in* "Topological Field Theory, Primitive Forms and Related Topics, Kyoto, 1996," Progr. Math., Vol. 160, pp. 35–77, Birkhäuser, Boston, MA, 1998; q-alg/9706008.

[C] P. J. Cassidy, Differential algebraic Lie algebras, *Trans. Amer. Math. Soc.* **247** (1979), 247–273.

[CP] V. Chari and A. Pressley, "A Guide to Quantum Groups," Cambridge Univ. Press, Cambridge, UK, 1995.

[CK] S.-J. Cheng and V. G. Kac, Conformal modules, *Asian J. Math.* **1**, No. 1 (1997), 181–193; Erratum, *Asian J. Math.* **2**, No. 1 (1998), 153–156.

[CKW] S.-J. Cheng, V. G. Kac, and M. Wakimoto, Extensions of conformal modules, *in* "Topological Field Theory, Primitive Forms and Related Topics, Kyoto, 1996," Progr. Math., Vol. 160, pp. 79–129, Birkhäuser, Boston, MA, 1998; q-alg/9709019.

[DK] A. D'Andrea and V. G. Kac, Structure theory of finite conformal algebras, *Selecta Math. (N.S.)* **4** (1998), 377–418.

[Do] I. Dorfman, "Dirac Structures and Integrability of Nonlinear Evolution Equations: Nonlinear Science: Theory and Applications," Wiley, Chichester, 1993.

[D] V. G. Drinfeld, Quantum groups, *in* "Proc. Intern. Congr. Math., Berkeley, 1986," pp. 798–820.

[DN1] B. A. Dubrovin and S. P. Novikov, Poisson brackets of hydrodynamic type, *Dokl. Akad. Nauk SSSR* **279** (1984), 294–297 [In Russian]; English translation, *Soviet Math. Dokl.* **30** (1984), 651–654.

[DN2] B. A. Dubrovin and S. P. Novikov, Hydrodynamics of weakly deformed soliton lattices: Differential geometry and Hamiltonian theory, *Uspekhi Mat. Nauk* **44** (1989), 29–98, 203 [In Russian]; English translation, *Russian Math. Surveys* **44** (1989), 35–124.

[ES] P. Etingof and O. Schiffmann, Lectures on the dynamical Yang-Baxter equations, preprint, math.QA/9908064.

[Fe] G. Felder, Conformal field theory and integrable systems associated to elliptic curves, *in* "Proceedings of the International Congress of Mathematicians," Vols. 1, 2, pp. 1247–1255, Birkhäuser, Basel, 1995.

[Fu] D. B. Fuchs, "Cohomology of Infinite-Dimensional Lie Algebras," Contemporary Soviet Mathematics, Consultants Bureau, New York, 1986.

[GD] I. M. Gelfand and I. Ja. Dorfman, Hamiltonian operators and infinite-dimensional Lie algebras, *Funktsional. Anal. i Prilozhen.* **15** (1981), 23–40 [In Russian]; English translation, *Funct. Anal. Appl.* **15** (1981), 173–187.

[GK] M. Golenishcheva-Kutuzova and V. G. Kac, Γ-conformal algebras, *J. Math. Phys.* **39** (1998), 2290–2305.

[G1] V. Guillemin, A Jordan–Hölder decomposition for a certain class of infinite dimensional Lie algebras, *J. Differential Geom.* **2** (1968), 313–345.

[G2] V. Guillemin, Infinite-dimensional primitive Lie algebras, *J. Differential Geom.* **4** (1970), 257–282.

[GS] V. Guillemin and S. Sternberg, An algebraic model of transitive differential geometry, *Bull. Amer. Math. Soc.* **70** (1964), 16–47.

[K1] V. G. Kac, "Infinite-Dimensional Lie Algebras," 3rd ed., Cambridge Univ. Press, Cambridge, UK, 1990.

[K2] V. G. Kac, "Vertex Algebras for Beginners," University Lecture Series, Vol. 10, Amer. Math. Soc., Providence, RI, 1996, 2nd ed., 1998.

[K3] V. G. Kac, The idea of locality, *in* "Physical Applications and Mathematical Aspects of Geometry, Groups and Algebras" (H.-D. Doebner *et al*, Eds.), pp. 16–32, World Scientific, Singapore, 1997; q-alg/9709008.

[K4] V. G. Kac, Formal distribution algebras and conformal algebras, *in* "Proc. XIIth International Congress of Mathematical Physics (ICMP '97) (Brisbane)," pp. 80–97, Internat. Press, Cambridge, MA, 1999; q-alg/9709027.

[Ka] C. Kassel, Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra, *J. Pure Appl. Algebra* **34** (1984), 265–275.

[Ki] A. A. Kirillov, Local Lie algebras, *Uspekhi Mat. Nauk* **31** (1976), 57–76 [In Russian]; English translation, *Russian Math. Surveys* **31** (1976), 55–75, Erratum, *Uspekhi Mat. Nauk* **32** (1977), 268.

[Ko] I. A. Kostrikin, Representations of height 1 of infinite-dimensional Lie algebras of the series K_n, *Uspekhi Mat. Nauk* **34** (1979), 229–230 [In Russian]; English translation, *Russian Math. Surv.* **34** (1979), 225–226.

[L] J. Lambek, Deductive systems and categories. II. Standard constructions and closed categories, *in* "Lecture Notes in Math.," Vol. 86, pp. 76–122, Springer-Verlag, Berlin, 1969.

[M] O. I. Mokhov, Poisson brackets of Dubrovin-Novikov type (DN-brackets), *Funktsional. Anal. i Prilozhen.* **22** (1988), 92–93 [In Russian]; English translation, *Funct. Anal. Appl.* **22** (1988), 336–338.

[NW] W. Nichols and B. Weisfeiler, Differential formal groups of J. F. Ritt, *Amer. J. Math.* **104** (1982), 943–1003.

[R1] J. F. Ritt, Associative differential operations, *Ann. of Math. (2)* **51** (1950), 756–765.

[R2] J. F. Ritt, Differential groups and formal Lie theory for an infinite number of parameters, *Ann. of Math (2)* **52** (1950), 708–726.

[R3] J. F. Ritt, Differential groups of order two, *Ann. of Math. (2)* **53** (1951), 491–519.

[R4] J. F. Ritt, Subgroups of differential groups, *Ann. of Math. (2)* **54** (1951), 110–146.

[Ru1] A. N. Rudakov, Irreducible representations of infinite-dimensional Lie algebras of Cartan type, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 835–866 [In Russian]; English translation, *Math. USSR Izv.* **8** (1974), 836–866.

[Ru2] A. N. Rudakov, Irreducible representations of infinite-dimensional Lie algebras of types S and H, *Izv. Akad. Nauk SSSR Ser. Mat.* **39** (1975), 496–511 [In Russian]; English translation, *Math. USSR Izv.* **9** (1976), 465–480.

[S] J.-P. Serre, "Lie Algebras and Lie Groups," Benjamin, Elmsford, NY, 1965.

[So] Y. Soibelman, Meromorphic tensor categories, preprint, `q-alg/9709030`.

[Sw] M. Sweedler, "Hopf Algebras," Math. Lecture Note Series, Benjamin, New York, 1969.

[X] X. Xu, Equivalence of conformal superalgebras to Hamiltonian superoperators, *Algebra Colloq.*, in press.

[Z] E. I. Zelmanov, A class of local translation-invariant Lie algebras, *Dokl. Akad. Nauk SSSR* **292** (1987), 1294–1297 [In Russian]; English translation, *Soviet Math. Dokl.* **35** (1987), 216–218.

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