JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 28, 383–399 (1969)

Lie Theory and Difference Equations. I

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A factorization method is constructed for sequences of second-order linear difference equations in analogy with the factorization method for differential equations. Six factorization types are established and recursion relations are obtained for various classes of special functions, among which are the hypergeometric functions and their limits, and the classical polynomials of a discrete variable: Tchebycheff, Krawtchouk, Charlier, Meixner, and Hahn. It is shown that the factorization method is a disguised form of Lie algebra representation theory.

INTRODUCTION

The factorization method for second-order differential equations is a useful technique for solving certain eigenvalue problems in mathematical physics and for deriving recursion relations of special functions [1]. This method is known to be Lie algebraic in nature: it is essentially a technique for finding realizations of a class of Lie algebras in terms of differential operators, [2], [3]. Here the factorization method is applied to the study of systems of second order difference equations. Six factorization types are isolated and the analysis of these types leads to recursion relations for various kinds of special functions, e.g., the contiguous function relations of Gauss for the hypergeometric functions. Furthermore, all of the classical polynomials of a discrete variable appear: The polynomials of Tchebycheff, Krawtchouk, Charlier, Meixner, and Hahn. Just as its counterpart for differential equations, the factorization method for difference equations is a disguised application of the representation theory of Lie algebras. Indeed an analysis of the factorization types based on representation theory leads to much more information about the corresponding special functions than does the factorization method alone.

1. The Factorization Method

We begin with an abstract algebraic formulation of the factorization method [1]. Let $\{X_m\}$, $m \in S = \{m_0, m_0 \pm 1, m_0 \pm 2, \ldots\}$ where m_0 is a complex number, be a sequence of linear operators defined on the complex vector space $\mathscr{V}$. (It is assumed that X_m and all other operators mentioned in this section have domain $\mathscr{V}$.) We wish to solve the eigenvalue problem

$$X_m Y_\lambda(m) = \lambda Y_\lambda(m) \tag{1.1}$$

simultaneously for all $m \in S$.

Definition. The operators X_m admit a *factorization* if there exist sequences of linear operators $\{L_m{}^+\}$, $\{L_m{}^-\}$, on $\mathscr{V}$ and constants $\{a_m\}$ such that

$$X_m \equiv L_m{}^+ L_m{}^- + a_m \equiv L_{m+1}^- L_{m+1}^+ + a_{m+1} \tag{1.2}$$

for all $m \in S$.

If the X_m admit a factorization then Equation (1.1) is equivalent to the two equations

$$L_m{}^+ L_m{}^- Y_\lambda(m) = (\lambda - a_m)\, Y_\lambda(m),$$
$$L_{m+1}^- L_{m+1}^+ Y_\lambda(m) = (\lambda - a_{m+1})\, Y_\lambda(m) \tag{1.3}$$

for all $m \in S$. Indeed we have the following result.

Lemma *Let $Y_\lambda(l)$ be a solution of* (1.1) *for $m = l$. Then $L_{l+1}^+ Y_\lambda(l)$ is a solution of* (1.1) *for $m = l + 1$ and $L_l{}^- Y_\lambda(l)$ is a solution for $m = l - 1$.*

The following statements are simple consequences of Equation (1.3) and the lemma: (1) The "raising operators" L_{l+1}^+ map a solution of (1.1) for $m = l$ into a solution for $m = l + 1$; (2) The "lowering operators" $L_l{}^-$ map a solution for $m = l$ into a solution for $m = l - 1$; and (3) If we first raise and the lower (or vice versa) we obtain the original vector multiplied by a fixed constant. (We do not exclude the possibility that $L_{l+1}^+ Y(l) = 0$ or $L_l{}^- Y(l) = 0$.) These results show that the existence of a factorization implies the existence of recurrence relations for the vectors $Y_\lambda(m)$. Indeed, if $Y_\lambda(l)$ is a solution of (1.1) for $m = l$, a ladder of solutions $Y_\lambda(l + n)$ can be defined recursively by

$$Y_\lambda(l + n + 1) = L_{l+n+1}^+ Y_\lambda(l + n), \qquad n = 0, 1, 2, \ldots .$$

Then Equations (1.2) imply

$$(\lambda - a_{l+n+1})\, Y_\lambda(l + n) = L_{l+n+1}^- Y_\lambda(l + n + 1), \qquad n = 0, 1, 2, \ldots .$$

Similarly, we can derive recurrence relations for functions $Y_\lambda(l-n)$ with $n \geqslant 0$.

The raising and lowering operators $L_m^\pm$ take on special importance if there is an $m_0 \in S$ such that $a_{m_0} = \lambda$. In that case the equation $X_{m_0} Y_\lambda(m_0) = \lambda Y_\lambda(m_0)$ becomes $L_{m_0}^+ L_{m_0}^- Y_\lambda(m_0) = 0$. Then any solution of

$$L_{m_0}^- \, Y_\lambda(m_0) = 0 \tag{1.4}$$

is a solution of the original equation. By applying the raising operators to a nonzero solution of (1.4) we can construct a ladder of solutions $Y_\lambda(m_0 + n)$, $n = 0, 1, 2, \ldots$. Because of (1.4) we say that this ladder is *bounded below* (it may also be bounded above). Similarly, if there is an $m_1 \in S$ such that $a_{m_1+1} = \lambda$ we can find a ladder of solutions *bounded above.*

The utility of the above remarks is that Equation (1.4) may be easier to solve than the original Equations (1.1). For the Infeld-Hull theory Equations (1.1) are second-order differential equations while (1.4) is a first order differential equation, [1]. In the present paper Equations (1.1) will be second-order difference equations while (1.4) will be a first-order difference equation. In fact, we choose

$$X_m \equiv E^2 + D_m(x)\, E + V_m(x) \tag{1.5}$$

where D_m, V_m are functions of the complex variable x and $Ef(x) = f(x+1)$ for any function f. For $\mathscr{V}$ we take a vector space of functions of x. The variable x takes discrete values of the form $x_0 + k$ where $x_0 \in \mathscr{C}$ and k is an integer such that $n_1 < k < n_2$. (The integers n_1, n_2 may be infinite.) At this point we need not be specific about the domains of the functions in $\mathscr{V}$. We simply assume that all of the following operations make sense for these functions. Finally, we assume that the operators L_m^+, L_m^- take the form

$$L_m^+ = f_m(x)\, E + q_m(x), \qquad L_m^- = F_m(x)\, E + Q_m(x). \tag{1.6}$$

To find all factorizations of the X_m by operators of the form (1.6) it is necessary to determine all solutions of Equations (1.2). In order to most conveniently obtain factorizations which are of importance in special function theory, we restrict ourselves to solutions such that $f_m(x)$, $q_m(x)$, $F_m(x)$, $Q_m(x)$ are polynomials in m and x. (In fact, we will soon restrict ourselves to polynomials which contain no powers of m higher than one.)

Under these hypotheses, substitution of Equations (1.5) and (1.6) into (1.2) leads to the following necessary and sufficient conditions for a factorization:

(A) $f_m(x) = c$, $F_m(x) = c^{-1}$, c a nonzero constant. In fact, we can require

$$f_m(x) \equiv F_m(x) \equiv 1.$$

(B) $q_{m+1}(x+1) + Q_{m+1}(x) = Q_m(x+1) + q_m(x) = D_m(x)$

(C) $Q_{m+1}(x)\, q_{m+1}(x) + a_{m+1} = Q_m(x)\, q_m(x) + a_m = V(x). \qquad (1.7)$

Equations (1.7) are still rather cumbersome to solve in general and for simplicity we restrict ourselves to solutions of the form

$$Q_m(x) = G(x) + mH(x), \qquad q_m(x) = g(x) + mh(x) \tag{1.8}$$

where G, H, g, h are all polynomials in x.

Under the hypothesis (1.8), Equations (1.7) can be solved. The general solution for the factorization operators $L_m^{\pm}$ and the constants a_m is

$$\begin{aligned} L_m^+ &= E + k(-x + b_1 + m) \\ L_m^- &= E + k(sx + b_2 + sm) \\ a_m &= -k^2[sm^2 + (sb_1 + b_2)m] \end{aligned} \tag{1.9}$$

where k, s, b_1, b_2 are constants. The operators X_m are given by the expressions

$$\begin{aligned} X_m = E^2 &+ k[(s-1)x + s + m(s+1) + b_1 + b_2]E \\ &+ k^2[(-x + b_1 + m)(sx + b_2 + s) - sm^2 - (sb_1 + b_2)m]. \end{aligned} \tag{1.10}$$

We will always assume $k \neq 0$ since otherwise the factorizations are trivial. These factorizations can conveniently be divided into three types depending on the constants s, b_2.

Type α': $s \neq 0$

Type β': $s = 0$, $b_2 \neq 0$

Type γ': $s = b_2 = 0$.

The following three sections are concerned with an analysis of these factorization types.

NOTE. There exist nontrivial solutions of Equations (1.7) where the hypotheses (1.8) is not satisfied. For example, $q_m(x) = 2x^2 - 2xm + m^2$, $Q_m(x) = 2x^2 + 2xm + m^2$ is such a solution. Furthermore, in the degenerate case $s = -1$ in (1.9) another solution appears which we will not consider.

2. TYPE α' FACTORIZATIONS

The solutions $Y_\lambda(m, x)$ of Equation (1.1) corresponding to the type α' factorization are hypergeometric functions. In fact, if $k = 1 - z$, $b_1 = -1$, $ks = -1$, $kb_2 = -\beta$, $\lambda = 0$ the functions

$$Y(m, x) = \Gamma(x + \beta)\,{}_2F_1(-m, -x; \beta; z) \tag{2.1}$$

satisfy the equations

$$X_m Y(m, x) = 0, \qquad L^+_{m+1} Y(m, x) = (\beta + m)\, Y(m + 1, x)$$
$$L_m^- Y(m, x) = (z - 1)\, m Y(m - 1, x), \qquad m \in S, \tag{2.2}$$

where Γ is the gamma function and ${}_2F_1$ is a hypergeometric function, [4]. The recurrence formulas (2.2) are easily shown to be two of the contiguous function relations of Gauss. It is left to the reader to verify that the solutions of the general type α' equations can always be expressed in terms of hypergeometric functions. (For example, the functions

$$Y'(m, x) = \Gamma(x + 1)\, {}_2F_1(- m - \beta + 1, - x - \beta + 1; 2 - \beta; z)$$

satisfy the equations

$$X_m Y'(m, x) = 0, \qquad L^+_{m+1} Y'(m, x) = (m + 1)\, Y'(m + 1, x)$$
$$L_m^- Y'(m, x) = (z - 1)(m + \beta - 1)\, Y'(m - 1, x)$$

where X_m, $L_m^{\pm}$ are the operators defined at the beginning of this section.)

In Section 1 it was shown that corresponding to a complex number $m_0 \in S$ such that $a_{m_0} = \lambda$, there exists a ladder of solutions $Y_\lambda(m_0 + n)$, $n = 0, 1, 2, \ldots$. In particular $Y_\lambda(m_0)$ is a solution of $L^-_{m_0} Y_\lambda(m_0) = 0$. For the problem at hand $a_m = (1 - z)\, m(m + \beta - 1)$, $\lambda = 0$. Thus, if we set $m_0 = 0$ we obtain a ladder of solutions bounded below:

$$L_0^- Y(0, x) = [E - x - \beta]\, Y(0, x) = 0 \tag{2.3}$$

$$L^+_{n+1} Y(n, x) = (\beta + n)\, Y(n + 1, x), \qquad n = 0, 1, 2, \ldots. \tag{2.4}$$

Equation (2.3) has the general solution

$$Y(0, x) = c\Gamma(x + \beta) \tag{2.5}$$

where c is a constant. It is now clear that the ladder of solutions is

$$\begin{aligned} Y(n, x) &= \Gamma(x + \beta)\, {}_2F_1(- n, - x; \beta; z) \\ &= \frac{\Gamma(x + \beta)\, \Gamma(\beta)\, n!}{\Gamma(\beta + n)} P_n^{(\beta-1, -\beta-n-x)}(1 - 2z), \qquad n = 0, 1, 2, \ldots, \end{aligned}$$

where the $P_n^{(\alpha,\beta)}$ are Jacobi polynomials [4]. Except for the factor $\Gamma(x + \beta)$ these solutions are polynomials in x, the Meixner polynomials,

$$m_n(x; \beta, c) = n!\, P_n^{(\beta-1, -\beta-n-x)}\left(\frac{2}{c} - 1\right), \qquad (c^{-1} = 1 - z).$$

Similarly, it follows from (2.2) that the Krawtchouk polynomial

$(p^n/n!)\, m_n(x; -N, -p/(1-p))$, N a positive integer, $0 < p < 1$, are associated with a ladder of solutions bounded above and below, [5].

3. Type β' Factorizations

The solutions of Equation (1.1) corresponding to a type β' factorization are confluent hypergeometric (Laguerre) functions. Indeed, for $s = 0$, $k = a^{-1}$, $b_1 = -1$, $b_2 = -a$, in (1.9) and $\lambda = 0$ we find that the functions

$$Y(m, x) = L_m^{(x-m)}(a) = \frac{\Gamma(x+1)}{\Gamma(m+1)\,\Gamma(1+x-m)}\, {}_1F_1(-m; x-m+1; a) \tag{3.1}$$

satisfy the equations

$$X_m Y(m, x) = 0, \qquad L_{m+1}^{+} Y(m, x) = -(m+1)\, a^{-1} Y(m+1, x)$$
$$L_m^{-} Y(m, x) = Y(m-1, x), \qquad m \in S.$$

The $L_\nu^{(\alpha)}$ are Laguerre functions and the ${}_1F_1$ are confluent hypergeometric functions [4]. An independent set of solutions is

$$Y'(m, x) = (-a)^{m-x}\, \Gamma(x+1)\, L_x^{(m-x)}(a) \tag{3.2}$$

which satisfy the equations

$$X_m Y'(m, x) = 0, L_{m+1}^{+} Y'(m, x) = -a^{-1} Y'(m+1, x)$$
$$L_m^{-} Y'(m, x) = m Y'(m-1, x).$$

Returning to the form of type β' factorization defined at the beginning of this section, we note that $a_m = a^{-1}m$. Since $\lambda = 0$ there exists a ladder of solutions $Y(n, x)$, $n = 0, 1, 2, \ldots$, bounded below. In particular $Y(0, x)$ is a solution of

$$L_0^{-} Y(0, x) = (E - 1)\, Y(0, x) = 0.$$

Thus, $Y(0, x) = c$ where c is a constant. Setting $c = 1$, we can define a ladder of solutions recursively by

$$L_{n+1}^{+} Y(n, x) = -(n+1)\, a^{-1} Y(n+1, x), \qquad n = 0, 1, 2, \ldots$$

where the factor $-(n+1)\, a^{-1}$ has been chosen for convenience. The solutions

$$Y(n, x) = L_n^{(x-n)}(a), \qquad n = 0, 1, 2, \ldots .$$

are polynomials in x, the Charlier polynomials

$$c_n(x, a) = (-a)^{-n}\, n!\, L_n^{(x-n)}(a),$$

[5].

4. Type γ' Factorizations

Corresponding to type γ' factorizations the solutions of (1.1) are Bessel functions. In fact, for $s = b_2 = 0$, $k = 2a^{-1} \neq 0$, $b_1 = -1$ in Equation (1.9) and $\lambda = -a$, the Bessel functions, [4],

$$Y(m, x) = J_{x-m}(a) \tag{4.1}$$

satisfy the equations

$$X_m Y(m, x) = -aY(m, x), \qquad L_{m+1}^{+} Y(m, x) = -Y(m+1, x)$$
$$L_m^{-} Y(m, x) = Y(m-1, x).$$

A linearly independent set of solutions is given by

$$Y'(m, x) = J_{m-x}(-a).$$

Since $a_m \equiv 0$ while $\lambda \neq 0$ for all nontrivial type γ' factorizations, it follows that all ladders of solutions are completely unbounded.

5. A New Class of Factorizations

We now investigate factorizations of the form

$$X_m \equiv D_m(x)\, E + V_m(x) + W_m(x)\, L \tag{5.1}$$

$$L_m^{+} = f(x)\, E + q_m(x), \qquad L_m^{-} = F(x)\, L + Q_m(x). \tag{5.2}$$

Here $Eg(x) = g(x+1)$, $Lg(x) = g(x-1)$ for any function g. (Again we assume that the undetermined functions in expressions (5.1) and (5.2) are polynomials in x and m.) Substitution of these expressions into (1.2) leads to the following necessary and sufficient conditions for a factorization

(A) $f(x)\, Q_{m+1}(x) = f(x)\, Q_m(x+1) = D_m(x)$

(B) $F(x)\, q_{m+1}(x-1) = F(x)\, q_m(x) = W_m(x)$

(C) $F(x+1) f(x) - F(x) f(x-1) = Q_{m+1}(x)\, q_{m+1}(x) - Q_m(x)\, q_m(x) + a_{m+1} - a_m$.

Just as in Section 1, we restrict our attention to solutions of the form

$$Q_m(x) = G(x) + mH(x), \qquad q_m(x) = g(x) + mh(x) \tag{5.3}$$

where G, H, g, h are polynomials in x. Also, in order to guarantee that the X_m are second-order difference operators we require that $F(x)f(x-1) \neq 0$. Finally, it is easy to see that conditions (A), (B), (C) do not enable us to solve for $f(x)$ and $F(x)$ separately, but only for the product $F(x)f(x-1)$. This indeterminancy does no harm, however, for if $\{Y_\lambda(m, x)\}$ is a ladder of solutions of (1.1) corresponding to a system with factorization (5.1), (5.2) then $\{h(x)\, Y_\lambda(m, x)\}$ is a ladder of solutions to a similar system of equations with factorization

$$L_m^- = L + Q_m(x), \qquad L_m^+ = F(x+1)f(x)\, E + q_m(x)$$

and a_m unchanged. Here, $h(x)$ is any nonzero solution of the equation $h(x-1) = h(x)F(x)$. Since $F(x)$ is a polynomial, such solutions $h(x)$ always exist. For example, if $F(x+1) = c(x-c_1) \cdots (x-c_n)$ then

$$h(x) = \frac{c^{-x}}{\Gamma(x-c_1) \cdots \Gamma(x-c_n)}$$

has the required properties. Therefore, without loss of generality we set $F(x) \equiv 1$. The factorization equations can be solved:

$$\begin{aligned} L_m^+ &= [k_1k_2x(x+2) + (s + k_1b_2 + k_2b_1)\, x + c]\, E + k_1(x+m) + b_1 \\ L_m^- &= L + k_2(x+m) + b_2 \\ a_m &= -\, k_1k_2m(m-1) + ms \end{aligned} \tag{5.4}$$

where k_1, k_2, b_1, b_2, s, c are constants. The operators X_m can be computed from the relations

$$X_m \equiv L_{m+1}^- L_{m+1}^+ + a_{m+1} \equiv L_m^+ L_m^- + a_m\,, \qquad m \in S.$$

We devide the factorizations into three types depending on k_1k_2 and s:

Type α'': $k_1k_2 \neq 0$

Type β'': $k_1k_2 = 0$, $s \neq 0$.

Type γ'': $k_1k_2 = s = 0$.

6. Type α'' Factorizations

The solutions of a system of equations (1.1) admitting type α'' factorizations are of the form ${}_3F_2$, [4]. In fact, if we choose $k_1 = -k_2 = 1$, $b_1 = \gamma + q - 1$,

$b_2 = 0$, $s = q$, $c = -\gamma$ and normalize the raising and lowering operators (5.2) such that $F(x) \equiv x$ we obtain

$$L_m^+ = -(x+\gamma)E + (x+m+\gamma+q-1)$$

$$L_m^- = xL - x - m, \qquad a_m = m(m+q-1)$$

$$X_m = (x+\gamma)(x+m+1)E - [2x^2 + x(2m+2\gamma+q) + \gamma(m+1)] + x(x+m+\gamma+q-1)L.$$

For $\lambda = n(n-q+1)$ the functions

$$Y_\lambda(m,x) = \frac{\Gamma(m+n+1)\,\Gamma(\gamma+n)}{\Gamma(m+1)\,\Gamma(\gamma)\,\Gamma(n+1)}\, {}_3F_2(-n, -x, n-q+1; m+1, \gamma; 1) \tag{6.1}$$

satisfy the equations

$$X_m Y_\lambda(m,x) = n(n-q+1)\, Y_\lambda(m,x)$$

$$L_{m+1}^+ Y_\lambda(m,x) = (q+m-n)\, Y_\lambda(m+1,x)$$

$$L_m^- Y_\lambda(m,x) = -(m+n)\, Y_\lambda(m-1,x), \qquad m \in S.$$

To guarantee convergence of the ${}_3F_2$ we can require that x takes only non-negative integer values. These recursion relations can be verified from Rainville's contiguous function relations for generalized hypergeometric functions, [6].

There is a ladder of solutions, bounded below, for the type α'' factorizations which is of special interest. To obtain these solutions choose n to be a non-negative integer and note that $a_{-n} = \lambda = n(n-q+1)$. It follows that there is a ladder of solutions $Y_\lambda(-n+k, x)$, $k = 0, 1, 2,\ldots$ such that

$$L_{-n}^- Y_\lambda(-n, x) = 0$$

$$L_{-n+k+1}^+ Y_\lambda(-n+k, x) = (q+m-n)\, Y_\lambda(-n+k+1, x).$$

Comparing with (6.1) we see that

$$\begin{aligned} Y_\lambda(m,x) &= \frac{(m+1)_n\,(\gamma)_n}{n!}\, {}_3F_2(-n, -x, n-q+1; m+1, \gamma; 1) \\ &= p_n(x; m+1, \gamma, \gamma+m+q), \qquad m = -n+1, -n+2,\ldots \end{aligned} \tag{6.3}$$

where the p_n are Hahn polynomials, [5]. (Note that expression (6.3) makes sense even for m a negative integer.) For $m = q = 0$, $\gamma = 1 - N$, N a positive integer, the Hahn polynomial $Y_\lambda(0, x)$ is a Tchebycheff polynomial, [5]. It follows from the recursion relations (6.2) that the Tchebycheff polynomials are contained in ladders of solutions which are bounded both above and below.

7. Type β'' Factorizations

Type β'' factorizations generate ladders of hypergeometric functions as solutions. To see this, set $k_1 = z - 1$, $k_2 = 0$, $b_1 = \alpha(z - 1)$, $b_2 = -1$, $s = -1$, $c = z\beta$ in (5.4) and obtain

$$L_m{}^+ = z(-x + \beta)E + (z - 1)(x + m + \alpha)$$

$$L_m{}^- = L - 1, \qquad a_m = -m$$

$$X_m = z(x - \beta)E + [z(-2x + \beta - \alpha - m) + x + \alpha] + (z - 1)(x + m + \alpha)L.$$

For $\lambda = 0$ the functions

$$Y(m, x) = \frac{\Gamma(m + x + \alpha + 1)}{\Gamma(x + \alpha + 1)} {}_2F_1(-m, \alpha + \beta + 1; x + \alpha + 1; z) \tag{7.2}$$

satisfy the equations

$$X_m Y(m, x) = 0, \qquad L^+_{m+;}Y(m, x) = - Y(m + 1, x)$$

$$L_m{}^- Y(m, x) = - mY(m - 1, x).$$

Since $a_0 = \lambda = 0$ there is a ladder of solutions bounded below: $Y(n, x)$, $n = 0, 1, 2, \ldots$. These solutions are given by

$$Y(n, x) = n!\, P_n^{(x+\alpha, -x-n+\beta)}(1 - 2z)$$

where the $P_n^{(\alpha,\beta)}$ are Jacobi polynomials.

Another family of solutions for the type β'' factorizations is given by the functions

$$Y'(m, x) = \frac{(-z)^{-x}(z - 1)^{x+m}\Gamma(x + \alpha)}{\Gamma(x - \beta)} {}_2F_1(m + 1, -\alpha - \beta; 1 - x - \alpha; z). \tag{7.3}$$

8. Type γ'' Factorizations

The type γ'' factorizations have Laguerre (confluent hypergeometric) functions as solutions. To verify this, set $k_1 = 1$, $k_2 = 0$, $s = 0$, $b_1 = 0$, $b_2 = -1$, $c = -1$ in (5.4) and obtain

$$L_m^+ = -(x+1)E + (x+m), \qquad L_m^- = L - 1, \qquad a_m = 0$$
$$X_m = (x+1)E - (2x+m+1) + (x+m)L. \tag{8.1}$$

If $\lambda = -z \neq 0$ the functions

$$Y_z(m, x) = L_x^m(z) \tag{8.2}$$

satisfy the relations

$$X_m Y_z(m, x) = -zY_z(m, x)$$
$$L_{m+1}^+ Y_z(m, x) = zY_z(m+1, x)$$
$$L_m^- Y_z(m, x) = -Y_z(m-1, x).$$

Another set of solutions is given by

$$Y_z'(m, x) = \frac{\Gamma(m+x+1)}{\Gamma(x+1)} L_{m+x}^{-m}(z). \tag{8.3}$$

Here,

$$X_m Y_z'(m, x) = -zY_z'(m, x)$$
$$L_{m+1}^+ Y_z'(m, x) = -Y_z'(m+1, x), \qquad L_m^- Y_z(m, x) = aY_z'(m-1, x).$$

Since $\lambda \neq 0$, $a_m \equiv 0$ it follows that all ladders of solutions are completely unbounded.

9. Factorizations as Representations

Given any pair of complex numbers (a, b) define the four-dimensional complex Lie algebra $\mathscr{G}(a, b)$ with basis $\mathscr{J}^+$, $\mathscr{J}^-$, $\mathscr{J}^3$, $\mathscr{I}$ by the commutation relations

$$[\mathscr{J}^+, \mathscr{J}^-] = 2a^2\mathscr{J}^3 - b\mathscr{I}, \qquad [\mathscr{J}^3, \mathscr{J}^\pm] = \pm\mathscr{J}^\pm,$$
$$[\mathscr{J}^\pm, \mathscr{I}] = [\mathscr{J}^3, \mathscr{I}] = \mathscr{O}.$$

(Here, $\mathscr{O}$ is the additive identity element of the Lie algebra.) As shown in [3] the following isomorphisms are valid:

$$\mathscr{G}(a, b) \approx \mathscr{G}(1, 0) \approx sl(2) \oplus (\mathscr{I}) \qquad \text{if} \quad a \neq 0$$
$$\mathscr{G}(a, b) \approx \mathscr{G}(0, 1) \qquad \text{if} \quad a = 0, b \neq 0$$
$$\mathscr{G}(a, b) \approx \mathscr{G}(0, 0) \approx \mathscr{T}_3 \oplus (\mathscr{I}) \qquad \text{if} \quad a = b = 0, \tag{9.2}$$

where $sl(2)$ is the Lie algebra of the 2×2 unimodular group and $\mathscr{T}_3$ is the Lie algebra of the complex Euclidean group in three-space.

There is an intimate connection between the representation theory of the Lie algebras $\mathscr{G}(a, b)$ and the factorization method as presented in this paper. To see this, define operators

$$J^+ = t\left(E + k(-x + b_1 + 1) + kt\frac{\partial}{\partial t}\right)$$
$$J^- = t^{-1}\left(E + k(sx + b_2) + kst\frac{\partial}{\partial t}\right)$$
$$J^3 = t\frac{\partial}{\partial t}, \tag{9.3}$$

in analogy with Equation (1.9). These operators act on a space $\mathscr{V}'$ of functions $f(x, t)$ of two complex variables. It is easy to verify that the J-operators and the identity operator I satisfy the following commutation relations on $\mathscr{V}'$:

$$[J^+, J^-] = -2k^2 sJ^3 - k^2(s + sb_1 + b_2) I$$
$$[J^3, J^\pm] = \pm J^\pm, \qquad [J^\pm, I] = [J^3, I] = 0. \tag{9.4}$$

Comparing (9.4) with (9.1) we see that $J^\pm$, J^3, I generate a Lie algebra isomorphic to $\mathscr{G}(a, b)$ where $a^2 = -k^2 s$ and $b = k^2(s + sb_1 + b_2)$. Furthermore, if we set $f_m(x, t) = Y_\lambda(m, x)\, t^m$ where $Y_\lambda(m, x)$ is a solution of Equation (1.1) we find

$$J^+ f_m(x, t) = L^+_{m+1} Y_\lambda(m, x)\, t^{m+1}$$
$$J^- f_m(x, t) = L_m^- Y_\lambda(m, x)\, t^{m-1}$$
$$J^3 f_m(x, t) = mY_\lambda(m, x)\, t^m \tag{9.5}$$

$$(J^+J^- + a^2 J^3 J^3 - a^2 J^3 - bJ^3 I) f_m(x, t) = X_m Y_\lambda(m, x)\, t^m = \lambda f_m(x, t). \tag{9.6}$$

Here the operator

$$C_{a,b} = J^+J^- + a^2 J^3 J^3 - a^2 J^3 - bJ^3 E \tag{9.7}$$

has the property

$$[C_{a,b}, J^\pm] = [C_{a,b}, J^3] = [C_{a,b}, I] = 0.$$

It follows from this analysis that in constructing the factorization types of Section 1 we were actually constructing irreducible representations of the Lie algebras $\mathscr{G}(a, b)$. In particular, the type α' factorizations correspond to representations of $\mathscr{G}(1, 0)$, the type β' factorizations correspond to representations of $\mathscr{G}(0, 1)$ and the type γ' factorizations correspond to representations of $\mathscr{G}(0, 0)$.

Similarly, in analogy with Equations (5.4) we can define operators

$$J^+ = t \left\{(k_1 k_2 x(x+2) + (s + k_1 b_2 + k_2 b_1)\, x + c)\, E + k_1(x+1) + b_1 + k_1 t \frac{\partial}{\partial t}\right\}$$

$$J^- = t^{-1} \left\{L + k_2 x + b_2 + k_2 t \frac{\partial}{\partial t}\right\}. \tag{9.8}$$

Then,

$$[J^+, J^-] = -2k_1 k_2 J^3 + sI$$
$$[J^3, J^\pm] = \pm J^\pm, \qquad [J^\pm, I] = [J^3, I] = 0$$

and $J^\pm, J^3, I$ generate a Lie algebra isomorphic to $\mathscr{G}(a, b)$ where $a^2 = -k_1 k_2$, $b = -s$. (It is also easy to verify that the transformation of these operators by the function $h(x)$, as presented in Section 5, doesn't affect the commutation relations.) Setting $f_m(x, t) = Y_\lambda(m, x)\, t^m$ where $Y_\lambda(m, x)$ is a solution of (1.1), we find that Equations (9.5) and (9.6) again hold, where now $L_m{}^\pm$ are given by expressions (5.4). Thus, the type α'' factorizations correspond to representations of $\mathscr{G}(1, 0)$, type β'' factorizations correspond to representations of $\mathscr{G}(0, 1)$, and type γ'' factorizations correspond to representations of $\mathscr{G}(0, 0)$.

A great deal is known about the irreducible representations of the Lie algebras $\mathscr{G}(a, b)$. In particular the matrix elements of these representations have been computed [3]. An examination of the six factorization types from the viewpoint of representation theory leads to much additional information about special functions. It is left to the reader to verify that each of the irreducible representations of $\mathscr{G}(a, b)$ listed in [3] has two models in terms of the factorizations given in this paper.

10. Hahn Polynomials and Representations of $\mathscr{O}_4$

We denote by $\mathscr{O}_4$ the Lie algebra of the group of 4×4 complex orthogonal matrices. This Lie algebra is six-dimensional and has a basis $\mathscr{J}^\pm$, $\mathscr{J}^3$, $\mathscr{K}^\pm$, $\mathscr{K}^3$ with commutation relations

$$\begin{aligned}
&[\mathscr{J}^3, \mathscr{J}^\pm] = \pm \mathscr{J}^\pm, \qquad [\mathscr{J}^+, \mathscr{J}^-] = 2\mathscr{J}^3 \\
&[\mathscr{J}^+, \mathscr{K}^+] = [\mathscr{J}^-, \mathscr{K}^-] = [\mathscr{J}^3, \mathscr{K}^3] = 0 \\
&[\mathscr{K}^+, \mathscr{K}^3] = \mathscr{K}^+, \qquad [\mathscr{K}^-, \mathscr{K}^3] = -\mathscr{K}^-, \qquad [\mathscr{K}^+, \mathscr{K}^-] = -2\mathscr{K}^3 \\
&[\mathscr{K}^3, \mathscr{J}^+] = [\mathscr{J}^3, \mathscr{K}^+] = \mathscr{K}^+ \\
&[\mathscr{K}^3, \mathscr{J}^-] = [\mathscr{J}^3, \mathscr{K}^-] = -\mathscr{K}^- \\
&[\mathscr{J}^+, \mathscr{K}^-] = [\mathscr{K}^+, \mathscr{J}^-] = 2\mathscr{K}^3.
\end{aligned} \tag{10.1}$$

The operators $\mathscr{J}^\pm$, $\mathscr{J}^3$ generate a subalgebra of $\mathscr{O}_4$ isomorphic to $sl(2)$. (Note: This choice of basis is not the usual one for $\mathscr{O}_4$ but it is the most convenient for showing the relationship between the representation theory of $\mathscr{O}_4$ and Hahn polynomials.)

As usual we look for linear operators $J^\pm$, J^3, $K^\pm$, K^3 on a complex vector space V which satisfy the commutation relations (10.1). In particular, the following irreducible representations ρ of $\mathscr{O}_4$ on V are of interest:

(1) $\rho_1(c, u_0)$; $2u_0$ not an integer, $c^2 \neq (u_0 + n)^2$ for any integer n

$$u = u_0, u_0 + 1, u_0 + 2, \ldots, u_0 + n, \ldots$$
$$m = -u, -u + 1, -u + 2, \ldots .$$

(2) $\rho_2(c, u_0)$; $2u_0$ a nonnegative integer, $u_0 \pm c$ not negative integers

$$u = u_0, u_0 + 1, u_0 + 2, \ldots$$
$$m = -u, -u + 1, \ldots, u - 1, u.$$

For each representation ρ there is a basis $\{f_m^{(u)}\}$ of the representation space V such that

$$\begin{aligned}
J^+ f_m^{(u)} &= (m - u) f_{m+1}^{(u)} \\
J^- f_m^{(u)} &= -(m + u) f_{m-1}^{(u)} \\
J^3 f_m^{(u)} &= m f_m^{(u)}
\end{aligned} \tag{10.2}$$

$$\begin{aligned}
2K^3 f_m^{(u)} = {} & \frac{(u - c + 1)(u + u_0 + 1)}{(u + 1)(2u + 1)} f_m^{(u+1)} - m\left(1 + \frac{u_0 c}{u(u + 1)}\right) f_m^{(u)} \\
& - \frac{(u^2 - m^2)(u - u_0)(u + c)}{u(2u + 1)} f_m^{(u-1)}
\end{aligned} \tag{10.3}$$

$$\begin{aligned}
2K^+ f_m^{(u)} = {} & \frac{(u - c + 1)(u + u_0 + 1)}{(u + 1)(2u + 1)} f_{m+1}^{(u+1)} + (u - m)\left(1 + \frac{u_0 c}{u(u + 1)}\right) f_{m+1}^{(u)} \\
& + \frac{(u - m)(u - m - 1)(u - u_0)(u + c)}{u(2u + 1)} f_{m+1}^{(u-1)}
\end{aligned} \tag{10.4}$$

$$
\begin{aligned}
2K^- f_m^{(u)} = & - \frac{(u-c+1)(u+u_0+1)}{(u+1)(2u+1)} f_{m-1}^{(u+1)} \\
& + (u+m)\left(1 + \frac{u_0 c}{u(u+1)}\right) f_{m-1}^{(u)} \\
& - \frac{(u+m)(u+m-1)(u-u_0)(u+c)}{u(2u+1)} f_{m-1}^{(u-1)}
\end{aligned} \tag{10.5}
$$

$$Q_1 \equiv K^+K^- + K^3K^3 + K^3 = \tfrac{1}{4}(c+u_0+1)(c+u_0-1) \tag{10.6}$$

$$Q_2 \equiv K^+J^- + K^-J^+ + 2K^3J^3 + J^+J^- + J^3J^3 - J^3 = -u_0c. \tag{10.7}$$

As the reader can verify, the Casimir operators Q_1 and Q_2 commute with $J^\pm$, J^3, $K^\pm$, K^3, and these invariant operators reduce to constants on V. The above classes of irreducible representations can be derived using the Gelfand-Naimark method [8, 9]. These representations have been selected for convenience in the applications to follow and by no means exhaust the irreducible representations of $\mathcal{O}_4$. Note that on restriction to $sl(2)$, we have

$$\rho_1(c, u_0)\,|_{sl(2)} \cong \sum_{k=0}^{\infty} \oplus \uparrow^0_{u_0+k}$$

$$\rho_2(c, u_0)\,|_{sl(2)} \cong \sum_{k=0}^{\infty} \oplus D^0(2k).$$

where the representations $\uparrow_u{}^0$, $D^0(2u)$ are defined in [3].

The justification of our introduction of the Lie algebra $\mathcal{O}_4$ follows from the fact that the type α'' operators, which define representations of $sl(2)$, can be extended to $\mathcal{O}_4$. Indeed the operators

$$
\begin{aligned}
& J^+ = t\left(-(x+\gamma)E + (x+\gamma+q) + t\frac{\partial}{\partial t}\right), \\
& J^- = t^{-1}\left(xL - x + q - t\frac{\partial}{\partial t}\right), \qquad J^3 = t\frac{\partial}{\partial t}
\end{aligned} \tag{10.8}
$$

$$K^+ = t(x+\gamma)E, \qquad K^- = -t^{-1}xL, \qquad K^3 = x + \frac{\gamma}{2} \tag{10.9}$$

satisfy the commutation relations (10.1). Note that the J-operators here are equivalent to those in (9.8). Forming the Casimir operators Q_1, Q_2 from (10.8), (10.9) we find

$$Q_1 = \frac{\gamma}{4}(\gamma - 2), \qquad Q_2 = q(q+\gamma-1). \tag{10.10}$$

Based on this result we set $q = -u_0$, $\gamma = u_0 + c + 1$.

To construct models of the representations ρ using the operators (10.8), (10.9) we must find functions $f_m^{(u)}(x, t) = g_m^{(u)}(x)\, t^m$ such that the recursion relations

$$(-(x + u_0 + c + 1)\, E + (x + c + 1 + m))\, g_m^{(u)}(x) = (m - u)\, g_{m+1}^{(u)}(x)$$

$$(xL - x - u_0 - m)\, g_m^{(u)}(x) = -(m + u)\, g_{m-1}^{(u)}(x) \tag{10.11}$$

$$\begin{aligned}(2x + u_0 + c + 1)\, g_m^{(u)}(x) &= \frac{(u - c + 1)(u + u_0 + 1)}{(u + 1)(2u + 1)}\, g_m^{(u+1)}(x) - m\left(1 + \frac{u_0 c}{u(u + 1)}\right) g_m^{(u)}(x) \\ &\quad - \frac{(u^2 - m^2)(u - u_0)(u + c)}{u(2u + 1)}\, g_m^{(u-1)}(x)\end{aligned} \tag{10.12}$$

hold for values of the parameters u_0, c, m, u corresponding to the representations $\rho_i(c, u_0)$, $i = 1, 2$, given above. Relations obtained from (10.4), (10.5) are consequences of (10.11) and (10.12). As is almost obvious from the computations in Section 6, the Hahn polynomials satisfy these relations. In fact, the functions

$$g_m^{(u)}(x) = \frac{(u - u_0)!}{\Gamma(u - c + 1)}\, p_{u-u_0}(x, m + u_0 + 1, u_0 + c + 1, m + c + 1)$$

are solutions for each of the representations listed above. Relation (10.12) is the recurrence relation for Hahn polynomials given by Weber and Erdélyi [7]. If formally we restrict ourselves to the representation $\rho_2(0, -N)$, N a nonnegative integer, then for $m = 0$, (10.12) reduces to the well-known recursion relation for Tchebycheff polynomials [5]. Note, however, from (10.3)–(10.5) that for $u_0 = 0$, $c = -N$ we get a reducible representation of $\mathcal{O}_4$.

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