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On the S-transform over a Banach algebra[☆]

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Abstract

The S-transform is shown to satisfy a specific twisted multiplicativity property for free random variables in a B-valued Banach noncommutative probability space, for an arbitrary unital complex Banach algebra B. Also, a new proof of the additivity of the R-transform in this setting is given.

Keywords: Free convolution; S-transform

1. Introduction and statement of the main result

Let B be a unital complex Banach algebra. (In this paper, all Banach algebras will be over the complex numbers.) A B-valued Banach noncommutative probability space is a pair (A, E), where A is a unital Banach algebra containing an isometrically embedded copy of B as a unital subalgebra and where $E : A \to B$ is a bounded projection

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satisfying the conditional expectation property

$$E(b_1 a b_2) = b_1 E(a) b_2 \quad (a \in A, b_1, b_2 \in B).$$

In the free probability theory of Voiculescu, see [7,10], elements x and y of A are said to be free if their mixed moments $E(b_1 a_1 \cdots b_n a_n)$, where $a_j \in \{x, y\}$ and $b_j \in B$, are determined in a specific way from the moments of x and of y. Of particular interest, for example to garner spectral data, are the symmetric moments

$$E(bxybxy \cdots bxy) \tag{1}$$

of the product xy, for $b \in B$.

In the case $B = \mathbf{C}$, Voiculescu [8] invented the S-transform of an element $x \in A$ satisfying $E(x) \neq 0$. The S-transform can be used to find the generating function for the symmetric moments (1) of xy in terms of those for x and y individually, when x and y are free and when $E(x) \neq 0$ and $E(y) \neq 0$. In particular, Voiculescu showed that the S-transform is multiplicative:

$$S_{xy} = S_x S_y \tag{2}$$

when x and y are free.

In [9], Voiculescu gave a definition of an S-transform in the context of an arbitrary noncommutative probability space. However, this definition was quite complicated and involved differential equations.

Recently, Aagaard [1] took the straightforward extension of Voiculescu's definition [8] of the scalar-valued S-transform to the Banach algebra situation and generalized Voiculescu's result (2) to the case when B is a commutative unital Banach algebra and $E(x)$ and $E(y)$ are invertible elements of B.

In this paper, we treat the case when B is an arbitrary unital Banach algebra. We make an improvement in Aagaard's definition of the S-transform. For us, S_x is a B-valued analytic function defined in a neighborhood of 0 in B. We write S_{xy} in terms of S_x and S_y (again assuming $E(x)$ and $E(y)$ are invertible). Instead of simple multiplicativity (2), we have in general a twisted multiplicativity, as stated in our main theorem immediately below, which reduces to (2) when B is commutative.

Theorem 1.1. *Let B be a unital complex Banach algebra and let (A, E) be a B-valued Banach noncommutative probability space. Let $x, y \in A$ be free in (A, E) and assume both $E(x)$ and $E(y)$ are invertible elements of B. Then*

$$S_{xy}(b) = S_y(b) S_x(S_y(b)^{-1} b S_y(b)). \tag{3}$$

Our definition of the S-transform and our proof of Theorem 1.1 rely on the theory of analytic functions between Banach spaces—see, for example, Chapters III and XXVI of Hille and Phillips [5] and papers cited there.

In [3], Haagerup gave two new proofs of the multiplicativity of the S-transform in the case $B = \mathbf{C}$. Our proof of Theorem 1.1 is very much inspired by one of Haagerup's proofs, namely Theorem 2.3 of Haagerup [3], which uses creation and annihilation operators in the full Fock space. In particular, we consider a B-valued Banach algebra analog of the full Fock space and we construct random variables having arbitrary moments up to a given finite order, using analogs of the creation and annihilation operators. These are reminiscent of, though slightly different from, Voiculescu's constructions in [9].

In §2 below, we define the S-transform S_a (assuming the expectation of a is invertible). Then, considering Taylor expansions about zero, we show that the nth order term in the expansion for S_a depends only on the moments up to nth order of a. In §3, we construct operators analogous to the creation and annihilation operators on full Fock space, and we use these to prove the main result, Theorem 1.1. In §4, we offer a new proof of additivity of the R-tranfrom over a Banach space, using the operators and techniques introduced in the preceding sections.

2. The S-transform in a Banach noncommutative probability space

Let B be a unital Banach algebra. For $n \geqslant 1$ we will let $\mathcal{B}_n(B)$ denote the set of all bounded n-multilinear maps

$$\alpha_n : \underbrace{B \times \cdots \times B}_{n \text{ times}} \to B,$$

where multilinearity means over $\mathbf{C}$ and a multilinear map α_n is bounded if

$$\|\alpha_n\| := \sup\{\|\alpha_n(b_1, \ldots, b_n)\| \mid b_j \in B, \|b_1\|, \ldots, \|b_n\| \leqslant 1\} < \infty.$$

We say α_n is *symmetric* if it is invariant under arbitrary permutations of its n arguments.

From the theory of analytic functions between complex Banach spaces, any B-valued analytic function F defined on a neighborhood of zero in B has an expansion

$$F(b) = F(0) + \sum_{n=1}^{\infty} F_n(b, \ldots, b) \tag{4}$$

for some symmetric multilinear functions $F_n \in \mathcal{B}_n(B)$, with $\limsup_{n\to\infty} \|F_n\|^{1/n} < \infty$; see, for example, Theorem 3.17.1 of Hille and Phillips [5] and its proof. Here, F_1 is just the Fréchet derivative of F at 0 and the multilinear function F_n appearing in (4) is $1/n!$ times the nth variation of F, i.e. $n!F_n(h_1, \ldots, h_n)$ is the n-fold Fréchet derivative taken with respect to increments $h_1, \ldots, h_n$. For convenience we will write F_0 for $F(0)$. We will refer to (4) as the *power series expansion* of $F(b)$ around 0 and to $F_n(b, \ldots, b)$ as the nth term in this power series expansion. Note that the full symmetric multilinear function F_n can be recovered from knowing its diagonal $b \mapsto F_n(b, \ldots, b)$; for example, $n!F_n(b_1, \ldots, b_n)$ is the obvious partial derivative of

$$F_n(t_1b_1 + \cdots + t_nb_n, \ldots, t_1b_1 + \cdots + t_nb_n)$$

at $(0, \ldots, 0)$, where $t_1, \ldots, t_n$ are real variables.

Let (A, E) be a Banach noncommutative probability space over B, let $a \in A$ and suppose $E(a)$ is an invertible element of B. Consider the function

$$\Psi_a(b) = E((1 - ba)^{-1}) - 1 = \sum_{n=1}^{\infty} E((ba)^n), \tag{5}$$

defined for $\|b\| < \|a\|^{-1}$. Then Ψ_a is Fréchet differentiable on its domain, i.e. is analytic there. We also have

$$\Psi_a(b) = b\Phi_a(b), \tag{6}$$

where

$$\Phi_a(b) = E(a(1 - ba)^{-1}); \tag{7}$$

clearly Φ_a is analytic on the domain of Ψ_a. The Fréchet differential of Ψ_a at $b = 0$ is easily found to be the bounded linear map

$$h \mapsto hE(a) \tag{8}$$

from B to itself. By hypothesis, this linear map has bounded inverse $h \mapsto hE(a)^{-1}$. By the usual Banach space inverse function theorem, there are neighborhoods U and V of zero in B, such that U lies in the domain of Ψ_a and the restriction of Ψ_a to U is a homeomorphism onto V. Moreover, letting $\Psi_a^{\langle -1 \rangle}$ denote the inverse with respect to composition of the restriction of Ψ_a to U, the function $\Psi_a^{\langle -1 \rangle}$ is Fréchet differentiable on its domain and is, therefore, analytic there.

Lemma 2.1. *Assuming $E(a)$ is invertible, there is an open neighborhood of* 0 *in B and unique analytic B-valued function H_a defined there, such that $\Psi_a^{\langle -1 \rangle}(b) = bH_a(b)$.*

Proof. Uniqueness of H_a is clear by uniqueness of power series expansions about zero. Let us show existence. Using (6), we seek H_a, such that $bH_a(b)\Phi_a(bH_a(b)) = b$, and it will suffice to find H_a, such that

$$H_a(b)\Phi_a(bH_a(b)) = 1. \tag{9}$$

The existence of H_a follows from an easy application of the implicit function theorem for functions between Banach spaces, which is a result of Hildebrandt and Graves [4] (see also the discussion on p. 655 of Graves [2]). Indeed, $H_a(0) = E(a)^{-1}$ is a solution of (9) at $b = 0$ and the Fréchet differential of the function $x \mapsto x\Phi_a(bx)$ at $b = 0$ is the map (8), which has bounded inverse. □

Definition 2.2. Let $a \in A$ and assume $E(a)$ is invertible. The *S-transform* of a is the B-valued analytic function

$$S_a(b) = (1 + b)H_a(b), \tag{10}$$

which is defined in some neighborhood of 0 in B, where H_a is the function from Lemma 2.1.

Note that $S_a(0) = E(a)^{-1}$.

We may write

$$S_a(b) = (1 + b)b^{-1}\Psi_a^{\langle -1\rangle}(b), \tag{11}$$

which is the same formula given by Voiculescu [8] and used by Aagaard [1]. In the case $B = \mathbf{C}$, the definition (10) yields, of course, the same function as Voiculescu's S-transform. Moreover, the only difference between the definition (10) and the one appearing in [1] is that we have used the implicit function theorem to show that (11) makes sense for all b in a neighborhood of zero.

If F, G and H are B-valued analytic functions defined on neighborhoods of 0 in B, then the product FG is analytic and, if $H(0) = 0$, also the composition $F \circ H$ is analytic in some neighborhood of 0 in B. Straightforward asymptotic analysis yields the following formulas for the diagonals of the multilinear functions appearing in the power series expansions of FG and $F \circ H$.

Lemma 2.3. *We have for* $n \geqslant 0$

$$(FG)_n(b, \ldots, b) = \sum_{k=0}^{n} F_k(b, \ldots, b)G_{n-k}(b, \ldots, b) \tag{12}$$

and for $n \geqslant 1$

$$(F \circ H)_n(b, \ldots, b) = \sum_{k=1}^{n} \sum_{\substack{p_1,\ldots,p_k \geqslant 1 \\ p_1+\cdots+p_k=n}} F_k(H_{p_1}(b, \ldots, b), \ldots, H_{p_k}(b, \ldots, b)). \tag{13}$$

Lemma 2.4. *Let F be analytic in a neighborhood of 0. If $F(0)$ is an invertible element of B, then $G(b) = F(b)^{-1}$ defines a function that is analytic in a neighborhood of 0, and the nth term of its power series expansion is $G_0 = F_0^{-1}$ and, for $n \geqslant 1$,*

$$G_n(b, \dots, b) = -F_0^{-1} \sum_{k=1}^{n} F_k(b, \dots, b) G_{n-k}(b, \dots, b). \tag{14}$$

On the other hand, if $F(0) = 0$ and if F_1 has a bounded inverse, then F has an inverse with respect to composition, denoted $F^{\langle -1 \rangle}$, that is analytic in a neighborhood of 0. Taking $H = F^{\langle -1 \rangle}$, we have $H_1 = (F_1)^{\langle -1 \rangle}$ and, for $n \geqslant 2$,

$$H_n(b, \dots, b) = -(F_1)^{\langle -1 \rangle} \left(\sum_{k=2}^{n} \sum_{\substack{p_1, \dots, p_k \geqslant 1 \\ p_1 + \dots + p_k = n}} F_k(H_{p_1}(b, \dots, b), \dots, H_{p_k}(b, \dots, b)) \right). \tag{15}$$

Proof. Assuming $F(0)$ is invertible, that $G(b) = F(b)^{-1}$ is analytic is clear, and we have $(FG)_0 = 1$ and $(FG)_n = 0$ for $n \geqslant 1$. Now the expression (14) results from solving (12) for G_n.

If $F(0) = 0$ and the Fréchet derivative F_1 of F at 0 has bounded inverse, then by the inverse function theorem for Banach spaces, F has an inverse with respect to composition $F^{\langle -1 \rangle}$ that is analytic in a neighborhood of 0. Taking $H = F^{\langle -1 \rangle}$, we have $(F \circ H)_1 = \mathrm{id}_B$ and $(F \circ H)_n = 0$ for all $n \geqslant 2$. Solving in (13) for H_n yields the expression (15). □

Consider an element $a \in A$ as at the beginning of this section. We say the *nth moment function* of a is the multilinear function $\mu_{a,n} \in \mathcal{B}_n(B)$ given by

$$\mu_{a,n}(b_1, \dots, b_n) = E(b_1 a b_2 a \cdots b_n a).$$

Proposition 2.5. *Assume $E(a)$ is an invertible element of B. Then the nth term $(S_a)_n (b, \dots, b)$ in the power series expansion of the S-transform S_a of a about zero depends only on the first n moment functions $\mu_{a,1}, \mu_{a,2}, \dots, \mu_{a,n}$ of a.*

Proof. The symmetric n-multilinear function $(\Psi_a)_n$ appearing in the power series expansion of Ψ_a is the symmetrization of $\mu_{a,n}$. Using Lemma 2.4, we see that the nth term $(\Psi_a^{\langle -1 \rangle})_n(b, \dots, b)$ in the power series expansion of $\Psi_a^{\langle -1 \rangle}(b)$ around 0 depends only on $\mu_{a,1}, \dots, \mu_{a,n}$. But

$$(\Psi_a^{\langle -1 \rangle})_n(b, \dots, b) = b(H_a)_n(b, \dots, b),$$

$$(S_a)_n(b, \dots, b) = (1 + b)(H_a)_n(b, \dots, b)$$

and the result is proved. □

3. Twisted multiplicativity of the S-transform

Let B be a unital Banach algebra over $\mathbf{C}$ and let I be a set. Let $D = \ell^1(I, B)$ be the Banach space of all functions $d : I \to B$, such that $\|d\| := \sum_{i\in I} \|d(i)\| < \infty$. For $i \in I$, $\delta_i \in D$ will denote the function taking value 1 at i and 0 at all other elements of I. We have the obvious left action of B on D by $(bd)(i) = bd(i)$, and the resulting algebra homomorphism $B \to \mathcal{B}(D)$ is isometric. (Whenever X is a Banach space, we denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from X to itself.) For $k \geqslant 1$, let $D^{\hat{\otimes} k} = D\hat{\otimes} \cdots \hat{\otimes} D$ be the k-fold Banach space projective tensor product of D with itself (over the complex field). Consider the Banach space

$$\mathcal{F} = B\Omega \oplus \bigoplus_{k=1}^{\infty} D^{\hat{\otimes} k} \hat{\otimes} B, \tag{16}$$

where also $\hat{\otimes} B$ is the Banach space projective tensor product and where the we take the direct sum with respect to the ℓ^1-norm. Here, $B\Omega$ signifies just a copy of B and Ω denotes the identity element of this copy of B, considered as a vector in $\mathcal{F}$. Let $\lambda : B \to \mathcal{B}(\mathcal{F})$ be the map defined by

$$\lambda(b)(b_0\Omega) = (bb_0)\Omega,$$

$$\lambda(b)(d_1 \otimes \cdots \otimes d_k \otimes b_0) = (bd_1) \otimes d_2 \otimes \cdots \otimes d_k \otimes b_0$$

for $k \in \mathbf{N}$, $d_1, \ldots, d_k \in D$ and $b_0 \in B$. Then λ is an isometric algebra homomorphism. We will often omit to write λ, and just think of B as included in $\mathcal{B}(\mathcal{F})$ by this left action.

Remark 3.1. For specificity, we took the ℓ^1 norms in the definitions of D and $\mathcal{F}$, but we actually have considerable flexibility. For D we need only a Banach space completion of the set of all functions $d : I \to B$ vanishing at all but finitely many elements in I with the property $\|b\delta_i\| = \|b\|$, and similarly for $\mathcal{F}$. Moreover, we could replace the projective tensor norm $\hat{\otimes} B$ in (16) with any tensor norm so that $\|d \otimes B\| = \|d\|\,\|b\|$ for all $d \in D^{\hat{\otimes} k}$ and $b \in B$.

Let $P : \mathcal{F} \to B$ be the projection onto the summand $B\Omega = B$ that sends all summands $D^{\hat{\otimes} k}\hat{\otimes} B$ to zero and let $\mathcal{E} : \mathcal{B}(\mathcal{F}) \to B$ be $\mathcal{E}(X) = P(X\Omega)$. Then $\mathcal{E}$ has norm 1 and satisfies $\mathcal{E} \circ \lambda = \mathrm{id}_B$. Let $\rho : B \to \mathcal{B}(\mathcal{F})$ be the map defined by

$$\rho(b)(b_0\Omega) = (b_0 b)\Omega,$$

$$\rho(b)(d_1 \otimes \cdots \otimes d_k \otimes b_0) = d_1 \otimes \cdots \otimes d_k \otimes (b_0 b).$$

Then ρ is an isometric algebra isomorphism from the opposite algebra B^{op} into $\mathcal{B}(\mathcal{F})$. Let $\mathcal{B}(\mathcal{F}) \cap \rho(B)'$ denote the set of all bounded operators on $\mathcal{F}$ that commute with $\rho(b)$ for all $b \in B$. Note that $\lambda(B) \subseteq \mathcal{B}(\mathcal{F}) \cap \rho(B)'$.

Proposition 3.2. *The restriction of $\mathcal{E}$ to $\mathcal{B}(\mathcal{F}) \cap \rho(B)'$ satisfies the conditional expectation property*

$$\mathcal{E}(b_1 X b_2) = b_1 \mathcal{E}(X) b_2 \quad (X \in \mathcal{B}(\mathcal{F}) \cap \rho(B)', b_1, b_2 \in B).$$

Proof. We have

$$\begin{aligned}\mathcal{E}(b_1 X b_2) &= P(\lambda(b_1) X \lambda(b_2)\Omega) = P(\lambda(b_1) X \rho(b_2)\Omega) \\ &= P(\rho(b_2)\lambda(b_1) X\Omega) = P(\lambda(b_1) X\Omega) b_2 = b_1 P(X\Omega) b_2 = b_1 \mathcal{E}(X) b_2. \qquad \square\end{aligned}$$

For $i \in I$, let $L_i \in \mathcal{B}(\mathcal{F})$ be defined by

$$L_i(b_0\Omega) = \delta_i \otimes b_0,$$

$$L_i(d_1 \otimes \cdots \otimes d_k \otimes b_0) = \delta_i \otimes d_1 \otimes \cdots \otimes d_k \otimes b_0.$$

Thus,

$$b_1\delta_{i_1} \otimes b_2\delta_{i_2} \otimes \cdots \otimes b_k\delta_{i_k} \otimes b_0 = b_1 L_{i_1} b_2 L_{i_2} \cdots b_k L_{i_k} b_0 \Omega.$$

Recall that $\mathcal{B}_n(B)$ denotes the set of all bounded multilinear functions from the n-fold product of B to B. We will also let $\mathcal{B}_0(B) = B$. If $i \in I$, $n \in \mathbf{N}$ and $\alpha_n \in \mathcal{B}_n(B)$, define $V_{i,n}(\alpha_n)$ and $W_{i,n}(\alpha_n)$ in $\mathcal{B}(\mathcal{F})$ by

$$V_{i,n}(\alpha_n)(b_0\Omega) = 0,$$

$$V_{i,n}(\alpha_n)(d_1 \otimes \cdots \otimes d_k \otimes b_0) = \begin{cases} 0, & k < n, \\ \alpha_n(d_1(i), \ldots, d_n(i)) b_0 \Omega, & k = n, \\ \alpha_n(d_1(i), \ldots, d_n(i)) d_{n+1} \otimes \cdots \otimes d_k \otimes b_0, & k > n \end{cases}$$

and

$$W_{i,n}(\alpha_n)(b_0\Omega) = 0,$$

$$\begin{aligned} &W_{i,n}(\alpha_n)(d_1 \otimes \cdots \otimes d_k \otimes b_0) \\ &\quad = \begin{cases} 0, & k < n, \\ \alpha_n(d_1(i), \ldots, d_n(i)) \delta_i \otimes b_0, & k = n, \\ \alpha_n(d_1(i), \ldots, d_n(i)) \delta_i \otimes d_{n+1} \otimes \cdots \otimes d_k \otimes b_0, & k > n. \end{cases}\end{aligned}$$

Finally, taking $n=0$ and $\alpha_0 \in B$, let

$$V_{i,0}(\alpha_0)=\alpha_0, \quad W_{i,0}(\alpha_0)=\alpha_0 L_i.$$

These formulas are guaranteed to define bounded operators on $\mathcal{F}$, because we took the projective tensor product in $D^{\hat{\otimes}k}$. The expression $V_{i,n}(\alpha_n)$, $n \geqslant 1$, is a sort of n-fold annihilation operator, while $W_{i,n}(\alpha_n)$ is n-fold annihilation combined with single creation, and, of course, $W_{i,0}$ is a single creation operator. Note that in all cases we have $V_{i,n}(\alpha_n),\ W_{i,n}(\alpha_n) \in \mathcal{B}(\mathcal{F}) \cap \rho(B)'$.

The relations gathered in the following lemma are easily verified.

Lemma 3.3. *Let $n, m \in \mathbf{N}$ and $\alpha_n \in \mathcal{B}_n(B)$, $\beta_m \in \mathcal{B}_m(B)$ and take $b \in B$. Then*

(i)

$$V_{i,n}(\alpha_n)\lambda(b)=V_{i,n}(\tilde{\alpha}_n), \quad W_{i,n}(\alpha_n)\lambda(b)=W_{i,n}(\tilde{\alpha}_n),$$

where

$$\tilde{\alpha}_n(b_1,\ldots,b_n)=\alpha_n(bb_1,b_2,\ldots,b_n);$$

(ii) *if $n=1$, then*

$$V_{i,1}(\alpha_1)L_i=\lambda(\alpha_1(1)), \quad W_{i,1}(\alpha_1)L_i=\lambda(\alpha_1(1))L_i$$

and for $n \geqslant 2$ we have

$$V_{i,n}(\alpha_n)L_i=V_{i,n-1}(\tilde{\alpha}_{n-1}), \quad W_{i,n}(\alpha_n)L_i=W_{i,n-1}(\tilde{\alpha}_{n-1}),$$

where here

$$\tilde{\alpha}_{n-1}(b_1,\ldots,b_{n-1})=\alpha_n(1,b_1,\ldots,b_{n-1});$$

(iii) *we have*

$$V_{i,n}(\alpha_n)V_{i,m}(\beta_m)=V_{i,n+m}(\gamma_{n+m}), \quad W_{i,n}(\alpha_n)V_{i,m}(\beta_m)=W_{i,n+m}(\gamma_{n+m}),$$

where

$$\gamma_{n+m}(b_1,\ldots,b_{m+n})=\alpha_n(\beta_m(b_1,\ldots,b_m)b_{m+1},b_{m+2},\ldots,b_{m+n});$$

(iv)

$$V_{i,n}(\alpha_n)W_{i,m}(\beta_m) = V_{i,n+m-1}(\gamma_{n+m-1}),$$

$$W_{i,n}(\alpha_n)W_{i,m}(\beta_m) = W_{i,n+m-1}(\gamma_{n+m-1}),$$

where

$$\gamma_{n+m-1}(b_1, \ldots, b_{m+n-1}) = \alpha_n(\beta_m(b_1, \ldots, b_m), b_{m+1}, b_{m+2}, \ldots, b_{m+n-1});$$

(v)

$$\lambda(b)V_{i,n}(\alpha_n) = V_{i,n}(b\alpha_n),$$

(vi) *if* $i' \neq i$ *and* $n \geqslant 1$, *then*

$$V_{i,n}(\alpha_n)L_{i'} = 0 = W_{i,n}(\alpha_n)L_{i'}.$$

Proposition 3.4. *For* $i \in I$ *let* $\mathfrak{A}_i \subseteq \mathcal{B}(\mathcal{F}) \cap \rho(B)'$ *be the subalgebra generated by*

$$\lambda(B) \cup \{L_i\} \cup \{V_{i,n}(\alpha_n) \mid n \in \mathbf{N}, \alpha_n \in \mathcal{B}_n(B)\} \cup \{W_{i,n}(\alpha_n) \mid n \in \mathbf{N}, \alpha_n \in \mathcal{B}_n(B)\}.$$

Then the family $(\mathfrak{A}_i)_{i \in I}$ *is free with respect to* $\mathcal{E}$.

Proof. Using Lemma 3.3, we see that every element of $\mathfrak{A}_i$ can be written as a sum of finitely many terms of the following forms:

(i) $\lambda(b)$,
(ii) $\lambda(b_0)L_i\lambda(b_1)\cdots L_i\lambda(b_n)$,
(iii) $V_{i,n}(\alpha_n)$,
(iv) $\lambda(b_0)L_i\lambda(b_1)L_i\cdots\lambda(b_k)L_iV_{i,n}(\alpha_n)$,
(v) $\lambda(b)W_{i,n}(\alpha_n)$,
(vi) $\lambda(b_0)L_i\lambda(b_1)L_i\cdots\lambda(b_{k-1})L_i\lambda(b_k)W_{i,n}(\alpha_n)$.

Now all terms of the forms (ii)–(vi) lie in $\ker\mathcal{E}$, while $\mathcal{E}(\lambda(b)) = b$. Therefore, $\mathfrak{A}_i \cap \ker\mathcal{E}$ is the set of all finite sums of terms of the forms (ii)–(vi).

Let $p \in \mathbf{N}$ with $p \geqslant 2$ and take $i_1, \ldots, i_p \in I$ with $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{p-1} \neq i_p$. Suppose $a_j \in \mathfrak{A}_{i_j} \cap \mathcal{E}$ $(1 \leqslant j \leqslant p)$ and let us show $\mathcal{E}(a_1 \cdots a_p) = 0$. From Lemma 3.3 part (vi), we see $a_1a_2\cdots a_p = 0$ unless either $\forall j\ a_j$ is of the form (ii) or $\forall j\ a_j$ is of the form (iii) or (v). But $V_{i,n}(\alpha_n)\Omega = 0 = W_{i,n}(\alpha_n)\Omega$ when $n \geqslant 1$, so if a_p is of the form (iii) or (v), then $\mathcal{E}(a_1 \cdots a_p) = 0$. We are left to consider the case when $a_1 \cdots a_p$

can be written as

$$(\lambda(b_0)L_{i_1}\lambda(b_1^{(1)})L_{i_1}\lambda(b_2^{(1)})\cdots L_{i_1}\lambda(b_{k(1)}^{(1)}))(L_{i_2}\lambda(b_1^{(2)})\cdots L_{i_2}\lambda(b_{k(2)}^{(2)}))$$
$$\cdots(L_{i_p}\lambda(b_1^{(p)})\cdots L_{i_p}\lambda(b_{k(p)}^{(p)})),$$

where all $k(j)\geqslant 1$. But in this case, clearly $\mathcal{E}(a_1\cdots a_p)=0$. □

Lemma 3.5. *Let $N\in\mathbf{N}$ and for every $n\in\{0,1,\ldots,N\}$ let $\alpha_n\in\mathcal{B}_n(B)$. Fix $i\in I$ and let*

$$X=\sum_{n=0}^{N-1}(V_{i,n}(\alpha_n)+W_{i,n}(\alpha_n)),$$

$$Y=X+V_{i,N}(\alpha_N)+W_{i,N}(\alpha_N).$$

Then for any $b_0,\ldots,b_N\in B$, we have

$$\mathcal{E}(b_0Yb_1Y\cdots b_NY)=b_0\alpha_N(b_1\alpha_0,b_2\alpha_0,\ldots,b_N\alpha_0)+\mathcal{E}(b_0Xb_1X\cdots b_NX).$$

Proof. To evaluate $\mathcal{E}(b_0Yb_1Y\cdots b_NY)$, first write

$$Y=\sum_{n=0}^{N}(V_{i,n}(\alpha_n)+W_{i,n}(\alpha_n))$$

and distribute. Now using the creation and annihilation properties of the $W_{i,n}(\alpha_n)$ and $V_{i,n}(\alpha_n)$ operators, we see that the only term involving α_N to contribute a possibly nonzero quantity to $\mathcal{E}(b_0Yb_1Y\cdots b_NY)$ is

$$\mathcal{E}(b_0V_{i,N}(\alpha_N)b_1W_{i,0}(\alpha_0)\cdots b_NW_{i,0}(\alpha_0)),$$

whose value is $b_0\alpha_N(b_1\alpha_0,b_2\alpha_0,\ldots,b_N\alpha_0)$. The other terms involve only $\alpha_0,\ldots,\alpha_{N-1}$ and their sum is $\mathcal{E}(b_0Xb_1X\cdots b_NX)$. □

Proposition 3.6. *Let (A,E) be a B-valued Banach noncommutative probability space and let $a\in A$, $N\in\mathbf{N}$. Suppose $E(a)$ is an invertible element of B. Let $\alpha_0=E(a)$. Then there are $\alpha_1,\ldots,\alpha_N$, with $\alpha_n\in\mathcal{B}_n(B)$, such that if*

$$X=\sum_{n=0}^{N}(V_{i,n}(\alpha_n)+W_{i,n}(\alpha_n))\in\mathcal{B}(\mathcal{F}),$$

then

$$\mathcal{E}(b_0 X b_1 X \cdots b_k X) = E(b_0 a b_1 a \cdots b_k a) \tag{17}$$

for all $k \in \{1, \ldots, N\}$ *and all* $b_0, \ldots, b_N \in B$.

Proof. Using Lemma 3.5, The maps α_k can be chosen recursively in k so that (17) holds. □

For the remainder of this section, we take $I = \{1, 2\}$.

Lemma 3.7. *Let* $\alpha_0 \in B$ *be invertible. Let* $N \in \mathbf{N}$ *and choose* $\alpha_n \in \mathcal{B}_n(B)$ *for* $n \in \{1, \ldots, N\}$, *and let*

$$F(b) = \alpha_0 + \sum_{n=1}^{N} \alpha_n(b, \ldots, b).$$

Note that $F(b)$ *is invertible for* $\|b\|$ *sufficiently small. Let*

$$X = \sum_{n=0}^{N} \big(V_{1,n}(\alpha_n) + W_{1,n}(\alpha_n)\big) \in \mathcal{B}(\mathcal{F}). \tag{18}$$

Then the S-transform of X is $S_X(b) = F(b)^{-1}$.

Proof. For $b \in B$, $\|b\| < 1$, let

$$\omega_b = \Omega + \sum_{k=1}^{\infty} (b\delta_1)^{\otimes k} \otimes 1 \in \mathcal{F}.$$

We have $V_{1,0}(\alpha_0)\omega_b = \alpha_0\omega_b$ and, for $n \geqslant 1$,

$$V_{1,n}(\alpha_n)\omega_b = \alpha_n(b, \ldots, b)\Omega + \sum_{k=n+1}^{\infty} \alpha_n(b, \ldots, b)(b\delta_1)^{\otimes(k-n)} \otimes 1 = \alpha_n(b, \ldots, b)\omega_b.$$

Moreover, $W_{1,0}(\alpha_0)\omega_b = \alpha_0 L_1 \omega_b$ and, for $n \geqslant 1$,

$$\begin{aligned} W_{1,n}(\alpha_n)\omega_b &= \alpha_n(b, \ldots, b)\delta_1 \otimes 1 + \sum_{k=n+1}^{\infty} \alpha_n(b, \ldots, b)\delta_1 \otimes (b\delta_1)^{\otimes(k-n)} \otimes 1 \\ &= \alpha_n(b, \ldots, b) L_1 \omega_b. \end{aligned}$$

Thus,

$$X\omega_b = F(b)(1+L_1)\omega_b.$$

For $\|b\|$ sufficiently small, we get

$$F(b)^{-1}X\omega_b = \omega_b + L_1\omega_b,$$

$$bF(b)^{-1}X\omega_b = b\omega_b + (\omega_b - \Omega),$$

$$\Omega = (1+b)\omega_b - bF(b)^{-1}X\omega_b,$$

$$\Omega = (1 - bF(b)^{-1}X(1+b)^{-1})(1+b)\omega_b,$$

$$(1 - bF(b)^{-1}X(1+b)^{-1})^{-1}\Omega = (1+b)\omega_b,$$

$$\begin{aligned}\mathcal{E}((1 - bF(b)^{-1}X(1+b)^{-1})^{-1}) &= P((1+b)\omega_b)\\ &= 1+b.\end{aligned}$$

Conjugating with $(1+b)$ yields

$$1+b = \mathcal{E}((1-(1+b)^{-1}bF(b)^{-1}X)^{-1}) = 1 + \Psi_X((1+b)^{-1}bF(b)^{-1}).$$

Hence,

$$\Psi_X^{\langle -1\rangle}(b) = (1+b)^{-1}bF(b)^{-1}$$

and $S_X(b) = F(b)^{-1}$. $\square$

Lemma 3.8. *Let* $\alpha_0, \ldots, \alpha_N$, F *and* X *be as in Lemma* 3.7. *Let* $\beta_0 \in B$ *be invertible and let* $\beta_n \in \mathcal{B}_n(B)$ *for* $n \in \{1, \ldots, N\}$. *Let*

$$G(b) = \beta_0 + \sum_{n=1}^{N} \beta_n(b, \ldots, b)$$

and let

$$Y = \sum_{n=0}^{N} (V_{2,n}(\alpha_n) + W_{2,n}(\alpha_n)) \in \mathcal{B}(\mathcal{F}). \tag{19}$$

Then the S-transform of XY is

$$S_{XY}(b) = G(b)^{-1}F(G(b)bG(b)^{-1})^{-1} = S_Y(b)S_X(S_Y(b)^{-1}bS_Y(b)). \tag{20}$$

Proof. From Lemma 3.7, we have $S_Y(b) = G(b)^{-1}$ and $S_X(b) = F(b)^{-1}$, so the right-most equality in (20) is true. For $b \in B$ let

$$Z_b = bL_2 + bG(b)^{-1}L_1G(b) + bG(b)^{-1}L_1G(b)L_2 \in \mathcal{B}(\mathcal{F})$$

and insist that $\|b\|$ be so small that $\|Z_b\| < 1$. Let

$$\sigma_b = (1 - Z_b)^{-1}\Omega = \Omega + \sum_{k=1}^{\infty} Z_b^k\Omega.$$

Using Lemma 3.3, we find for $n, k \geqslant 0$,

$$V_{2,n}(\beta_n)Z_b^k = \begin{cases} V_{2,n-k}(\tilde{\beta}_{n-k}), & k < n, \\ \beta_n(b, \ldots, b), & k = n, \\ \beta_n(b, \ldots, b)Z_b^{k-n}, & k > n \end{cases}$$

and

$$W_{2,n}(\beta_n)Z_b^k = \begin{cases} W_{2,n-k}(\tilde{\beta}_{n-k}), & k < n, \\ \beta_n(b, \ldots, b)L_2, & k = n, \\ \beta_n(b, \ldots, b)L_2Z_b^{k-n}, & k > n, \end{cases}$$

where

$$\tilde{\beta}_{n-k}(b_1, \ldots, b_{n-k}) = \beta_n(\underbrace{b, \ldots, b}_{k}, b_1, \ldots, b_{n-k}).$$

Therefore,

$$V_{2,n}(\beta_n)Z_b^k\Omega = \begin{cases} 0, & k < n, \\ \beta_n(b, \ldots, b)\Omega, & k = n, \\ \beta_n(b, \ldots, b)Z_b^{k-n}\Omega, & k > n \end{cases}$$

and

$$W_{2,n}(\beta_n)Z_b^k\Omega = \begin{cases} 0, & k < n, \\ \beta_n(b, \ldots, b)L_2\Omega, & k = n, \\ \beta_n(b, \ldots, b)L_2Z_b^{k-n}\Omega, & k > n \end{cases}$$

and we get

$$Y\sigma_b = G(b)(1+L_2)\sigma_b.$$

Letting $b' = G(b)bG(b)^{-1}$, we similarly find for $n, k \geqslant 0$,

$$V_{1,n}(\alpha_n)G(b)Z_b^k = \begin{cases} V_{1,n-k}(\tilde{\alpha}_{n-k})G(b), & k < n, \\ \alpha_n(b', \ldots, b')G(b)(1+L_2), & k = n, \\ \alpha_n(b', \ldots, b')G(b)(1+L_2)Z_b^{k-n}, & k > n \end{cases}$$

and

$$W_{1,n}(\alpha_n)G(b)Z_b^k = \begin{cases} W_{1,n-k}(\tilde{\alpha}_{n-k})G(b), & k < n, \\ \alpha_n(b', \ldots, b')L_1G(b)(1+L_2), & k = n, \\ \alpha_n(b', \ldots, b')L_1G(b)(1+L_2)Z_b^{k-n}, & k > n, \end{cases}$$

where

$$\tilde{\alpha}_{n-k}(b_1, \ldots, b_{n-k}) = \alpha_n(\underbrace{b', \ldots, b'}_{k}, b_1, \ldots, b_{n-k}).$$

Therefore, we get

$$XY\sigma_b = F(b')(1+L_1)G(b)(1+L_2)\sigma_b.$$

Thus, for $\|b\|$ sufficiently small we get

$$F(b')^{-1}XY = (1+L_1)G(b)(1+L_2)\sigma_b,$$

$$F(b')^{-1}XY = G(b)\sigma_b + (G(b)L_2 + L_1G(b) + L_1G(b)L_2)\sigma_b,$$

$$bG(b)^{-1}F(b')^{-1}XY\sigma_b = b\sigma_b + Z_b\sigma_b,$$

$$bG(b)^{-1}F(b')^{-1}XY\sigma_b = b\sigma_b + (\sigma_b - \Omega),$$

$$\Omega = ((1+b) - bG(b)^{-1}F(b')^{-1}XY)\sigma_b,$$

$$\Omega = (1 - bG(b)^{-1}F(b')^{-1}XY(1+b)^{-1})(1+b)\sigma_b,$$

$$(1 - bG(b)^{-1}F(b')^{-1}XY(1+b)^{-1})^{-1}\Omega = (1+b)\sigma_b,$$

$$\begin{aligned}\mathcal{E}((1-bG(b)^{-1}F(b')^{-1}XY(1+b)^{-1})^{-1}) &= P((1+b)\sigma_b)\\ &= 1+b.\end{aligned}$$

Conjugating with $(1+b)$ yields

$$\Psi_{XY}((1+b)^{-1}bG(b)^{-1}F(b')^{-1})=\mathcal{E}((1-(1+b)^{-1}bG(b)^{-1}F(b')^{-1}XY)^{-1})-1=b.$$

Hence,

$$\Psi_{XY}^{\langle -1\rangle}(b) = (1+b)^{-1}bG(b)^{-1}F(b')^{-1}$$

and (20) holds. □

Proof of Theorem 1.1. The formula (3) asserts the equality of the germs of two analytic B-valued functions. This is equivalent to asserting the equality of the nth terms in their respective power series expansions around zero, for every $n\geqslant 0$. By Lemmas 2.3 and 2.4, the nth term, call it RHS_n, in the expansion for the right-hand side of (3) depends only on the 0th through the nth terms of the power series expansions for $S_x(b)$ and $S_y(b)$. Hence, by Proposition 2.5, RHS_n depends only on the moment functions $\mu_{x,1},\ldots,\mu_{x,n}$ and $\mu_{y,1},\ldots,\mu_{y,n}$. On the other hand, again by Proposition 2.5, the nth term in the power series expansion for the left-hand side of (3), call it LHS_n, depends only on $\mu_{xy,1},\ldots,\mu_{xy,n}$. But by freeness of x and y, for each $k\geqslant 1$ the moment function $\mu_{xy,k}$ depends only on $\mu_{x,1},\ldots,\mu_{x,k}$ and $\mu_{y,1},\ldots,\mu_{y,k}$. Thus, both LHS_n and RHS_n depend only on $\mu_{x,1},\ldots,\mu_{x,n}$ and $\mu_{y,1},\ldots,\mu_{y,n}$.

Hence, in order to prove (3) at the level of the nth terms in the power series expansion, it will suffice to prove (3) for some free pair X and Y of elements in a Banach noncommutative probability space over B, whose first n moment functions agree with those of x and y, respectively. However, by Propositions 3.4 and 3.6, such X and Y can be chosen of the forms (18) and (19). By Lemma 3.8, the equality (3) holds for these operators. □

4. A proof of the additivity of the R-transform over a Banach algebra

The R-transform over a general unital algebra B has been well understood since Voiculescu's work [9] (and see also Speicher's approach in [6]). However, for completeness, in this section we offer a new proof, using the techniques and constructions of the previous two sections, of the additivity of the R-transform for free random variables in a Banach noncommutative probability space. This proof is, of course, analogous to Haagerup's proof of Theorem 2.2 of Haagerup [3] in the scalar-valued case.

Let (A,E) be a Banach noncommutative probability space over B and let $a\in A$. Consider the function

$$\mathcal{C}_a(b) = E((1-ba)^{-1}b) = \sum_{n=0}^{\infty} E((ba)^n b),$$

defined and analytic for $\|b\| < \|a\|^{-1}$. We have $\mathcal{C}_a(b) = b + b\Phi_a(b)b$, where Φ_a is as in (7). Since the Fréchet differential of $\mathcal{C}_a$ at $b = 0$ is the identity map, $\mathcal{C}_a$ is invertible with respect to composition in a neighborhood of zero.

Proposition 4.1. *There is a unique B-valued analytic function R_a, defined in a neighborhood of 0 in B, such that*

$$\mathcal{C}_a^{\langle -1\rangle}(b) = (1 + bR_a(b))^{-1}b = b(1 + R_a(b)b)^{-1}. \tag{21}$$

Proof. Again, uniqueness is clear by the power series expansions.

The right-most equality in (21) holds for any analytic function R_a. We seek a function R_a, such that

$$\mathcal{C}_a((1 + bR_a(b))^{-1}b) = b.$$

But

$$\mathcal{C}_a((1 + bR_a(b))^{-1}b) = (1 + bR_a(b))^{-1}b + (1 + bR_a(b))^{-1}b\Phi_a \times \Big((1 + bR_a(b))^{-1}b\Big)\,(1 + bR_a(b))^{-1}b,$$

so it will suffice to find R_a so that any of the following hold:

$$(1 + bR_a(b))^{-1} + (1 + bR_a(b))^{-1}b\,\Phi_a\Big((1 + bR_a(b))^{-1}b\Big)(1 + bR_a(b))^{-1} = 1,$$

$$1 + b\Phi_a\Big((1 + bR_a(b))^{-1}b\Big)(1 + bR_a(b))^{-1} = 1 + bR_a(b),$$

$$b\Phi_a\Big((1 + bR_a(b))^{-1}b\Big)(1 + bR_a(b))^{-1} = bR_a(b),$$

$$\Phi_a\Big((1 + bR_a(b))^{-1}b\Big)(1 + bR_a(b))^{-1} = R_a(b). \tag{22}$$

However, $R_a(0) = E(a)$ is a solution of (22) at $b = 0$, and the Fréchet differential of the function $x \mapsto \Phi_a((1+bx)^{-1}b)(1+bx)^{-1} - x$ at $b = 0$ is the negative of the identity map, hence is invertible. The implicit function theorem of Hildebrandt and Graves [4] (see also the discussion on p. 655 of Graves [2]) guarantees the existence of R_a. $\square$

The *R-transform* of a is defined to be the analytic function R_a from Proposition 4.1. Analogously to Proposition 2.5, we have the following.

Proposition 4.2. *The nth term $(R_a)_n(b, \ldots, b)$ in the power series expansion for R_a about zero depends only on the first $n + 1$ moment functions $\mu_{a,1}, \ldots, \mu_{a,n+1}$ of a.*

Here is the analog to Lemma 3.5, which can be proved similarly.

Lemma 4.3. *Let $N \in \mathbf{N}$ and for every $n \in \{0, 1, \ldots, N\}$ let $\alpha_n \in \mathcal{B}_n(B)$. Fix $i \in I$ and let*

$$X = L_i + \sum_{n=0}^{N-1} V_{i,n}(\alpha_n),$$

$$Y = X + V_{i,N}(\alpha_N).$$

Then for any $b_0, \ldots, b_N \in B$, we have

$$\mathcal{E}(b_0 Y b_1 Y \cdots b_N Y) = b_0 \alpha_N(b_1, b_2, \ldots, b_N) + \mathcal{E}(b_0 X b_1 X \cdots b_N X).$$

We immediately get the following analog of Proposition 3.6.

Proposition 4.4. *Let (A, E) be a B-valued Banach noncommutative probability space and let $a \in A$, $N \in \mathbf{N}$. Then there are $\alpha_0, \alpha_1, \ldots, \alpha_N$, with $\alpha_n \in \mathcal{B}_n(B)$, such that if*

$$X = L_i + \sum_{n=0}^{N} V_{i,n}(\alpha_n) \in \mathcal{B}(\mathcal{F}),$$

then

$$\mathcal{E}(b_0 X b_1 X \cdots b_k X) = E(b_0 a b_1 a \cdots b_k a)$$

for all $k \in \{1, \ldots, N\}$ and all $b_0, \ldots, b_N \in B$.

Now we have the following analogos of Lemmas 3.7 and 3.8.

Lemma 4.5. *Let $N \in \mathbf{N}$ and choose $\alpha_n \in \mathcal{B}_n(B)$ for $n \in \{0, 1, \ldots, N\}$, and let*

$$F(b) = \alpha_0 + \sum_{n=1}^{N} \alpha_n(b, \ldots, b).$$

Let

$$X = L_1 + \sum_{n=0}^{N} V_{1,n}(\alpha_n) \in \mathcal{B}(\mathcal{F}).$$

Then the R-transform of X is $R_X(b) = F(b)$.

Proof. With ω_b defined as in the proof of Lemma 3.7, we have

$$X\omega_b = L_1\omega_b + F(b)\omega_b,$$

$$bX\omega_b = (\omega_b - \Omega) + bF(b)\omega_b,$$

$$(1 + bF(b) - bX)\omega_b = \Omega,$$

$$(1 - bX(1 + bF(b))^{-1})^{-1}\Omega = (1 + bF(b))\omega_b,$$

$$\begin{aligned}\mathcal{E}((1 - bX(1 + bF(b))^{-1})^{-1}) &= P((1 + bF(b))\omega_b) \\ &= 1 + bF(b).\end{aligned}$$

Conjugating yields

$$\mathcal{E}((1 - (1 + bF(b))^{-1}bX)^{-1}) = 1 + bF(b),$$

so

$$\mathcal{C}_X((1 + bF(b))^{-1}b) = \mathcal{E}((1 - (1 + bF(b))^{-1}bX)^{-1})(1 + bF(b))^{-1}b = b.$$

Thus,

$$\mathcal{C}_X^{\langle -1\rangle}(b) = (1 + bF(b))^{-1}b$$

and $R_a(b) = F(b)$. $\square$

Lemma 4.6. *Let* $\alpha_0, \ldots, \alpha_n$, F *and* X *be as in Lemma* 4.5. *Let* $\beta_n \in \mathcal{B}_n(B)$ *for* $n \in \{0, 1, \ldots, N\}$. *Let*

$$G(b) = \beta_0 + \sum_{n=1}^{N} \beta_n(b, \ldots, b)$$

and let

$$Y = L_2 + \sum_{n=0}^{N} V_{2,n}(\alpha_n) \in \mathcal{B}(\mathcal{F}).$$

Then the R-transform of $X+Y$ is

$$R_{X+Y}(b) = F(b) + G(b) = R_X(b) + R_Y(b).$$

Proof. For $b \in B$ with $\|b\| < \frac{1}{2}$, let

$$\sigma_b = (1 - b(L_1 + L_2))^{-1}\Omega = \Omega + \sum_{k=1}^{\infty} (b\delta_1 + b\delta_2)^{\otimes k} \otimes 1 \in \mathcal{F}.$$

Then

$$(X+Y)\sigma_b = (L_1 + L_2)\sigma_b + (F(b) + G(b))\sigma_b,$$

$$b(X+Y)\sigma_b = (\sigma_b - \Omega) + b\,(F(b) + G(b))\sigma_b.$$

Now arguing as in the proof of Lemma 4.5 above yields $R_{X+Y}(b) = F(b) + G(b)$. □

Finally, we get a proof, which is analogous to our proof of Theorem 1.1, of the additivity of the R-transform in a Banach noncommutative probability space.

Theorem 4.7 (*Voiculescu [9]*). *Let B be a unital complex Banach algebra and let (A, E) be a B-valued Banach noncommutative probability space. Let $x, y \in A$ be free in (A, E). Then*

$$R_{x+y}(b) = R_x(b) + R_y(b).$$

Proof. By Proposition 4.2, it will suffice to show that given $n \in \mathbf{N}$ we have $R_{X+Y} = R_X + R_Y$ for some free pair X and Y of elements in a Banach noncommutative probability space over B whose first n moment functions agree with those of x and y, respectively. Precisely this fact follows from Propositions 4.4, 3.4 and Lemmas 4.5 and 4.6. □

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