$P$-chaos implies distributional chaos and chaos in the sense of Devaney with positive topological entropy

Tatsuya Arai $^a$, Naotsugu Chinen $^b,*$

$^a$ Department of General Education for the Hearing Impaired, Tsukuba College of Technology, Ibaraki 305-0005, Japan
$^b$ Liberal Arts (Science and Mathematics), Okinawa National College of Technology, Okinawa 905-2192, Japan

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Abstract

Let $f$ be a continuous map from a compact metric space $X$ to itself. The map $f$ is called to be $P$-chaotic if it has the pseudo-orbit-tracing property and the closure of the set of all periodic points for $f$ is equal to $X$. We show that every $P$-chaotic map from a continuum to itself is chaotic in the sense of Devaney and exhibits distributional chaos of type 1 with positive topological entropy.

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1. Introduction

Devaney’s definition of chaos is one of the most popular and widely known. A continuous map $f$ from a compact metric space $(X, d)$ to itself is chaotic in the sense of Devaney if

(1) $f$ is transitive,
(2) the set of all periodic points of $f$ is dense in $X$, and
(3) $f$ has sensitive dependence on initial conditions; there exists $\delta > 0$ such that for each $x \in X$ and each neighborhood $U$ of $x$, there exist $y \in U$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.

In [5], it has been shown that $f$ is chaotic in the sense of Devaney if and only if $f$ satisfies the above conditions (1) and (2).

R. Bowen has introduced the definition of the specification property. It is shown in [4] that if a map $f$ has the specification property, then $f$ is topologically mixing, the set of all periodic points for $f$ is dense and $f$ has positive topological entropy, thus, $f$ is chaotic in the sense of Devaney. However, there exists a chaotic map in the sense of Devaney which has no specification property (see Example 2.6). The specification property is very strong.
R. Bowen [9] has also introduced the concept of the pseudo-orbit-tracing property and proved that expansive homeomorphisms with this property are topologically stable. In [4, p. 733], it has been shown that if a homeomorphism $f$ is topologically mixing, expansive and has the pseudo-orbit-tracing property, then $f$ has the specification property. From the above facts, by replacing the condition (1) in the definition of Devaney’s chaos by the pseudo-orbit-tracing property, we introduce the $P$-chaos; a continuous map $f$ from a compact metric space $X$ to itself is called to be $P$-chaotic if it has the pseudo-orbit-tracing property and the closure of the set of all periodic points for $f$ is equal to $X$. We see that every Anosov diffeomorphism $f$ on a compact metric space and the tent map $f: [0, 1] \to [0, 1]$ are $P$-chaotic (see Examples 2.12 and 2.13).

Distributional chaos has been introduced in [16]. In the class of continuous functions mapping $[0, 1]$ into itself, the subclass of distributionally chaotic functions is equal to the subclass of continuous functions with positive topological entropy. However, it is known that there exists a continuous map from a 0-dimensional compact metric space to itself which exhibits distributional chaos of type 1 with zero topological entropy by [14]. A. Sklar and J. Smítal in [17] show that if a continuous map $f$ from a compact metric space to itself has the specification property, then $f$ exhibits distributional chaos of type 3.

In this paper, after showing that every $P$-chaotic from a continuum to itself is transitive, we prove the following:

**Theorem 1.1.** Every $P$-chaotic map from a continuum $X$ to itself is chaotic in the sense of Devaney and exhibits distributional chaos of type 1 with positive topological entropy.

2. Definitions

**Definition 2.1.** A *continuum* is a nondegenerate compact connected metric space.

**Notation 2.2.** Let $Y$ be a subspace of a metric space $(X, d)$. Denote the interior, the closure and the diameter of $Y$ in $X$ by $\text{Int}(Y)$, $\text{Cl}(Y)$ and $\text{diam}(Y)$, respectively. Let $x$ be a point of $X$ and $\epsilon > 0$. Set $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$.

The cardinality of a set $P$ will be denoted by $\text{Card}(P)$.

Let $f$ be a continuous map from a compact metric space $X$ to itself. We denote the $n$-fold composition $f^n$ of $f$ by $f \circ \cdots \circ f$ and $f^0$ the identity map.

Let $f$ be a continuous map from $X$ to itself. A point $x$ is a *periodic point* for $f$ with *period* $n$ if $f^n(x) = x$ and $f^k(x) \neq x$ for $1 \leq k < n$. Let $P(f)$ denote the set of all periodic points for $f$. For $x \in X$, denote $\text{Orb}(x, f) = \{f^n(x) : n \geq 0\}$.

**Definition 2.3.** A continuous map $f : X \to X$ is transitive if for nonempty open sets $U$ and $V$, there exists $n > 0$ such that $U \cap f^n(V) \neq \emptyset$.

A continuous map $f : X \to X$ is topologically mixing if for nonempty open sets $U$ and $V$, there exists $n_0 > 0$ such that $U \cap f^n(V) \neq \emptyset$ for any $n \geq n_0$.

**Definition 2.4.** Let $f$ be a continuous map from a compact metric space $(X, d)$ to itself. For $x, y \in X$, denote $\delta_{x,y}(m) = d(f^m(x), f^m(y))$ for $m \geq 0$. We define the functions $F_{x,y}^{(n)}$ on the real line by

$$F_{x,y}^{(n)}(t) = n^{-1} \text{Card}\{m : 0 \leq m \leq n - 1 \text{ and } \delta_{x,y}(m) < t\}.$$  

And let us define the upper and lower distribution functions as:

$$F_{x,y}^{*}(t) = \limsup_{n \to \infty} F_{x,y}^{(n)}(t) \quad \text{and} \quad F_{x,y}^{*}(t) = \liminf_{n \to \infty} F_{x,y}^{(n)}(t).$$

The map $f$ is said to exhibit *distributional chaos of type 1*, briefly $DC1$, if there exist $x, y \in X$ such that $F_{x,y}^{*}(t) = 1$ for all $t > 0$ and $F_{x,y}(t_0) = 0$ for some $t_0 > 0$. The map $f$ is said to exhibit *distributional chaos of type 3*, briefly $DC3$ if there exist $x, y \in X$ such that $F_{x,y}^{*}(t) > F_{x,y}(t)$ for all $t$ in an interval. See [7] for the distributional chaos of types 1 and 3.

**Definition 2.5.** A continuous map $f$ from a nondegenerate compact metric space $X$ to itself has the *specification property* if for any $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that for any $k \geq 2$, for any $k$ points $x_1, x_2, \ldots, x_k \in X$, for any
nonnegative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_i - b_{i-1} \geq M$ for each $i = 2, 3, \ldots, k$ and for any $p \geq M + b_k - a_1$, there exists a point $y \in X$ such that $f^p(y) = y$ and $d(f^n(y), f^n(x_i)) \leq \varepsilon$ for all $a_i \leq n \leq b_i$, $1 \leq i \leq k$.

By [4, pp. 733–734], if a map $f$ has the specification property, then $P(f)$ is dense in $X$, $f$ is a topologically mixing map with positive topological entropy, thus, $f$ is chaotic in the sense of Devaney. And for every continuous map $f$ from the unit interval to itself, it is known by [8] or [10] that if $f$ is topologically mixing, then $f$ has the specification property.

**Example 2.6.** We show that there exists a map $f : [0, 1] \to [0, 1]$ with no specification property which is chaotic in the sense of Devaney.

Let $f : [0, 1] \to [0, 1]$ be the piecewise linear function defined by

$$
f(0) = 1/2, \quad f(1/4) = 1, \quad f(3/4) = 0, \quad \text{and} \quad f(1) = 1/2.
$$

Let $I_0 = [0, 1/2]$ and $I_1 = [1/2, 1]$. We see that $f^2(I_0) = I_0$ and $f^2(I_1) = I_1$. Let $f^2|I_i : I_i \to I_i$ be a restriction map of $f$ for $i = 1, 2$. It is not difficult to show that $Cl(P(f^2|_{I_i})) = I_i$ for $i = 1, 2$, thus, $Cl(P(f)) = [0, 1]$. And we see that $f$ is transitive but $f^2$ is not transitive. We have that $f$ is chaotic in the sense of Devaney, but is not topologically mixing by [6, Theorem VI 46, p. 158]. Thus by [8] or [10], $f$ has no specification property.

**Definition 2.7.** Let $f$ be a continuous map from a compact metric space $X$ to itself. A sequence of points $\{x_i : i \geq 0\}$ is called a $\delta$-pseudo-orbit for $f$ if $d(f(x_i), x_{i+1}) < \delta$ for each $i$. A sequence $\{x_i : i \geq 0\}$ is said to be $\varepsilon$-traced by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ holds for each $i \geq 0$. A map $f$ is said to have the pseudo-orbit-tracing property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that each $\delta$-pseudo-orbit for $f$ is $\varepsilon$-traced by some point of $X$.

**Example 2.8.** We show that there exists a map $f : [0, 1] \to [0, 1]$ with the pseudo-orbit-tracing property which is not transitive, is not DC3 and has zero topological entropy, thus, $f$ does not have the specification property.

Let us define $f$ by

$$
f(t) = \begin{cases} 
2t & \text{if } t \leq 1/2, \\
1 & \text{if } 1/2 \leq t \leq 1.
\end{cases}
$$

Since the set of all nonwandering points for $f$ is equal to $P(f) = \{0, 1\}$, we see that the topological entropy of $f$ is equal to zero by [6, Theorem VIII 6, p. 193]. And we have from [16] that $f$ is not DC3. We have the following property; for each $\varepsilon > 0$ with $\varepsilon < 1/2$, there exist $n \in \mathbb{N}$ and $\delta > 0$ with $\delta < 1/2$ such that for each $\delta$-pseudo-orbit $\{x_i : i \geq 0\}$ with $x_1 \geq \varepsilon$, $x_k \geq 1 - \varepsilon/2$ for all $k \geq n$. By [4, Lemma 4.4, p.668], this shows that $f$ has the pseudo-orbit-tracing property.
Example 2.9. There exists a continuous map $f : [0, 1] \to [0, 1]$ with $\text{Cl}(\text{P}(f)) = [0, 1]$ which is DC1, is not transitive and has no pseudo-orbit-tracing property. Let $f : [0, 1] \to [0, 1]$ be the piecewise linear function defined by

$$ f(0) = 0, \quad f(1/6) = 1/2, \quad f(1/3) = 0, \quad f(2/3) = 1, \quad f(5/6) = 1/2, \quad \text{and} \quad f(1) = 1. $$

Let $I_0 = [0, 1/2]$ and $I_1 = [1/2, 1]$. We see that $f(I_0) = I_0$ and $f(I_1) = I_1$. Let $\delta > 0$ with $\delta < 1/3$. There exist $x_0, x_1 \in [0, 1]$ such that $x_0 < 1/3$, $1/2 < x_1 < 1/2 + \delta/2$ and $\text{Cl}(\text{Orb}(x_i, f)) = I_i$ for $i = 1, 2$. We have $n \in \mathbb{N}$ such that $1/2 - \delta/2 < f^n(x_0) < 1/2$. Let us define $x_i$ by

$$ x_i = \begin{cases} 
  f^i(x_0) & \text{if } 0 \leq i < n, \\
  f^{i-n}(x_1) & \text{if } n \leq i.
\end{cases} $$

We see that $\{x_i : i \in \mathbb{N}\}$ is a $\delta$-pseudo-orbit, but is traced by no point in $[0, 1]$.

It follows from Examples 2.8 and 2.9 that the pseudo-orbit-tracing property is independent of the denseness of the set of all periodic points.

Definition 2.10. A continuous map $f$ from a compact metric space $X$ to itself is said to be $P$-chaotic if $f$ has the pseudo-orbit-tracing property and $\text{Cl}(\text{P}(f)) = X$.

Example 2.11. Every Anosov diffeomorphism $f$ on a compact metric space $X$ is $P$-chaotic. Since $f$ is an Anosov diffeomorphism, we have $\text{Cl}(\text{P}(f)) = X$. By [19], $f$ is topologically stable. It follows from [20, Theorem 11, p. 243] that $f$ has the pseudo-orbit-tracing property, thus, $f$ is $P$-chaotic.

Example 2.12. Let $f : [0, 1] \to [0, 1]$ be the tent map which is defined by

$$ f(x) = \begin{cases} 
  2x & \text{if } 0 \leq x \leq 1/2, \\
  -2x + 2 & \text{if } 1/2 \leq x \leq 1.
\end{cases} $$

We show that $f$ has the pseudo-orbit-tracing property. Let $\varepsilon > 0$, $n \in \mathbb{N}$ and let $\{x_i : i \geq 0\}$ be a $\varepsilon/4$-pseudo-orbit for $f$. Since $\text{Cl}(B(x_{i+1}, \varepsilon/2)) \subseteq \text{Cl}(f(B(x_i, \varepsilon/2))) = f(\text{Cl}(B(x_i, \varepsilon/2)))$, we have $f^{-i}(\text{Cl}(B(x_{i+1}, \varepsilon/2))) \cap \text{Cl}(B(x_i, \varepsilon/2)) \neq \emptyset$. Thus, there exists a point $x \in \bigcap_{i=0}^{n} f^{-i}(\text{Cl}(B(x_i, \varepsilon/2))) \neq \emptyset$ and $\{x_i : i \geq 0\}$ is $\varepsilon$-traced by $x$. Hence we see that $f$ has the pseudo-orbit-tracing property. Since for each open set $U$ there exists $k$ such that $f^k(U) = [0, 1]$, we have $\text{Cl}(\text{P}(f)) = [0, 1]$. Thus $f$ is $P$-chaotic.
3. Lemmas

Proposition 3.1. Let \( f \) be a continuous map from a compact metric space \( X \) to itself. If \( f \) is \( P \)-chaotic, then \( f^k \) is \( P \)-chaotic for each \( k > 0 \). Moreover, if \( f^k \) is \( P \)-chaotic for some \( k > 0 \), then \( f \) is \( P \)-chaotic.

Proof. Suppose that \( f \) is \( P \)-chaotic. We see \( \text{Cl}(P(f^k)) = \text{Cl}(P(f)) = X \) for each \( k \in \mathbb{N} \). By [4, Theorem 4.3, p. 667], \( f^k \) has the pseudo-orbit-tracing property for each \( k \in \mathbb{N} \), thus, \( f^k \) is \( P \)-chaotic for each \( k \in \mathbb{N} \).

Suppose that \( f^k \) is \( P \)-chaotic for some \( k > 0 \). We see \( \text{Cl}(P(f)) = \text{Cl}(P(f^k)) = X \). Since \( f^k \) has the pseudo-orbit-tracing property, by the proof of [4, Theorem 4.5, p. 668], \( f \) has the pseudo-orbit-tracing property, thus, \( f \) is \( P \)-chaotic.

Lemma 3.2. Let \( f \) be a \( P \)-chaotic map from a continuum \((X, d)\) to itself. For any \( \varepsilon > 0 \) there exists \( M \in \mathbb{N} \) such that for any \( k \geq 2 \), for any \( k \) points \( x_1, x_2, \ldots, x_k \in X \), for any nonnegative integers \( a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k \) with \( a_i - b_i - 1 \geq M \) for each \( i = 2, 3, \ldots, k \) and for any \( p \geq M + b_k - a_1 \), there exists a periodic point \( y \in X \) for \( f \) such that \( d(y, f^p(y)) < \varepsilon \) and \( d(f^n(y), f^{ni}(x_i)) \leq \varepsilon \) for all \( a_i \leq n < b_i \), \( 1 \leq i \leq k \).

Proof. We may assume that \( a_1 = 0 \). Since \( f \) has the pseudo-orbit-tracing property, there is \( \delta > 0 \) such that each \( \delta \)-pseudo-orbit for \( f \) is \( \varepsilon/2 \)-traced by some point of \( X \). Since \( X \) is compact, there exist \( y_1, y_2, \ldots, y_t \in X \) such that \( B(y_1, \delta/2) \cup B(y_2, \delta/2) \cup \cdots \cup B(y_t, \delta/2) = X \). If \( B(y_i, \delta/2) \cap B(y_j, \delta/2) \neq \emptyset \) for any \( i < j \), we can choose a point \( z_{i,j} \in B(y_i, \delta/2) \cap B(y_j, \delta/2) \cap P(f) \) and let \( Z = \{ z_{i,j} : B(y_i, \delta/2) \cap B(y_j, \delta/2) \neq \emptyset \} \). Let \( p_i = \text{Card}(\text{Orb}(z_{i,j})) \) for \( i = 1, 2, \ldots, m \), \( M_0 = \prod_{i=1}^{m} p_i \), and \( M = 2M_0 \). Also, for \( i = 1, 2, \ldots, k \), denote \( d_i = b_i - a_i \) and let \( n_i = n_i'/M_0 + r_i \), where \( n_i' \in \mathbb{N} \). Since \( X \) is connected, for each \( i = 1, 2, 3, \ldots, k \) there exist \( z_{i,1}, z_{i,2}, \ldots, z_{i,m(i)} \in Z \) such that \( z_{i,j} \neq z_{i,j'} \) for any \( j 
eq j' \) and that

\[
\max \left\{ d(f^{b_i+r_i}(z_i), z_{i,j}), d(z_{i,j}, z_{i,j}), \ldots, d(z_{i,m(i)-1}, z_{i,m(i)}), d(z_{i,m(i)}, f^{a_i+1}(z_i+1)) \right\} < \delta,
\]

where let denote \( a_{k+1} = p \) and \( f^{a_{k+1}}(x_{k+1}) = x_1 \). Let \( n_i = n_i'/M_0 + \sum_{i=1}^{m(i)} p_{i,t} \) \( i \in \mathbb{N} \). Define a sequence \( \{ x'_j \} \) for \( j \leq p \) by

\[
x'_j = \begin{cases} 
  f^{j}(x_1) & \text{if } a_1 \leq j < b_1 + r_1, \\
  f^{j-b_1+r_1}(z_{1,j}) & \text{if } b_1 + r_1 \leq j < b_1 + r_1 + n_1, \\
  f^{j-b_1-r_1-n_1p_{i,1}}(z_{i,j}) & \text{if } b_1 + r_1 + n_1p_{i,1} \leq j < b_1 + r_1 + n_1(p_{i,1} + p_{i,2}), \\
  \vdots & \\
  f^{j-b_1-r_1-\sum_{i=1}^{m(i)-1} p_{i,t}}(z_{i,m(i)}) & \text{if } b_1 + r_1 + n_1 \sum_{i=1}^{m(i)-1} p_{i,t} \leq j < b_1 + r_1 + n_1 \sum_{i=1}^{m(i)} p_{i,t}.
\end{cases}
\]

Thus, we have the \( \delta \)-pseudo-orbit \( \{ x'_i : i \in \mathbb{N} \} \) for \( f \) such that \( x_i = x'_i \), where \( i = pp' + r \) for some \( p' \geq 0 \) and \( 0 \leq r < p \). By the definition of \( \delta \), there exists an element \( z \in X \) such that \( d(f^i(z), x_i) < \varepsilon/2 \) for each \( i \geq 0 \). Thus, we have a periodic point \( y \in X \) for \( f \) such that \( d(f^i(y), x_i) < \varepsilon/2 \) for each \( 0 \leq i \leq p \).

Corollary 3.3. Let \( f \) be a \( P \)-chaotic map from a continuum \((X, d)\) to itself and \( \varepsilon > 0 \). There exists \( M \in \mathbb{N} \) such that for each \( x, y \in X \) and each \( k \geq M \), \( d(x, z_k) < \varepsilon \) and \( d(y, f^k(z_k)) < \varepsilon \), for some periodic point \( z_k \in X \).

Corollary 3.4. Every \( P \)-chaotic map from a continuum to itself is mixing.

Proof. Let \( f \) be a \( P \)-chaotic map from a continuum \( X \) to itself and let \( U, V \) be nonempty open subsets of \( X \). There exist \( x \in U \), \( y \in V \) and \( \varepsilon > 0 \) such that \( B(x, \varepsilon) \subset U \) and \( B(y, \varepsilon) \subset V \). By Corollary 3.3, we have \( M \in \mathbb{N} \) such that for each \( k \geq M \), there exists \( z_k \in X \) such that \( d(x, z_k) < \varepsilon \) and \( d(y, f^k(z_k)) < \varepsilon \). We see that \( f^k(U) \cap V \neq \emptyset \) for all \( k \geq M \), thus, \( f \) is mixing.

By [2], we have the following corollary.
Corollary 3.5. Let $f$ be a continuous map from a finite graph to itself. If $f$ is $P$-chaotic, then $f$ has the specification property.

Example 3.6. Let $f : [0, 1] \to [0, 1]$ be the piecewise linear function defined by
\[
\begin{align*}
  f(0) &= 1/2, & f(1/8) &= 0, & f(1/4) &= 1, & f(3/8) &= 0, \\
  f(5/8) &= 1, & f(3/4) &= 0, & f(7/8) &= 1, & f(1) &= 1/2.
\end{align*}
\]

We can show that $f$ has the specification property, but it is not $P$-chaotic.

Fix $\varepsilon > 0$ with $\varepsilon < 4^{-3}$. Let $\delta > 0$ with $\delta < \varepsilon$, $x_0 = 0$, $x_1 = 1/2 + \delta/2$ and $x_i = f^{-1}(x_1)$ for each $i \geq 2$. We see that \{\{x_i : i \geq 0\}\} is a $\delta$-pseudo-orbit for $f$. Suppose that there exists a point $x$ which $\varepsilon$-traces \{\{x_i : i \geq 0\}\}. We see that $x \in [0, \varepsilon)$, thus, $f(x) \in (1/2 - 4\varepsilon, 1/2]$ and $f^2(x) \in (1/2 - 4^2\varepsilon, 1/2]$. There exists $i_0 = \min\{i \geq 2 : f^i(x) \leq 1/2 - 3/4^3\}$. Also, there exists $i_1 = \min\{i \geq 2 : x_i \geq 1/2 + 3/4^3\}$. Let $i_2 = \min\{i_0, i_1\}$. We have $d(f^{i_2}(x), x_{i_2}) \geq 3/4^3$, thus, this is a contradiction. We see that $f$ has not the pseudo-orbit-tracing property.

We see that $f$ is topologically mixing, by [8] or [10], it has the specification property.

4. Main theorem

By Corollary 3.4 and the definition of $P$-chaos, we can obtain the following:

Theorem 4.1. Every $P$-chaotic map from a continuum to itself is chaotic in the sense of Devaney.

Theorem 4.2. Every $P$-chaotic map from a continuum to itself is DC1.

Proof. Let $f$ be a $P$-chaotic map from a continuum $X$ to itself. Since $X$ is a continuum and $\text{Cl}(P(f)) = X$, there exist two points $x_0, y_0 \in P(f)$ such that $\text{Orb}(x_0, f) \cap \text{Orb}(y_0, f) = \emptyset$. Denote $m = \text{Card}(\text{Orb}(x_0, f))$ and $n = \text{Card}(\text{Orb}(y_0, f))$. Choose $\varepsilon > 0$ with $\varepsilon < 1/3d(\text{Orb}(x_0, f), \text{Orb}(y_0, f))$. Since $f$ has the pseudo-orbit-tracing property, there is $\delta_n > 0$ such that each $\delta_n$-pseudo-orbit for $f$ is $\varepsilon/2^n$-traced by some point of $X$.

From Corollary 3.3, there exist $z_1 \in B(x_0, \delta_1)$, $z_1' \in B(y_0, \delta_1)$ and $k_1, k_1' \in \mathbb{N}$ such that $f^{k_1m}(z_1) \in B(y_0, \delta_1)$ and $f^{k_1'm}(z_1') \in B(x_0, \delta_1)$. Choose $n_1 \in \mathbb{N}$ such that $k_1m + k_1'm + mnn_1 < 2mn_n$. Denote $\ell_1 = k_1m + k_1'm + mnn_1$. Set a sequence $\{x_{1, i} : i \geq 0\}$ of points by
\[
x_{1, i} = \begin{cases}
  f^{i}(z_1) & \text{if } 0 \leq i < k_1m, \\
  f^{i-k_1m}(y_0) & \text{if } k_1m \leq i < k_1m + mnn_1, \\
  f^{i-k_1m-mnn_1}(z_1') & \text{if } k_1m + mnn_1 \leq i < k_1m + k_1'm + mnn_1 = \ell_1, \\
  f^{i-\ell_1}(x_0) & \text{if } \ell_1 \leq i.
\end{cases}
\]

Since $\{x_{1, i} : i \geq 0\}$ is a $\delta_1$-pseudo-orbit for $f$, there exists $x_1 \in X$ which $\varepsilon/2$-traces $\{x_{1, i} : i \geq 0\}$. We notice that
\[
\frac{1}{\ell_1} F_{x_0, x_1}(\varepsilon) \leq \frac{\ell_1 - mnn_1}{\ell_1} = 1 - \frac{mnn_1}{\ell_1} < \frac{1}{2}.
\]
There exists $m_1 \in \mathbb{N}$ such that $\ell'_1 = \ell_1 + m_1 < 2^1m m_1$. We have

$$\frac{1}{\ell'_1} F_{x_0,x_1}(\varepsilon/2) \geq \frac{m m_1}{\ell'_1} \geq \frac{1}{2^2}.$$

From Corollary 3.3, there exist $z_2 \in B(f^{\ell_1}(x_1), \delta_2)$, $z'_2 \in B(y_0, \delta_2)$ and $k_2, k'_2 \in \mathbb{N}$ such that $f^{k_2m}(z_2) \in B(y_0, \delta_2)$ and $f^{k'_2m}(z'_2) \in B(x_0, \delta_2)$. We have $n_2 \in \mathbb{N}$ such that

$$1 - \frac{m n n_2}{\ell'_1 + \ell_2} < \frac{1}{2^2},$$

where we denote $\ell_2 = k_2m + k'_2m + m n n_2$. Set a sequence $\{x_{2,i} : i \geq 0\}$ of points by

$$x_{2,i} = \begin{cases} f^i(x_1) & \text{if } 0 \leq i < \ell'_1, \\ f^{i-\ell_1}(z_2) & \text{if } \ell'_1 \leq i < \ell'_1 + k_2m, \\ f^{i-\ell'_1-k_2m}(y_0) & \text{if } \ell'_1 + k_2m \leq j < \ell'_1 + k_2m + m n n_2, \\ f^{i-\ell'_1-k_2m-m n n_2}(z'_2) & \text{if } \ell'_1 + k_2m + m n n_2 \leq j < \ell' + \ell_2, \\ f^{i-\ell'_1-\ell_2}(x_0) & \text{if } \ell'_1 + \ell_2 \leq i. \end{cases}$$

Since $\{x_{2,i} : i \geq 0\}$ is a $\delta_2$-pseudo-orbit for $f$, there exists $x_2 \in X$ which $\varepsilon/2^2$-traces $\{x_{2,i} : i \geq 0\}$. We notice that

$$1 - \frac{m n n_2}{\ell'_1 + \ell_2} < \frac{1}{2^2}.$$

There exists $m_2 \in \mathbb{N}$ such that $\ell'_2 = \ell'_1 + \ell_2 + m n n_2 < 2^2m m_2/(2^2 - 1)$. We have

$$\frac{1}{\ell'_2} F_{x_0,x_2}(\varepsilon/2^2) \geq \frac{m m_1}{\ell'_2} > 1 - \frac{1}{2^2}.$$

By induction, for $s \geq 2$ we have $m_s \in \mathbb{N}$ with $\ell'_s = \ell'_{s-1} + \ell_s + m_s < 2^s m m_s/(2^s - 1)$, $z'_s \in B(f^{\ell'_{s-1}}(x_{s-1}), \delta_s)$, $z'_s \in B(y_0, \delta_s)$ and $k_s, k'_s, n_s \in \mathbb{N}$ such that

$$f^{k_s m}(z'_s) \in B(y_0, \delta_s), \quad f^{k'_s m}(z'_s) \in B(x_0, \delta_s) \quad \text{and} \quad 1 - \frac{m n n_s}{\ell'_{s-1} + \ell_s} < \frac{1}{2^s},$$

where we denote $\ell_s = k_s m + k'_s m + m n n_s$. Also set a sequence $\{x_{s,i} : i \geq 0\}$ of points by

$$x_{s,i} = \begin{cases} f^i(x_{s-1}) & \text{if } 0 \leq i < \ell'_{s-1}, \\ f^{i-\ell'_{s-1}}(z_s) & \text{if } \ell'_{s-1} \leq i < \ell'_{s-1} + k_s m, \\ f^{i-\ell'_{s-1}-k_s m}(y_0) & \text{if } \ell'_{s-1} + k_s m \leq j < \ell'_{s-1} + k_s m + m n n_s, \\ f^{i-\ell'_{s-1}-k_s m-m n n_s}(z'_s) & \text{if } \ell'_{s-1} + k_s m + m n n_s \leq j < \ell'_{s-1} + \ell_s, \\ f^{i-\ell'_{s-1}-\ell_s}(x_0) & \text{if } \ell'_{s-1} + \ell_s \leq i. \end{cases}$$

Moreover, there exists $x_s \in X$ which $\varepsilon/2^s$-traces $\{x_{s,i} : i \geq 0\}$. We notice that

$$1 - \frac{m n n_s}{\ell'_{s-1} + \ell_s} < \frac{1}{2^s} \quad \text{and} \quad 1 - \frac{m n n_s}{\ell'_{s-1} + \ell_s} < \frac{1}{2^s}.$$

Since $\{x_i : i \in \mathbb{N}\}$ is a Cauchy sequence, there exists $x_\infty = \lim_{i \to \infty} x_i$. By the above, we see that

$$1 - \frac{m n n_\infty}{\ell'_{s-1} + \ell_\infty} < \frac{1}{2^s} \quad \text{and} \quad 1 - \frac{m n n_\infty}{\ell'_{s-1} + \ell_\infty} < \frac{1}{2^s}.$$
Let $A = (a_1, a_2, \ldots, a_k)$ and $B = (b_1, b_2, \ldots, b_\ell)$ be sequences consisting of points of a space $X$ of the lengths $k$ and $\ell$, respectively. Denote $AB = (a_1, \ldots, a_k, b_1, \ldots, b_\ell)$ and $m \times A = A_1A_2 \cdots A_m$, where $A_i = A$ for each $i \in \{1, 2, \ldots, m\}$.

**Theorem 4.3.** Every $P$-chaotic map from a continuum to itself has positive topological entropy.

**Proof.** Let $f$ be a $P$-chaotic map from a continuum $X$ to itself. And let $\text{Orb}(p, f)$ and $\text{Orb}(q, f)$ be periodic orbits with periods $m$ and $n$, respectively. Since $f$ has the pseudo-orbit-tracing property, for any $\epsilon > 0$ with $\epsilon < 1/3d(\text{Orb}(p, f), \text{Orb}(q, f))$, there exists $\delta > 0$ such that each $\delta$-pseudo-orbit is $\epsilon$-traced. By the transitivity of $f$, there exists $x \in X$ such that $\{x, f^\ell(x)\} \in B(p, \delta)$ and $f^k(x) \in B(q, \delta)$ for some positive integers $k < \ell$. Denote $P = (p, f(p), \ldots, f^{m-1}(p))$ and $Q = (x, f(x), \ldots, f^{k-1}(x), q, f(q), \ldots, f^{n-1}(q), f^k(x), f^{k+1}(x), \ldots, f^{\ell-1}(x))$. And put $P' = (\ell + n) \times P$ and $Q' = m \times Q$. Since the length of $P$ is $m$ and the length of $Q$ is $\ell + n$, the length $m(\ell + n)$ of $P'$ is equal to that of $Q'$. Let $A$ be the set of all sequences $A_1A_2 \cdots$, where each $A_i$ is an element of $\{P', Q'\}$ for $i \geq 1$. Note that each element of $A$ is a $\delta$-pseudo-orbit for $f$. Since $d(P', Q') > 3\epsilon$, any distinct elements $A_1A_2 \cdots$ and $A_1' A_2' \cdots$ of $A$ are $\epsilon$-traced by distinct points $x$ and $x'$ satisfying $d(f^i(x), f^i(x')) > \epsilon$ for some $i \geq 0$, respectively. Hence for each $k \geq 1$, there exists an $(km(\ell + n), \epsilon)$-separated set for $f$ with cardinality at least $2^k$. Therefore $h(f) = \lim_{k \to \infty} 1/km(\ell + n) \log sk_{km(\ell + n)}(\epsilon, X) \geq \lim_{k \to \infty} 1/km(\ell + n) \log 2^k = 1/m(\ell + n) \log 2 > 0$. □

**Corollary 4.4.** Let $f$ be a continuous map from a compact metric space $X$ to itself. If $f$ is $P$-chaotic and dim $X \geq 1$, then it is DC$1$ with positive topological entropy.

**Proof.** Since Cl($P(f)$) = $X$ and dim $X \geq 1$, there exists a component $Y$ of $X$ and $k \in \mathbb{N}$ such that $f^k(Y) = Y$. Set $g = f^k|_Y : Y \to Y$. We see that $g$ is a $P$-chaotic, thus, $g$ is DC$1$ with positive topological entropy by Theorem 1.1. It follows from the proof of Lemma 3 in [11] that $f$ is DC$1$ with positive topological entropy. □

**Example 4.5.** There exists a $P$-chaotic map $f$ from a 0-dimensional compact metric space $X$ to itself which is not DC3 with zero topological entropy.

Let $X = \{1/n \in [0, 1]: n \in \mathbb{N}\} \cup \{0\}$ and $f$ the identity map. It is not difficult to show that $f$ has the pseudo-orbit-tracing property and $P(f) = X$, thus, $f$ is $P$-chaotic. Moreover, we see that $f$ is distributionally nonchaotic with zero topological entropy.

**References**


