On the local convergence of a deformed Newton’s method under Argyros-type condition

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Received 11 June 2005
Available online 13 September 2005
Submitted by William F. Ames

Abstract


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Keywords: Deformed Newton’s method; Local convergence; Radius of convergence ball; Argyros-type condition

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^*$ of a nonlinear operator equation

$$F(x) = 0,$$  \hspace{1cm} (1.1)

where $F$ is a Fréchet-differentiable operator defined on an open convex domain $D$ of a Banach space $X$ with values in a Banach space $Y$.

This work is supported by NNSF (Grant no. 10471128).

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0022-247X/$ – see front matter © 2005 Published by Elsevier Inc.
doi:10.1016/j.jmaa.2005.08.057
There are kinds of method to find a solution of (1.1). Iterative methods are often used to solve this problem. If we use the famous Newton’s method (see [1]), we can do as

\[ x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n \geq 0, \quad x_0 \in D. \]

(1.2)

Newton’s method is often modified in two directions: one is to simplify and the other is to accelerate. But after modifications they have all some good and bad aspects between the cost of computation and efficiency. King [2] and Werner [3] proposed independently the following modification:

\[ x_{n+1} = x_n - F'(x_n + z_n)\frac{1}{2}F(x_n), \]

\[ z_{n+1} = x_{n+1} - F'(x_n + z_n)\frac{1}{2}F(x_{n+1}), \quad n \geq 0, \quad x_0, z_0 \in D. \]

(1.3)

Although the number of the evaluations of the function value increases by one at each step, the convergence order is raised from 2 to \(1 + \sqrt{2}\). Hence it is no doubt to its advantage.

The convergence of (1.3) to a solution of (1.1) has been studied in [2–5,7]. Reference [4] established a Kantorovich-type semilocal convergence theorem of this method and assumed the conditions

\[ \|F''(x)\| \leq M \text{ and } \|F'''(x)\| \leq N. \]

Reference [5] established a Smale-type semilocal convergence theorem of this method, and assumed \(F\) to be completed analytic according to the famous theory of point estimate (see [6]). Ref. [7] proposed a type of weak condition named as the second \(\gamma\)-condition, and a semilocal convergence theorem of this method was also established.

In all above works, the conditions were mainly added to iterative initial points. But in this paper, we study this method from different way, we suppose \(x^*\) is a solution of (1.1). An interesting problem is to estimate the radius of the convergence ball of the deformed Newton’s method. An open ball \(B(x^*, r) \subset X\) with center \(x^*\) and radius \(r\) is called a convergence ball of an iterative method, if the sequence generated by this iterative method starting from any initial points in it converges.

Under the hypotheses that \(x^*\) is a zero of \(F\), i.e., \(F(x^*) = 0, F'(x^*)^{-1}\) exists, and the derivative of \(F\) satisfying the Lipschitz condition

\[ |F'(x^*)^{-1}(F(x) - F'(y))| \leq K|x - y|, \quad \forall x, y \in D, \quad \text{for some } K > 0. \]

(1.4)

Refs. [8,9] gave an exact estimate of radius \(r^* = \frac{2}{3K}\) respectively for Newton’s method. Reference [10] generalized this result for Newton’s method to a type of Hölder condition.

To our problem, it is natural that we can go on the study under the same condition. We prove that, if the Lipschitz condition (1.4) holds, the radius \(r^*\) of the convergence ball of the deformed Newton’s method (1.3) is \(\frac{s^*}{K}\) at least, where \(s^* \approx 0.496613\) is the minimum positive zero of the following equation:

\[ 17s^3 - 72s^2 + 96s - 32 = 0. \]

(1.5)

We also give the error analysis which shows the convergence of the sequence \(\{x_n\} \quad (n \geq 0)\) generated by the deformed Newton’s method is quadratic at least. In fact, we obtain a similar result under the Hölder condition.

Recently, Argyros [11,12] proposed the more general conditions as follow:

(a) There exists a nondecreasing function \(f_0\) such that

\[ |F'(x^*)^{-1}(F'(x) - F'(y))| \leq f_0(\|x - y\|), \quad \forall x, y \in D. \]

(1.6)
(b) There exists a nondecreasing function $g_0$ such that

$$\left| F'(x^*)^{-1}\left(F'(x) - F'(x^*)\right) \right| \leq g_0\left(\|x - x^*\|\right), \quad \forall x \in D. \quad (1.7)$$

Under these kinds of conditions, the convergence ball study is given for some Newton-like methods which include Newton’s method as its special case, and the radius of convergence ball is enlarged. Reference [13] applies this new idea to study a kind of secant-type method, and the similar result is obtained. In this paper, we also succeed in applying this new idea to study the deformed Newton’s method (1.3).

Without loss of generality, we assume that the functions $f_0$ and $g_0$ are all strict increasing.

2. Convergence ball study under Argyros-type condition

In this section, the Argyros-type condition is considered, and we obtain

**Theorem 1.** Suppose $F(x^*) = 0$, $F'(x^*)^{-1}$ exists, conditions (1.6) and (1.7) hold. In addition, let us assume that:

(a) Equation

$$q(r) \int_0^1 f_0\left((1 + tq(r))r\right) dt + g_0(r) = 1, \quad (2.1)$$

has a minimum positive zero $R$ and $g_0(R) < 1$, where

$$q(r) = \frac{\int_0^1 f_0\left((\frac{1}{2} - t)\|x_n - x^*\| + \frac{1}{2}\|z_n - x^*\|\right) dt}{1 - g_0\left(\frac{\|x_n - x^*\| + \|z_n - x^*\|}{2}\right)}. \quad (2.2)$$

(b) $B(x^*, R) = \{x \in X : \|x - x^*\| < R\} \subset D$.

Then the sequence $\{x_n\}$ generated by the deformed method (1.3) is well defined, remains in $B(x^*, R)$ for all $n > 0$ and converges to $x^*$ provided that $x_0, z_0 \in B(x^*, R)$. Moreover, the following error bounds hold for all $n \geq 0$:

$$\|x_{n+1} - x^*\| \leq \frac{\int_0^1 f_0\left(\frac{1}{2} - t\|x_n - x^*\| + \frac{1}{2}\|z_n - x^*\|\right) dt}{1 - g_0\left(\frac{\|x_n - x^*\| + \|z_n - x^*\|}{2}\right)} \|x_n - x^*\|, \quad (2.3)$$

$$\|z_{n+1} - x^*\| \leq \frac{\int_0^1 f_0\left(\frac{1}{2}\|x_n - x^*\| + \frac{1}{2}\|z_n - x^*\| + t\|x_{n+1} - x^*\|\right) dt}{1 - g_0\left(\frac{\|x_n - x^*\| + \|z_n - x^*\|}{2}\right)} \|x_{n+1} - x^*\|. \quad (2.4)$$

**Proof.** Given $x, y \in B(x^*, R)$, using (1.7) and (2.1), we obtain

$$\left\| I - F'(x^*)^{-1}F'\left(\frac{x + y}{2}\right) \right\| = \left\| F'(x^*)^{-1}\left(F'(x^*) - F'\left(\frac{x + y}{2}\right)\right) \right\| \leq g_0\left(\left\|\frac{x + y}{2} - x^*\right\|\right)$$
\[
\leq g_0 \left( \frac{\|x - x^*\| + \|y - x^*\|}{2} \right) \\
\leq g_0(R) \\
< 1.
\] (2.5)

By Banach lemma on invertible operators (see [1]) and (2.5), it follows that \( F' \left( \frac{x + y}{2} \right)^{-1} \) exists and
\[
\left\| F' \left( \frac{x + y}{2} \right)^{-1} F' \left( x^* \right) \right\| \leq \frac{1}{1 - g_0 \left( \frac{\|x - x^*\| + \|y - x^*\|}{2} \right)}.
\] (2.6)

Let us choose \( x_0, z_0 \in B(x^*, R) \). By the above discussion, we can see \( F' \left( \frac{x_0 + z_0}{2} \right)^{-1} \) exists, and by (1.3), \( x_1 \) and \( z_1 \) are well defined. Moreover, (2.6) holds for \( x = x_0 \) and \( y = z_0 \). Now from (1.3), (1.6) and (2.6), we obtain
\[
\left\| x_1 - x^* \right\| = \left\| x_0 - x^* - F' \left( \frac{x_0 + z_0}{2} \right)^{-1} F(x_0) \right\|
\leq \left\| F' \left( \frac{x_0 + z_0}{2} \right)^{-1} \left( F' \left( \frac{x_0 + z_0}{2} \right) \left( x_0 - x^* \right) - \left( F(x_0) - F(x^*) \right) \right) \right\|
\leq \frac{\int_0^1 f_0 \left( \frac{\|x_0 + z_0 - t x_0 - (1 - t)x^*\|}{\|x_0 - x^*\| + \|z_0 - x^*\|} \right) dt}{1 - g_0 \left( \frac{\|x_0 - x^*\| + \|z_0 - x^*\|}{2} \right)} \|x_0 - x^*\|
\leq \frac{\int_0^1 f_0 \left( \frac{\|x_0 + z_0 - t x_0 - (1 - t)x^*\|}{\|x_0 - x^*\| + \|z_0 - x^*\|} \right) dt}{1 - g_0 \left( \frac{\|x_0 - x^*\| + \|z_0 - x^*\|}{2} \right)} \|x_0 - x^*\|.
\] (2.7)

That is to say, (2.3) holds for \( n = 0 \). Furthermore, by (2.1) and (2.2), it is easy to see that
\[
q(R) < 1.
\] (2.8)

Then, by (2.1) and \( x_0, z_0 \in B(x^*, R) \), we have
\[
\left\| x_1 - x^* \right\| \leq \frac{\int_0^1 f_0 \left( \frac{\|x_0 + z_0 - t x_0 - (1 - t)x^*\|}{\|x_0 - x^*\| + \|z_0 - x^*\|} \right) dt}{1 - g_0(R)} \|x_0 - x^*\|
= q(R) \|x_0 - x^*\|
< R.
\] (2.9)

So \( x_1 \in B(x^*, R) \).

Similarly, by (1.3), (1.6) and (2.6), we have
\[
\left\| z_1 - x^* \right\| = \left\| x_1 - x^* - F' \left( \frac{x_0 + z_0}{2} \right)^{-1} F(x_1) \right\|
\leq \left\| F' \left( \frac{x_0 + z_0}{2} \right)^{-1} \left( F' \left( \frac{x_0 + z_0}{2} \right) \left( x_1 - x^* \right) - \left( F(x_1) - F(x^*) \right) \right) \right\|
\leq \frac{\int_0^1 f_0 \left( \|x_0 + z_0 - t x_1 - (1 - t)x^*\| \right) dt}{1 - g_0 \left( \frac{\|x_0 - x^*\| + \|z_0 - x^*\|}{2} \right)} \|x_1 - x^*\|.
\]
\[
\begin{align*}
&= \left\| F'\left(\frac{x_0 + z_0}{2}\right)^{-1} \int_0^1 \left( F'\left(\frac{x_0 + z_0}{2}\right) - F'(t x_1 + (1-t)x^*)\right) dt (x_1 - x^*) \right\| \\
&\leq \frac{\int_0^1 f_0\left(\|x_0 + z_0 - t x_1 - (1-t)x^*\|\right) dt}{1 - g_0\left(\|x_0 - x^*\| + \|z_0 - x^*\|\right)} \|x_1 - x^*\| \\
&\leq \frac{\int_0^1 f_0\left(\|x_0 - x^*\| + \|z_0 - x^*\| + t\|x_1 - x^*\|\right) dt}{1 - g_0\left(\|x_0 - x^*\| + \|z_0 - x^*\|\right)} \|x_1 - x^*\|.
\end{align*}
\]

Then (2.4) holds for \(n = 0\). Moreover, by (2.1), (2.9), (2.10) and \(x_0, z_0 \in B(x^*, R)\), we get
\[
\|z_1 - x^*\| \leq \frac{\int_0^1 f_0(R + t q(R) R) dt}{1 - g_0(R)} q(R) \|x_0 - x^*\| \\
= \|x_0 - x^*\| < R.
\]

So \(z_1 \in B(x^*, R)\).

Generally, let us assume that for some integer \(k > 0\), \(x_n, z_n\) are well defined, and \(x_n, z_n \in B(x^*, R)\) for all \(n = 0, 1, \ldots, k\). Then, by the similar technique, we can see \(x_{k+1}\) and \(z_{k+1}\) are well defined, (2.3) and (2.4) hold for \(n = k\), and \(x_{k+1}, z_{k+1} \in B(x^*, R)\). By induction, the sequence \(\{x_n\}\) generated by the deformed method (1.3) is well defined, and \(x_n, z_n \in B(x^*, R)\) for all \(n \geq 0\).

Moreover, by the definition of \(f_0\) and \(g_0\), (2.3) and (2.4), we get
\[
\frac{\int_0^1 f_0\left(\|x_n - x^*\| + \|z_n - x^*\|\right) dt}{1 - g_0\left(\|x_n - x^*\| + \|z_n - x^*\|\right)} \leq q(R) < 1.
\]

Hence, there exists a positive constant \(c \in [0, 1]\) such that
\[
\|x_{n+1} - x^*\| \leq c \|x_n - x^*\| \leq \cdots \leq c^{n+1} \|x_0 - x^*\| \leq c^{n+1} R.
\]

then, we can see \(\{x_n\}\) converges to \(x^*\) linearly at least, the proof is completed. \(\square\)

Set \(f_0(t) = K t^p\), \(g_0(t) = K' t^p\), we obtain

**Corollary 2.1.** Suppose \(F(x^*) = 0\), \(F'(x^*)^{-1}\) exists, let us assume that the following Hölder conditions hold:

\[
\begin{align*}
&\left| F'(x^*)^{-1} (F'(x) - F'(y)) \right| \leq K |x - y|^p, \quad \forall x, y \in D, \text{ for some } K > 0, \quad (2.14) \\
&\left| F'(x^*)^{-1} (F'(x) - F'(x^*)) \right| \leq K' |x - x^*|^p, \quad \forall x \in D, \text{ for some } K' > 0, \quad (2.15)
\end{align*}
\]

with \(0 < p \leq 1\) and \(K' \leq K\). In addition, let us assume that:

(a) **Equation**

\[
Kr^p\left((1 - K' r^p)2^p(1 + p) + Kr^p\left(2^{1+p} - 1\right)\right)^{1+p} \\
- (1 + p - ((1 + p)K' - K)r^p)((1 - K' r^p)2^p(1 + p))^{1+p} = 0.
\]

has a minimum positive zero \(R\).
The minimum positive zero

Table 1

<table>
<thead>
<tr>
<th>p</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
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<tr>
<td>s*</td>
<td>0.499089</td>
<td>0.498333</td>
<td>0.497721</td>
<td>0.497241</td>
<td>0.496883</td>
</tr>
<tr>
<td>p</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
</tr>
<tr>
<td>s*</td>
<td>0.496638</td>
<td>0.496496</td>
<td>0.496449</td>
<td>0.496491</td>
<td>0.496613</td>
</tr>
</tbody>
</table>

(b) \( B(x^*, R) = \{ x \in X : \| x - x^* \| < R \} \subset D \).

Then the sequence \( \{ x_n \} \) generated by the deformed method (1.3) is well defined, remains in \( B(x^*, R) \) for all \( n > 0 \) and converges to \( x^* \) provided that \( x_0, z_0 \in B(x^*, R) \). Moreover, the following error bounds hold for all \( n \geq 0 \):

\[
\| x_{n+1} - x^* \| \leq \frac{2K(\| x_n - x^* \| + \| z_n - x^* \|)^{1+p}(\| x_n - x^* \| + \| z_n - x^* \|)^{1+p}}{(1 + p)(1 - K'(\| x_n - x^* \| + \| z_n - x^* \|)^{1+p})},
\]

(2.17)

\[
\| z_{n+1} - x^* \| \leq \frac{K((\| x_n - x^* \| + \| z_n - x^* \|) + \| x_{n+1} - x^* \|)^{1+p} - (\| x_n - x^* \| + \| z_n - x^* \|)^{1+p})}{(1 + p)(1 - K'(\| x_n - x^* \| + \| z_n - x^* \|)^{1+p})}.
\]

(2.18)

**Proof.** By (2.2), the Hölder conditions (2.14) and (2.15), \( q(r) \) can be expressed by

\[
q(r) = \frac{K r^p \int_0^1 (| \frac{1}{2} - t | + \frac{1}{2})^{1+p} \, dt}{1 - K' r^p} = \frac{K r^p (2^{1+p} - 1)}{(1 - K' r^p) 2^p (1 + p)}.
\]

(2.19)

It is easy to see that Eq. (2.1) can be simplified to Eq. (2.16). Denote \( s = K' r^p, \lambda = K/K' \geq 1 \). Then, (2.16) can be changed to

\[
\lambda s((1 - s) 2^p (1 + p) + \lambda s(2^{1+p} - 1))^{1+p} - (1 + p - ((1 + p)s - \lambda s)(1 - s) 2^p (1 + p))^{1+p} = 0.
\]

(2.20)

Denote the left of (2.20) by a function \( h_p(s) \). Then, \( h_p(0) = -(1 + p)(2^p (1 + p))^{1+p} < 0 \), and \( h_p(1) = \lambda(\lambda(2^{1+p} - 1))^{1+p} > 0 \). So (2.20) has a minimum positive zero \( s^* < 1 \), and the minimum positive zero \( R \) of (2.16) satisfies \( g_0(R) = K' R^p = s^* < 1 \). The rest of the proof follows by Theorem 1. \( \square \)

**Remark 2.** Let us assume that \( K = K' \), i.e., \( \lambda = 1 \), then Eq. (2.20) can be simplified further to

\[
\lambda s((1 - s) 2^p (1 + p) + s(2^{1+p} - 1))^{1+p} - (1 + p - (s - \lambda s)(1 - s) 2^p (1 + p))^{1+p} = 0.
\]

(2.21)

Specially, when \( p = 1 \), (2.21) becomes (1.5).

In Table 1, we list the values of \( s^* \) for \( p = 0.1, 0.2, \ldots, 1.0 \).

**Remark 3.** It needs further study to decide whether the estimate of radius of the convergence ball of the deformed Newton’s method (1.3) established in Theorem 1 is optimal.
3. Further study on convergence under the Hölder condition

In Section 2, we have established a local convergence theorem for the deformed method (1.3), and under the Argyros-type condition, we proved that the sequence \( \{x_n\} \) generated by the deformed method (1.3) converges to a solution \( x^* \) of (1.1) linearly at least. In this section, we assume that the Hölder condition holds \((0 < p \leq 1)\), and we prove that the convergence order can be improved to \(1 + p\) at least. Specially, if the Lipschitz condition (1.4) holds, the convergence order is quadratic at least. We have

**Theorem 4.** Suppose \( F(x^*) = 0, F'(x^*)^{-1} \) exists, the Hölder condition holds with \(0 < p \leq 1\), i.e., (1.6) and (1.7) hold for \( f_0(t) = Kr^p, g_0(t) = K'r^{p} \) \((K' \leq K)\), and \( R \) is the minimum positive zero of (2.16). Let us assume that \( B(x^*, R) \subset D \). Then the sequence \( \{x_n\} \) generated by the deformed method (1.3) is well defined, remains in \( B(x^*, R) \) for all \( n > 0 \) and converges to \( x^* \) provided that \( x_0, z_0 \in B(x^*, R) \). Moreover, the following error bound holds for all \( n > 0 \):

\[
\|x_{n+1} - x^*\| \leq \frac{K((1 + \frac{1}{q(R)})^{1+p} - (\frac{1}{q(R)})^{1+p})}{2p(1+p)(1 - K'R^p)}\|x_n - x^*\|^{1+p}, \quad \text{(3.1)}
\]

where \( q(R) \) is defined by (2.19). That is to say, the convergence order of the sequence \( \{x_n\} \) \((n \geq 0)\) generated by the deformed Newton’s method (1.3) is at least \(1 + p\).

**Proof.** By Theorem 1, it is easy to see

\[
\|x_{n+1} - x^*\| \leq q(R)\|x_n - x^*\|, \quad n \geq 0, \quad \text{(3.2)}
\]

\[
\|z_{n+1} - x^*\| \leq \frac{\|x_{n+1} - x^*\|}{q(R)}, \quad n \geq 0. \quad \text{(3.3)}
\]

Substituting (3.2) to (2.3), for any \( n \geq 1 \), we obtain

\[
\|x_{n+1} - x^*\| \leq \frac{\int_0^1 f_0\left(||\frac{1}{2} - t|| + \frac{1}{2q(R)}\|x_n - x^*\|\right) dt}{1 - g_0(R)}\|x_n - x^*\|
\]

\[
= \frac{K\int_0^1 \left(||\frac{1}{2} - t|| + \frac{1}{2q(R)} \right)^p dt}{1 - K'R^p}\|x_n - x^*\|^{1+p}
\]

\[
= \frac{K(1 + \frac{1}{q(R)})^{1+p} - (\frac{1}{q(R)})^{1+p}}{2p(1+p)(1 - K'R^p)}\|x_n - x^*\|^{1+p}. \quad \text{(3.4)}
\]

So (3.1) holds, and that shows the convergence order of the sequence \( \{x_n\} \) \((n \geq 0)\) generated by the deformed Newton’s method (1.3) is at least \(1 + p\). Specially, if the Lipschitz condition (1.4) holds, the convergence order is quadratic at least. This completes the proof. \( \square \)

4. Some numerical examples

**Example 4.1.** Let \( X = Y = \mathbb{R}, D = (-1, 1) \) and define function \( F \) on \( D \) by

\[
F(x) = e^x - 1. \quad \text{(4.1)}
\]

It is easy to see, \( x^* = 0, F'(x^*) = 1, \) and \( \|F'(x) - F'(y)\| \leq e\|x - y\| \), for any \( x, y \in D \). Hence we set \( K = e \) in Corollary 2.1. Such as in [12], since \( x^* = 0 \), we obtain in turn
\[ F'(x) - F'(x^*) = e^x - 1 = x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \]
\[ = \left( 1 + \frac{x}{2!} + \cdots + \frac{x^{n-1}}{n!} + \cdots \right) (x - x^*) \quad (4.2) \]

and for any \( x \in D \),
\[ \left\| F'(x) - F'(x^*) \right\| \leq (e - 1) \left\| x - x^* \right\|. \quad (4.3) \]

That is, we can set \( K' = e - 1 \) in Corollary 2.1. By (2.16), we obtain the convergence radius of the deformed method (1.3) is \( R \approx 0.223532 \) at least. If we only use the Lipschitz condition (1.4), i.e., we set \( K = K' = e \), then we only obtain the convergence radius \( R' \approx 0.182694 < R \).

**Example 4.2.** Let \( X = Y = \mathbb{R} \), \( D = (-1, 1) \) and define function \( F \) on \( D \) by
\[ F(x) = \sin x. \quad (4.4) \]

It is easy to see, \( x^* = 0 \), \( F'(x^*) = 1 \), and \( \left\| F'(x) - F'(y) \right\| \leq \sin 1 \left\| x - y \right\| \), for any \( x, y \in D \). Hence we set \( K = \sin 1 \) in Corollary 2.1. Since \( x^* = 0 \), we obtain in turn
\[ \left\| F'(x) - F'(x^*) \right\|
\[ = 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots + (-1)^{n-1} \frac{x^{2n}}{(2n)!} + \cdots 
\[ = \left\| \frac{x}{2} \right\| \left( 1 - \frac{2x^2}{4!} + \frac{2x^4}{6!} - \cdots + (-1)^{n-1} \frac{2x^{2n-2}}{(2n)!} + \cdots \right) \right\| x - x^* \right\| \quad (4.5) \]

and for any \( x \in D \),
\[ \left\| F'(x) - F'(x^*) \right\| \leq \frac{1}{2} \left\| x - x^* \right\|. \quad (4.6) \]

That is, we can set \( K' = \frac{1}{2} \) in Corollary 2.1. By (2.16), we obtain the convergence radius of the deformed method (1.3) is \( R \approx 0.739126 \) at least. If we only use the Lipschitz condition (1.4), i.e., we set \( K' = K = \sin 1 \), then we only obtain the convergence radius \( R' \approx 0.590173 < R \).

**References**

