Algebras of Linear Growth, the Kurosh–Levitzky Problem and Large Independent Sets

A. Y. Samet-Vaillant

A description of algebras of linear growth is given. This leads to a new invariant which is similar to the number of ends of a group. This note is a further step in developing of a geometric study of infinite algebras and C*-algebras which should lead to a common geometric framework for infinite discrete groups, algebras and manifolds.

© 2002 Elsevier Science Ltd. All rights reserved.

INTRODUCTION

Let $A$ be an infinite-dimensional associative algebra over a field $K$. The algebra $A$ is said to be finitely generated or affine if it admits a finite set $S$ of elements such that $A = \text{span} \bigcup_{1 \leq j \leq m} S^j$ where $\text{span} \bigcup_{1 \leq i \leq j} S^i$ denotes the $K$-linear span on the monomials of length less than $j$ on $S$. The growth function of the finitely generated algebra $A$ with respect to the generating set $S$ is defined as $f_S(n) = \dim \text{span} \bigcup_{1 \leq i \leq n} S^i$. The algebra $A$ has linear growth if there exist a finite generating set $S$ in $A$ and a constant $a$ such that, for all $n > 0$, $f_S(n) \leq an$. This condition implies it has Gelfand–Kirillov dimension one [6, 10]. The aim of this note is to give a description of algebras of linear growth. Our description is in a way similar to Gromov’s Theorem on groups of polynomial growth.

1. Preliminaries

Let $A$ be an associative algebra over a field $K$. In the following, all algebras are associative and the base field $K$ is fixed and no longer mentioned. In particular, all linear spans and dimensions are over $K$. Suppose $A$ is a finitely generated algebra and $S = \{s_1, \ldots, s_m\}$ a finite generating set for $A$. Suppose $S$ is minimal in the sense that the $s_i$ are linearly independent. Consider the free semigroup $G$ on $m$ generators $x_1, \ldots, x_m$ and order $G$ as follows.

- $(1)$ $x_1 < x_2 < \cdots < x_m$.
- $(2)$ If $x$ and $y$ are elements of $G$, set $x < y$ if
  - (a) length $x <$ length $y$
  - or (b) $x$ precedes $y$ lexicographically in cases of equal length.

There is a canonical semigroup homomorphism $\pi$ from $G$ into the multiplicative semigroup of $A$ given by $\pi(x_{i(1)} \cdots x_{i(k)}) = s_{i(1)} \cdots s_{i(k)} (*)$, where $1 \leq i(j) \leq m$. A subset $W$ of $G$ is constructed recursively by setting

- $(1)$ $W_1 = \{x_1, \ldots, x_m\}$.
- $(2)$ Assume $W_n$ has been obtained for $n \geq 1$ and consists of words of length $\leq n$. List all words of length $n + 1$ in lexicographical order, and, starting from the left, remove any word $w$ from the list for which $\pi(w)$ is a linear combination of monomials $\pi(y)$ in $A$, $y$ in $G$, with $y < w$. Define $W_{n+1}$ as the union of $W_n$ with the set of words of length $n + 1$ which remain after this process has been completed.

$W = \bigcup_{n=1}^{\infty} W_n$. 

© 2002 Elsevier Science Ltd. All rights reserved.
Lemma 1. Let $W$ be the set of words constructed as above. Then

1. the set $\mathcal{W} = \pi(W)$ is a basis for $A$.
2. Every subword of a word in $W$ belongs to $W$.

A proof of this lemma appears for example in [6, Lemma 2.2]. In the following, such a set in a finitely generated algebra is called a monomial basis.

Define the length function on $A$ associated with $S$ to be the map $\mathcal{L}_S : A \to \mathbb{N}$ defined via $\mathcal{L}_S(w) = \inf\{n \in \mathbb{N} \mid w \in \text{span}\left(\bigcup_{1 \leq \ell \leq n} S^\ell\right)\}$. Let $W$ be a monomial basis in an algebra $A$ with generating set $S$. Not only is the map $\pi$ in Lemma 1 one-to-one on $\text{span}(W)$, the length of elements of $W$ as defined previously is the algebraic word length of their preimage words in $W$. Hence the elements in $W$ have a canonical writing as monomials on $S$ and can be identified with the corresponding words in $W$. Hence, by (1), the second property of $W$ in Lemma 1 also holds for the set $W = \pi(W)$ using this canonical writing.

Let $S$ be a finite alphabet. An infinite chain over $S$ is a map $w : \mathbb{N}^+ \rightarrow S$. The set of infinite chains over $S$ can be considered as a direct product $S^\omega$ with Tychonov’s topology. The finite segment $w(1\ldots n)$ of length $n$ of a chain $w$ is the word consisting of its first $n$ letters and is defined via $w(1\ldots n) = w(1) \cdots w(n)$.

Let $A$ be a finitely generated algebra, $S$ a generating set for $A$ and $\mathcal{W}$ an associated monomial basis over $S$. A chain $w$ over $S$ is non-zero if all its finite segments $w(1\ldots n)$ are non-zero. An infinite chain $w$ over $S$ is said to be in $\mathcal{W}$ if all its finite segments $w(1\ldots n)$ belongs to $\mathcal{W}$. Such chains are automatically non-zero. In the following, all considered chains will belong to a monomial basis.

Compactness Lemma 2. If $A$ is infinite dimensional, then the set of chains in $\mathcal{W}$ is non-empty.

Proof. If there are no infinite chains in a monomial basis $\mathcal{W}$, then $\mathcal{W}$ must be a finite set, since the Tychonov topology is compact and the set of chains is the intersection of decreasing sequence of non-empty closed sets (all infinite $w$ with $w(1\ldots n)$ in $\mathcal{W}$). But, by Lemma 1, $\mathcal{W}$ is a basis of $A$. Hence, the algebra $A$ must be finite dimensional.

The $k$-tail $w_{|k}$ of a chain $w$ is the chain $w$ truncated by removing its first $k$ letters and is defined via $w_{|k}(n) = w(n + k)$. The finite segment $w_{|k}(1\ldots n)$ of the $k$-tail $w_{|k}$ of a chain $w$ is then defined as $w_{|k}(1\ldots n) = w(1 + k) \cdots w(n + k)$. A chain is periodic with period $k$ and preperiod $h$ if $w_{|k} = w_{|h+k}$. A non-periodic chain $w$ is called aperiodic; then, for all $h, k$, one has $w_{|h} \neq w_{|k}$, that is there exists an integer $N$ such that $w_{|h}(1\ldots n) \neq w_{|k}(1\ldots n)$ for $n > N$. Recall that an element $x$ in an algebra $A$ is non-algebraic or transcendent, if it generates an infinite-dimensional cyclic subalgebra in $A$.

Lemma 3. Let $w$ be an infinite chain in a monomial basis $\mathcal{W}$.

1. $w_{|k}(1\ldots n)$ is in $\mathcal{W}$.
2. $w_{|k}(1\ldots n)$ has length $n$.
3. $w_{|k}(1\ldots n) = w_{|k}(1\ldots m)$ implies $n = m$.
4. If $w_{|h}(1\ldots N) \neq w_{|k}(1\ldots N)$ for some integer $N$, then $w_{|h}(1\ldots n) \neq w_{|k}(1\ldots n)$ for all $n \geq N$.
5. If the chain $w$ is periodic, then there exists a transcendent element in $\mathcal{W}$.

Proof. (1)–(4) follow from the definitions of chains in $\mathcal{W}$ and their finite segments together with Lemma 1 and the remarks which follow (5). Suppose $w$ is periodic with period $k$ and preperiod $h$. The equality of chains $w_{|h} = w_{|h+k}$ means that for any integer $n$ one has $w_{|h}(1\ldots n) = w_{|h+k}(1\ldots n)$. Consider $w_{|h}(1\ldots k)$. For all non-zero integer $m$, $w_{|h}(1\ldots mk)$ is its $m$th power. All the monomials $w_{|h}(1\ldots mk)$ are distinct by (2) and (3). By (1), they belongs to $\mathcal{W}$. But by (1) of Lemma 1, $\mathcal{W}$ defines a basis of $A$. Hence they are linearly independent. It follows that $w_{|h}(1\ldots k)$ is a transcendent element in $\mathcal{W}$.
2. **Algebras of Linear Growth Have Transcendent Elements**

The first result we shall need for our description is the existence of transcendent elements in infinite-dimensional algebras of linear growth. This is a consequence of the structure theory, and we present hopefully new proofs.

**Theorem 4.** If an infinite-dimensional algebra has linear growth, then all chains in its monomial basis are periodic. In particular, it has transcendent elements.

**Proof.** Let $A$ be a finitely generated infinite-dimensional algebra with generating set $S$ and $W$ a monomial basis on $S$ in $A$. Suppose that the growth function $f_S$ of $A$ with respect to $S$ satisfies $f_S(n) \leq an$, for $n > 0$. Let $w$ be an infinite chain in $W$ whose existence is ensured by Lemma 2. Suppose $w$ is an infinite aperiodic chain in $W$. All the $w_{|M}$ are different; that is, for every $h, k$ there exists $N$ such that $w_{|M}(1 . . . N) \neq w_{|M}(1 . . . N)$. For all $h$ and $k$, there exists a $M = M(h, k)$ such that $w_{|M}(1 . . . M) \neq w_{|M}(1 . . . M)$. In particular, this implies that $w_{|M}(1 . . . n) \neq w_{|M}(1 . . . n)$, $\forall n \geq M$.

Take the first $N$ such that $w_{|M}(1 . . . N) \neq w_{|M}(1 . . . N)$, and $\forall h, k \leq a + 1$. Define $W_a(n) = \{w_{|M}(1 . . . m) | N \leq m \leq n, k \leq a + 1\}$. The elements in $W_a(n)$ are of length less than or equal to $n$. They belong to $W$. Then the elements in $W_a(n)$ are all different. But any monomial basis defines a basis of the considered algebra. Hence the elements in $W_a(n)$ are linearly independent. It follows that

$$f_S(n) \geq \dim \text{span } W_a(n)$$

$$\geq | W_a(n) |$$

$$\geq (n - N + 1) \cdot (a + 1),$$

which for large values of $n$ contradicts the linear growth of $A$ (we supposed that for all $n$, the growth function $f_S$ of $A$ with respect to $S$ satisfies $f_S(n) \leq an$ for all $n > 0$). Hence all chains in $W$ are periodic chains. Then, there exist transcendent elements in $A$ which in particular belongs to $W$.

Here is another argument. A basic result of symbolic dynamics asserts that the number of (different) segments of length $n$ of an aperiodic chain is greater than or equal to $n + 1$. Applying this to an aperiodic chain in a monomial basis of an algebra $A$, it follows that any of its growth functions satisfy $f_S(n) \geq \frac{1}{2}n^2 + \frac{1}{2}n$. \(\square\)

In 1902, Burnside formulated his famous problem for periodic groups: is every finitely generated group of bounded exponent finite? This is the so-called Ordinary Burnside Problem; a restricted version formulated by Magnus was solved by Zelmanov, while Golod had given a negative answer in 1964 to the General Burnside Problem [11]. Kurosh and Levitzky formulated independently problems for algebras (actually associative and nil Lie algebras) which are similar to the mentioned various Burnside problems. The general formulation of these problems reads: is every affine algebraic algebra finite dimensional?

**Corollary 5.** Affine algebraic algebras of subquadratic growth are finite dimensional. Finely generated torsion groups having subquadratic growth are finite.

**Proof.** The first assertion follows from the end of the last proof.

Let $G$ be a group and $K$ a field. Then $K[G]$ is finitely generated (as an associative algebra) if and only if $G$ is a finitely generated group. For strictly subquadratic growth, this is an application of the previous result to the group ring of a finitely generated group. First, the image of a group $G$ in its group ring $K[G]$ defines a monomial basis in $K[G]$. Then $G$ and $K[G]$ have the same asymptotic growth. \(\square\)
The next corollary is a special case of Gromov’s Theorem on groups of polynomial growth [2], which also can be proved along the lines of our approach.

**Corollary 6.** Let $K[G]$ be a finitely generated group algebra with finite generating set $S$. If there exists a constant $a$ such that $f_2(n) \leq an$, then $G$ contains a subgroup of finite index isomorphic to $\mathbb{Z}$. In particular, groups having linear growth contain a subgroup of finite index isomorphic to $\mathbb{Z}$.

**Lemma 7.** For any pair of elements $x, y$ of infinite order in a group $G$ having linear growth, there exist two integers $m$ and $n$ such that $x^m = y^n$. Moreover, the subgroup in $G$ generated by the elements of infinite period has finite index in $G$.

**Proof.** If infinite-order elements $x, y$ in $G$ are independent in the sense that $x^m \neq y^n$, $\forall m, n \neq 0$ then the elements in $\{x^m y^n\}$ are all different. These elements generate quadratic growth, so that the group $G$ itself has at least quadratic growth, which is impossible. Denote by $H$ the subgroup generated by the elements in $G$ of infinite period. $H$ is characteristic in $G$. $G/H$ is infinitely generated and has linear growth. Then, since it cannot have elements of infinite period, $G/H$ is finite.

**Proof of the Corollary.** Let $K[G]$ be a finitely generated algebra having linear growth. Consider a minimal generating set of $K[G]$ consisting of generators of $G$. The image of the group $G$ in its group ring defines a monomial basis $\mathcal{V}$ in $K[G]$. By Theorem 4, there exists a transcendent element in $\mathcal{V} = G \subset K[G]$. Denote by $H$ the subgroup generated by the elements in $G$ of infinite period. By the last Lemma 7, for all elements $x, y$ of $G$ of infinite order, there exist constant $m, n$ such that $x^m = y^n$. Now the subgroup $H$ has finite index in $G$ by Lemma 7 and so is finitely generated. But $G/H$ is finitely generated with linear growth and has no elements of infinite order, hence it is finite by Theorem 4 and its Corollary. Finally, $H$ is a finite extension of $\mathbb{Z}$. Indeed it is finitely generated and its infinite-order generators have some power equal to some $x$ which is centralized by them, so that it is central in $H$. But $H/\langle x \rangle$ has no elements of infinite order, hence again it is finite and the result follows. □

3. **Independent Sets**

Let $A$ be a finitely generated algebra with generating set $S$. Let $\mathcal{F} = \{x_i\}_{i \in I}$ be a set of transcendental monomials on $S$ and denote by $I_{\mathcal{F}}$ the ideal in $A$ generated by $\mathcal{F}$. The set $\mathcal{F}$ is called independent if, for all $i \in I$, the image of $x_i$ in $A/I_{\mathcal{F}} \backslash \{x_i\}$ is transcendental.

**Proposition 8.** Let $\mathcal{F}$ be an independent set in an algebra $A$ of linear growth. Then the cardinality of $\mathcal{F}$ is finite.

Let $\mathcal{F} = \{x_i\}_{i \in I}$ be a set of transcendental monomials in $A$. Denote by $x_i^{\infty} = x_i x_i x_i \ldots$ the associated infinite chains.

**Lemma 9.** The set $\mathcal{F} = \{x_i\}_{i \in I}$ is independent if and only if, for all $\alpha$, there is an integer $N$ such that the set $\{x_i^{\infty}(1 \ldots n) \mid n > N, i = 1, \ldots, \alpha\}$ is linearly independent.

**Proof.** Suppose that, for each $\alpha$, there exists an $N$ such that $\{x_i^{\infty}(1 \ldots n) \mid n > N, i = 1, \ldots, \alpha\}$ is linearly dependent, that is there is a couple $(i, m)$ such that $x_i^{\infty}(1 \ldots m)$ is linearly dependent on the $x_j^{\infty}(1 \ldots p)$. One may suppose that $m \geq \ell_S(x_i)$ and $p \geq \ell_S(x_j)$ with $i \neq j$. Hence one has $x_i^{\infty}(1 \ldots m) = x_j^{\ell_S(x_i)} \cdot x_j^{\infty}(q \cdot \ell_S(x_i) \ldots m)$ for some $q \geq 1$ and $x_i^{q+1} = x_i^{\infty}(1 \ldots m) \cdot x_i^{\infty}(m + 1 \ldots (q + 1) \cdot \ell_S(x_i))$, but $x_i^{\infty}(1 \ldots m)$ is linearly dependent on the
Algebras of linear growth

349

\(x_i^\infty(1 \ldots p)\) for \(p \geq \ell_\mathcal{S}(x_i)\) with \(i \neq j\) whose image is zero in \(A/I_{\mathcal{F}\{x_i\}}\), so that the image of \(x_i\) in \(A/I_{\mathcal{F}\{x_i\}}\) is algebraic.

Suppose now that, for all \(\alpha\), there exists an \(N\) such that \(\{x_i^\infty(1 \ldots n) \mid n > N, i = 1, \ldots, \alpha\}\) is linearly independent. Then, since \(x_i^n = x_i^\infty(1 \ldots q \cdot \ell_\mathcal{S}(x_i))\) for all \(i, \pi_\mathcal{F}(x_i)\) is transcendent, for all \(i\).

The following observation is a straightforward consequence of the previous Lemma.

**Lemma 10.** Let \(\mathcal{F}_\alpha = \{x_1, x_2, \ldots, x_\alpha\}\) be a finite independent set in an algebra \(A\) and \(x_\alpha^\infty\) the associated infinite chains, \(1 \leq i \leq \alpha\). Denote by \(f_\mathcal{F}_\alpha\) the function defined by \(f_\mathcal{F}_\alpha(n) = \dim \text{span}\{x_i^\infty(1 \ldots m) \mid 1 \leq i \leq \alpha, 1 \leq m \leq n\}\). Then for some positive constants \(C\) and \(N\), one has \(f_\mathcal{F}_\alpha(n) \geq an + C\) for \(n > N\).

**Proof of the Proposition.** Suppose the growth function \(f_\mathcal{F}\) of \(A\) with respect to the considered generating set \(S\) satisfies \(f_\mathcal{F}(n) \leq an\). Suppose there exists an independent infinite set \(\mathcal{F} = \{x_1, x_2, \ldots\}\) of transcendental monomials in \(A\). Denote by \(\mathcal{F}_\alpha\) the finite subset \(\{x_1, x_2, \ldots, x_\alpha\}\) which is also independent.

Let \(x_i^\infty\) and \(f_\mathcal{F}_\alpha\) as in Lemma 10. Since the set \(\mathcal{F}_\alpha\) is independent, by Lemma 10, one has \(f_\mathcal{F}_\alpha(n) \geq an + C\) for some positive constants \(C\) and \(n > N\). Since the words \(x_i^\infty(1 \ldots n)\) are of length \(n\), one has the inequality \(f_\mathcal{F}(n) \geq f_\mathcal{F}_\alpha(n) \geq an + C\). Hence for large values of \(\alpha\) and \(n\) this contradicts the linear growth of \(A\) and the cardinality of any independent set in \(A\) is finite.

Call a finite independent set \(\mathcal{F}\) in an algebra \(A\) maximal, if, for all element \(x\) in \(A\), the set \(\mathcal{F} \cup \{x\}\) is not independent. The corresponding invariant for \(A\) is \(\text{Max}_{\mathcal{F} \text{ indep.}} |\mathcal{F}|\).

**Proposition 8 Revisited.** Let \(A\) be a finitely generated algebra with generating set \(S\). If the growth function of \(A\) with respect to \(S\) satisfies \(f_\mathcal{F}(n) \leq an\) for some constant \(a\) and \(n > 0\), then, considering finite sets of transcendental monomials on \(S\), one has \(\text{Max}_{\mathcal{F} \text{ indep.}} |\mathcal{F}| \leq a\).

4. **Large Sets and Their Rank**

There can be no meaningful satisfactory general notion of index for a subring of a rings. All possible definitions lack some basic property one would require, see [4]. Let \(A\) be a finitely generated algebra with generating set \(S\) and \(\mathcal{F} = \{x_\ell\}_{\ell \in T}\) a finite set in \(A\). Let \(I_{\mathcal{F}}\) be the ideal in \(A\) generated by \(\mathcal{F}\). Define the rank of \(\mathcal{F}\) in \(A\) to be \(\text{Rank}(\mathcal{F}, A) = \max(\dim \text{Alg}(x) \mid x \in A/I_{\mathcal{F}})\). The set \(\mathcal{F}\) is large in \(A\) if it has finite rank in \(A\). The rank of finite independent sets does not define an invariant of \(A\). The cardinality of large sets in \(A\) also does not define an invariant in general. The related invariants are

\[
\text{Rank}_{\text{min}}(A) = \min_{\mathcal{F}} \text{Rank}(\mathcal{F}, A)
\]

\[
\text{Min}_{\text{large}}(A) = \min_{\mathcal{F}} |\mathcal{F}|
\]

\[
\text{Max}_{\text{large}}(A) = \max_{\mathcal{F}} |\mathcal{F}|
\]

over all large finite sets \(\mathcal{F}\) in \(A\). On the other hand, the quantity \(\text{Rank}_{\text{max}}(A)\) over all large finite sets \(\mathcal{F}\) in \(A\) being in general infinite is not so interesting.

**Theorem 11.** Let \(A\) be a finitely generated algebra. The algebra \(A\) has linear growth if and only if it admits a finite large independent set generating a subalgebra of linear growth. The cardinality of such sets defines an invariant.
Let \( A \) be a finitely generated algebra with generating set \( S \) and \( \mathcal{W} \) a monomial basis on \( S \) in \( A \). Recall first that by [9], algebras of linear growth are PI-algebras and that Amitsur and Small [1, Theorem 5] proved that if \( R \) is an affine PI ring over \( K \) and \( I \subset R \) is a left ideal, then \( R/I \) is finite dimensional if and only if every element of \( R \) is algebraic over \( I \).

If our algebra \( A \) has linear growth, then, by Theorem 4, there exists a transcendental element \( x_1 \) in \( \mathcal{W} \). If \( A/I_{\{x_1\}} \) is algebraic, then \( \mathcal{F}_1 = \{x_1\} \) defines a finite large independent set in \( A \) and we are done.

If not, consider now \( A/I_{\mathcal{F}_1} \), which is also finitely generated of linear growth and then, again by Theorem 4, there exists some transcendental element \( x_2 \) such that \( \mathcal{F}_2 = \mathcal{F}_1 \cup \{x_2\} = \{x_1, x_2\} \) is independent. Then again, if this set is large we are done and if not use Theorem 4 once more. Proposition 8 ensures that there are only finitely many such steps, that is any independent set in an algebra of linear growth is necessarily finite. Then let \( \mathcal{F} \) be a maximal (finite by Proposition 8) independent set in \( A \); it is by construction large in \( A \). Note that one may take the large independent set in the monomial basis \( \mathcal{W} \). In particular, this large independent set in \( A \) generates an infinite-dimensional subalgebra whose growth is less than that of the ambient algebra, that is linear.

On the other hand, let \( A \) be a finitely generated algebra admitting a finite large independent set \( \mathcal{F} \) generating a subalgebra of linear growth. Then the algebra \( A/I_{\mathcal{F}} \) is actually finite dimensional. Taking a base over this ideal, the growth of \( A \) must be linear, since the growth of the algebra generated by \( \mathcal{F} \) is linear.

Let us prove the last affirmation by induction. Consider the property \((P_n)\) to be the invariance of the cardinality of large independent sets in algebras of linear growth for sets of cardinality less than \( n \).

For \( n = 0 \), this is a bit formal, since it means that \( A \) is finite dimensional (\( A \) has linear growth and is algebraic, hence it is finite dimensional by Corollary 5). Then all large independent sets in \( A \) are trivial (that is empty sets).

Let us consider the case \( n = 1 \). Let \( A \) be an algebra of linear growth and \( \mathcal{F} \) be a large independent set in \( A \) with \( |\mathcal{F}| = 1 \). This implies that \( A \) is finite dimensional. Assume that \( \mathcal{F}' \) is another large independent set in \( A \) with \( |\mathcal{F}'| \neq 1 \). If \( |\mathcal{F}'| = 0 \), then \( A \) is finite dimensional and this contradicts our hypothesis. Suppose now \( |\mathcal{F}'| > 1 \), say \( \mathcal{F}' = \{x\} \) and \( \mathcal{F}' = \{y_1, \ldots, y_m\}_{m>1} \), then, for \( i = 1, \ldots, m \), there exists \( 0 \neq p_i(t) \in K[t] \) such that \( p_i(y_i) \in K[x] \), so that the image of \( x \) in \( A/I_{\{y_i\}} \) for \( i = 1, \ldots, m \) is algebraic. But this implies that \( A/I_{\{y_i\}} \) is algebraic itself and this contradicts the independence of the set \( \mathcal{F}' \).

Assume now the property \((P_n)\) is true and let us show that \((P_{n+1})\) is true. Let \( A \) be an algebra of linear growth and \( \mathcal{F} \) a large independent set in \( A \) with \( |\mathcal{F}| = n + 1 \). Assume \( \mathcal{F}' \) is a large independent set in \( A \) with \( |\mathcal{F}'| \neq n + 1 \). The case \( |\mathcal{F}'| < n + 1 \) contradicts the validity of \((P_n)\), so it remains the case \( |\mathcal{F}'| > n + 1 \). Denote \( \mathcal{F} = \{x_1, \ldots, x_{n+1}\} \) and \( \mathcal{F}' = \{y_1, \ldots, y_{n+1}, \ldots, y_p\} \). Some of the \( x_i \) have to be different from the \( y_j \), since if not one would have \( \mathcal{F} \subset \mathcal{F}' \), and then, since \( A/I_{\mathcal{F}} \) is algebraic this would contradict the independence of \( \mathcal{F}' \).

So say \( x_q \notin \mathcal{F}' \), and consider the associated map \( \pi: A/I_{\mathcal{F}} \to A/I_{\mathcal{F}\setminus\{x_q\}} \). One has \( \text{dim Alg}(\pi(\mathcal{F}')) = \infty \). Now if \( \text{Alg}(\pi(\mathcal{F}')) \) is generated by more than two elements then one can conclude as at the end of the proof of \((P_1)\).

It remains just to prove that \( \text{Alg}(\pi(\mathcal{F}')) \) is at least two-generated. If not this would mean that, for some \( x_i, y_j \), say \( y_r \) and \( y_s \), would have algebraic images in \( A/I_{\{x_i\}} \). But this contradicts the independence of the subset \( \{y_r, y_s\} \) as at the end of the proof of \((P_1)\). \(\square\)

5. Examples

Let us first make some remarks.
(1) If an algebra $A$ is finitely generated, then, for any generating finite set $S$, $\mathcal{F} = S$ defines a (not necessarily independent) large set in $A$. For finitely generated algebras, $\text{Min}_{\text{large}}(A)$ is bounded above by the minimal number of generators for $A$. One may ask whether there is a link to deficiency.

(2) Let $A = K[x]$. Then, for all $i \in \mathbb{N}^*$, the set $\mathcal{F}_i = \{x^i\}$ is large in $A$ and its rank in $A$ depends on $i$. In this simple example, one already has $\text{Rank}_{\text{max}}(A) = \infty$ and $\text{Rank}_{\text{min}}(A) = 0$.

(3) Let $\mathcal{F} = \{x_1^{p_1}, \ldots, x_m^{p_m}\}$ be a finite set in an algebra $A$. If $\mathcal{F}$ is large in $A$, then the set $\{x_1, \ldots, x_m\}$ are also large in $A$. If $\mathcal{F} = \{x_1^{p_1q_1}, \ldots, x_m^{p_1q_m}\}$ is large in $A$, then the rank of the set $\{x_1^{p_1r_1}, \ldots, x_m^{p_1r_m}\}$, with $r_i \leq q_i$, depends on $[r_i]$. For a fixed $[x_i]$, it seems reasonable that the rank may be evaluated as a function of the $[r_i]$ and the minimal rank estimated.

(4) Let $A$ be a finitely generated algebra having linear growth. Define $a_{\text{min}}(A) = \min \{a \mid f_\mathcal{F}(n) \leq an + C\}$ over all finite generating sets $S$ for $A$. Then one has $\text{Min}_{\text{large}}(A) \leq a_{\text{min}}(A)$.

In Proposition 8, we show that the cardinality of independent sets in a finitely generated algebra $A$ of linear growth is finite. Moreover, if $S$ is a finite generating set for $A$ and $\mathcal{F}$ is a large set in $A$ such that its associated growth function $f_\mathcal{F}$ satisfies $f_\mathcal{F}(n) \leq an$ for $n > 0$, then the cardinality of $\mathcal{F}$ is less than or equal to $a$. Indeed, in the proof of Proposition 8, one gets a contradiction as soon as $a > a$.

(5) Let $A' \subset A$ be algebras generated by finite sets $S'$ and $S$ respectively, where $S' \subset S$. Assume that $A$ has linear growth and that, for some real number $C$ and all sufficiently large $n$, one has $\frac{f_\mathcal{F}(n)}{f_\mathcal{S}(n)} \leq C$. If $\mathcal{F}$ is a finite large set in $A'$, then it is not necessarily large in $A$. (Consider for example, $A = K[x] \oplus K[y]$, $A' = K[x]$ and $\mathcal{F} = \{x\}$).

(6) Let $A$ be an algebra of linear growth with generating set $S$ and $\mathcal{F}$ be a finite set of monomials on $S$.

1—If $\mathcal{F}$ is large in $A$ and minimal, then it is independent.

2—If $\mathcal{F}$ is independent in $A$ and maximal, then it is large in $A$.

3—If $\mathcal{F}$ is large in $A$, then $A$ and $\text{Alg}(\mathcal{F})$ have same growth.

(7) The cardinality of large independent sets in algebras of linear growth defines a new invariant for algebras of linear growth. This invariant plays to some extent a role similar to that of the number of ends for groups and this point is one of the motivations of this note. This is the minimal number of monomials one should kill in an algebra of linear growth to get an algebraic algebra. One has the following observations: (1) if $A$ is finite dimensional or algebraic, then it has only trivial large independent sets. (2) Let $A$ and $B$ be two algebras of linear growth. Then the cardinality of large independent sets in $A \oplus B$ is the sum of the cardinalities of large independent sets in $A$ and $B$. (3) Any cardinality may be reached (the algebra $A = \oplus_{i \leq n} K[x_i]$ has a large independent set of cardinality $n$).

PROPOSITION 12. Let $A$ be an infinite-dimensional algebra with finite generating set $S$. Suppose there exists an integer $N$ such that for $n > N$ one has $f_S(n + 1) - f_S(n) \leq 1$. Then it admits a transcendent generator which defines a large set in $A$. More generally, any transcendent monomial $x$ on $S$ in $A$ defines a large set in $A$.

Note that if $f_S(n + 1) - f_S(n) \leq a$ for all $n$ greater than some fixed $N$ (let us call it finite step asymptotic behaviour of order $a$), then it admits a large set $\{u_i\}_{i \leq k}$ of cardinality less than $a$. One then has $A = \text{span}(\bigcup_{j=1}^m S_j') + P(u_i)$ for some $m$. In the case $f_S(n + 1) - f_S(n) \leq 1$ for all $n$ greater than some fixed $N$, one has $A = \text{span}(\bigcup_{j=1}^m S_j') + P(s)$ for some integer $m$ and generator $s$. In the case $f_S(n + 1) - f_S(n) \leq 2$ for all $n$ greater than some fixed $N$, one can show that (1) there is no alternation of step-one and step-two growth for $f_S$ and (2) there
exist \( u \) and \( v \) such that the considered algebra is of the form \( A = A_m + P(u) + P(v) \) or of the form \( A = A_m + P(u) + P(u)v \). The proofs are left to the reader.

**Proof of the Proposition.** First remark that if an algebra \( A \) satisfying, for \( n > N \), \( f_S(n+1) - f_S(n) \leq 1 \), is infinite dimensional, then it actually satisfies \( f_S(n+1) = 1 + f_S(n) \), for \( n > N \). Indeed, if, for some \( n_0 \), \( \dim f_S(n_0 + 1) = f_S(n_0) \) then, denoting by \( (A_n)_{n \in \mathbb{N}} \) the filtration associated with the generating set \( S \), one has \( A_m \subseteq A_{n_0} \) for \( m > n_0 \), but since \( A_{n_0} \subseteq A_m, m > n_0 \), one gets \( A = A_{n_0} \).

Let \( w \) be a monomial of length \( N \) (using the length \( \ell_S \) associated with the finite generating set \( S \)) and a generator \( s \) in \( S \) such that \( ws \) has length \( N + 1 \). Then there is a \( t \) in \( S \) such that \( wst \) has length \( N + 2 \). But, if \( w = u_1 \cdots u_n \), then the monomial \( u_2 \cdots u_{n+1}st \) have length \( N + 1 \) so that it is linearly dependent on \( ws \) up to shorter words. Hence, up to shorter words, \( wst, u_1ws = u_1u_2 \cdots u_n s \) and \( wss \) are linearly dependent. Hence \( s^2 \) (and also \( u_1^2 \)) has length 2. In a similar way, considering elements in \( A_{N+j} \), one shows that \( s^j \) has length \( j \) and hence does not belong to \( A_{j-1} \). Hence \( s \) is a transcendental generator for \( A \) and \( A/I[I] \) is algebraic. In particular, \( \{ s \} \) defines a large set in \( A \) of rank less than \( f_S(N) = \dim A_N \).

Now, for any transcendental monomial \( x \) and \( m \) big enough (say the least \( m \) such that \( m \cdot \ell_S(x) \geq N \)), \( x^m \) will be linearly independent of \( A_m \cdot \ell_S(x) - 1 \) and hence \( \{ x \} \) defines a large set in \( A \). Note that since \( x \) is transcendental, the set \( \{ x \} \) is independent. In particular, one has \( \text{Rank}(\{ x \}, A) \leq f_S(m \cdot \ell_S(x)). \)

**Proposition 13.** Let \( A \) be a finitely generated infinite-dimensional algebra of linear growth. If \( A \) is a group algebra, then \( \text{Min}_{\text{large}}(A) = 2 \). In particular, if \( A \) admits a large independent set of cardinality different from 2, then \( A \) is not a group algebra.

**Proof.** If \( A = K[G] \) has linear growth, then \( G \) also has linear growth. By Corollary 6, \( G \) admits a subgroup of finite index isomorphic to \( \mathbb{Z} \). Then the two generators \( \{ u^{\pm 1} \} \) of \( \mathbb{Z} \) seen as elements in \( A \) define a large independent set in \( A \). Suppose there exists an independent set \( \mathcal{F} = \{ x \} \) large in \( A \). Then, with \( \{ u^{\pm 1} \} \) as previously, there exists \( n, m \) such that \( u^n, u^{-m} \in K[x] \). Hence \( u^m u^{-nm} = 1 \in K[x] \). But \( x \) is transcendental. Hence we get a contradiction. \( \square \)

By a graph we mean an oriented graph where loops are allowed as well as multiple edges. A path of length \( n \) in a graph is a set of vertices \( v_i \) and the edges \( e_j: v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} v_n \) such that, for every edge \( e_j \), its ending is the vertex \( v_j \) and the beginning the vertex \( v_{j-1} \). A path is called cyclic if its last vertex \( v_n \) coincides with the first. If, in addition, all the edges, except the first and the last, are mutually different, then we can consider a subgraph made up of the vertices \( v_i \) and the edges \( e_j \), called a cycle. A path is called simple or a chain, if all the vertices \( v_i \) are different. The growth of a graph \( \mathcal{G} \) is the function \( f_{\mathcal{G}}(n) \) equal to the number of paths of length not greater than \( n \).

Let \( A \) be a finitely presented monomial algebra with generating set \( S \), that is an algebra whose set \( X \) of defining relations is of the form \( x = 0 \) where \( x \) is a word. Let \( m + 1 \) be the maximum of the lengths of words in \( X \) and let \( V \) be the set of normal words of length \( m \). Construct a graph \( \mathcal{G}(A) \) whose vertices are the elements in \( V \) and the edge \( x \rightarrow y \) is placed if and only if there exist words \( u, v \in S \) such that \( xu = vy \). There is a bijective correspondence between the set of normal words of length \( \geq m \) and the paths in the graph. In particular they have the same growth. Let the diameter \( \text{Diam}(A) \) of \( A \) be the maximal length of simple paths in \( \mathcal{G}(A) \) which do not contain any cyclic subpath.

**Proposition 14.** Let \( A \) be a finitely presented monomial algebra of linear growth. Then there is a large independent set in \( A \) of cardinality the number of cycles in \( \mathcal{G}(A) \) and such that \( \text{Rank}(\mathcal{F}, A) \leq \text{Diam}(A) \).
Algebras of linear growth

PROOF. Ufnarovski [10, Theorem 1, p. 90] showed that the growth of every graph is alternating. It is exponential if and only if there are two different cycles in the graph with a common vertex. Otherwise it is polynomial of degree $d$, where $d$ is the maximal number of cycles in line. In particular, the graph of a finitely presented monomial algebra of linear growth has only single cycles through which one can pass. Note that $A$ is infinite dimensional if and only if its graph contains a cycle. It follows immediately that a finitely presented monomial algebra $A$ is algebraic if and only if it is finite dimensional. The monomials representing the maximal singly oriented simple paths in $G(A)$ which end by the cycles define a finite set $\mathcal{F}$ with the required properties. $A/I\mathcal{F}$ is algebraic since its graph contains no cycles. Actually, it is finite dimensional and one has $\text{Rank}(\mathcal{F}, A) \leq \text{Diam}(A)$.

We finish with a result on $C^*$-algebras. A $C^*$-algebra is said to be finitely generated if it admits a finitely generated dense subalgebra and is said to have a given growth, if it admits a finitely generated dense subalgebra having this growth [7]. In [8, Observation A] we observed that a $C^*$-algebra is infinite dimensional if and only if it has a singly generated infinite-dimensional commutative $C^*$-subalgebra. Hence the Kurosh–Levitzky problem always has a positive answer in $C^*$-algebras. We showed [8, Theorem E] that $C^*$-algebras of linear growth are subhomogeneous, that is representable. Note that associative algebras of linear growth need not be representable in general [3]. In the case of step-one linear growth, the size of its irreducible representations is bounded by $1 + \sqrt{\dim(\text{span}(S))}$, where $S$ denotes the generating set [5, Theorem 4.4].

PROPOSITION 15. Let $A$ be an infinite dimensional $C^*$-algebra of linear growth. The cardinality of large independent sets in $A$ does not define an invariant.

The corresponding invariant for $C^*$-algebras is the minimum of the cardinality of large independent sets over all dense subalgebras.

PROOF. Let $A$ be a finitely generated $C^*$-algebra with self-adjoint finite generating set $S$. Suppose $A$ has linear growth with respect to $S$. Assume that $A$ is not finite dimensional then, by [8, Lemma 1], there exists an Abelian $C^*$-subalgebra $B$ of $A$ such that $c_0$ is a quotient of $B$. Lemma 2 in [8] asserts that for every positive integer $m$, there exists a generating system $X^{(m)}$ of $c_0$ such that $GK \dim(c_0, X^{(m)}) = m$. Let $Z = \{z_i\}$ be preimages of $\{x_i\} = X^{(m)} \subset c_0$ with $m > 1$ in $A$. Then consider the new finite generating system $\tilde{S} = S \cup \{Z \cup Z^*\}$. The growth function associated with $\tilde{S}$ in $A$ has polynomial growth of degree greater than $m$. If $\mathcal{F}$ is a large independent set for the dense subalgebra defined by $S$ then it is clearly not large in the dense subalgebra defined by $\tilde{S}$ for which large independent sets are larger.

ACKNOWLEDGEMENT

This note was written during a stay at I.H.E.S. in 1997. We thank them for their hospitality.

REFERENCES


