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Some results on intersecting families of subsets

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Abstract

Let $K = \{k_1, k_2, \ldots, k_r\}$, $L = \{l_1, l_2, \ldots, l_s\}$ be two sets of non-negative integers, $\mathscr{F} \subset 2^{[n]}$, $[n] := \{1, 2, \ldots, n\}$, such that $|F_i \cap F_j| \in L$, $|F_i| \in K$ for all $F_i, F_j \in \mathscr{F}$, $i \neq j$; $x_0 \in [n]$, and $\mathscr{F}_1 = \{F_i \colon x_0 \in F_i \in \mathscr{F}\}$, $\mathscr{F}_2 = \{F_i \colon x_0 \notin F_i \in \mathscr{F}\}$, $K_1 = \{|F_i| \colon F_i \in \mathscr{F}_1\} \cup \{|F_i| + 1 \colon F_i \in \mathscr{F}_2\}$ and $t = |K_1|$. Suppose that no set in $\mathscr{F}_1$ contains any set in $\mathscr{F}_2$. We will prove that $|\mathscr{F}| \leqslant \sum_{i=s_1}^{s} \binom{n-1}{i}$, where $s_1 = \min_{1 \leqslant i \leqslant r}(s - t + 1, k_i - 1)$.

This extends the results of Alon et al. [1] and Snevily [9]. The proof we give here is much easier. Noting that one conjecture of Frankl and Füredi [3] is as follow: Let $L = \{1, 2, \ldots, s\}$, $\mathscr{F}$ is L-intersecting family, then $|\mathscr{F}| \leqslant \sum_{i=0}^{s} \binom{n-1}{i}$.

Using the above results, we prove this conjecture is true for $s = 3$. Combining the result of Snevily [9], we know that the above conjecture is true for $s \leqslant 3$.

In the last part of this paper, we consider the case $L = \{l, l+1, \ldots, l+s-1\}$. We prove the following result.

Let $l < s$, $L = \{l, l+1, \ldots, l+s-1\}$ and $\mathscr{F} \subset 2^{[n]}$ be an L-intersecting family such that $|F| \geqslant l + r$ for any $F \in \mathscr{F}$, where $r > 0$. Then $|\mathscr{F}| \leqslant \sum_{i=r}^{s} \binom{n-l}{i} + 1$ for $n > n_0(s)$. Equality holds iff $r \leqslant s$ and $\mathscr{F}(A, B) = \{F \in \binom{[n]}{l+r \leqslant i \leqslant l+s} \colon A \subset F\} \cup ([n] \setminus \{B\})$ for some l-element set A and its subset $B \subset A$ such that $0 < |B| \leqslant r$.

We also consider some related problems and raise several questions.

1. Introduction

In this paper, we will use the same notation and terminology as that of [1], including $[n] := \{1, 2, \ldots, n\}$.

Let $L = \{l_1, l_2, \ldots, l_s\}$, $K = \{k_1, k_2, \ldots, k_r\}$ be two sets of non-negative integers, $\mathscr{F} = \{F_1, F_2, \ldots, F_m\} \subset 2^{[n]}$. We say that $\mathscr{F}$ is (K,L)*-intersecting* if $|F_i \cap F_j| \in L$, $|F_i| \in K$ for all $F_i, F_j \in \mathscr{F}$, $i \neq j$; (K,L)*-intersecting* (mod p) if $|F_i \cap F_j| \in L + p\mathbb{Z}$, $|F_i| \in K + p\mathbb{Z}$ for all $F_i, F_j \in \mathscr{F}$, $i \neq j$; and that $\mathscr{F}$ is (K,L)*-cointersecting* if $|F_i \setminus F_j| \in L$, $|F_i| \in K$ for all $F_i, F_j \in \mathscr{F}$, $i \neq j$; (K,L)*-cointersecting* (mod p) if

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$|F_i \setminus F_j| \in L + p\mathbb{Z}$, $|F_i| \in K + p\mathbb{Z}$ for all $F_i, F_j \in \mathscr{F}$, $i \neq j$. The following results were proved by Alon et al. [1].

Theorem 1.1 (Alon et al. [1]). *Let $\mathscr{F}$ be a (K,L)-intersecting family of $2^{[n]}$ and assume that $k_i > s - r$ for every i. Then $|\mathscr{F}| \leqslant \sum_{i=s-r+1}^{s} \binom{n}{i}$.*

Theorem 1.2 (Alon et al. [1]). *Let p be a prime and K, L two disjoint subsets of $\{0, 1, \ldots, p-1\}$. Let $|K| = r$, $|L| = s$ and assume $r(s - r + 1) \leqslant p - 1$ and $n \geqslant s + k_r$, where k_r is the maximal element of K. $\mathscr{F} \subset 2^{[n]}$ is (K,L)-intersecting* (mod p). *Then $|\mathscr{F}| \leqslant \sum_{i=s-r+1}^{s} \binom{n}{i}$.*

Snevily has proved the following results in [9].

Theorem 1.3 (Snevily [9]). *Let $L = \{l_1, \ldots, l_s\}$ be a collection of positive integers $\mathscr{F} \subset 2^{[n]}$ and $\mathscr{C} = \{F_i\colon F_i \in \mathscr{F},\ |F_i| \in L\}$. If $|F_i \cap F_j| \in L$ for all $F_i, F_j \in \mathscr{F}$ and if $|\mathscr{C}| = 0$ or $\bigcap_{F_i \in \mathscr{C}} F_i \neq \emptyset$, then $|\mathscr{F}| \leqslant \sum_{i=0}^{s} \binom{n-1}{i}$.*

Theorem 1.4 (Snevily [9]). *Let p be a prime and K, L two disjoint subsets of $\{0, 1, \ldots, p-1\}$. Let $|L| = s$. $\mathscr{F} \subset 2^{[n]}$ is (K,L)-intersecting* (mod p). *Then $|\mathscr{F}| \leqslant \sum_{i=0}^{s} \binom{n-1}{i}$.*

By counting method, Frankl and Füredi have proved the next result.

Theorem 1.5 (Frankl and Füredi [3]). *Let $L = \{1, 2, \ldots, s\}$. $\mathscr{F}$ is an L-intersecting family, then $|\mathscr{F}| \leqslant \sum_{i=0}^{s} \binom{n-1}{i}$ for $n \leqslant 2s + 2$ or $n > 100s^2/[\ln(s+1)]$.*

They also made the following

Conjecture A. Theorem 1.5 is true for all n.

Snevily conjectured a general result in [9].

Conjecture B. Let $L = \{l_1, l_2, \ldots, l_s\}$, where $0 < l_1 < l_2 < \cdots < l_s$. $\mathscr{F}$ is an L-intersecting family, then $|\mathscr{F}| \leqslant \sum_{i=0}^{s} \binom{n-1}{i}$.

He also proved in [9] that Conjecture A is true for $s \leqslant 2$ and Conjecture B is true for $n > n_0(l_s)$.

The purpose of this paper is to establish some stronger results in more general form. The structure of this paper is as follow: In Sections 2 and 3, we prove some results about intersecting families for all n, which improves the result of [1,9]. As an application of this result, we give a simpler proof of a result of Snevily [9], which claims that Conjecture B is true for $n > n_0(l_s)$. This proof also gives a little stronger result. In Section 4, we prove much stronger results for sufficiently large n, which lead us to conjecture some stronger results. Using a result in this section, we will also prove that Conjecture A is true for $s = 3$.

2. Results on intersecting families

Theorem 2.1. *Let* $L=\{l_1,l_2,\dots,l_s\}$, $K=\{k_1,k_2,\dots,k_r\}$, $\mathscr{F}\subset 2^{[n]}$ *is* (K,L)*-intersecting. Let* $x_0\in[n]$, $\mathscr{F}_1=\{F_i\colon x_0\in F_i\in\mathscr{F}\}$, $\mathscr{F}_2=\{F_i\colon x_0\notin F_i\in\mathscr{F}\}$, *and* $K_1=\{|F_i|\colon F_i\in\mathscr{F}_1\}\cup\{|F_i|+1\colon F_i\in\mathscr{F}_2\}$ *(or* $K_1'=\{|F_i|-1\colon F_i\in\mathscr{F}_1\}\cup\{|F_i|\colon F_i\in\mathscr{F}_2\}$*),* $t=|K_1|=|K_1'|$. *Suppose that no set in* $\mathscr{F}_1$ *contains any set in* $\mathscr{F}_2$. *Then* $|\mathscr{F}|\leqslant\sum_{i=s_1}^{s}\binom{n-1}{i}$, *where* $s_1=\min_{1\leqslant i\leqslant r}(s-t+1,k_i-1)$.

In the following context, all the polynomial involved will have domain in the n-cube $\{0,1\}^n\subset\mathbb{R}^n$. Therefore, we can identify x_i^2 with x_i for every variable. So, every polynomial can be written uniquely as a multilinear polynomial. All the multilinear polynomials in n variables constitute a linear space over $\mathbb{R}$ of dimension 2^n.

Lemma 1. *Let* $L=\{l_1,l_2,\dots,l_s\}$, $K=\{k_1,k_2,\dots,k_r\}$, $\mathscr{F}\subset 2^{[n]}$ *is* (K,L)*-intersecting. If there exists an a-element subset* A *of* $[n]$ *such that either* $A\subset F_i$ *or* $F_i\cap A=\emptyset$ *for any* $F_i\in\mathscr{F}$. *Set* $\mathscr{F}_1=\{F_i\colon A\subset F_i\in\mathscr{F}\}$, $\mathscr{F}_2=\{F_i\colon A\cap F_i=\emptyset,\ F_i\in\mathscr{F}\}$. *Suppose that no set in* $\mathscr{F}_1$ *contains any set in* $\mathscr{F}_2$. *Then* $|\mathscr{F}|\leqslant\sum_{i=0}^{s}\binom{n-a}{i}$.

Proof. We may suppose that $A=\{1,2,\dots,a\}$. Let $\mathscr{F}_1=\{F_1,F_2,\dots,F_u\}$, $\mathscr{F}_2=\{F_{u+1},\dots,F_m\}$ such that $|F_1|\leqslant|F_2|\leqslant\cdots\leqslant|F_u|$, $|F_{u+1}|\leqslant\cdots\leqslant|F_m|$. Let v_i denote the characteristic vector of F_i. For $x=(x_1,x_2,\dots,x_n)$, let $x^*=(x_{a+1},\dots,x_n)$ be the truncated vector of x. Define,

$$f_i(x^*)=\prod_{t,\,l_t<|F_i|}(v_i^*\cdot x^*+a-l_t)\quad\text{for } 1\leqslant i\leqslant u,$$

$$f_i(x^*)=\prod_{t,\,l_t<|F_i|}(v_i^*\cdot x^*-l_t)\quad\text{for } u+1\leqslant i\leqslant m.$$

Then

$$f_i(v_i^*)\neq 0\quad\text{for } 1\leqslant i\leqslant m$$

and

$$f_i(v_j^*)=0\quad\text{for } 1\leqslant j<i\leqslant m.$$

So $\{f_i(x^*)\}$ is linearly independent over $\mathbb{R}$ as multilinear polynomials. But $f_i(x^*)$'s are all multilinear polynomials in variables $x_{a+1},\dots,x_n$ and of degree at most s. Therefore, our result follows. □

Proof of Theorem 2.1. We may suppose that $x_0 = 1$. Use the same notation as in the Lemma 1 with $a = 1$. Let $\binom{[n]-\{1\}}{0} \cup \cdots \cup \binom{[n]-\{1\}}{s_1-1} = \{T_1,\dots,T_e\}$ such that $|T_1| \leqslant \cdots \leqslant |T_e|$. It can be seen that $e = \binom{n-1}{s_1-1} + \cdots + \binom{n-1}{0}$. Let

$$f(x^*) = \prod_{k\in K_1} (x_2 + \cdots + x_n + 1 - k) \quad \left(\text{or } f(x^*) = \prod_{k\in K_1'} (x_2 + \cdots + x_n - k)\right),$$

$$h_i(x^*) = \prod_{j\in T_i} x_j f(x^*), \quad 1 \leqslant i \leqslant e.$$

It is easily seen that

$$h_i(t_i^*) = f(t_i^*) \neq 0 \quad \text{for } 1 \leqslant i \leqslant e, \qquad h_i(t_j^*) = 0 \quad \text{for } j < i \leqslant e,$$

where t_i is the characteristic vector of T_i. So, $h_1, h_2, \dots, h_e$ are linearly independent. Now, note that $f(v_i^*) = 0$ for $1 \leqslant i \leqslant m$, it can easily be checked that $f_1,\dots,f_m,h_1,\dots,h_e$ are linearly independent. This proves our results. □

Let $\binom{X}{a\leqslant i\leqslant b} = \{F\colon F \subseteq X, a \leqslant |F| \leqslant b\}$, $\mathscr{F} + x = \{F \cup \{x\}\colon F \in \mathscr{F}\}$. Theorem 2.1 is sharp as the following families show. For $0 \leqslant i \leqslant t-1$, $L = \{0,1,\dots,s-1\}$, $K = \{s-t+2,\dots,s\}$, the families $\mathscr{F} = \binom{[n]-\{1\}}{s-i\leqslant j\leqslant s} \cup (\binom{[n]-\{1\}}{s-t+1\leqslant j\leqslant s-i-1} + 1)$ attain the bound given in the Theorem 2.1, but they do not satisfy the conditions of Theorem 1.3.

Corollary 1. *If $K = \{k,\dots,k+r-1\}$, L is the same as in Theorem 2.1, $L \cap K = \emptyset$, then $|\mathscr{F}| \leqslant \sum_{i=s_1}^{s} \binom{n-1}{i}$, where $s_1 = \min(s-r, k-1)$.*

As an application of Theorem 2.1, we can prove the following result, Theorem 2.2. It is stronger than the results obtained in [9], but the proof here is simpler. We will make use of a result of Frankl and Wilson.

F–W Theorem (Frankl and Wilson [5]). Let L be a collection of s non-negative integers. $\mathscr{F} \subset 2^{[n]}$ such that $|F_i \cap F_j| \in L$ for all $F_i, F_j \in \mathscr{F}$ and $i \neq j$, then $|\mathscr{F}| \leqslant \sum_{i=0}^{s} \binom{n}{i}$.

Theorem 2.2. *Conjecture B is true if $n > s\binom{l_s(l_s+1)}{l_1+1}$.*

Remark. A little more careful computation shows that Conjecture B is true for $n > s\left(\binom{2l_s-l_1}{l_1+1} + (l_s-l_1)^2\binom{l_s-1}{l_1}\right)$.

Proof of Theorem 2.2. By Theorem 2.1, if conjecture B is false, then there exist at least three set $F_1,F_2,F_3 \in \mathscr{F}$ such that $|F_i| \leqslant l_s$ for $i = 1,2,3$, moreover, for any $x \in B = F_1 \cap F_2$, there exists $F_x \in \mathscr{F}$ such that $x \notin F_x$ and $|F_x| \leqslant l_s$.

Let $F_1, F_2 \in \mathscr{F}$ such that $|F_i| \leqslant l_s$ for $i = 1,2$ and that $|B| = |F_1 \cap F_2|$ is as smaller as possible. Let $B = \{x_1,\dots,x_t\}$ and $F_{x_i} \in \mathscr{F}$ such that $x_i \notin F_{x_i}$. Set $Q = F_1 \cup F_2 \cup F_{x_1} \cup \cdots \cup F_{x_{t-l_1+1}}$, then $|Q \cap F| \geqslant l_1 + 1$ for any $F \in \mathscr{F}$. But $|Q| \leqslant (t - l_1$

$+1)(l_s - l_1) + 2l_s - t < l_s(l_s+1)$. Therefore,

$$|\mathscr{F}| \leqslant \sum_{H \in \binom{Q}{l_1+1}} |\mathscr{F}(H)|,$$

where $\mathscr{F}(H) = \{F \in \mathscr{F}\colon H \subseteq F\}$. By F–W Theorem, we have

$$|\mathscr{F}(H)| \leqslant \sum_{i=0}^{s-1} \binom{n-l_1-1}{i} \leqslant \sum_{i=0}^{s-1} \binom{n-2}{i} < \frac{s}{n-1} \sum_{i=1}^{s} \binom{n-1}{i}.$$

Hence,

$$|\mathscr{F}| < \frac{s}{n-1} \binom{l_s(l_s+1)}{l_1+1} \sum_{i=1}^{s} \binom{n-1}{i} < \sum_{i=0}^{s} \binom{n-1}{i},$$

if $n > s\binom{l_s(l_s+1)}{l_1+1}$. This is a contradiction. □

Theorem 2.3. *Let $L = \{0, l_1, l_2, \ldots, l_s\}$, $K = \{k_1, k_2, \ldots, k_r\}$, $\mathscr{F} \subset 2^{[n]}$ is (K,L)-cointersecting. $k_i > s - r$ for every i. Then $|\mathscr{F}| \leqslant \sum_{i=s-r+1}^{s} \binom{n}{i}$.*

Proof. Let us label the family $\mathscr{F} = \{F_1, \ldots, F_m\}$ such that $|F_1| \geqslant |F_2| \geqslant \cdots \geqslant |F_m|$. Define

$$f_i(x) = \prod_{t=1}^{s} (\overline{v_i} \cdot x - l_t), \quad 1 \leqslant i \leqslant m,$$

where $\overline{v_i}$ is the characteristic vector of $[n] \setminus F_i$, $x = (x_1, \ldots, x_n)$. Let $\binom{[n]}{0} \cup \cdots \cup \binom{[n]}{s-t} = \{T_1, \ldots, T_w\}$ such that $|T_1| \leqslant \cdots \leqslant |T_w|$. Let

$$f(x) = \prod_{k \in K} (x_1 + \cdots + x_n - k),$$

$$h_i(x) = \prod_{j \in T_i} x_j f(x), \quad 1 \leqslant i \leqslant w.$$

It can easily be proved that $\{f_1, \ldots, f_m; h_1, \ldots, h_w\}$ is linearly independent. Hence, our result follows. □

Theorem 2.4. *Let $L = \{0, l_1, l_2, \ldots, l_s\}$, $K = \{k_1, k_2, \ldots, k_r\}$, $\mathscr{F} \subset 2^{[n]}$ is (K,L)-cointersecting. Let $x_0 \in [n]$, $\mathscr{F}_1 = \{F_i\colon x_0 \in F_i \in \mathscr{F}\}$, $\mathscr{F}_2 = \{F_i\colon x_0 \notin F_i \in \mathscr{F}\}$, $K_1 = \{|F_i| - 1\colon F_i \in \mathscr{F}_1\} \cup \{|F_i|\colon F_i \in \mathscr{F}_2\}$ (or $K_1' = \{|F_i|\colon F_i \in \mathscr{F}_1\} \cup \{|F_i| + 1\colon F_i \in \mathscr{F}_2\}$). $t = |K_1| = |K_1'|$. Suppose that no set in $\mathscr{F}_1$ contains any set in $\mathscr{F}_2$. Then $|\mathscr{F}| \leqslant \sum_{i=s_1}^{s} \binom{n-1}{i}$, where $s_1 = \min_{1 \leqslant i \leqslant r}(s - t + 1, k_i - 1)$.*

Note that we have $|L| = s + 1$ in Theorem 2.4, but the bound is still the same as that of other results.

Proof. The proof goes almost the same as that of Theorem 2.1 except several changes indicated below. Here we label the family $\mathscr{F}$ such that $1 \in F_i$ iff $i \leqslant u$, $|F_1| \geqslant |F_2| \geqslant \cdots$

$\geqslant |F_u|$, $|F_{u+1}| \geqslant \cdots \geqslant |F_m|$. Define

$$f_i(x^*) = \prod_{t=1}^{s} (\overline{v_i}^* \cdot x^* - l_t), \quad 1 \leqslant i \leqslant m,$$

where $\overline{v_i}$ is the characteristic vector of $[n] \setminus F_i$, $x^* = (x_2, \ldots, x_n)$. Then

$$f_i(v_i^*) \neq 0 \quad \text{for } 1 \leqslant i \leqslant m,$$

$$f_i(v_j^*) = 0 \quad \text{for } 1 \leqslant j < i \leqslant u \text{ or } i \leqslant u < j \text{ or } u < j < i \leqslant m.$$

Let

$$f(x^*) = \prod_{k \in K_1} (x_2 + \cdots + x_n - k) \quad \left(\text{or } f(x^*) = \prod_{k \in K_1'} (x_2 + \cdots + x_n + 1 - k)\right),$$

$$h_i(x^*) = \prod_{j \in T_i} x_j f(x^*) \quad 1 \leqslant i \leqslant e. \qquad \square$$

Corollary 2. *If* $K = \{k, \ldots, k + r - 1\}$, *L and* $\mathscr{F}$ *are the same as Theorem* 2.4, *then* $|\mathscr{F}| \leqslant \sum_{i=s_1}^{s} \binom{n-1}{i}$, *where* $s_1 = \min(s - r + 1, k - 1)$.

3. Modular version

We can prove the following modular version of Theorems 2.1 and 2.4. In this case, we need the following definition and a lemma of Alon, et al. [1]. Since the proofs of Theorems 3.1 and 3.2 are exactly the same as that of Theorems 2.1 and 2.4, respectively, except that we consider the problems over finite field $\mathbb{F}_p$, so we omit all the details.

Definition. We say that a set $H = \{h_1, \ldots, h_m\} \subset [n]$ has a *gap* of size $\geqslant g$ (where the h_i are arranged in increasing order) if $h_1 \geqslant g - 1$, $n - h_m \geqslant g - 1$, or $h_{i+1} - h_i \geqslant g$ for some i.

Lemma 2 (Alon et al. [1]). Let $K \subset \{0, 1, \ldots, p - 1\}$ be a set of integers and assume that the set $(K + p\mathbb{Z}) \cap \{0, 1, \ldots, n\}$ has a gap $\geqslant s + 1$, where $s \geqslant 0$. Define

$$f(x) = \prod_{k \in K} (x_1 + \cdots + x_n - k).$$

Then the set of multilinear polynomials $\{\prod_{i \in I} x_i f \colon |I| \leqslant s - 1\}$ is linearly independent over $\mathbb{F}_p$.

Theorem 3.1. *Let p be a prime,* $K = \{k_1, \ldots, k_r\}$, $L = \{l_1, \ldots, l_s\}$ *be two disjoint subsets of* $\{0, 1, \ldots, p-1\}$. *Let* $\mathscr{F} = \{F_1, \ldots, F_m\} \subset 2^{[n]}$ *is (K,L)-intersecting* (mod p).

Let x_0 be any element of $[n]$ such that $x_0 \in F_i$ iff $1 \leqslant i \leqslant h$. Let $K_1 = \{|F_i|: 1 \leqslant i \leqslant h\} \cup \{|F_i|+1: i > h\} + p\mathbb{Z} = \{k'_1, \dots, k'_t\} + p\mathbb{Z}$. If $K_1 \cap \{0,1,\dots,n\}$ has a gap of size $\geqslant g$, $s^* = \min(s-t+1, g-1)$, Then

$$|\mathscr{F}| \leqslant \sum_{i=s^*}^{s} \binom{n-1}{i}. \qquad \square$$

As pointed in [1], if $t(s-t+1) \leqslant p-1$, $n \geqslant s+k'_t$, where k'_t is the maximal element of the set $\{k'_1, \dots, k'_t\}$, then $K_1 \cap \{0,1,\dots,n\}$ has a gap of size $\geqslant s-t+2$. Hence, we have

Corollary 3. *Under the assumption of Theorem 3.1, if $t(s-t+1) \leqslant p-1$, $n \geqslant s+k'_t$, then $|\mathscr{F}| \leqslant \sum_{i=s-t+1}^{s} \binom{n-1}{i}$.*

In particular, if $K = \{k, k+1, \dots, k+r-1\}$, $(r+1)(s-r) \leqslant p-1$ and $n \geqslant k+s+r$, then $|\mathscr{F}| \leqslant \sum_{i=s-r}^{s} \binom{n-1}{i}$. $\square$

Theorem 3.2. *Let p be a prime, $K = \{k_1, \dots, k_r\}$, $L = \{l_1, \dots, l_s\}$ be two disjoint subsets of $\{0,1,\dots,p-1\}$ and $l_i > 0$ for every i. Let $\mathscr{F} = \{F_1, \dots, F_m\} \subset 2^{[n]}$ is (K,L)-cointersecting* (mod p). *Let x_0 be any element of $[n]$ such that $x_0 \in F_i$ iff $1 \leqslant i \leqslant h$. Let $K_1 = \{|F_i|-1: i \leqslant h\} \cup \{|F_i|: i > h\} + p\mathbb{Z} = \{k'_1, \dots, k'_t\} + p\mathbb{Z}$. If $K_1 \cap \{0,1,\dots,n\}$ has a gap of size $\geqslant g$, $s^* = \min(s-t+1, g-1)$, Then*

$$|\mathscr{F}| \leqslant \sum_{i=s^*}^{s} \binom{n-1}{i}. \qquad \square$$

Corollary 4. *Under the assumption of Theorem 3.2, if $t(s-t+1) \leqslant p-1$, $n \geqslant s+k'_t$, then $|\mathscr{F}| \leqslant \sum_{i=s-t+1}^{s} \binom{n-1}{i}$.*

In particular, if $K = \{k, k+1, \dots, k+r-1\}$, $(r+1)(s-r) \leqslant p-1$, $k > 0$, $n \geqslant k+s+r-1$ (or $k=0$, $n \geqslant p+s-1$), then $|\mathscr{F}| \leqslant \sum_{i=s-r}^{s} \binom{n-1}{i}$. $\square$

4. Stronger version of Theorem 2.1

If we assume that $k_i > l_j$ for every i,j, then we can prove much stronger results than that of Section 2.

Theorem 4.1. *Let $\mathscr{F} \subset 2^{[n]}$ such that $|F_i| \geqslant k$ and $|F_i \cap F_j| \in \{0,1,\dots,t-1\}$ for $i \neq j$, if $t \leqslant k$. Then*

$$|\mathscr{F}| \leqslant \frac{\binom{n}{t}}{\binom{k}{t}}$$

with equality iff $\mathscr{F}$ is a Steiner system $S(n,k,t)$.

Proof. It is easily seen that $\bigcup_{F \in \mathscr{F}} \binom{F}{t} \subset \binom{[n]}{t}$ and $\binom{F_i}{t} \cap \binom{F_j}{t} = \emptyset$ for different F_i, F_j of $\mathscr{F}$. Now $|\binom{F}{t}| \geqslant \binom{k}{t}$, and the result follows directly. $\square$

As an application of Theorems 4.1 and 2.1, we prove that Conjecture A is true for $s \leqslant 3$.

Theorem 4.2. *Conjecture A is true for* $s \leqslant 3$.

To prove Theorem 4.2, we need the following results.

Lemma 3 (Pyber [8]). Let $L = \{1,2,\dots,s\}$. $\mathscr{F}$ is a maximum L-intersecting family, then $F \in \mathscr{F}_{\leqslant s}$, $F \subset F'$ and $|F'| = s+1$ implies $F' \in \mathscr{F}$.

Corollary 5 (Pyber [8]). *If* $\mathscr{F}$ *is a maximum* $\{1,2,\dots,s\}$*-intersecting family, then no in* $\mathscr{F}_{\geqslant s+2}$ *contains any set in* $\mathscr{F}_{\leqslant s}$.

H-M Theorem (Hilton and Milner [6]. *Let* $\mathscr{F} \subset \binom{[n]}{k}$ *be an intersecting family. If* $\bigcap_{F \in \mathscr{F}} F = \emptyset$, *then* $|\mathscr{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. *Equality holds iff for some* $x \in [n]$, $A \subset [n]$, $|A| = k$, $x \notin A$, *and* $\mathscr{F} = \{F \in \binom{[n]}{k}: x \in F,\ F \cap A \neq \emptyset\} \cup \{A\}$.

Proof of Theorem 4.2. Let $c_s(n) = \sum_{i=0}^{s} \binom{n}{i}$. Suppose that $\mathscr{F}$ is a maximum $\{1,2,3\}$-intersecting family. To prove the result, we may assume that $n \geqslant 9$ and $\bigcap \mathscr{F}_{\leqslant 3} = \emptyset$. We may also assume that $\mathscr{F}_1 = \emptyset$, $\mathscr{F}_2 \cup \mathscr{F}_3 \neq \emptyset$, $\mathscr{F}_{\geqslant 5} \neq \emptyset$ by Theorem 2.1 and Erdös–Ko–Rado theorem. Now by Lemma 3, we have $\bigcap \mathscr{F}_4 = \emptyset$.

(1) Suppose that $|\mathscr{F}_2| \geqslant 2$. Let $F_1 = \{x_1,x_2\}$, $F_2 = \{x_1,x_3\}$ and $F_3 \in \mathscr{F}_{\leqslant 3}$ such that $x_1 \notin F_3$, say, $F_3 = \{x_2,x_3,x_4\}$. Set $\mathscr{F}^B_{\geqslant k,A} = \{F \in \mathscr{F}_{\geqslant k}: A \subset F, B \cap F = \emptyset\}$. Then $|\mathscr{F}^B_{\geqslant k,A}| \leqslant c_{s-|A|+1}(n - |A \cup B|)$ or $\leqslant \binom{n-|A\cup B|}{s-|A|+1} / \binom{k-|A|}{s-|A|+1}$ by F–W Theorem and Theorem 4.1. Now we have

$$\mathscr{F} \subset \mathscr{F}^{\{x_1\}}_{\{x_2,x_3\}} \cup \mathscr{F}^{\{x_3\}}_{\{x_1,x_2\}} \cup \mathscr{F}^{\{x_2\}}_{\{x_1,x_3\}} \cup \mathscr{F}^{\{x_2,x_3\}}_{\{x_1,x_4\}}.$$

So $|\mathscr{F}| \leqslant 3c_2(n-3) + c_2(n-4) \leqslant c_3(n-1)$ for $n \geqslant 9$.

(2) Suppose that $|\mathscr{F}_2| = 1$. Let $F_1 = \{x_1,x_2\}$, $F_2 = \{x_1,x_3,x_4\}$, $F_3 = \{x_2,x_3,x_5\}$ with $x_1 \neq x_5$. Then

$$\mathscr{F}_{\geqslant 5} \subset \mathscr{F}^{\{x_1\}}_{\geqslant 5,\{x_2,x_3\}} \cup \mathscr{F}^{\{x_1,x_3\}}_{\geqslant 5,\{x_2,x_4\}} \cup \mathscr{F}^{\{x_2,x_4\}}_{\geqslant 5,\{x_1,x_3\}} \cup \mathscr{F}^{\{x_2,x_3\}}_{\geqslant 5,\{x_1,x_5\}}.$$

By Theorem 4.1, we have $|\mathscr{F}_{\geqslant 5}| \leqslant \frac{1}{3}(\binom{n-3}{2} + 3\binom{n-4}{2})$. We also have $|\mathscr{F}_3| \leqslant 2n-5$ and $|\mathscr{F}_4| \leqslant \binom{n-1}{3} - \binom{n-5}{3} + 1 = 2n^2 - 16n + 35$. So, $|\mathscr{F}| \leqslant 2n^2 - 14n + 30 + \frac{1}{3}(n-4)(2n-9) \leqslant C_3(n-1)$ for $n \geqslant 9$.

(3) Suppose that $\mathscr{F}_2 = \emptyset$.

If $|F_i \cap F_j| \geqslant 2$ for all $F_i, F_j \in \mathscr{F}_3$, then $\mathscr{F}_3 = \binom{\{x_1,x_2,x_3,x_4\}}{3}$. By Corollary 5 and Theorem 4.1, we have $|\mathscr{F}_{\geqslant 5}| \leqslant \frac{4}{3}\binom{n-4}{2}$. Therefore, $|\mathscr{F}| \leqslant 2n^2 - 16n + 39 + \frac{2}{3}(n-4)(n-5) \leqslant c_3(n-1)$.

Now let $F_1 = \{x_1,x_2,x_3\}$, $F_2 = \{x_1,x_4,x_5\}$, $F_2 = \{x_2,x_4,x_6\}$, then

$$|\mathscr{F}_{\geqslant 5}| \leqslant |\mathscr{F}^{\{x_1,x_6\}}_{\geqslant 5,\{x_2,x_4\}}| + |\mathscr{F}^{\{x_1,x_4\}}_{\geqslant 5,\{x_2,x_5\}}| + |\mathscr{F}^{\{x_1,x_2\}}_{\geqslant 5,\{x_3,x_4\}}| + |\mathscr{F}^{\{x_1,x_2,x_4\}}_{\geqslant 5,\{x_3,x_5\}}|$$

$$+|\mathscr{F}^{\{x_3,x_4\}}_{\geqslant 5,\{x_1,x_2\}}| + |\mathscr{F}^{\{x_2,x_3,x_5\}}_{\geqslant 5,\{x_1,x_4\}}| + |\mathscr{F}^{\{x_2,x_4\}}_{\geqslant 5,\{x_1,x_6\}}|.$$

So, $|\mathscr{F}_{\geqslant 5}| \leqslant \frac{5}{3}\binom{n-4}{2} + \frac{2}{3}\binom{n-5}{2}$. We also have $|\mathscr{F}_3| \leqslant 3n-8$ and $|\mathscr{F}_4| \leqslant 2n^2 - 16n + 35$. Therefore, we still have $|\mathscr{F}| \leqslant c_3(n-1)$ for $n \geqslant 9$. □

Theorem 4.3. *Let* $K = \{k_1 < k_2 < \cdots < k_r\}$, $L = \{l_1 < l_2 < \cdots < l_s\}$, $k_1 > l_s$, $l = \max_{1 \leqslant i \leqslant s-1}\{l_1, l_{i+1} - l_i - 1\}$, $\mathscr{F} \subset 2^{[n]}$ *is* (K,L)*-intersecting. Then*

$$|\mathscr{F}| \leqslant \prod_{i=1}^{s} \frac{n-l_i}{k_1 - l_i}$$

holds for $n \geqslant 2k_r\binom{2k_r}{l+1}$.

Proof. Apply induction on $|L| = s$. The case $s = 0$ is trivial. Suppose that the theorem is true for $s-1$, we prove the case $|L| = s$.

(i) $l_1 = 0$. For $x \in [n]$ the family $\mathscr{F}(x) = \{F \in \mathscr{F}: x \in F\}$ is an $\{l_2,\ldots,l_s\}$-intersecting family. Thus, by induction

$$|\mathscr{F}(x)| \leqslant \prod_{2 \leqslant i \leqslant s} \frac{n-l_i}{k_1 - l_i}.$$

Since $\sum_{\substack{x\in F\\ F\in\mathscr{F}}} (x,F) = \sum_{x\in[n]} |\mathscr{F}(x)| = \sum_{F\in\mathscr{F}} \sum_{x\in F} 1 \geqslant k_1|\mathscr{F}|$ holds, we obtain

$$|\mathscr{F}| \leqslant \frac{n}{k_1} \prod_{2 \leqslant i \leqslant s} \frac{n-l_i}{k_1 - l_i} = \prod_{1 \leqslant i \leqslant s} \frac{n-l_i}{k_1 - l_i}.$$

(ii) $l_1 > 0$. If $|F_i \cap F_j| \neq l_1$ holds for all $F_i, F_j \in \mathscr{F}$ then $\mathscr{F}$ is actually an $\{l_2,\ldots,l_s\}$-intersecting family. Suppose that l_t is the minimum element of L such that $|F_i \cap F_j| = l_t$ for some F_i, $F_j \in \mathscr{F}$, $t > 1$. Then, by the same arguments below, we have either $|\mathscr{F}| \leqslant \prod_{t \leqslant i \leqslant s}[(n-l_i)/(k_1 - l_i)]$ or $|\mathscr{F}| \leqslant 2\binom{2k_r}{l_t+1}\prod_{t+1 \leqslant i \leqslant s}[(n-l_i)/(k_1-l_i)]$. Since $\binom{2k_r}{l_t+1} < \binom{2k_r}{l+1}^t$, we have a stronger result .

Suppose that $|F_1 \cap F_2| = l_1$ for some $F_1, F_2 \in \mathscr{F}$. Let $B = F_1 \cap F_2$. If $B \subset F$ for all $F \in \mathscr{F}$, then replacing $\mathscr{F}$ by $\{F - B: F \in \mathscr{F}\}$, k_i by $k_i - l_1$ and L by $\{0, l_2 - l_1, \ldots, l_s - l_1\}$ brings us back to case (a).

Suppose finally that $B \not\subseteq F_3$ holds for some $F_3 \in \mathscr{F}$. Then it can be seen that either $|F \cap (F_1 \cup F_2)| > l_1$ or $|F \cap (F_1 \cup F_3)| > l_1$. Let $\mathscr{F}(H) = \{F \in \mathscr{F}: H \subseteq F\}$. Then $\mathscr{F}(H)$ is an $\{l_2,\ldots,l_s\}$-intersecting family. We have

$$|\mathscr{F}| \leqslant \sum_{\substack{H \in \binom{F_1 \cup F_2}{l_1+1} \cup \binom{F_1 \cup F_3}{l_1+1} \\ |H| = l_1+1}} |\mathscr{F}(H)|.$$

Thus by induction the result follows. □

Corollary 5. Let $K = \{k_1 < \cdots < k_r\}$, $L = \{l, l+1, \ldots, l+s-1\}$, $\mathscr{F} \subset 2^{[n]}$ such that $|F_i| \in K$, $|F_i \cap F_j| \in L$ for $F_i, F_j \in \mathscr{F}$ $i \neq j$. Then

$$|\mathscr{F}| \leqslant \frac{\binom{n-l}{s}}{\binom{k_1-l}{s}}$$

holds for $n \geqslant 2k_r\binom{2k_r}{l+1}$, with equality iff $\mathscr{F} - F = \{F_i - F\colon F_i \in \mathscr{F}\}$ constitutes a Steiner system $S(n-l, k_1-l, s)$ for some l-element set $F \subseteq [n]$. □

Theorem 4.4. *Let $L = \{l, l+1, \ldots, l+s-1\}$ and $l < s$. $\mathscr{F} \subset 2^{[n]}$ such that $|F_i \cap F_j| \in L$ for all $F_i, F_j \in \mathscr{F}$, $i \neq j$. If $|F_i| \geqslant l+s$ for $F_i \in \mathscr{F}$, then $|\mathscr{F}| \leqslant \binom{n-l}{s} + 1$ for $n > n_0(s)$. Equality holds iff $\mathscr{F} = \{F \in \binom{[n]}{s+l}\colon A \subset F\} \cup ([n] \setminus B)$, where $|A| = l$, $\emptyset \neq B \in A$.*

Proof. Let us write $\mathscr{F} = \bigcup_{i \geqslant l} \mathscr{F}_i$, where $\mathscr{F}_i = \{F \in \mathscr{F}\colon |F_i| = i\}$. Denote $\mathscr{F}_{\geqslant k} = \bigcup_{i \geqslant k} \mathscr{F}_i$. If there exists an l-element set A such that $A \subset \cap \mathscr{F}$, then we are done by Theorem 4.1.

Now we suppose that $|\mathscr{F}| \geqslant \binom{n-l}{s}$ and $|\cap \mathscr{F}| < l$. Let $\mathscr{H}(F_0, F) = \{H \in \binom{F}{l+s}\colon |H \cap F_0| \geqslant l\}$ for $F_0, F \in \mathscr{F}$. It can be seen that

$$\mathscr{H}(F_0, F_i) \cap \mathscr{H}(F_0, F_j) = \emptyset \quad \text{for } F_i \neq F_j$$

and

$$\cup_{F \in \mathscr{F}_{\geqslant k}} \mathscr{H}(F_0, F) \subset \left\{H \in \binom{[n]}{l+s}\colon |H \cap F_0| \geqslant l\right\}.$$

Hence,

$$|\mathscr{F}_{\geqslant k}| \leqslant \frac{\binom{|F_0|}{l}}{\binom{k-l}{s}} \binom{n-l}{s}.$$

Since if we let $|F_0| = k$ and k satisfies $\binom{k}{l} < \binom{k-l}{s}$, then $|\mathscr{F}_{\geqslant k}| < \binom{n-l}{s}$. Let k_1 be the minimal integer satisfying $\binom{k}{l} < \binom{k-l}{s}$, then we have $|\mathscr{F}_{l+s \leqslant i \leqslant k_1}| > 0$.

Now let $\mathscr{G} = \bigcup_{l+s \leqslant i \leqslant k_2} \mathscr{F}_i$, where k_2 is some large integers depending on l, s only. By Corollary 5, we have $|\mathscr{G}| \leqslant \binom{n-l}{s}$ for sufficiently large n. Moreover, if $|\cap \mathscr{G}| < l$, then there exist at most $t (\leqslant l+s)$ sets $F_1, F_2, \ldots, F_t$ of $\mathscr{G}$ such that $|\bigcap_{1 \leqslant i \leqslant t} F_i| < l$ and any set F in $\mathscr{F}$ intersects $Q = \bigcup_{1 \leqslant i \leqslant t} F_i$ at least $l+1$ elements, so by the method in the proof of Theorems 4.3 and 4.1, we will have $|\mathscr{F}| < \binom{n-l}{s}$. This contradicts our assumption. So, we may assume that $|\cap \mathscr{G}| \geqslant l$. Let A be an l-element set of $\cap \mathscr{G}$ and $F' \in \mathscr{F}_{\geqslant k_2+1}$ be of minimal size not containing A, denote $t = |F' \cap A| < l$. Then

$$\bigcup_{A \subset F \in \mathscr{F}_{\geqslant l+s}} \mathscr{H}_A(F', F) \subset \left\{H \in \binom{[n]}{l+s}\colon A \subset H, |H \cap F'| \geqslant t+1\right\},$$

where $\mathscr{H}_A(F', F) = \{H \in \binom{F}{l+s}\colon A \subset H, |H \cap F'| \geqslant t+1\}$. Note that $\mathscr{H}_A(F', F) \geqslant 1$ for $A \subset F$, so

$$|\{F \in \mathscr{F}_{\geqslant s+1}\colon A \subset F\}| \leqslant \left|\binom{[n] \setminus A}{s}\right| - \left|\left\{H \in \binom{[n]}{l+s}\colon A \subset H, |H \cap F'| \leqslant t\right\}\right|.$$

Since, $|F' \cap A| = t$ ($\leqslant l-1$), then

$$\binom{n-|F'|-l+t}{s} \leqslant |\{F \in \mathscr{F}_{\geqslant l+s}\colon A \not\subseteq F\}| \leqslant |\mathscr{F}_{\geqslant |F'|}| \leqslant \frac{\binom{k_1}{l}}{\binom{|F'|-l}{s}}\binom{n-l}{s}.$$

But if we let k_2 be large enough, then the above inequality implies $|F'| > (n+l+s)/2$ for sufficiently large n. Therefore, $|\{F \in \mathscr{F}_{\geqslant l+s}\colon A \not\subseteq F\}| \leqslant 1$. Hence, $|\mathscr{F}_{\geqslant l+s}| \leqslant \binom{n-l}{s} + 1$ for sufficiently large n. The extremal set must be as described in the Theorem. □

Note that in the proof of Theorem 4.4, we only use the assumption $l < s$ to prove that there exist some set of relatively small size as compare to $l+s$. So, combining Theorem 4.1 with the proof of Theorem 4.4, we can actually prove the following:

Theorem 4.5. *Let $L = \{l, l+1, \ldots, l+s-1\}$. $\mathscr{F} \subset \binom{[n]}{\geqslant k}$ is an L-intersecting family such that $\mathscr{F}_k \neq \emptyset$ and $k \geqslant l+s$, then $|\mathscr{F}| \leqslant \binom{n-l}{s}/\binom{k-l}{s} + 1$ for $n > n_0(k)$. Equality may occur only if $\mathscr{F}_k \backslash A = \{F \backslash A\colon F \in \mathscr{F}_k\}$ is a Steiner system $S(n-l, k-l, s)$ for some l-element A of $[n]$.*

Using the method as in [7], we can prove that there exists a computable constant c independent of l, s, k such that $n_0(k) \leqslant c \max(l(k-l)^2, (k-l)^2 \binom{k}{l}^{1/s})$.

Theorem 4.6. *Let $l < s$, $L = \{l, l+1, \ldots, l+s-1\}$ and $\mathscr{F} \subset 2^{[n]}$ be an L-intersecting family. Then $|\mathscr{F}| \leqslant \sum_{i=0}^{s} \binom{n-l}{i}$ for $n > n_0(s)$. Equality holds iff $\mathscr{F}$ is $\mathscr{F}(A) = \{F \in \binom{[n]}{l \leqslant i \leqslant l+s}\colon A \subset F\}$ or $\mathscr{F}(A,x) = \{F \in \binom{[n]}{l+1 \leqslant i \leqslant l+s}\colon A \subset F\} \cup ([n] \backslash \{x\})$ for some l-element set A and $x \in A$.*

Proof. By the Erdös–Ko–Rado Theorem, we have $|\mathscr{F}_i| \leqslant \binom{n-l}{i-l}$ for $n \geqslant (i-l+1)(l+1)$. So, $|\mathscr{F}_{\leqslant l+s-1}| \leqslant \sum_{i=0}^{s-1} \binom{n-l}{i}$, with equality iff $\mathscr{F}_{\leqslant l+s-1} = \{F \in \binom{[n]}{\leqslant l+s-1}\colon A \subset F\}$ for some l-element set A. Now Theorem 4.6 follows from Theorem 4.4. □

Corollary 6. *Let $l, s, L, \mathscr{F}$ be the same as in Theorem 4.6. If $|F| \geqslant l+r$ for any $F \in \mathscr{F}$ and $r > 0$, then $|\mathscr{F}| \leqslant \sum_{i=r}^{s} \binom{n-l}{i} + 1$ for $n > n_0(s)$. Equality holds iff $r \leqslant s$ and $\mathscr{F}$ is $\mathscr{F}(A,B) = \{F \in \binom{[n]}{l+r \leqslant i \leqslant l+s}\colon A \subset F\} \cup ([n] \backslash \{B\})$ for some l-element set A and its subset $B \subset A$ such that $0 < |B| \leqslant r$.*

Actually, by more careful computation, we can prove that $n_0(s) \leqslant 16s^5/ln^2 s$ for Theorem 4.6 and Corollary 6.

5. Some open problems

First, it would be interesting to find the exact minimal value of $n_0(l, k_r, s)$ such that Corollary 5 holds for all $n \geqslant n_0(l, k_r, s)$. Considering the symmetric designs on $[n]$ and projective planes of order k, it leads us to make the following conjecture.

Conjecture C. If $l > 0$, for given k_r, s, then there exist positive constants c_1, c_2 not depending on l, s such that $c_1 k_r^2 \leqslant n_0(l, k_r, s) \leqslant c_2 k_r^2$.

Actually, by using the method as in [7] in the proof of Theorem 4.3 we can show that $n_0(l, k_r, s) \leqslant (k_r - l + 1)^2 \max(l, \binom{k_r}{l}^{1/s})$.

The last two theorems in Section 4 lead us to make the following stronger conjecture as compared to Conjecture B.

Conjecture D. Let $L = \{l_1 < l_2 < \cdots < l_s\}$, $s > 1$. $\mathscr{F}$ is an L-intersecting family such that $|F| > l_s$ for all $F \in \mathscr{F}$, then

$$|\mathscr{F}| \leqslant \prod_{i=1}^{s} \frac{n - l_i}{l_s - l_i + 1} + 1 \quad \text{for } n > n_0(l_s).$$

Moreover, equality may occur only if $L = \{l, l+1, \ldots, l+s-1\}$.

Conjecture E. Let $L = \{l_1 < l_2 < \cdots < l_s\}$, $s > 1$. $\mathscr{F}$ is an L-intersecting family, then

$$|\mathscr{F}| \leqslant n - l_1 + 1 + \sum_{i=2}^{s} \prod_{j \leqslant i} \frac{n - l_i}{l_{i+1} - l_j}, \text{ where } l_{s+1} = l_s + 1 \quad \text{for } n > n_0(l_s).$$

Equality may occur only if $L = \{l, l+1, \ldots, l+s-1\}$.

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