Limit Theorems for Decomposable Multi-Dimensional Galton-Watson Processes

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1. INTRODUCTION

In this paper we consider a decomposable $k$-dimensional Galton-Watson process. Specifically, we consider a temporally homogeneous, $k$-vector-valued Markov chain, $\{Z_n; n = 0, 1,\ldots\}$, with the following properties.

(i) For each $i$, $1 \leq i \leq k$, let $e_i = (\delta_{i,1}, \ldots, \delta_{i,k})$. Then $Z_0$ is taken to be one of the vectors $e_i$.

(ii) Let $P$ denote the probability measure of the process; let $Z_n = (Z_n^1, \ldots, Z_n^k)$, $n \geq 0$; and let

\[ F_{i,j}(x) = P\{Z_{i,j}^1 \leq x \mid Z_0 = e_i\}, \quad 1 \leq i, j \leq k. \]

Then $Z_n^j$, $1 \leq j \leq k$, $n \geq 0$, takes only non-negative integer values and for each $n, n \geq 0$,

\[ P\{Z_{n+1} \leq x \mid Z_0, \ldots, Z_n\} = F_{1,j} \ast F_{2,j} \ast \cdots \ast F_{k,j}(x), \]

where the right-hand side is the convolution of $Z_n^i$ times $F_{i,j}$ for $1 \leq i \leq k$.

(iii) Let $E$ denote the expectation functional; let $m_{i,j} = E\{Z_{1}^i \mid Z_0 = e_i\}$, $1 \leq i, j \leq k$; and let $M$ denote the matrix $(m_{i,j})$. Then

\[ m_{i,j} = \int_0^\infty x dF_{i,j}(x) < \infty \quad \text{for all} \quad 1 \leq i, j \leq k. \quad (1.1) \]

(iv) Let $\rho$ denote the largest positive eigenvalue of $M$. Then

\[ \rho > 1. \quad (1.2) \]

We will prove limit theorems for this process that differ in several aspects...
from those obtained in [1] and in Part 3 of [2]. For instance in [1] and [2] we showed that if \( M \) is irreducible, i.e., if there exist integers \( t(i,j) \) such that 
\[
(M^t(i,j))_{i,j} > 0, \quad 1 \leq i, j \leq k,
\]
then for some integer \( d, \rho^{-nd-t}Z_{nd+t} \) converges as \( n \to \infty \) with probability one to a random vector with a fixed direction depending on \( t; 0 \leq t \leq d - 1 \). Here we will encounter cases when the \( \rho^{-n} \)'s are "correct" normalizing constants but for a subset of components \( j \) \( \rho^{-nd-t}Z_{nd+t}^{(j)} \) converges to zero with probability one, for \( j \) in an other subset \( \rho^{-nd-t}Z_{nd+t}^{(j)} \) converges to a random vector with fixed direction, and for \( j \) in a third subset it converges to a random vector whose direction is not fixed.

If the limit of \( \rho^{-n}Z_n \) has the direction of the fixed vector \( v \), we can write the limit as \( w \cdot v \) for a one-dimensional random variable \( w \). In [1] and [2] we showed that for an irreducible matrix \( M \) the distribution of this random variable \( w \) was either concentrated at one point or had a jump at the origin and a continuous density function away from the origin. There are no other possibilities as long as \( M \) is irreducible. However, if \( M \) does not satisfy this condition, there is still a third possibility, even when the \( \rho^{-n}Z_n \)'s converge to a random vector with fixed direction. Specifically, if
\[
\lim_{n \to \infty} \rho^{-n}Z_n = w \cdot v \quad \text{w.p.1},
\]
then there are cases where the distribution of \( w \) away from the origin has a discrete part not concentrated at one point. The distribution of \( w \) may even be entirely discrete.

There are even more significant differences between the limiting behavior of indecomposable and decomposable processes than those suggested above. When \( M \) is not irreducible, \( \rho^{-n}, n \geq 0 \), need not be correct normalizing constants. In fact, as we will show, different subsets of the components of the \( Z \)-process have, in general, different normalizing constants. These constants always have the form, \( n^{-\gamma}\lambda^{-n} \), where \( \gamma \) is a non-negative integer and \( \lambda \) is a positive real number greater than or equal to one. Moreover, if \( \gamma > 0, \lambda > 1 \), and if \( n^{-\gamma}\lambda^n, n > 0 \), are correct normalizing constants for a particular subset of the components of the \( Z \)-process, then the corresponding limit vector may be zero with probability one, it may have a fixed direction or a random direction, it is even possible that there is no limiting vector at all along any subsequence of the form \( nd + t \), where \( d \) and \( t \) are finite integers with \( t \geq 0 \) and \( d \geq 1 \). Thus we see that when we drop the assumption that \( M \) is irreducible, we obtain a richer theory of the limiting properties of multivariate branching processes.

To simplify the reading of the paper we will state and discuss our main results in Section 2 and develop the proofs of the theorems in Section 3.

\(^2\) w.p.1 stands for "with probability one."
For the convenience of the reader and also for easy reference we will state the main result of [1] below. In the statement of this theorem \(u\) and \(v\) are respectively positive right and left eigenvectors of \(M\) corresponding to \(\rho\) chosen so that

\[
u \cdot v' = \sum_{i=1}^{k} u_i v_i = 1.
\]

Also

\[
q_i = P\{Z_n = 0 \text{ eventually } | \ Z_0 = e_i\}, \quad 1 \leq i \leq k. \quad (1.3)
\]

We shall use the fact that the \(q_i\)'s satisfy the equations,

\[
q = f(q) \quad \text{or} \quad q_i = f_i(q_1, \ldots, q_k), \quad 1 \leq i \leq k, \quad (1.4)
\]

where

\[
f^i(s) = \sum_{r_1 \geq 0, \ldots, r_k \geq 0} P\{Z_1^1 = r_1, \ldots, Z_1^k = r_k | Z_0 = e_i\} s_1^r \ldots s_k^r.
\]

(See Theorem II.7.1 of [3].) In most cases which we shall encounter, the \(q_i\) satisfy in addition

\[
q_i < 1. \quad (1.5)
\]

**Theorem 1.1.** Let \(Z_n, n \geq 0, \) be a \(k\)-dimensional branching process that satisfies Assumption (i)-(iv) above and in addition the condition

\[
(M^t)_{i,j} > 0 \quad \text{for a sufficiently large } t \text{ and all } 1 \leq i, j \leq k. \quad (1.6)
\]

Then there exists a random vector \(W\) and a one-dimensional random variable \(\omega\), such that

\[
\lim_{n \to \infty} \frac{Z_n}{\rho^n} = W \quad \text{w.p.1} \quad (1.7)
\]

and

\[
W = \omega \cdot v \quad \text{w.p.l.} \quad (1.8)
\]

If

\[
E[Z_1^i \log Z_1^j | Z_0 = e_i] < \infty \quad \text{for all} \quad 1 \leq i, j \leq k, \quad (1.9)
\]

then

\[
E[\omega | Z_0 = e_i] = u_i, \quad 1 \leq i \leq k, \quad (1.10)
\]

and if (1.9) fails to hold for some pair \(i, j\) then

\[
\omega = 0 \quad \text{w.p.l.} \quad (1.11)
\]

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Finally, if \( Z_0 = e_{i_0} \), \( 1 \leq i_0 \leq k \), if (1.9) holds, and if there is at least one \( j_0 \) such that

\[
\sum_{m=1}^{k} Z_1^m u_m \text{ can take at least two values with positive probability,}
\]

then the distribution of \( w \) has a jump of magnitude \( q_{i_0} < 1 \) at the origin and a continuous density function on the set of positive real numbers. If (1.12) fails to hold for all \( j_0 \), \( 1 \leq j_0 \leq k \), then the distribution of \( w \) is concentrated at one point.

We shall have occasion to use a generalization of Theorem 1.1 where (1.6) is replaced by

\[
(M^{(i,j)})_{t+j} > 0 \quad \text{for some set of integers} \quad t(i,j) \geq 1, \quad 1 \leq i, j \leq k.
\]

In this case there exists an integer \( d \), positive vectors \( v(1), \ldots, v(d) \), and a random variable \( w \), such that

\[
\lim_{n \to \infty} \frac{Z_{n+t+p}}{\rho^{n+t+p}} = w \cdot v(q(i,p)) \quad \text{w.p.1},
\]

where \( q(\cdot, \cdot) \) can be given explicitly. The properties of \( w \) are entirely analogous to those listed in Theorem 1.1. (See Section 3 of [2] for a full statement and proof of these assertions.)

**Remark.** Some of our results resemble a number of results in the literature for continuous time processes. For instance Lemma 3.3 with its proof can be viewed as a result about branching processes with immigration, which where treated in [4] by B. A. Sevastyanov. Also, an analogue to formula (2.37) below was announced by V. P. Chistyakov in [5]. However, Chistyakov requires existence of second moments and only obtains convergence in the mean for his random variables, rather than convergence with probability one. Cases which correspond to \( \rho = 1 \) and \( \rho < 1 \) were treated by A. A. Savin and V. P. Chistyakov in [6] and by V. P. Chistyakov in [7].

2. PRELIMINARY REMARKS AND STATEMENT OF RESULTS

We will begin by showing how the components of the \( Z \)-process in a natural way can be grouped into equivalence classes, \( \{C_a\}_{1 \leq a \leq 1} \), and how these
equivalence classes can be partially ordered in a way that allows us to rearrange the rows and columns of $M$ so that $M$ takes the form

$$
\begin{pmatrix}
M(1) & \cdots & 0 \\
M(2, 1) & M(2) & \cdots \\
\vdots & \vdots & \ddots \\
M(m, 1) & M(m, 2) & \cdots & M(l)
\end{pmatrix},
$$

(2.1)

where $M(a, b) = (m_{i,j})_{i \in C_a, j \in C_b}$ and $M(a) = M(a, a)$. We shall show that if $Z_0 = e_i$ for some $i \in C_a$ and if $M(a) \neq 0$, then the subprocess

$$Z_n(a) = \{Z_n^i; i \in C_a\}, \quad n \geq 0,$$

(2.2)

is an indecomposable branching process to which our results in [1] and in Section 3 of [2] apply. This takes care of the limit behavior of $Z_n(a)$ when $Z_0 = e_i$, $i \in C_a$. We shall then state three general theorems for the case when there are only two blocks along the diagonal in (2.1). These theorems allow us to determine the behavior of the subprocesses $(Z_n(a), Z_n(b))$, $n \geq 0$, $b > a$, whenever $Z_0 = e_i$ for some $i \in C_b$. Moreover these theorems together with the lemmas used to prove them in Section 3 of the paper provide the necessary arguments for an inductive proof of the corresponding limit theorems for the case when there are more than two blocks along the diagonal in (2.1). We conclude the discussion in this section by stating several of our results for the general case.

To define the equivalence classes $C_a$ we proceed as follows. We say that the $i$th and $j$th component of the $Z$-process communicate ($i \sim j$) if and only if there exist non-negative integers $n_1$ and $n_2$ such that $(M^{n_1})_{i,j} > 0$ and $(M^{n_2})_{i,j} > 0$. In particular since we allow $n_1 = n_2 = 0$, the $i$th component of $Z$, $1 \leq i \leq k$, always communicates with itself. Moreover, if $i \neq j$, then $i \sim j$ if and only if there exist positive integers $n_1$ and $n_2$ such that

$$P\{Z_{n_1} > 0 \mid Z_0 = e_i\} > 0 \text{ and } P\{Z_{n_2} > 0 \mid Z_0 = e_j\} > 0.$$

Clearly, $\sim$ is an equivalence relation according to which we can divide the components of $Z$ into equivalence classes, $\{C_a\}_{1 \leq a \leq m}$.

We shall say that $a$ follows $b$ if there exists an $i \in C_b$ and a $j \in C_a$ such that for some non-negative integer $n$, $(M^n)_{i,j} > 0$. The relation follows induces in a natural way a partial ordering of the $C_a$'s. Since $a$ follows $b$ evidently implies that for each $i \in C_b$ and $j \in C_a$ we can find an integer $n \geq 0$ such that $(M^n)_{i,j} > 0$, we can use this partial ordering to renumber the components of $Z$ such that the relations, $i \geq j$, $i \in C_b$ and $j \in C_a$ imply that either $a$ follows $b$ or $a$ is not comparable to $b$. After this rearrangement $M$ takes the
form (2.1), and we assume from now on that $M$ has been brought into this form.

An immediate consequence of (2.1) is

$$(M^n)_{i,i} = (M^n(a))_{i,i}, \quad i, j \in C_\alpha. \quad (2.3)$$

From this and the definition of equivalence classes it follows that either

$$C_\alpha = \{i\} \text{ (one element only)} \quad \text{and} \quad m_{i,i} = 0, \quad (2.4)$$

or, for each pair $i, j \in C_\alpha$ there is an integer $t = t(i, j)$ such that

$$(M^t(a))_{i,i} > 0. \quad (2.5)$$

If (2.5) holds, $M(a)$ has a positive eigenvalue $\rho_\alpha$ such that no eigenvalue of $M(a)$ exceeds $\rho_\alpha$ in absolute value whereas (2.4) implies that

$$(M^n(a))_{i,i} = 0 \quad \text{for all} \quad n \geq 0. \quad (2.6)$$

In the last case we put $\rho_\alpha = 0$.

In general there may be several eigenvalues of $M(a)$ with absolute value $\rho_\alpha$. If $M(a)$ satisfies (2.5) $\rho_\alpha$ exceeds all other eigenvalues of $M(a)$ in absolute value only if $M(a)$ is aperiodic in the sense of Section 3 of [2]. Section 3 of [2] also shows that "$M(a)$ is aperiodic" is equivalent to "$M(a)$ is positively regular," i.e., the integer $t$ in (2.5) is independent of $i, j$.

From the above remarks one immediately sees that when $\pi_i = e_i$ with $i \in C_\alpha$, the subprocess $Z_n(a), n \geq 0$, is a branching process with expectation matrix $M(a)$, whose asymptotic behavior is easily determined. In fact, if (2.4) holds then $Z_n(a) = 0, n \geq 1$, and if (2.5) holds Section 3 of [2] applies to $Z_n(a)$. In particular, if $\rho_\alpha < 1$ and if $Z(a)$ is not singular, then

$$\lim_{n \to \infty} Z_n(a) = 0 \quad \text{w.p.1},$$

and if $\rho_\alpha > 1$, the behavior of $\rho_\alpha^n Z_n(a)$ is described in Theorem 1.1 above in the aperiodic case and in Theorem 3.1 of [2] in the periodic case.

For the study of $Z_n(a)$ when $Z_0 = e_i, i \notin C_\alpha$, we introduce the random variables $T_m^{p,a}$, which represent the number of descendants of type $q$ in the $m$th generation of the $Z_{m-1}^p$ particles of type $p$ in the $(m - 1)$th generation. We will number these particles $r = 1, 2, \cdots, T_m^{p,a}$ and denote the descendants in class $C_\alpha$ in the $n$th generation of the $r$th of these particles by the vector, $U_r(m, n, a, p)$. If a particle is of a type in $C_\alpha$ at time $n$, there must have been a first generation in which the ancestor of the particle had a type belonging to $C_\alpha$. Let the first generation be the $m$th one (we allow $m = n$, in which
case we consider the particle as an "ancestor" of itself). When we separate the particles at time $n$ according to the value of $m$, we find that

$$Z_n(a) = \sum_{m=1}^{n} \sum_{q \in C_a} \sum_{a < b \leq l} \sum_{p \in C_0} \sum_{r=1}^{T_{m}^{p,q}} U_r^{q}(m, n, a, p). \quad (2.7)$$

From (2.7) we see that the study of the behavior of $Z_n(a)$ reduces to a study of sums of random vectors of the form

$$\sum_{m=1}^{n} \sum_{r=1}^{T_{m}^{p,q}} U_r^{q}(m, n, a, p), \quad n \geq 0, \quad p \in C_b, \quad q \in C_a, \quad a < b \leq l. \quad (2.8)$$

For given $Z_{m-1}^{p}$, the distribution of $T_{m}^{p,q}$, $q \in C_a$, depends only on the distribution of $Z_1(a)$, given $Z_0 = e_p$. Therefore, all essential features of the study of (2.7) and (2.8) already occur in the study of subprocesses $(Z_n(a), Z_n(b))$ with $Z_0 = e_i$, $i \in C_0$. The corresponding expectation matrix is

$$M = \begin{pmatrix}
M(a) & 0 \\
M(b, a) & M(b)
\end{pmatrix},$$

and without loss of generality we can take $a = 1$, $b = 2$. We shall, therefore, be content to state and prove precise theorems only when there are two blocks along the diagonal in (2.1). However, the proofs of the theorems are broken down into lemmas that are stated in such a way as to provide the reader with the necessary tools for an inductive argument in the general case.

We thus turn our attention to processes with expectation matrix,

$$M = \begin{pmatrix}
M(1) & 0 \\
M(2, 1) & M(2)
\end{pmatrix}. \quad (2.9)$$

Clearly, if $Z_0 = e_i$ with $i \in C_1$, then $Z_n(2) = 0$, $n \geq 0$, and one only has to study $Z(1)$. Similarly, if $Z_0 = e_i$, $i \in C_2$, but $M(2, 1) = 0$, then $Z_n(1) = 0$. These situations have already been considered in the beginning of this section, so we may and shall assume that $Z_0 = e_i$, $i \in C_2$, and $M(2, 1) \neq 0$. If $p_2 \leq 1$ and $Z(2)$ is not singular, then we know (Remark 3.1 of [2]) that $T_{m}^{p,q}$ ($p \in C_2$ and $q \in C_1$) will be zero eventually w.p.l. because this is so for $Z_{m-1}(2)$. If in addition $p_1 \leq 1$ and $Z(1)$ is not singular, then each $U_r^{q}(m, n, 1, p)$ will be zero eventually since $U_r^{q}(m, n, 1, p)$ has the distribution of $Z_{n-m}(1)$ given $Z_0 = e_q$. Thus, by (2.7)

$$\lim_{n \to \infty} Z_n = 0 \quad \text{w.p.1}$$

whenever $\rho_0 \leq 1$ and $Z(a)$ is nonsingular for $a = 1, 2$. This is the reason
why we shall only be interested in the case where \( \rho_a > 1 \) for some \( a \) (even though there are interesting aspects to the case where \( \rho_2 = 1 \), \( Z(2) \) is singular, and \( \rho_1 \leq 1 \)). Since the spectrum of \( M \) is precisely the union of the spectra of the \( M(a) \), \( \rho_a > 1 \) for some \( a \) is equivalent to \( \rho > 1 \).

In the discussion that follows we disregard the cases where \( M(1) = 0 \) or \( M(2) = 0 \). If \( M(1) = 0 \), then \( C_1 \) consists of one element only, say \( q \), and \( U_r^q(m, n, 1, p) = 0 \) for \( m < n \) (see (2.6)). Thus by (2.7)

\[
Z_n(1) = \sum_{p \in C_a} T_n^{p,a},
\]

and the asymptotic behavior of the \( T_n^{p,a} \) is covered by Remark 3.1 below. If \( M(2) = 0 \), then \( T_n^{p,q} = 0 \) for \( m \geq 2 \), \( p \in C_2 \), \( q \in C_1 \), and one is essentially back in the one-block case again. From now on we therefore assume

\[
M(1) \neq 0, \quad M(2) \neq 0, \quad \text{and} \quad M(2, 1) \neq 0. \tag{2.10}
\]

Then both \( M(1) \) and \( M(2) \) satisfy (2.5) and by Section 3 of [2] there exist unique (up to positive multiplicative constants) vectors \( v(a) \) and \( u(a) \) satisfying

\[
v(a) M(a) = \rho_a v(a), \quad v_i(a) > 0, \quad i \in C_a,
M(a) u'(a) = \rho_a u'(a), \quad u_i(a) > 0, \quad i \in C_a,
\sum_{i \in C_a} u_i(a) v_i(a) = 1. \tag{2.11}
\]

To study branching processes with moment matrix \( M \) satisfying (2.9) and (2.10) we begin with the simplest case where \( \rho \) is simple and exceeds all other eigenvalues of \( M \) in absolute value. An interesting aspect of our results for this case is that they hold irrespective of whether or not the block, \( M(a) \), with \( \rho_a < \rho \) is periodic in the sense of Section 3 of [2]. Of course, when \( \rho \) is simple, \( M \) has unique (up to positive multiplicative constants) non-negative left and right eigenvectors \( v \) and \( u \) corresponding to \( \rho \) such that \( vu' = 1 \).

One easily sees that if \( \rho = \rho_1 > \rho_2 \), then

\[
v = (v(1), 0) \quad \text{and} \quad u' = (u'(1), (\rho I - M(2))^{-1} M(2, 1) u'(1)), \tag{2.12}
\]

whereas for \( \rho_1 < \rho_2 = \rho \),

\[
v = (v(2) M(2, 1) (\rho I - M(1))^{-1}, v(2)) \quad \text{and} \quad u' = (0, u'(2)). \tag{2.13}
\]

(The 0 in \( v \) in (2.12) and in \( u \) in (2.13) stands for a zero vector of the same
limit theorems

dimension as \( M(2) \) and \( M(1) \), respectively. \( u(1) \) and \( v(1) \) are the same vectors as defined in (2.11). With these notations we have

**Theorem 2.1.** If \( \rho \) is simple and larger in absolute value than all the other eigenvalues of \( M \), if (1.2) and (2.10) hold, and if \( Z_0 = e_i \) for some \( i \in C_2 \), then there exists a random vector \( W \) and a one-dimensional random variable \( w \), such that

\[
\lim_{n \to \infty} \frac{Z_n}{\rho^n} = W \quad \text{w.p.1} \tag{2.14}
\]

and

\[
W = w \cdot v \quad \text{w.p.l.} \tag{2.15}
\]

Also one has either

\[
E\{w \mid Z_0 = e_i\} = u_i > 0 \tag{2.16}
\]

or

\[
w = 0 \quad \text{w.p.l.} \tag{2.17}
\]

Moreover (2.16) holds if and only if

\[
E(Z_1^a \log Z_1^a \mid Z_0 = e_p) < \infty \tag{2.18}
\]

for all pairs \( p, q \in C_{a(p)} \), where \( a(p) \) denotes the unique index with \( \rho_a = \rho \) (\( a = 1 \) or \( 2 \)). Finally if (2.18) holds and if there is at least one \( j_0 \in C_{a(p)} \) such that

\[
Z_1(a(p)) u'(a(p)) \text{ can take at least two values with positive probability, given } Z_0 = e_{j_0}, \tag{2.19}
\]

then the distribution of \( w \) has a jump of magnitude \( q_i \) at the origin and a continuous density function on the set of positive real numbers.

The reader should note that if \( \rho = \rho_2 \) in Theorem 2.1 and if (2.19) is not satisfied, then it follows from Theorem 1.1 and (3.36) below that the distribution of \( w \) is concentrated on one point only. More interesting is the case where \( \rho = \rho_1 > \rho_2 \) and (2.19) fails to hold. By (2.7), (2.8), and (2.15)

\[
w \cdot v(1) = \lim_{n \to \infty} \frac{1}{\rho^n} \sum_{m=1}^{n} \sum_{p \in C_2} \sum_{q \in C_1} \sum_{r=1}^{T_{mn}^p} U_r^q(m, n, 1, p),
\]

and if

\[
T_{mn}^{p, q} = 0 \tag{2.20}
\]
for all \( p, q \) and \( m > M \) for \( M \) sufficiently large, this limit reduces to

\[
\mathbf{w} \cdot \mathbf{v}(1) = \sum_{m=1}^{M} \sum_{p \in C_{q}} \sum_{q \in C_{l}} \lim_{n \to \infty} \rho^{-m} U_{r,q}(m, n, 1, p).
\]

If (2.19) is not satisfied, it follows from Theorem 1.1 applied to the process \( U_{r,q}(m, n, 1, p) \), \( n \geq m \), that

\[
\lim_{n \to \infty} \rho^{m-n} U_{r,q}(m, n, 1, p) = E \{ \mathbf{w} \mid Z_{0} = e_{q} \} \cdot \mathbf{v}(1) = u_{q}(1) \cdot \mathbf{v}(1) \quad \text{w.p.1.}
\]

Hence

\[
\mathbf{w} = \sum_{m=1}^{M} \sum_{p \in C_{q}} \sum_{q \in C_{l}} \rho^{-m} T_{m}^{p,q} u_{q}(1) \cdot \mathbf{v}(1).
\]

We conclude immediately that if (2.20) can occur with positive probability, then the distribution of \( \mathbf{w} \) has a discrete component even on \((0, \infty)\). In particular this will be the case if

\[
\tau_{i} = P\{ Z_{n}(2) = 0 \text{ eventually} \mid Z_{0} = e_{i} \} > 0
\]

since \( Z_{M}(2) = 0 \) implies \( T_{m}^{p,q} = 0 \) for all \( p, q \) and \( m > M \). If \( \rho_{2} \leq 1 \) and if \( Z(2) \) is nonsingular, then we even have \( \tau_{i} = 1 \) (Theorem II.7.1 of [3]) so that (2.20) holds for some \( M \) w.p.1 and the distribution of \( \mathbf{w} \) is discrete in these circumstances.

Next we will consider the case when only one of the two blocks has \( \rho \) as eigenvalue but this block is periodic. If \( \rho \) is an eigenvalue of \( M(2) \) and \( Z(2) \) has period \( d \) in the sense of Section 3 of [2], then by Theorem 3.1 of [2] for a suitable random variable \( \mathbf{w} \) and certain vectors \( \vec{v}(2, t), 0 < t < d - 1 \),

\[
\lim_{n \to \infty} \frac{Z_{n+d+t}(2)}{\rho^{n+d+t}} = \mathbf{w} \cdot \vec{v}(2, t) \quad \text{w.p.1} \quad (2.21)
\]

if \( Z_{0} = e_{i} \) for some \( i \in C_{2} \). The properties of \( \mathbf{w} \) are quite similar to those of \( \mathbf{w} \) in Theorem 2.1. It also follows from Lemma 3.4, just as in the proofs of Theorems 2.1 and 2.2 below, that for \( Z_{0} = e_{i}, i \in C_{2} \),

\[
\lim_{n \to \infty} \rho^{-nd-t} Z_{n+d+t}(1) = \mathbf{w} \cdot \vec{v}(1, t) \quad \text{w.p.1}
\]

for the same \( \mathbf{w} \) as in (2.21).

The vectors \( \vec{v} \) above have a rather complicated expression and we shall
not give the details here. Instead we turn to the case where \( \rho = \rho_1 > \rho_2 \).

When \( i \in C_0 \):

\[
\sum_{n=0}^{\infty} \rho^{-n} E[|Z_n(2)| \mid Z_0 = e_i] = \sum_{n=0}^{\infty} 0 \left( \frac{\rho_2}{\rho} \right)^n < \infty.
\]

Hence

\[
\lim_{n \to \infty} \frac{Z_n(2)}{\rho^n} = 0 \quad \text{w.p.1.} \quad (2.22)
\]

Thus in this case we need only study the behavior of the appropriate subsequences of the random vectors, \( \rho^{-n}Z_n(1) \), \( n \geq 0 \), when \( Z_0 = e_i \) for some \( i \in C_0 \). Suppose that \( M(1) \) has period \( d \) and let \( D_a \), \( a = 1, \ldots, d \) denote the corresponding subclasses of components of \( Z(1) \) as in Section 3 of [2] with \( M \) replaced by \( M(1) \). Let \( v(1, a) \) and \( u(1, a) \) be the corresponding vectors for \( M(1) \) defined as in (3.9) and (3.10) of [2] for \( M \). We assume that the \( D_a \)'s have been ordered as in (3.1) of [3] and take \( D_{a_1} = D_{a_2} \), \( v(1, a_1) = v(1, a_2) \), \( u(1, a_1) = u(1, a_2) \) if \( a_1 = a_2 \) (mod \( d \)). It will not matter whether or not \( M(2) \) is aperiodic. Our results for this situation are summarized in

**Theorem 2.2.** If \( \rho = \rho_1 > \rho_2 \), if (1.2) and (2.10) hold, and if \( M(1) \) has period \( d \), then for all \( i \in C_0 \), \( 1 \leq b \leq d \), and \( 0 \leq t < d \), there exist random variables, \( x_i(b, t) \), such that for \( Z_0 = e_i \)

\[
\lim_{n \to \infty} \frac{Z_{nd+t}(1, b)}{\rho^{nd+t}} = x_i(b, t) \cdot v(1, b) \quad \text{w.p.1,} \quad (2.23)
\]

where \( Z_m(1, b) \) denotes the vector, \( \{ Z_{m, j} : j \in D_b \} \). If

\[
E\{z_1^p \log z_1^q \mid Z_0 = e_2\} < \infty \quad \text{for all} \quad p, q \in C_1, \quad (2.24)
\]

then

\[
E\{x_i(b, t)\} = \sum_{m=1}^{\infty} \sum_{p \in C_1} \sum_{q \in D_{b-t+m}} \rho^{-m}(M(2))_{i, p}^{m-1} m_{p, q} u_q(1, b - t + m),
\]

whereas if (2.24) fails to hold, then

\[
x_i(b, t) = 0 \quad \text{w.p.1.} \quad (2.26)
\]

\footnote{For a \( k \)-vector \( Y \) we define \(|Y| = (\sum_{i=1}^{k} |Y_i|)^{1/2} \).}
If (2.24) holds, and if there exists a pair \((j, f)\) such that \(j \in D\), and such that
\[ Z_1(1, f + 1) u'(1, f + 1) \]
can take at least two values with positive probability
when \(Z_0 = e_j\),
\[ (2.27) \]
then for each fixed \(i \in C_2\) and \(t\) the joint distribution of \((x_i(1, t), \ldots, x_i(d, t))\) has a jump at the origin of magnitude \(q_i\) and for \(1 \leq j_1 < \cdots < j_t \leq d\) it is absolutely continuous in \(x_i(j_1, t), \ldots, x_i(j_t, t)\) on the set where \(x_i(j, t) > 0\) for \(j \in \{j_1, \ldots, j_t\}\) and \(x_i(j, t) = 0\) for \(j \notin \{j_1, \ldots, j_t\}\). Moreover (still if (2.24) and
(2.27) hold) the marginal distribution of each \(x_i(b, t)\) has a jump at the origin and a continuous density function on \((0, \infty)\).

We remark that the situation of this theorem provides the first example where
\[
\lim_{n \to \infty} \rho^{-nd-t} Z_{nd+t} 
\]
does have a random direction even when the limit is not zero. The simplest example of this phenomenon is a three-dimensional branching process with an expectation matrix with two blocks along the diagonal,
\[
\begin{pmatrix}
0 & \alpha & 0 \\
\beta & 0 & 0 \\
m_{3,1} & m_{3,2} & \gamma
\end{pmatrix},
\]
\(0 < \rho_2 = \gamma < \sqrt{\alpha \beta} = \rho, \alpha \beta > 1, m_{3,1} > 0\) and \(m_{3,2} > 0\). If \(Z_0 = e_3\), both
\[
\lim_{n \to \infty} \rho^{-2n} Z_{2n} \quad \text{and} \quad \lim_{n \to \infty} \rho^{-2n-1} Z_{2n+1}
\]
will be vectors whose first two components have in general a genuine two-dimensional distribution.

Lastly we shall study the case when \(\rho_1 = \rho_2 = \rho\) and \(M(1)\) and \(M(2)\) are aperiodic. Here for the first time we run into normalizing constants of the form, \(n^k p^n\), with \(k > 0\).

**Theorem 2.3.** Let \(M(1)\) and \(M(2)\) be aperiodic with \(\rho_1 = \rho_2 > 1\), and let (2.10) hold. If \(Z_0 = e_i\) for some \(i \in C_2\), and if
\[ E\{Z_1^q \log Z_1^q \mid Z_0 = e_p\} < \infty \]
(2.28)
for all pairs \(p, q \in C_1\), then there exists a random variable \(w\) such that
\[
\lim_{n \to \infty} \frac{Z_n(2)}{\rho^n} = w \cdot \nu(2) \quad \text{w.p.1} \quad (2.29)
\]
and
\[
\lim_{n \to \infty} \frac{Z_n(1)}{n \rho^n} = \left( \frac{\sigma}{\rho} \nu(2) M(2, 1) u'(1) \right) \nu(1) \quad \text{w.p.1.} \tag{2.30}
\]

If (2.28) holds for all pairs \( p, q \in C_2 \) and all pairs \( p, q \in C_1 \), then
\[
E\{w | Z_0 = e_1\} = u_4(2), \tag{2.31}
\]
whereas if (2.28) holds for all \( p, q \in C_1 \) but fails for some pair \( p, q \in C_2 \), then
\[
w = 0 \quad \text{w.p.1.} \tag{2.32}
\]
Moreover if (2.28) holds for all pairs \( p, q \in C_2 \) and \( p, q \in C_1 \), and if there is an index \( j \in C_2 \) such that
\[
Z_1(2) u'(2) \text{ can take at least two values with positive probability when } Z_0 = e_j, \tag{2.33}
\]
then the distribution of \( w \) has a jump of magnitude \( q_4 \) at the origin and a continuous density function on the set of positive real numbers.

The reader should note that in this case we have not managed to give both necessary and sufficient conditions for the convergence of the \( (n \rho^n)^{-1} Z_n(1) \)'s.

We can show that if (2.28) does not hold for all pairs \( p, q \in C_1 \), then
\[
\lim_{n \to \infty} \frac{Z_n(1)}{n \rho^n} = 0 \quad \text{w.p.1}
\]
regardless of whether or not (2.28) is satisfied for all pairs \( p, q \in C_1 \). On the other hand if (2.28) is satisfied by all pairs \( p, q \in C_2 \) but fails to hold for some pair \( p, q \in C_1 \), then the situation is not so clear cut. In fact in that case it can occur that the limit of the random variables, \( (n \rho^n)^{-1} Z_n(1) \), does not exist.

In two dimensions, i.e., if
\[
M = \begin{pmatrix} \rho & 0 \\ m & \rho \end{pmatrix}, \quad m \neq 0,
\]
we can show that if \( Z_0 = e_2 \), if \( \lim_{n \to \infty} \rho^{-n} Z_n^2 = w \), if
\[
E\{Z_1 Z_1 \log Z_1 | Z_0 = e_1\} = \infty,
\]
and if
\[
\lim_{n \to \infty} n \rho^{-1} \int_{n \rho}^{\infty} x dF_{1,1}(x) = B, \quad 0 \leq B \leq \infty, \tag{2.34}
\]
then

\[
\lim_{n \to \infty} \frac{Z_n^{-1}}{n \rho^n} = w \cdot \left( \frac{m}{\rho(B+1)} \right) \quad \text{w.p.1.} \tag{2.35}
\]

The proofs of these results are much too lengthy to be given here. We only state them for the sake of completeness.

If \( \rho \) is an eigenvalue of both \( M(1) \) and \( M(2) \) and if one or both of these matrices are periodic, then Theorem 3.1 of [2], Theorems 2.2 and 2.3, Lemma 3.2, and Remark 3.1 below will suffice to determine the behavior of appropriate subsequences of the \( \{Z_n^{-1} Z_n(1)\} \)'s. We shall not attempt to write out the details.

The results obtained in Theorems 2.1-2.3 can be extended to the case when there are more than two blocks along the diagonal in (2.1). In fact one can show by arguments indicated at the end of this paper that Theorems 2.1 and 2.2 with easy reinterpretations are true regardless of the value of \( l \) as long as only one \( \rho_a \) equals \( \rho \). In the case of Theorem 2.3 the situation is slightly more complicated. If the matrices that have \( \rho \) as eigenvalue are aperiodic, then the following general results are true. For a given pair \( a, b \), \( 1 \leq a < b \leq l \), let a chain be a sequence \( \delta_1, \ldots, \delta_r \) such that \( a = \delta_1 < \delta_2 < \cdots < \delta_r = b \) and such that \( M(\delta_{r+1}, \delta_1) \neq 0 \). Let \( k(a,b) \) be the maximal number of matrices with eigenvalue \( \rho \) in any chain from \( b \) to \( a \) and suppose that

\[
E[Z_1^q \log Z_1^q \mid Z_0 = \epsilon_p] < \infty \tag{2.36}
\]

for all pairs \( p, q \) belonging to groups associated with matrices that have \( \rho \) as eigenvalue in all the chains that connect \( b \) with \( a \). Then if \( Z_0 = \epsilon_i \) for some \( i \in C_a \) and if either \( M(a) \) or \( M(b) \) or both have eigenvalue \( \rho \), there exists a random variable \( w \) and a vector \( v^* \) such that

\[
\lim_{n \to \infty} \frac{Z_n(a)}{n^{k(a,b)-1} \rho^n} = w \cdot v^* \quad \text{w.p.1.} \tag{2.37}
\]

Moreover, if \( \rho \) is an eigenvalue of \( M(b) \) and if there exists an integer \( j \in C_b \) such that \( Z_j(b) \epsilon'(b) \) can take at least two values with positive probability when \( Z_0 = \epsilon_j \), then the distribution of \( w \) has a jump at the origin and a continuous density function on the set of positive real numbers. If \( \rho \) is an eigenvalue of \( M(a) \) but not of \( M(b) \), a sufficient condition for the absolute continuity of the distribution of \( w \) can be stated in terms of properties of the components of \( Z \) associated with chains having the maximal number of matrices with eigenvalue \( \rho \). We will not attempt to state this condition precisely here. Instead we will point out that if (2.36) is satisfied
and if $\rho$ is not an eigenvalue of $M(a)$ and $M(b)$, then it is possible that the $\{(n^{k(a,b)} - \rho^n)^{-1} Z_n(a)\}$'s converge to a random vector whose direction is not fixed even if all the $M(c)$'s are aperiodic. Consider for example a $k$-dimensional branching process with moment matrix,

$$M = \begin{pmatrix}
M(1) & 0 & 0 & 0 \\
M(2, 1) & M(2) & 0 & 0 \\
M(3, 1) & 0 & M(3) & 0 \\
M(4, 1) & M(4, 2) & M(4, 3) & M(4)
\end{pmatrix},$$

where $M(a, b) \neq 0$ if $a \neq b$ and $(a, b) \neq (3, 2)$. If $Z_0 = e_i$ for some $i \in C_4$ and if $\rho$ is an eigenvalue of $M(2)$ and $M(3)$ but not of $M(1)$ and $M(4)$, then the $\{\rho^{-n}Z_n(1)\}$'s converge to a random vector whose direction need not be fixed.

We shall give a full proof of Theorems 2.1 and 2.3 and give an outline of the proof of Theorem 2.2 after a number of lemmas. This will be followed by an indication of the steps one has to follow if more than two blocks occur along the diagonal in (2.1).

3. PROOF OF THEOREMS

We will begin by establishing a useful property of positively regular, $k$-dimensional, branching processes.

**Lemma 3.1.** If $\{Z_n : n \geq 0\}$ is a positively regular, $k$-dimensional, branching process with largest eigenvalue $\rho > 1$, then for each fixed $1 \leq i, j < k$

$$E \left\{ \sup_{n \geq 0} \frac{Z_n^{ij}}{\rho^n} \mid Z_0 = e_i \right\} < \infty \quad (3.1)$$

if and only if

$$E\{Z_1^{ij} \log Z_1 \mid Z_0 = e_i\} < \infty \quad \text{for all pairs} \quad 1 \leq i, j \leq k. \quad (3.2)$$

**Proof.** Let

$$w = (v_j)^{-1} \lim_{n \to \infty} \rho^{-n}Z_n^{ij}$$

as in Theorem 1.1 and suppose that (3.2) holds. We shall first show that there exist positive constants $A$ and $B$ such that for all $x \geq 1$

$$P \left\{ \sup_{n \geq 0} \frac{Z_n^{ij}}{\rho^n} \geq x, Z_0 = e_i \right\} \geq B. \quad (3.3)$$
In fact the event,
\[
\left\{ \sup_{n \geq 0} \frac{Z_n^j}{\rho^n} \geq x, Z_0 = e_i \right\},
\]
is the disjoint union of the sets,
\[
\left\{ Z_0 = e_i, \frac{Z_n^j}{\rho^n} \geq x, \frac{Z_m^j}{\rho^m} < x, m = 0, 1, \ldots, n - 1 \right\},
\]
and on the set where \( Z_n^j \geq xp^n \), \( Z_{n+m}^j \) is at least the sum of \( xp^n \) independent variables each with the distribution of \( Z_n^j \) given \( Z_0 = e_j \). Thus
\[
P \left\{ w \geq Ax \mid \frac{Z_n^j}{\rho^n} \geq x, Z_0, \ldots, Z_{n-1} \right\} \geq P \left\{ \frac{1}{\rho^n} \sum_{r=1}^{\infty} w_r \geq Ax \right\},
\]
where each \( w_r \) has the distribution of \( w \) given \( Z_0 = e_j \) and the \( w_r \)'s are independent. Moreover by Theorem 1.1 (3.2) implies the existence of constants \( C, \delta > 0 \) such that
\[
P\{w_r > C\} \geq \delta. \tag{3.4}
\]
Thus, since
\[
P \left\{ \frac{1}{\rho^n} \sum_{r=1}^{\infty} w_r \geq \frac{\delta C x}{2} \right\}
\geq P \left\{ \text{number of } w_r, 1 \leq r \leq xp^n, \text{ which exceed } C \text{ is at least } \frac{\delta}{2} xp^n \right\},
\]
and since (3.4) together with the law of large numbers implies that the right-hand side of (3.5) is uniformly bounded below by some constant \( B > 0 \), (3.3) follows with \( A = (\delta C)/2 \).

But then
\[
P\{w \geq Ax \mid Z_0 = e_i\} \geq P \left\{ w \geq Ax, \sup_{n \geq 0} \frac{Z_n^j}{\rho^n} \geq x \mid Z_0 = e_i \right\}
\geq B \cdot P \left\{ \sup_{n \geq 0} \frac{Z_n^j}{\rho^n} \geq x \mid Z_0 = e_i \right\}.
\]
Hence
\[
E \left\{ \sup_{n \geq 0} \frac{Z_n^j}{\rho^n} \mid Z_0 = e_i \right\} = \int_0^\infty P \left\{ \sup_{n \geq 0} \frac{Z_n^j}{\rho^n} \geq x \mid Z_0 = e_i \right\} dx
\leq 1 + \frac{1}{B} \int_0^\infty P\{w \geq Ax \mid Z_0 = e_i\} dx < \infty,
\]
since \( E\{w \mid Z_0 = e_i\} < \infty \) (see (1.10)).
Conversely, if (3.1) holds, then by the dominated convergence theorem

\[ E \left\{ \lim_{n \to \infty} \frac{Z_n^j}{\rho_n} \mid Z_0 = e_t \right\} = \lim_{n \to \infty} E \left\{ \frac{Z_n^j}{\rho_n} \mid Z_0 = e_t \right\}, \quad (3.6) \]

and we know from Theorem 1.1 that (3.6) holds only if (3.2) holds. Q.E.D.

Before we prove our main lemmas we must introduce several random variables. In agreement with our previous definition

\[ U_r^q(m, n, a) = \{ U_r^{q,j}(m, n, a) : j \in C_a \}, \quad n = m, m + 1, ... \quad (3.7) \]

is for each pair \((r, m)\) a Galton-Watson process with particles of types \(j \in C_a\) such that \( U_r^q(m, n, a) \) has the same distribution as \( Z_{n-m}(a) \) given that \( Z_0 = e_q, q \in C_a \). Notice that we can write \( U_r^{q,j}(\cdot, \cdot, \cdot) \) instead of the former \( U_r^q(\cdot, \cdot, \cdot, p) \) since the distribution of \( U_r^q \) does not depend on \( p \). Moreover, still in agreement with our previous definition the family of random processes,

\[ \{ U_r^q(m, n, a) ; q \in C_a, r = 1, 2, ..., m = 1, 2, ..., \} \]

are independent. For a fixed pair \( q, j \in C_a \) let

\[ V_r(m, s) = \sup_{n \geq m + s} \left\{ \frac{U_r^{q,j}(m, n, a)}{\rho_n^{n-m}} - \lim_{N \to \infty} \frac{U_r^{q,j}(m, N, a)}{\rho_a^{N-m}} \right\}. \quad (3.8) \]

For each fixed \( s \), the \( V_r(m, s)'s, r = 1, 2, ..., m = 1, 2, ... \), are then independent random variables each with the same distribution as

\[ \sup_{n \geq s} \left\{ \frac{Z_n^j}{\rho_a^n} - \lim_{N \to \infty} \frac{Z_n^j}{\rho_a^N} \right\} \]

given \( Z_0 = e_q \). Thus by Theorem 1.1 for each fixed \( s \) the \( V_r(m, s)'s \) are finite with probability one and

\[ \lim_{s \to \infty} V_r(m, s) = 0 \quad \text{w.p.1.} \quad (3.9) \]

Moreover, if

\[ E\{Z_i^j \log Z_i^j \mid Z_0 = e_i\} < \infty \quad (3.10) \]

for all pairs \( i, j \in C_a \), then Lemma 3.1, (3.9), and the dominated convergence theorem imply that

\[ \lim_{s \to \infty} E\{V_r(m, s)\} = 0 \quad (3.11) \]

uniformly in \((r, m)\).
In lemmas 3.2-3.4 $T_m$, $m = 1, 2, \ldots$, will be a sequence of non-negative, integer-valued random variables which are independent of the $U_r^q(m, n, a)$'s.

**Lemma 3.2.** If $\rho_a > 1$, if $M(a)$ is aperiodic, if (3.10) holds, and if

$$B = \lim_{m \to \infty} \frac{T_m}{m^k \rho_a^m}$$

exists w.p.1, then for each $q \in C_a$

$$\lim_{n \to \infty} \frac{1}{n^{k+1} \rho_a^n} \sum_{m=1}^{n} \sum_{r=1}^{T_m} U_r^q(m, n, a) = \frac{B}{k + 1} u_q(a) v(a) \text{ w.p.1.} \quad (3.12)$$

**Proof.** To save space we drop the argument $a$ in $\rho_a$, $U_r^q(m, n, a)$, $Z_n(a)$, $u(a)$, and $v(a)$ throughout the proof. Since Theorem 1.1 applies to the $U_r^q(m, n)$'s and since these random vectors are non-negative, it is clear that on the set, $\{A_1 < B\}$,

$$\liminf_{n \to \infty} \frac{1}{n^{k+1} \rho_a^n} \sum_{m=1}^{n} \sum_{r=1}^{T_m} U_r^q(m, n, a) \leq \liminf_{n \to \infty} \frac{1}{n^{k+1} \rho_a^n} \sum_{m=1}^{n} \sum_{r=1}^{T_m} U_r^q(m, n).$$

A similar inequality holds on the set, $\{B < A_2\}$. Therefore to prove the lemma it suffices to show that for each rational $A > 0$ and for each pair $q, j \in C_a$

$$\lim_{n \to \infty} \frac{1}{n^{k+1} \rho_a^n} \sum_{m=1}^{n} \sum_{r=1}^{T_m} U_r^{q,j}(m, n) = \frac{A}{k + 1} u_q v_j \text{ w.p.1.} \quad (3.13)$$

Let $q, j$ be an arbitrarily chosen pair in $C_a$ and to simplify the notation let $U_r(m, n)$ denote the random variable, $U_r^{q,j}(m, n)$. Moreover let

$$u_r(m) = \lim_{n \to \infty} \frac{U_r(m, n)}{\rho^{n-m}}, \quad m = 1, 2, \ldots.$$ 

By (3.10) and Theorem 1.1 these limits exist and

$$E[u_r(m)] = u_q v_j.$$

Also all $\{u_r(m) : r = 1, 2, \ldots, m = 1, 2, \ldots\}$ are independent and have the same distribution as $\lim_{n \to \infty} \rho^{-n} Z_n^j$, given $Z_0 = e_0$. We now write

$$\frac{1}{n^{k+1} \rho_a^n} \sum_{m=1}^{n} \sum_{r=1}^{T_m} U_r(m, n) = \frac{1}{n} \sum_{m=1}^{n} \left(\frac{m}{n}\right)^k \frac{1}{m^k \rho_a^m} \sum_{r=1}^{T_m} u_r(m)$$

$$+ \frac{1}{n} \sum_{m=1}^{n} \frac{1}{n^k \rho_a^n} \sum_{r=1}^{T_m} \{U_r(m, n) \rho^{n-m} - u_r(m)\}, \quad (3.14)$$
and observe that the second sum on the right-hand side is bounded by

\[ \frac{1}{n} \sum_{m=1}^{n-s} \frac{1}{n^k p^m} \sum_{r=1}^{A m^k p^m} V_r(m, s) + \frac{1}{n} \sum_{m=n-s+1}^{n} \frac{1}{n^k p^m} \sum_{r=1}^{A m^k p^m} V_r(m, 0). \]

By Lemma 3.1 above and by Lemma 3.1 of [2]

\[ \lim_{m \to \infty} \frac{1}{m^k p^m} \sum_{r=1}^{A m^k p^m} V_r(m, s) = AE(V_1(1, s)), \quad w.p.1. \]

Hence, for each fixed \( s \)

\[ \lim_{m \to \infty} \sup_{m} \left| \frac{1}{n} \sum_{m=1}^{n} \frac{1}{n^k p^m} \sum_{r=1}^{A m^k p^m} \left\{ U_r(m, n) p^{m-n} - u_r(m) \right\} \right| \leq AE(V_1(1, s)) \quad w.p.1. \]

Since \( E(V_1(1, s)) \) can be made arbitrarily small by choosing \( s \) large (see (3.11)), we have shown that the second sum on the right-hand side of (3.14) can be ignored.

Another application of Lemma 3.1 of [2] shows that

\[ \lim_{m \to \infty} \frac{1}{m^k p^m} \sum_{r=1}^{A m^k p^m} u_r(m) = AE(u_1(1)), \quad w.p.1. \]

Thus with probability one

\[ \lim_{m \to \infty} \frac{1}{n^k p^m} \sum_{m=1}^{n} \sum_{r=1}^{A m^k p^m} U_r(m, n) \]

\[ = \lim_{m \to \infty} \frac{1}{n} \sum_{m=1}^{n} \left( \frac{m}{n} \right)^k \frac{1}{m^k p^m} \sum_{r=1}^{A m^k p^m} u_r(m) = \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \left( \frac{m}{n} \right)^k AE(u_1(1)) \]

\[ = \frac{A}{k+1} E(\lim_{n \to \infty} \rho^{-n} Z_n^j \mid Z_0 = \epsilon_q) = \frac{A u_q \psi_j}{k+1}. \]

This completes the proof of (3.13) and the Lemma. Q.E.D.

In the next lemma we make one important change in our assumptions of Lemma 3.2. Specifically we assume that the normalizing constants of the \( T_m \)'s are \( m^k \lambda^m \) rather than \( m^k \rho_a^m \) where \( \lambda < \rho_a \). The fact that \( \lambda < \rho_a \) leads to a significant change in the normalizing constants of the random vectors,

\[ \sum_{m=1}^{n} \sum_{r=1}^{A m^k p^m} U_r(m, n, a). \]
LEMMA 3.3. If $\rho_a > 1$, if (3.10) holds, if $M(a)$ is aperiodic, and if for some $\lambda < \rho_a$

$$\lim_{m \to \infty} \frac{T_m}{m^k \lambda^m} \hspace{3cm} (3.15)$$

exists w.p.l., then for each $q \in C_a$ there exists a one-dimensional random variable $x_q$, such that

$$\lim_{n \to \infty} \frac{1}{\rho_a n} \sum_{m=1}^{n} \sum_{r=1}^{T_m} U_r^q(m, n, a) = x_q(a) \text{ w.p.l.} \hspace{3cm} (3.16)$$

Moreover,

$$E(x_q) = \left(\sum_{m=1}^{\infty} \frac{E(T_m)}{\rho_a m} \right) u_q(a), \hspace{3cm} (3.17)$$

where the right-hand side may be $+\infty$ if $E(T_m) = \infty$ for some $m$ or if the series diverges.

The reader should note that (as is easily seen from the proof of the lemma) if the $T_m$'s are "periodic" with period $d$, i.e., if (3.15) is replaced by

$$\lim_{m \to \infty} \frac{T_{dm+j}}{(dm+j)^k \lambda^{dm+j}} \hspace{3cm} (3.18)$$

exists w.p.l. for $0 \leq j \leq d - 1$, then (3.16) and (3.17) still hold.

PROOF OF LEMMA 3.3. As in the proof of the previous lemma we drop the argument $a$. We note first that

$$\frac{1}{\rho^n} \sum_{m=1}^{n} \sum_{r=1}^{T_m} U_r^q(m, n) = \sum_{m=1}^{n} \left(\frac{1}{\rho^m} \sum_{r=1}^{T_m} \frac{U_r^q(m, n)}{\rho^{n-m}} \right)$$

$$+ \sum_{m=n+1}^{n} m^k \left(\frac{\lambda}{\rho}\right)^m \left\{ \frac{1}{m^k \lambda^m} \sum_{r=1}^{T_m} \frac{U_r^q(m, n)}{\rho^{n-m}} \right\}. \hspace{3cm} (3.19)$$

It follows from (3.15), Lemma 3.1 and the corollary to Lemma 3.1 of [2] that with probability one

$$\lim_{n_0 \to \infty} \lim_{n \to \infty} \sum_{m=n_0+1}^{n} m^k \left(\frac{\lambda}{\rho}\right)^m \frac{1}{m^k \lambda^m} \sum_{r=1}^{T_m} \frac{U_r^q(m, n)}{\rho^{n-m}}$$

$$\leq \left(\lim_{n \to \infty} \frac{T_n}{n^k \lambda^n} \right) \cdot \lim_{n_0 \to \infty} \sum_{m=n_0+1}^{n} m^k \left(\frac{\lambda}{\rho}\right)^m E \left\{ \sup_{n \geq 0} \frac{Z_n^q}{\rho^n} \mid Z_0 = \epsilon_q \right\} = 0.$$
Also as \( n \to \infty \) the first sum on the right-hand side of (3.19) converges to

\[
\sum_{m=1}^{n} \frac{1}{\rho_{m}^{m}} \sum_{r=1}^{T_{m}} w_{r}(m) v,
\]

(3.20)

where \( w_{r}(m) v = \lim_{n \to \infty} \rho_{m}^{m-n} U_{r}^{q}(m, n) \) (see (1.8)). This proves (3.16) with

\[
x_{q} = \sum_{m=1}^{\infty} \frac{1}{\rho_{m}^{m}} \sum_{r=1}^{T_{m}} w_{r}(m).
\]

(3.17) follows immediately since by Theorem 1.1

\[
E\{w_{r}(m)\} = u_{q}.
\]

Q.E.D.

In the next lemma we consider the case when the normalizing constants of the \( T_{m} \)'s are larger than \( m^{k} \rho_{a}^{m} \). The assumptions that \( M(a) \) is aperiodic and (3.10) now become superfluous and are therefore dropped.

**Lemma 3.4.** If \( \lambda > \rho_{a} \), if \( \lambda > 1 \), and if

\[
B = \lim_{m \to \infty} \frac{T_{m}}{m^{k} \lambda^{m}}
\]

(3.21)

exists w.p.1, then for each pair \( q, j \in \mathbb{C}_{a} \)

\[
\lim_{n \to \infty} \frac{1}{n^{k} \lambda^{n}} \sum_{m=1}^{n} \sum_{r=1}^{T_{m}} U_{r}^{q}(m, n, a) = B \sum_{s=0}^{\infty} \left( \frac{M^{s}(a)}{\lambda} \right)_{q,j} = B \left( \left( I - \frac{M(a)}{\lambda} \right)^{-1} \right)_{q,j}
\]

(3.22)

with probability one.

**Note.** This lemma also has a "periodic" version when each limit in (3.18) exists separately. We do not insist on this generalization here.

**Proof.** Again we drop the argument \( a \). As in Lemma 3.2 it suffices to show that for each fixed \( A > 0 \)

\[
\lim_{n \to \infty} \frac{1}{n^{k} \lambda^{n}} \sum_{m=1}^{n} \sum_{r=1}^{A m^{k} \lambda^{m}} U_{r}^{q}(m, n) = A \sum_{s=0}^{\infty} \left( \frac{M^{s}(a)}{\lambda} \right)_{q} \text{ w.p.1},
\]

(3.23)
where \((M(a))_q\) denotes the \(q\)th row of \(M(a)\). It follows from Lemma 3.1 of [2] and its corollary that for each fixed \(t\)
\[
\lim_{n \to \infty} \frac{1}{n^k \lambda^n} \sum_{m=n-t+1}^{n} \sum_{r=1}^{A(n-r)^k} U^q(m, n)
\]
\[
= \lim_{n \to \infty} \frac{1}{n^k \lambda n} \sum_{s=0}^{t-1} \sum_{r=1}^{A(n-s)^k} U^q(n - s, n)
\]
\[
= A \sum_{s=0}^{t-1} E\left\{U^q(n - s, n) \right\} = A \sum_{s=0}^{t-1} E\left\{Z^q(a) \big| Z_0 = e_q\right\} = A \sum_{s=0}^{t-1} \left\langle \left(\frac{M^q(a)}{\lambda s}\right)_{n}\right\rangle
\]
\[
\text{w.p.1. (3.24)}
\]

Also
\[
\sum_{m=1}^{n-t} \sum_{r=1}^{A(n-r)^m} U^q(m, n) \leq \sum_{m=1}^{n-t} \sum_{r=1}^{A(n-m)^m} \left\langle \sup_{n \geq m} \frac{U^q(m, n)}{\tau^n-m} \right\rangle
\]
for any \(\tau, \rho_a < \tau < \lambda\). Since for some constant \(C_0\)
\[
E\left\langle \sup_{n \geq m} \frac{U^q(m, n)}{\tau^n-m} \right\rangle \leq \sum_{n=0}^{\infty} E\left\{Z^q(a) \big| Z_0 = e_q\right\}
\]
\[
\leq C_0 \sum_{n=0}^{\infty} \left(\frac{\rho_a}{\tau}\right)^n = \frac{C_0 \tau}{\tau - \rho_a}, \quad (3.25)
\]

it follows from Lemma 3.1 of [2] that
\[
\lim_{m \to \infty} \frac{1}{m^k \lambda^m} \sum_{\tau=1}^{A(n-r)^m} \left\langle \sup_{n \geq m} \frac{U^q(m, n)}{\tau^n-m} \right\rangle \leq A \left(\frac{C_0 \tau}{\tau - \rho_a}\right) \quad \text{w.p.1.}
\]

Hence there exist constants \(C_1\) and \(C_2\) such that w.p.1
\[
\lim_{n \to \infty} \frac{1}{n^k \lambda^n} \sum_{m=1}^{n-t} \sum_{r=1}^{A(n-r)^m} U^q(m, n) \leq AC_1 \lim_{n \to \infty} \frac{1}{n^k \lambda^n} \sum_{m=1}^{n-t} \tau^n-m^k \lambda^m
\]
\[
\leq AC_1 \sum_{s=t}^{\infty} \left(\frac{\tau}{\lambda}\right)^s \leq AC_2 \left(\frac{\tau}{\lambda}\right)^{s}. \quad (3.26)
\]

The last term in (3.26) can be made arbitrarily small by choosing \(t\) sufficiently large and this fact together with (3.24) proves (3.23) and the lemma. Q.E.D.

The preceding lemmas all consider cases in which the condition (3.10), is satisfied or irrelevant. In the case when \(\rho_a > \lambda\) the following lemma shows what happens if (3.10) fails to hold for some pair of indices \(i, j \in C_a\). Speci-
fically we consider a $k$-dimensional branching process with moment matrix as in (2.9), where $\rho = \rho_1 > \rho_2 > 0$ and (2.10) holds. In this case one derives from (2.12) that $u_i > 0$ for all $i$. Since the $\rho^{-n}(Z_n u')$, $n = 0, 1, \ldots$, form a non-negative martingale with uniformly bounded means and thus converge to an integrable random variable, it follows that the $\rho^{-n}Z_n$'s are bounded. Therefore if the random process $Y_n$, $n \geq 0$, is defined as in section 2 of [1], then Lemma 1 of [1] remains valid. Thus the $Y$-process provides a good approximation of the $Z$-process. We will use this fact in the proof of the lemma. Notice that no periodicity assumptions for $M(1)$ or $M(2)$ are needed.

**Lemma 3.5.** Suppose that $M$ has the form (2.9) and assume that $\rho = \rho_1 > \rho_2$, $\rho > 1$. If for some pair $i, j \in C_1$$F\{Z_1 \log Z \mid Z_0 = e\} = \infty$, \hspace{1cm} (3.27)
then for all $q, 1 \leq q \leq k$,$$
\lim_{n \to \infty} \frac{Z_n}{\rho^n} = 0 \hspace{1cm} \text{w.p.1} \hspace{1cm} (3.28)
whenever $Z_0 = e_2$.

**Proof.** When $M(2, 1) = 0$, there is nothing to prove. (See (1.11) if $q \in C_1$ and (2.22) if $q \in C_2$.) Similarly, if $M(2) = 0$. Thus we may assume (2.10). Introduce $$(\epsilon(1, t), t) = \int_{B_0} x dF_{i,j}(x), \hspace{1cm} i, j \in C_1,$$and define $\epsilon(2, t)$ and $\epsilon(2, 1, t)$ correspondingly. If for all $n$ we define$$\lambda(n) = \sum_{j=0}^{n+1} \prod_{t=0}^{i-1} (M(2) - \epsilon(2, i))(M(2, 1) - \epsilon(2, 1, j)) \prod_{r=j+1}^{n} (M(1) - \epsilon(1, r)) \hspace{1cm} (3.29)$$and let$$M(n) = M - \begin{pmatrix} \epsilon(1, n) & 0 \\ \epsilon(2, 1, n) & \epsilon(2, n) \end{pmatrix},$$then$$M(0) \cdots M(n) \rho^{n+1} = \begin{pmatrix} \prod_{t=0}^{n} \frac{[M(1) - \epsilon(1, t)]}{\rho} & 0 \\ \rho^{-n-1}\lambda(n) \prod_{t=0}^{n} \frac{[M(2) - \epsilon(2, t)]}{\rho} & \end{pmatrix}. \hspace{1cm} (3.29)
As in the proof of Lemma 2 in [1] (3.27) implies that for each j
\[
\lim_{n \to \infty} \prod_{t=0}^{n} \frac{[M(1) - \epsilon(1, t)]}{\rho} = 0.
\] (3.30)

Furthermore since \( \rho_2 < \rho \),
\[
\prod_{t=0}^{n} \frac{[M(2) - \epsilon(2, t)]_{i,j}}{\rho} \leq \left( \frac{M^{n+1}(2)}{\rho^{n+1}} \right)_{i,j} = 0 \left( \left( \frac{\rho_2}{\rho} \right)^n \right), \quad \text{for all } i, j \in C_2.
\] (3.31)

Finally we can write
\[
\frac{\lambda(n)}{\rho^n} = \rho^{-n} \left( \sum_{j=0}^{n_0} + \sum_{j=n_0+1}^{n} \right) \prod_{t=0}^{j-1} (M(2) - \epsilon(2, i)) (M(2, 1) - \epsilon(2, 1, j))
\]
\[
\times \prod_{r=j+1}^{n} (M(1) - \epsilon(1, r))
\] .

Observe that the first sum in (3.32) over \( j \) from 0 to \( n_0 \) goes to zero as \( n \to \infty \) for each fixed \( n_0 \) and that the second sum is \( O((\rho_2/\rho)^{n_0}) \) (see (3.30) and (3.31)). This together with (3.30) and (3.31) implies that
\[
\lim_{n \to \infty} \frac{1}{\rho^n} E\{ Y_n \mid Z_0 = \epsilon_0 \} = \lim_{n \to \infty} \frac{(\overline{M}(0) \cdots \overline{M}(n))_{q,j}}{\rho^n} = 0, \quad 1 \leq q \leq k.
\] (3.33)

The remainder of the proof follows the proof of Lemma 2 of [1]. (3.33) implies that for each \( q, 1 \leq q \leq k \),
\[
\lim_{n \to \infty} \frac{Y_n}{\rho^n} = 0 \quad \text{in probability whenever} \quad Z_0 = \epsilon_0
\]
and hence that
\[
\lim_{n \to \infty} \frac{Z_n}{\rho^n} = 0 \quad \text{in probability}.
\]

From the last observation and from the fact that \( u_1 > 0 \) for all \( i \) and \( \rho^{-n}(Z_nu_1') \) converges with probability one to an integrable random variable it follows that
\[
\lim_{n \to \infty} \frac{Z_n}{\rho^n} = 0 \quad \text{w.p.1}.
\]

Q.E.D.
The next lemma shows that if $M$ is as in (2.9), if $M(2, 1) \neq 0$, and if $\rho = \rho_1 > \rho_2$, then the nonzero states of the $Z$-process are transient. This fact enables us to assert that there is only one solution to the equations (1.4) and (1.5). (See proof of Theorem II.7.2 in [3].) This in turn will be used to derive the required properties of the distribution of the random variable $\omega$ in Theorems 3.1 and 3.2.

**Lemma 3.6.** Let $M$ be as in (2.9) and assume that $\rho = \rho_1 > \rho_2$, $\rho > 1$. If $M(2, 1) \neq 0$, then the nonzero states of the $Z$-process are transient.

**Proof.** Since $\rho_1 > 1$, $Z(1)$ is nonsingular. Thus since all types in $C_1$ communicate with each other (see (2.5)), all states of the form, $r = (r_1, 0)$, $r_1 \neq 0$, are transient (see Remark 3.1 of [2]). On the other hand, if $r = (r_1, r_2)$ with $r_2 \neq 0$ is visited infinitely often, then it follows from $M(2, 1) \neq 0$ that infinitely many particles of type $i$ will be created for some $i \in C_1$ (outside a set of probability zero). Since $\rho_1 > 1$, $q_i = P\{Z_n = 0 \text{ eventually } | Z_0 = e_i\} < 1$ for all $i \in C_1$ and hence $Z_n(1)$ will eventually go to infinity. Thus $r = (r_1, r_2)$ with $r_2 \neq 0$ cannot be visited infinitely often with positive probability. Q.E.D.

Before we demonstrate how to use the lemmas to prove Theorems 2.1-2.3 we make one more remark which will be used repeatedly in the proofs below.

**Remark 3.1.** If one already knows that for some $\lambda > 1$

$$\lim_{n \to \infty} \frac{Z_n(b)}{n^\lambda} = W(b) \quad \text{exists} \quad \text{w.p.1},$$

then for $p \in C_b$, $q \notin C_b$,

$$\lim_{n \to \infty} \frac{T_n^{p,q}}{n^\lambda} = \frac{(W(b))_p m_{p,q}}{\lambda} \quad \text{w.p.1.}$$

This follows directly from the corollary to Lemma 3.1 of [2] if one takes into account that by definition,

$$T_n^{p,q} = \sum_{i=1}^{Z_{n-1}^p} X_i(n-1),$$

where $X_i(n-1)$ is the number of descendants of type $q$ in the $n$th generation of the $j$th particle of type $p$ in the $(n-1)$st generation. The $X_j(n-1)$'s are independent, independent of $Z_{n-1}^p$, and have the distribution of $Z_1^q$, given $Z_0 = e_p$.

This remark together with Lemmas 3.2-3.5 allows one to find the asymp-
The proof below indicates how to do this.

**Proof of Theorem 2.1.** Since \( p \) exceeds all eigenvalues of \( M \) in absolute value, \( M(a(p)) \) is positively regular. Thus if \( p \) is an eigenvalue of \( M(2) \), we can apply Theorem 1.1 to \( Z(2) \). We know then that there exists a random variable, \( w \), such that

\[
\lim_{n \to \infty} \frac{Z_n(2)}{\rho^n} = w_2(2) \quad \text{w.p.1.} \tag{3.36}
\]

Moreover, since \( u' = (0, u'(2)) \), for this \( w \) (2.18) implies (2.16) and if (2.18) does not hold, then (2.17) must hold. Also (2.18) and (2.19) imply that the distribution of \( w \) has the required properties. Finally by (3.36) and (3.35)

\[
\lim_{n \to \infty} \frac{T_n^{p,q}}{\rho^n} = \frac{w_2(p)(M(2))_{p,q}}{\rho}, \quad p \in C_2, \quad q \in C_1,
\]

so that Lemma 3.4 with \( \lambda = p, \alpha = 1, T_m = T_m^{p,q} \), and (2.7) show that

\[
\lim_{n \to \infty} \frac{1}{\rho^n} Z_n^j = w \cdot \sum_{p \in C_2} \sum_{q \in C_1} (\psi(2))_p m_{p,q}(\rho I - M(1))_{q,j}^{-1} = w_2[\psi(2)(M(2), 1)(\rho I - M(1))^{-1}], \quad j \in C_1.
\]

In view of (2.13) and (3.36) this proves (2.14) and (2.15) with the \( w \) from (3.36). The proof of the Theorem is now complete for the case when \( p = p_2 \).

If \( p = p_1 \), then (3.36) has to be replaced by

\[
\lim_{n \to \infty} \frac{Z_n(2)}{\lambda^n} = 0 \quad \text{w.p.1} \tag{3.37}
\]

for any \( \lambda > \max(1, p_2) \). (3.37) is an immediate consequence of

\[
\sum_{n=0}^{\infty} \frac{E | Z_n(2) |}{\lambda^n} = \sum_{n=0}^{\infty} 0 \left( \frac{p_2}{\lambda} \right)^n < \infty.
\]

For \( \lambda = p_1 \), (3.37) shows that \( \rho^{-n}Z_n(2) \to 0 \) w.p.1, in agreement with (2.14) and (2.15), since \( \psi = (\psi(1), 0) \) (see (2.12)). Also, by (3.35) and (3.37)

\[
\lim_{n \to \infty} \frac{T_n^{p,q}}{\lambda^n} = 0 \quad \text{w.p.1} \tag{3.38}
\]

for some \( \lambda < p_1 \) and all pairs \( p \in C_1, q \in C_2 \). Moreover, for all \( m \geq 1 \)

\[
E(T_n^{p,q} | Z_0 = e_i) = E(Z_n^{p,q} | Z_0 = e_i) m_{p,q} = (M^{n-1}(2))_{p,q}, \quad (M(2, 1))_{p,q}
\]
so that by Lemma 3.3 with \(a = 1\), (3.38) and (2.7), if (2.18) holds,

\[
\lim_{n \to \infty} \frac{1}{\rho^n} Z_n(1) = \left( \sum_{\rho \in C_1} \sum_{q \in C_2} X_{\rho,q} \right) \cdot v(1)
\]

for random variables \(X_{\rho,q}\) with

\[
E\{X_{\rho,q}\} = \sum_{m=1}^{\infty} \rho^{-m} E\{T_{m}^{p,q} | Z_0 = e_t\} u_q(1) = (\rho I - M(2))^{-1} M(2, 1)_{p,q} u_q(1).
\]

In view of (2.12) this proves (2.14)-(2.16) when (2.18) holds. If (2.18) does not hold, (2.14), (2.15), and (2.17) follow from Lemma 3.5. Finally we have to prove the properties of the distribution of \(w\) if \(\rho = \rho_1 > \rho_2\) and (2.18) and (2.19) hold. Lemma 3.6 together with the argument in the beginning of the proof of Theorem II.7.2 of [3] show that the equations (1.4) and (1.5) have the unique solution (1.3). But \(q_i^* = P\{w = 0 | Z_0 = e_t\}\) also, satisfies \(q_i^* = f(q^*)\) (Ch. I.8 Remark 1 of [3]) and by (2.16) \(q_i^* < 1\) for \(i \in C_2\) (see proof of (2.30) in [1]) whereas \(q_i^* < 1\) for \(i \in C_1\) follows from Theorem 1.1 applied to the \(Z(1)\) process. The fact that \(w\) has a continuous density on \((0, \infty)\) can now be copied verbatim from Lemmas 5-7 of [1].

**Proof of Theorem 2.2.** By (2.7) one has for \(0 \leq t \leq d - 1, j \in C_1\)

\[
Z_{nd+t}^j(1) = \sum_{m=1}^{nd+t} \sum_{\rho \in C_1} \sum_{q \in C_2} \sum_{r=1}^{T_{m}^{p,q}} U_{\rho,q}^{r,i}(m, nd + t, 1, p).
\]

Let \(j \in D_b, q \in D_a\). Then, since \(U_{\rho,q}^{r,i}(m, nd + t, 1, p)\) has the distribution of \(Z_{nd+t-m}^j\), given \(Z_0 = e_0\),

\[
U_{\rho,q}^{r,i}(m, nd + t, 1, p) = 0 \quad \text{w.p.1 if } b - a \equiv t - m (\text{mod } d).
\]

On the other hand, by Theorem 3.1 of [2]

\[
\lim_{n \to \infty} \frac{1}{\rho^{nd+t}} U_{\rho,q}^{r,i}(m, nd + t, 1, p) = w_r(q, t, m) v_j(1, b) \quad \text{w.p.1} \quad (3.39)
\]

for a suitable random variable \(w_r(q, t, m)\) if \(b - a \equiv t - m (\text{mod } d)\). Thus we can write for \(j \in D_b\)

\[
Z_{nd+t}^j(1, b) = \sum_{m=1}^{nd+t} \sum_{\rho \in C_1} \sum_{q \in D_2} \sum_{r=1}^{T_{m}^{p,q}} U_{\rho,q}^{r,i}(m, nd + t, 1, p)
\]
and go through the same argument as in the proof of Theorem 2.1 for the case \( \rho = \rho_1 > \rho_2 \), (3.38) is still valid and (3.39) shows that

\[
\lim_{n \to \infty} \frac{1}{\rho^{nd+t}} \sum_{r=1}^{T^{p,q}_n} U_r^q(m, nd + t, 1, \rho) = \left( \frac{1}{\rho} \sum_{r=1}^{T^{p,q}_n} \nu_r(q, t, m) \right) \nu(1, b) \quad \text{w.p.1.}
\]

This leads to an analogue of (3.20) and thence to (2.23)-(2.26) just as in Theorem 2.1. At the same time we can find some of the properties of the distribution of the \( x_i(b, t) \). However, the proof of all the stated properties of these distributions is rather lengthy and will not be given here.

**Proof of Theorem 2.3.** Again (2.29) follows from Theorem 1.1 applied to the \( Z_\tau(2) \) process. Thus for \( p \in C_\rho , q \in C_1 \), by (3.35)

\[
\lim_{n \to \infty} \frac{T^{p,q}_n}{\rho^n} = \nu(2)p_{\rho, q} \quad \text{w.p.1.}
\]

We can now apply Lemma 3.2 to conclude that

\[
\lim_{n \to \infty} \frac{1}{n \rho^n} \sum_{m=1}^{n} \sum_{r=1}^{T^{p,q}_n} U_r^q(m, n, 1, \rho) = \nu(2)p_{\rho, q, u_0(1)} \nu(1) \quad \text{w.p.1.}
\]

This result together with (2.7) shows that

\[
\lim_{n \to \infty} \frac{Z_n(1)}{n \rho^n} = \left( \frac{\nu}{\rho} \sum_{p \in C_\rho} \sum_{q \in C_1} p_{\rho, q, u_0(1)} \right) \nu(1)
\]

\[
= \left( \frac{\nu}{\rho} \nu(2) M(2, 1) \nu(1) \right) \nu(1)
\]

with probability one. Since \( \nu \) is determined by the \( Z(2) \) process, the properties of the distribution of \( \nu \) again follow from Theorem 1.1 applied to the \( Z(2) \) process.

To conclude the paper we indicate how to obtain limit theorems if there are more than two blocks along the diagonal in (2.1). For simplicity we assume that all \( M(c) \), \( 1 \leq c \leq l \) are aperiodic. Let \( Z_0 = e_i \), \( i \in C_0 \). Then, of course \( Z_n(c) = 0 \) for all \( n \geq 0 \) and \( c > b \). Also, if \( \rho_b > 1 \), then by Theorem 1.1 applied to \( Z_n(b) \), there exists a random variable \( \nu \) such that

\[
\lim_{n \to \infty} \frac{Z_n(b)}{\rho^n} = \nu(1) \quad \text{w.p.1.}
\]

(3.40)
If \( \rho_b \leq 1 \), then for any \( \lambda > 1 \) (compare (3.37))

\[
\lim_{n \to \infty} \frac{Z_n(b)}{\lambda^n} = 0 \quad \text{w.p.1.} \quad (3.41)
\]

We now proceed by induction. Assume that we have already proved the following facts: For \( d \geq a + 1 \),

\[
\lim_{n \to \infty} \frac{Z_n(d)}{n^\gamma(d)\lambda_d^n} = W(d) \quad \text{w.p.1}
\]

for suitable constants \( \gamma(d) \geq 0 \), \( 1 < \lambda_d \leq \rho_d \), and vectors \( W(d) \); if \( \rho_b = \rho \), then \( \lambda_d = \rho \) for all \( d \geq a + 1 \). By Remark 3.1 we then conclude for any \( p \in C_d \), \( q \in C_a \), \( d \geq a + 1 \),

\[
\lim_{n \to \infty} \frac{T_{n,p,q}}{n^\gamma(d)\lambda_d^n} = \frac{W(p)_{m,p,q}}{\lambda_d} \quad \text{w.p.1.}
\]

The Lemmas 3.2-3.4 now show that

\[
\lim_{n \to \infty} \frac{1}{n^{\mu}} \sum_{m=1}^{n} \sum_{r=1}^{r_{n,m}} U_r^q(m, n, a, p) = W(a, p, q) \quad \text{w.p.1},
\]

where \( W(a, p, q) \) is a (random) vector, \( \mu = \max(\lambda_d, \rho_a) \), and \( \delta = 0 \) if \( \lambda_d < \rho_a \), \( \delta = \gamma(d) \) if \( \lambda_d > \rho_a \), \( \delta = \gamma(d) + 1 \) if \( \lambda_d = \rho_a \). We also see from Lemmas 3.2 and 3.3 that if \( \rho_a \geq \lambda_d \), then \( W(a, p, q) \) is a (random) multiple of the fixed vector \( v(a) \). In case \( \lambda_d \geq \rho_a \), then Lemmas 3.2 and 3.4 imply that no new random variable occurs in the \( W(a, p, q) \). More precisely,

\[
W(a, p, q) = \left( \lim_{n \to \infty} \frac{T_{n,p,q}}{n^\gamma(d)\lambda_d^n} \right) \text{ times a fixed vector}
\]

and hence the random character of \( W(a, p, q) \) is already contained in the \( T_{n,p,q} \) or \( Z_n(a) \), \( d \geq a + 1 \). By (2.7) \( Z_n(a) \) is a sum of expressions (2.8) and the proper normalization factor of \( Z_n(a) \) is the highest one occurring among the normalization factors \( n^\gamma(a) \mu^n \) for the expressions (2.8) entering in \( Z_n(a) \). Thus we find, for some random vector \( W(a) \) and some constant \( \gamma(a) \)

\[
\lim_{n \to \infty} \frac{Z_n(a)}{n^\gamma(a)\lambda_a^n} = W(a) \quad \text{w.p.1}
\]

with

\[
\lambda_a = \max \{ \rho_a, \max [\lambda_c : M(c, a) \neq 0] \}.
\]

By the preceding remarks we see that if \( \rho_a = \rho \), then \( W(a) \) is a multiple of the
fixed vector $v(a)$ since for each $d \geq a + 1$ $\rho_\alpha \geq \lambda_d$ and hence $W(a, \rho, q)$ is a multiple of $v(a)$. Finally, if $\rho_\alpha = \rho$, then $\lambda_d = \rho$ for $d \geq a + 1$ and $W(a)$ is of the form,

$$W(a) = w \cdot \tilde{W}(a),$$

for some fixed vector $\tilde{W}(a)$, i.e., the only random element in all the limits is in the variable $w$ of (3.40).

References