JOURNAL OF COMBINATORIAL THEORY 1, 51-81 (1966)

Graphs and Composite Games

CEDRIC A. B. SMITH

Galton Laboratory, University College,

London, England

Communicated by F. Harary

1. DEFINITIONS

There is a well-known relationship between a certain type of two-player game without chance moves (including chess, nim, noughts and crosses) and directed graphs. (See, for example [4, 15, 21, 22].) Let the actual physical arrangement of pieces on the board in such a game, or of marks on paper, be called the *configuration* V_i. By a *position* V_i^A or V_i^B we mean a configuration V_i together with an indication of which of the two players (Abe or Barbara) will move next. To each game $\mathscr{G}$ of the kind we consider in this paper there corresponds a digraph (directed graph) $\Gamma(\mathscr{G})$ whose vertices or points are the positions V_a^A, V_b^B of the game, and whose edges or directed lines (V_a^A, V_b^B), (V_c^B, V_d^A) are the *moves* permitted by the rules of the game. We consider only games in which the players move alternately. The digraph is then bicolored, since the vertices (positions) fall into two classes (colors) marked by the superscripts A (amber) and B (blue), and by definition Abe always moves from Amber to Blue, and Barbara always from Blue to Amber. When there is a move (edge) from V_a^A to V_b^B we call V_b^B a *follower* of V_a^A. When it is necessary to make this relationship explicit we write V_b^B alternatively as V_{ab}^B, where the suffix a is superfluous except to show that this is a follower of V_a^A. Similarly a follower V_c^A of V_b^B may be writtens as V_{bc}^A or V_{abc}^A. A position V_t^A or V_u^B without followers is *terminal.* The class T of all terminal positions is divided by the rules of the game into three nonoverlapping subclasses, T_A (= "win for Abe"), T_B (= "win for Barbara") and T_O (= "draw").

We regard the game $\mathscr{G}$ as being completely specified when the graph $\Gamma(\mathscr{G})$ and the terminal classes T_A, T_B, and T_O are given.

Now the usual manner of playing such a game is as follows. Some *starting position* V_s^A, say, is either specified by the rules or agreed by the two players; to simplify the discussion we take this to be amber. If this is not terminal, Abe makes some move from V_s^A to V_{sb}^B; that is, he chooses at will some follower V_{sb}^B of V_s^A. If this in turn is not terminal, Barbara moves from V_{sb}^B to some follower V_{sbc}^A, and so on, with the players moving alternately. In this way the players generate a *play* or directed walk $(V_s^A, V_{sb}^B, V_{sbc}^A, \ldots)$ in the digraph. If at any point in the play a terminal position V_t^A $(= V_{sbc\ldots t}^A)$ or V_u^B is reached, the play necessarily ends, and the *outcome* is declared to be a win for Abe, a win for Barbara (= a loss for Abe) or a draw according to whether the end position V_t^A or V_u^B lies in T_A, T_B, or T_O, respectively. The *length* of the play is the number of moves in it (one less than the number of positions). (If the same position or move occurs repeatedly in the play, it must be counted repeatedly.) If a vertex V_i^X ($X = A$ or B) is such that plays starting from it have a finite maximum length $D(V_i^X)$, this maximum is the *terminal distance* of V_i^X. If the starting position V_s^A has finite terminal distance, the game falls within the definition of von Neumann and Morgenstern [21] as a "two-player game with perfect information and without chance moves." However, in some bicolored digraphs it is possible for a walk to have an arbitrarily long or infinite length, e.g., by repeatedly going round a cycle. Now, as we will emphasize again later, a real player cannot go on playing indefinitely long in the sense of moving physical pieces on a physical board, or naming nodes in a graph. To overcome this difficulty we introduce the idea of a *strategy*. Suppose that $(V_s^A, V_{sb}^B, \ldots, V_{sb\ldots k}^A)$ is any *incomplete play* from V_s^A ending at an amber vertex, where an incomplete play is a directed walk whose last vertex is not terminal. Then a *strategy in the wide sense* [13] for Abe is a rule specifying which position he should move to next, i.e., a function

$$W_A(V_s^A,\ V_{sb}^B,\ \ldots,\ V_{sb\ldots k}^A) = V_{sb\ldots kl}^B$$

whose argument ranges over the incomplete plays from V_s^A and whose value is a follower $V_{sb\ldots kl}^B$ of the last position $V_{sb\ldots k}^A$ of the play. A strategy W_B for Barbara is similarly defined. Each time the game is played we imagine that each player selects a strategy, not knowing the other's choice. These two strategies, together with the starting position V_s^A, evidently define a unique play $(V_s^A, V_{sb}^B, V_{sbc}^A, ...)$ in the obvious way:

$$V_{sb}^B = W_A(V); \qquad V_{sbc}^A = W_B(V_s^A, V_{sb}^B); \qquad \text{etc.}$$

If this play terminates, the outcome is determined by the class T_A, T_B, T_0 to which the last position belongs. If the play can be shown to be infinitely long, we declare it a "draw," or, more exactly, an "unterminated draw." (This is the usual convention. If an infinitely long play can be declared a win, draw, or loss for Abe, depending on the positions moved through, Gale and Stewart [7] showed that the property of "strict determination" [15, 21] can fail.) Conversely any play $(V_s^A, V_{sb}^B, V_{sbc}^A, ...)$, whether incomplete, terminated, or infinitely long, can always be regarded as arising from two suitably chosen strategies W_A, W_B, in the wide sense.

However, it is clear that at any point in a play the future possibilities depend only on the actual position then reached, and are quite uninfluenced by past moves. It is therefore intuitively reasonable that a good choice for the next move will depend only on the present position. We define a *strategy in the narrow sense* (or, simply, a *strategy* unless the wide sense is explicitly indicated) [13] as follows. A strategy σ^A for Abe is a function defined over the set of non-terminal amber positions, such that $\sigma^A\ (V_a^A) = V_{ab}^B$ is always a follower of V_a^A, and is interpreted as Abe's choice for the next position should the play arrive at V_a^A. A strategy σ^B for Barbara is similarly defined. Thus, when the starting position V_s^A is specified, any pair of strategy functions uniquely define a play $(V_s^A, V_{sb}^B, V_{sbc}^A, ...)$ in the obvious way.

At any point in a play the "previous player" and "next player" have obvious definitions; at V_a^A the next player is Abe, and the previous player Barbara (even if V_a^A is the starting or terminating position of the play). The terminal positions can therefore be divided into three non-overlapping classes T_P, T_N, T_O, meaning "previous player wins" (= last player wins), "next player wins" (= last player loses), and "draw," respectively. In this paper we place particular emphasis on *last-player-winning* games, in which $T_P = T$, and hence $T_N = T_O = \varnothing$, the null set.

2. Impartial Games

In some games, any physical move of the pieces which is permitted to one player is also permitted to the other, should opportunity occur;

e.g., this is so in Nim and many of its variants [1, 2, 6, 8–10, 12, 17–19]. That is to say, if (V_a^A, V_b^B) is a permitted move, so also is (V_a^B, V_b^A), and we may say that there is a permitted move from configuration (physical position of the pieces) V_a to configuration V_b. Such a game therefore defines a *configuration graph* $C(\mathscr{G})$, whose vertices are the configurations V_a and whose directed edges are the permitted moves (V_a, V_b). If, in addition, the terminal configurations can be divided into non-overlapping classes T_P, T_N, T_O, irrespective of which player has just moved, the game is called *impartial.* This is true of most forms of Nim, which are "last-player-winning" $(T_P = T)$ or "last-player-losing" $(T_N = T)$.

From a formal point of view, the distinction between a non-impartial and an impartial game is rather slight. If an impartial game $\mathscr{G}$ has a configuration digraph $C(\mathscr{G})$, the corresponding bicolored position digraph $\Gamma(\mathscr{G})$ is obtained by replacing each configuration V_a by a pair of positions V_a^A and V_a^B, and each move (V_a, V_b) by a pair of moves (V_a^A, V_b^B), (V_a^B, V_b^A). A position V_a^A or V_a^B is terminal if and only if the configuration V_a is terminal, and both are assigned to the same class T_P, T_N, or T_O. On the other hand, if $\Gamma(\mathscr{G})$ is the bicolored graph of any game $\mathscr{G}$, with the terminal positions classified into T_P, T_N, T_O, we can derive a configuration graph $C(\mathscr{G}')$ of an impartial game $\mathscr{G}'$ by simply omitting the colors. But this new game $\mathscr{G}'$ is exactly the same as the old $\mathscr{G}$ as far as the physical moves of the pieces on the board by the players are concerned, provided only that the same starting position is chosen. However, the definitions of "disjunctive" and "selective" compounds to be introduced later apply in a natural way only to impartial games. Although they can be formally extended to partial games, like chess, by using the uncolored digraphs $C(\mathscr{G}')$, this means that in general each player will sometimes find himself moving a white piece and sometimes a black one, contrary to the usual custom.

3. Finite Digraphs

For the present we restrict attention to finite digraphs of, say, ν vertices. In a bicolored graph we denote a typical vertex by V_i^X, where $X = A$ or B; the other color, B or A respectively, will then be denoted by Y. We have already defined the terminal distance $D(V_i^X)$ of a vertex V_i^X as the length of the longest directed path from V_i^X, if such a path exists. (A similar definition applies to the vertices of an uncolored graph; and, furthermore, corresponding vertices in the three digraphs $C(\mathscr{G})$, $\Gamma(\mathscr{G})$, and $C(\mathscr{G}')$, where they exist, all have the same terminal distance. This kind of remark will also apply to most of the other functions which we define later, so we confine the discussion to one type of graph, bicolored or uncolored according to convenience, leaving that for the other type to follow by analogy.) We see that $D(V_i^X)$, if finite, must be less than ν; for if there started from V_i^X a path of ν moves, and hence of $(\nu + 1)$ vertices, this must contain at least one vertex twice, i.e., it must contain a cycle, and so by traversing this cycle repeatedly we get an infinite walk from V_i^X. The values of $D(V_i^X)$ can be found by induction. Terminal positions have terminal distance 0. Non-terminal positions with all followers terminal have terminal distance 1. In general, suppose that all positions with terminal distance less than d have been found; then a position has terminal distance d if and only if all its followers have terminal distance less than d. All positions which do not have a terminal distance less than ν assigned to them are given $D(V_i^X) = \infty$.

For any set Σ of non-negative integers (x_i) it is convenient to define the "least greater number"

$$\begin{aligned} \text{super } \Sigma &= \text{super}_i\ x_i \\ &= \text{the smallest non-negative integer greater than} \\ &\quad\ \text{all } x_i \text{ in } \Sigma \end{aligned} \tag{1}$$

In particular, if Σ is empty, super $\Sigma = 0$. If Σ is not empty,

$$\text{super}_i\, x_i = \max_i x_i + 1,$$

but the "super" notation is more convenient for generalization to infinite sets later. It immediately follows that, when $D(V_i^X)$ is finite,

$$D(V_i^X) = \text{super}_j\, D(V_{ij}^Y) \tag{2}$$

while

$$D(V_i^X) = \infty \quad \text{if and only if, for some } j,\ D(V_{ij}^Y) = \infty \tag{3}$$

(Eq. (3) can be included in (2) by the convention that $\text{super}_i\, x_i = \infty$ whenever any $x_i = \infty$.)

4. The Remoteness Function

Consider a last-player-winning game, i.e., one in which $T_P = T$. Speaking intuitively, we may say that a position V_i^A is "winning for the next player" (Abe) if he has a "good strategy" σ_g^A which will guarantee him a win however Barbara chooses to move. If so, suppose that Abe tries to win as quickly as he can, while Barbara replies by trying to postpone defeat as long as possible. Then the number of moves in the resulting play is called the remoteness [20] $R(V_i^A)$ of V_i^A. Since Abe must move last to win, $R(V_i^A)$ must then be odd. On the other hand, if V_i^A is such that, however Abe moves from it, Barbara can win for sure by use of a suitable good strategy, then $R(V_i^A)$ is the maximum number of moves by which Abe can postpone defeat if Barbara plays well. Since Barbara, who wins, must move last, $R(V_i^A)$ is then even. More precisely, we will define the Steinhaus *remoteness function* R by the following inductive definition, and show that it has the required properties. (This method is analogous to that used by Kalmár [13], but the function he introduces is half the integral part of R.)

Conditions on the Remoteness Function

$$\text{If } V_i^X \text{ is terminal, } R(V_i^X) = 0. \tag{4a}$$

If V_i^X has at least one follower V_{ij}^Y with $R(V_{ij}^Y)$ even, then

$$R(V_i^X) = \text{minimum even } R(V_{ij}^Y) + 1 \tag{4b}$$

In this case $R(V_i^X)$ is odd.

If, for every follower V_{ij}^Y of V_i^X the remoteness $R(V_{ij}^Y)$ is odd, then

$$R(V_i^X) = \text{super}_j\, R(V_{ij}^Y) \tag{4c}$$

In this case $R(V_i^X)$ is even.

Also, in all cases,

$$R(V_i^X) \geqq 0 \tag{4d}$$

Note that (4d) taken in conjunction with (4b) and (4c), implies that $R(V_i^X) > 0$ whenever V_i^X is non-terminal. Hence the set of all V_i^X with $R(V_i^X) = 0$ is the terminal set T. Condition (4b) then defines uniquely the set of all V_i^X with $R(V_i^X) = 1$, condition (4c) the set with

$R(V_i^X) = 2$, and so on, using (4b) and (4c) alternately. If the set U_k of all positions of remoteness $k \geq 0$ is empty, so also is U_{k+1}, and hence so also are U_{k+2}, U_{k+3}, ... If ν is the number of positions, it follows that U_k is uniquely defined for $k \geq 0$ and empty for $k \geq \nu$. For all positions V_i^X not in any U_k $(0 \leq k < \nu)$ we conventionally set $R(V_i^X)$ to be ∞. It follows that

$$R(V_i^X) = \infty \text{ if and only if no follower } V_{ij}^Y \text{ has even remoteness,} \\ \text{and at least one follower has } R(V_{ij}^Y) = \infty. \qquad \text{(4e)}$$

Note that (4a), (4c), and (4e) can be compressed into a single condition:

If $R(V_{ij}^Y)$ is not even for any j, then

$$R(V_i^X) = \text{super}_j\, R(V_{ij}^Y) \qquad \text{(4f)}$$

Note also that for impartial games, with the last player winning, $R(V_i^A) = R(V_i^B) = R(V_i)$.

Now the definition of $R(V_i^X)$ shows that each non-terminal position V_i^X has at least one follower V_{ij}^X with

$$R(V_{ij}^Y) = R(V_i^X) - 1 \qquad \text{(5)}$$

with the convention that, if $R(V_i^X) = \infty$, then $R(V_{ij}^Y) = \infty$. For each non-terminal V_i^X choose such a V_{ij}^Y, and define "good" strategies σ_g^A, σ_g^B by

$$\sigma_g^X(V_i^X) = V_{ij}^Y. \qquad \text{(6)}$$

Suppose, first, that $R\{V_s^A\}$ is odd at the starting position V_s^A. Then if Abe adopts the strategy σ_g^A, it follows that he will move to a position V_{si}^B whose remoteness is even, and equal to $R(V_s^A) - 1$. Either this is terminal, in which case Abe wins, or else by (4c) Barbara must move to a position V_{sij}^A of still smaller odd remoteness. The argument then repeats. Since the remoteness decreases by at least one at each move, it follows that the play must result in a win for Abe in at most $R(V_s^A)$ moves. A consequence of this is that, in order to be sure of winning, Abe must use a strategy whereby any move from a position of odd remoteness goes to a position of even remoteness; for, if at any time Abe moved from odd to odd remoteness, then by an argument similar to the one above Barbara could be sure of winning by using σ_g^B. Now suppose that Barbara uses σ_g^B, and Abe uses any strategy guaranteeing a win; then Abe moves from odd to even remoteness, and hence by (4b) decreases the remoteness by at least 1, while Barbara decreases the remoteness by exactly 1 at each move, changing it from even to odd. Since the game ends only when remoteness 0 is reached, it follows that the play has at least $R(V_s^A)$ moves. Hence the strategy σ_g^A does ensure a guaranteed win for Abe in the smallest possible number of moves, i.e., at most $R(V_s^A)$, and the strategy σ_g^B does postpone defeat for Barbara for at least $R(V_s^A)$ moves. These are the properties we required of the remoteness function according to the intuitive definition.

If $R(V_s^A)$ is odd, a similar argument shows that Barbara can be sure to win in at most $R(V_s^A)$ moves by using a good strategy, while Abe can postpone defeat to at least this amount. If $R(V_s^A)$ is ∞, then either player can avoid defeat by using a good strategy; if both players use good strategies, the outcome will be an unterminated draw.

The positions can be divided into three non-overlapping classes, N, P, O, according as the remoteness is, respectively, odd, even, and ∞. If the starting position is in P, we have seen that the Previous player can force a win, and if it is in N, the Next player. (Sometimes the P positions are called "safe" positions, but the $N - P$ notation is more explicit, as it shows for which player, Previous or Next, the position is safe.) The use of a good strategy implies that a player will move from an N, O, P position to a P, O, N position respectively. As a partial converse, if the starting position has finite terminal distance, then this rule (N, O, P, to P, O, N) by itself is sufficient to ensure the best possible guaranteed outcome, though not necessarily in the smallest number of moves [17]. However, this can fail for games with infinite terminal distances, as is shown by the simple counterexample of the game with three positions and three moves

$$V_1^B \rightleftarrows V_2^A \rightarrow V_3^B \quad \text{(terminal)}.$$

We have the values:

Position V_i^X	V_1^B	V_2^A	V_3^B
Terminal distance $D(V_i^X)$	∞	∞	0
Remoteness $R(V_i^X)$	2	1	0
Outcome class	P	N	P

If V_2^A is the starting position, Abe can be sure to win (by moving to V_3^B); but a non-terminating oscillation between V_2^A and V_1^B is also possible, conforming to the (N, O, P to P, O, N) rule, but resulting in a draw instead of a win for Abe. This does not, of course, conform to a "good" strategy for Abe.

Note that, since there exists a play of length $R(V_i^X)$ starting from V_i^X,

$$R(V_i^X) \leq D(V_i^X) \tag{7}$$

with the obvious convention for infinite values.

5. Generalized Remoteness functions (Q-Functions)

Consider now a game $\mathscr{G}$ in which some terminal positions are "last-player-losing," i.e., in T_N, and the remainder (if any) are last-player-winning, i.e., in T_P. We modify the game by adding, for each position V_t^X in T_N, a new move (V_t^X, V_{tu}^Y) to a new last-player-winning terminal position V_{tu}^Y. These additions will not alter the outcome of any play in the game, since a player moving to V_t^X must still lose, but one move later. This modified game, $\mathscr{G}^*$, is now last-player-winning and can be analyzed by use of the R function. Define the function $Q(V_i^X; T_P)$ of the position V_i^X in the original game $\mathscr{G}$ to be equal to the value of $R(V_i^X)$ in the modified game $\mathscr{G}^*$. This function obeys relations analogous to (4b), (4c), (4d), (4e) (with R replaced by Q), but the value of $Q(V_t^X; T_P)$ at a terminal position V_t^X is 0 or 1 according as V_t^X lies in T_P or $T_N = T - T_P$, respectively. It is still true that, if $Q(V_s^A; T_P)$ is odd where V_s^A is the starting position, then Abe can force a win by using a strategy $\sigma_g^A(.\,; T_P)$ by which he diminishes the value of the $Q(.\,; T_P)$ function by one at each move. He will therefore win in not more than $Q(V_s^A; T_P)$ moves (and he will win in not more than $Q(V_s^A; T_P) - 1$ moves if the strategy ensures that the play will always terminate in T_N). Similarly, if $Q(V_s^A; T_P)$ is even, Barbara can force a win in not more than $Q(V_s^A; T_P)$ moves, while if $Q(V_s^A; T_P) = \infty$, neither player can force a win, and the outcome will be an unterminated draw if both players play well.

More generally, let us suppose that it is somehow advantageous to Abe for the play to terminate within a certain subclass S_A of T. We make this more explicit later on; for the moment let us call a termination in S_A a "success" for Abe. The negation of this, i.e., a termination in $T - S_A = S_B$, it is convenient to call a "success for Barbara," although it should be noted that this definition only implies that it deprives Abe of some advantage, and not that it necessarily benefits Barbara. We can now define terminal classes S_N (or S_P) as respectively "success for the next (previous) player," and hence a generalized remoteness function $Q(.\,; S_P)$ and strategy $\sigma_g^X(.\,; S_P)$, as above. If, for example, $Q(V_s^A; S_P)$ is odd, then Abe can be sure of achieving a success in not more than $Q(V_s^A; S_P)$ moves by using the strategy $\sigma_g^A(.\,; S_P)$.

These definitions enable us to analyze a game in which all three classes T_A, T_O, and T_B of terminal positions occur. First, let us take a "success" for Abe to be a win, i.e., set $S_A = T_A$, so that $S_B = T - T_A = T_O \cup T_B$, and a "success" for Barbara is a win or terminated draw for her. Denote the corresponding function $Q(.\,; S_P)$ by Q_1, and the corresponding "good" strategies $\sigma_g^X(.\,; S_P)$ by σ_1^X. Second, let us take a success for Abe to be a win or terminated draw, so that $S_A = T_A \cup T_0$, and $S_B = T_B$. Denote the corresponding Q function and strategies by Q_0 and σ_0^X. Then the following cases can occur:

(i) If $Q_1(V_s^A)$ is odd, Abe can force a win by strategy σ_1^A.

(ii) If $Q_0(V_s^A)$ is odd, but $Q_1(V_s^A)$ is not, Abe can force "at worst a terminated draw" (i.e., either a terminated draw or a win) by using σ_0^A, and Barbara can make sure that Abe does no better than draw by using σ_0^B.

(iii) If $Q_0(V_s^A)$ is even, Barbara can force a win by σ_0^B.

(iv) If $Q_1(V_s^A)$ is even, but $Q_0(V_s^A)$ is not, Barbara can force at worst a terminated draw by using σ_1^B, and Abe can prevent her doing better by using σ_1^A.

(v) if $Q_1(V_s^A) = Q_0(V_s^A) = \infty$, the best either player can ensure is a nonterminated draw, Abe by using σ_1^A, and Barbara by using σ_0^B.

These conclusions show that a player cannot do better by using a strategy in the wide sense, instead of the narrow sense. For in each case we have shown what is the best outcome available to each player by using a suitable narrow strategy, and we have also shown that his opponent has a narrow strategy which will prevent his doing any better.

A similar method can be used if, instead of having merely three outcomes, "win," "draw," or "lose," we have a set of 4 or more outcomes in decreasing order of preference, e.g., a set of money payments. We can then define a "success" for Abe to be an outcome at least as good as v, where v is some point in the scale of outcomes. We then know that, if $Q(V_s^A; S_P)$ is odd, Abe can be sure of obtaining at least v; if it is even, Barbara can force Abe to do worse than this, and if ∞, then Abe can keep the play going indefinitely rather than accept a worse outcome.

A comparison of the Q-functions for different values v will therefore show what is the best outcome that Abe can ensure.

6. Conjunctive Compounds

A *compound* or *composite* game may be defined as follows. We imagine that Abe and Barbara play simultaneously a number of impartial *component* games, $\mathcal{G}^1$, $\mathcal{G}^2$, ..., $\mathcal{G}^m$. Each player in turn (legally) makes a move in some or all of the components. More precisely, in a *conjunctive* compound the player makes a move in every component game. In a *disjunctive* compound he selects at will one of the games, and moves in that, leaving the positions in all other games unchanged. In a *selective* compound he chooses some non-empty set of component games, and moves in each of them, leaving the position unaltered in all other components. In each type of compound, the play continues until no further move is possible under the rules. Other forms of compound games, different from these, are considered by Berge [2] and Milnor [16].

More formally, let C^1, C^2, ..., C^m be a (finite) set of m uncolored directed graphs. Their conjunctive compound C^{cnj} is defined as follows. Let $V_a{}^1$ be an arbitrary vertex in C^1, $V_b{}^2$ a vertex in C^2, and so on; then the ordered set $(V_a{}^1, V_b{}^2, ..., V_e{}^m) = V_p^{cnj}$ (say) is a vertex of C^{cnj}. We call $V_a{}^1$, $V_b{}^2$, ..., $V_e{}^m$ the *component* vertices of V_p^{cnj}; and we will use the symbol $V_d{}^h$ for a typical component ($h = 1, ..., m$; $d = a, ..., e$). (It would be more systematic to use a double subscript notation $(V_{p_1}^1, V_{p_2}^2, ..., V_{p_m}^m) = V_p^{cnj}$; but the single subscript, though less tidy, is more legible.) There is a move (edge) in C^{cnj} from V_p^{cnj} to $V_{pP}^{cnj} = (V_{aA}^1, V_{bB}^2, ..., V_{eE}^m)$ if and only if each component position V_{dD}^h in V_{pP}^{cnj} is a follower of the corresponding component $V_d{}^h$ in V_p^{cnj} is a follower of the corresponding component $V_d{}^h$ in C_p^{cnj}. Hence V_p^{cnj} is terminal if and only if some component $V_d{}^h$ is terminal. A (possibly incomplete) play in C^{cnj} of length L consists in the obvious way of a set of (possibly incomplete) plays of length L, one in each C^h; and the play is complete if and only if one of these *component* plays is complete. Since the terminal distance of V_p^{cnj} is by definition the maximum length of any complete play beginning at V_p^{cnj}, it follows that

$$D(V_p^{cnj}) = \min[D(V_a{}^1),\ D(V_b{}^2),\ ...,\ D(V_e{}^m)] \tag{8}$$

as can also be formally verified inductively, using (2).

If the compound game $\mathscr{G}^{cnj}$ is taken to be last-player-winning, it is natural to take the components $\mathscr{G}^h$ also to be last-player-winning. This means that the winner of $\mathscr{G}^{cnj}$ is the player who is the first to win (= terminate) in any component game. The obvious strategy for any player is therefore to try to win as quickly as possible in those components in which he can force a win, and to delay defeat as long as possible in the others, i.e., to use a "good" strategy $\sigma_g^X(.)$ in each component game. Since $R(V_d^h)$ means the length of play from V_d^h in game $\mathscr{G}^h$ when both players use good strategies, this suggests that

$$R(V_p^{cnj}) = \min[R(V_a^1),\ R(V_b^2),\ ...,\ R(V_e^m)]. \tag{9}$$

A formal proof is, however, not entirely trivial. Denote the right-hand side of (9) by $R^*(V_p^{cnj})$; we have to verify that this satisfies relations of the form (4a) to (4e). Of these, (4a) and (4d) are immediate. We also have $R^*(V_p^{cnj}) = \infty$ if and only if $R(V_d^h) = \infty$ for every component V_d^h; then the truth of (4e) for V_p^{cnj} follows from its truth for each component position.

Suppose therefore that $R^*(V_p^{cnj}) \neq 0$ or ∞. From the definition it follows that $R^*(V_p^{cnj}) = R(V_c^j)$ for some component position V_c^j. Let $V_{c\gamma}^j$ be a follower of V_c^j with

$$R(V_{c\gamma}^j) < R(V_c^j). \tag{10}$$

We now assert

Lemma 1. *There exists a follower $V_{p\pi}^{cnj}$ of V_p^{cnj} in the compound game such that $R^*(V_{p\pi}^{cnj}) = R(V_{c\gamma}^j)$.*

Proof: For each h, $R(V_d^h) \geq R^*(V_p^{cnj}) = R(V_c^j) > R(V_{c\gamma}^j)$. According to whether $R(V_d^h)$ is odd, even, or ∞, it follows from (4b), (4c), (4e), respectively, that there exists a follower $V_{d\delta}^h$ with $R(V_{d\delta}^h) \geq R(V_{c\gamma}^j)$. Take the j-th component of $V_{p\pi}^{cnj}$ to be $V_{c\gamma}^j$, and for each $h \neq j$, the h-th component to be $V_{d\delta}^h$.

Lemma 2. *Let V_{pq}^{cnj}* be any follower of V_p^{cnj}. Then if $R^*(V_{pq}^{cnj})$ is less than $R(V_{c\gamma}^j)$, it is odd.

Proof: For suppose if possible that $R^*(V_{pq}^{cnj})$ is even and less than $R(V_{c\gamma}^j)$. Then there is some component of V_{pq}^{cnj}, say V_{fg}^k, for which $R^*(V_{pq}^{cnj}) = R(V_{fg}^k)$. Hence

$$\begin{aligned}
R^*(V_p^{cnj}) &\leq R(V_f^k) && \text{(by definition of } R^*) \\
&\leq R(V_{fg}^k) + 1 && \text{(by (4b))} \\
&= R^*(V_{pq}^{cnj}) + 1 && \\
&< R(V_{c\gamma}^j) + 1 && \text{(by hypothesis)} \\
&\leq R(V_c^j) && \text{(by (10))} \\
&= R^*(V_p^{cnj}) && \text{(by definition of } V_c^j)
\end{aligned}$$

and this is a contradiction. Then lemma is therefore proved.

There are now two possibilities. The first is that $R^*(V_p^{cnj}) = R(V_c^j)$ is odd. Then by (4b) there exists $V_{c\gamma}^j$ with $R(V_{c\gamma}^j) = R(V_c^j) - 1$, and hence by Lemma 1 there exists $V_{p\pi}^{cnj}$ with $R^*(V_{p\pi}^{cnj}) = R^*(V_p^{cnj}) - 1$, which is even; and by Lemma 2 there is no smaller $R^*(V_{pq}^{cnj})$ which is even. Hence $R^*(.)$ obeys a relation analogous to (4b). On the other hand if $R^*(V_p^{cnj}) = R(V_c^j)$ is even and greater than 0, then by (4c), if $V_{pq}^{cnj} = (V_{aA}^1, \ldots, V_{cC}^j, \ldots, V_{eE}^m)$ is any follower of V_p^{cnj}, we know that $R(V_{cC}^j)$ is odd and less than $R(V_c^j)$. Therefore, in the first place

$$R^*(V_{pq}^{cnj}) \leq R(V_{cC}^j) < R(V_c^j) = R^*(V_p^{cnj})$$

so that

$$R^*(V_p^{cnj}) \geq \text{super}_q\, R^*(V_{pq}^{cnj}).$$

But by Lemma 1 there exists $V_{p\pi}^{cnj}$ such that $R^*(V_{p\pi}^{cnj}) = R(V_{cC}^j)$, and hence

$$\begin{aligned}
\text{super}_q\, R^*(V_{pq}^{cnj}) &\leq \text{super}_\pi\, R^*(V_{p\pi}^{cnj}) = \text{super}_C\, R(V_{cC}^j) \\
&= R(V_c^j) = R^*(V_p^{cnj})
\end{aligned}$$

By combining these two inequalities

$$R^*(V_p^{cnj}) = \text{super}_q\, R^*(V_{pq}^{cnj}) \tag{11}$$

In order to complete the proof that $R^*(.)$ obeys the relation analogous to (4c), it is necessary to show that $R^*(V_{pq}^{cnj})$ is odd for all q. But since in this case $R(V_c^j)$ is even, by (4c) the follower $V_{c\gamma}^j$ of V_c^j defined by (10) can be taken to be any of the followers V_{cC}^j; so

$$R^*(V_{pq}^{cnj}) \leq R(V_{cC}^j) = R(V_{c\gamma}^j),$$

which is odd. If equality holds in this last relation, $R^*(V_{pq}^{cnj})$ is odd; if inequality, $R^*(V_{pq}^{cnj})$ is odd by Lemma 2.

We have thus shown that $R(V_p^{cnj})$ and $R^*(V_p^{cnj})$ are functions obeying the same initial conditions and inductive relations (4a) to (4e), and hence that they are equal; i.e., (9) is proved.

The definition of a *conjunctive* combination can be extended in a natural way to non-impartial games; it is only necessary to proceed from the bicolored digraph $\Gamma(\mathscr{G})$ to the uncolored digraph $C(\mathscr{G}')$, by omitting the colors, and then using the definitions in terms of the impartial game $\mathscr{G}'$. This does not affect the value of the remoteness function, so that the remoteness function for the compound position is the minimum of the remoteness functions of its components, as before. The theory also extends naturally to a conjunctive compound of an infinite number of games, though no real player could be expected to play an infinite number of games simultaneously in any physical sense.

7. Selective Compounds

We consider only the case of a *finite* number of *impartial* games $\mathscr{G}^1, ..., \mathscr{G}^m$. A compound position $V_u^{sel} = (V_a^1, ..., V_e^m)$ is now a follower of $V_U^{sel} = (V_A^1, ..., V_E^m)$ if and only if for each $h (1 \leq h \leq m)$ either $V_d^h = V_D^h$, or V_d^h is a follower of V_D^h, and, for at least one h, V_d^h is a follower of V_D^h. It follows that V_u^{sel} is terminal if and only if every component is terminal. Also, since a player is not compelled to move in more than one component game at a time,

$$D(V_u^{sel}) = D(V_a^1) + \cdots + D(V_e^m). \tag{12}$$

The formula for the remoteness function is as follows. If each component V_d^h of V_u^{sel} has even $R(V_d^h)$, then

$$R(V_u^{sel}) = R(V_a^1) + \cdots + R(V_e^m), \tag{13a}$$

which is also even. If, on the other hand, a positive number k of components have odd remoteness function, and the remainder even remoteness functions, then

$$R(V_u^{sel}) = R(V_a^1) + \cdots + R(V_e^m) - k + 1 \tag{13b}$$

which is odd. If any $R(V_d^h) = \infty$, then $R(V_u^{sel}) = \infty$. These assertions can be readily proved by showing that $R(V_u^{sel})$ so defined satisfies the conditions (4a) to (4e). It follows from them that V_u^{sel} lies in class P if and only if every component is in P, in class O if at least one component is in O, and in class N otherwise. If every component configuration has finite terminal distance, this gives a simple rule for getting the best possible outcome, namely, to leave components in classes P and O in the same class, if possible (i.e., except when all components are in P, when a move in at least one component game to N is obligatory), and to move from components in N to ones in P.

Because both conjunctive and selective compounds are analyzed in terms of the remoteness $R(.)$ function, it is easy to give an analysis of conjunctive compounds of components which are themselves selective compounds, or selective compounds of conjunctive compounds, etc.

8. Conjunctive and Selective Compounds With the Last-Player Losing

A last-player-losing game has $T = T_N$; hence it can be analyzed by the method of Section 5, using the function $Q(.\,; \emptyset) = R'(.)$, say, which obeys the relations (4b) to (4e), but which takes the value 1 at terminal configurations. A conjunctive compound with the last-player-losing is defined exactly like one with the last-player-winning, except that all terminal configurations are in T_N. It can readily be shown to obey the relation

$$R'(V_p^{cnj}) = \min[R'(V_a^1), \ldots, R'(V_e^m)]$$

analogous to (9).

There does not seem to be any simple analog for Eqs. (13a) and (13b) in the last-player-losing selective compound. But if no component has terminal distance ∞ it is easy to give a strategy which will give the best outcome possible. So long as at least two of the components are non-terminal, the moves to be made are exactly the same as if the game was last-player-winning. When a player would according to this rule move to a configuration with fewer than 2 non-terminal components, he must instead move as follows. In all component games except one he must make the same move to a terminal configuration as he would in the last-player-winning game. In the remaining component game, $\mathscr{G}^j$, say, he either moves, or leaves the configuration unchanged, in such a way that the configuration V_c^j left has even $R'(V_c^j)$. Thereafter, this player follows a good strategy in this game $\mathscr{G}^j$ and hence will ultimately win. The correctness of this rule follows from the fact that a player who can move to a terminal position in any component game can force a win in the last-player-winning compound (for, as the terminal distance is by hypothesis not ∞, there are no configurations of class O); and this rule ensures that he can also force a win in the last-player-losing compound (so long as at least two components were non-terminal before the move).

9. Disjunctive Compounds

We restrict ourselves to the disjunctive compound of a finite number of impartial games with the last player winning. (The case [9] with the last-player-losing is very complicated.) A compound configuration $V_U^{dsj} = (V_A{}^1, \ldots, V_C{}^j, \ldots, V_E{}^m)$ is a follower of $V_u^{dsj} = (V_a{}^1, \ldots, V_c{}^j, \ldots, V_e{}^m)$ if and only if in one of the games, say $\mathscr{G}^j$, the component $V_C{}^j$ is a follower of V_c^j, while in every other component $\mathscr{G}^h$, $h \neq j$, $V_D{}^h = V_d{}^h$. That is, a move in the compound game consists of a move in just one component game, other component configurations remaining unchanged. A terminal configuration is one in which every component is terminal, and hence

$$D(V_u^{dsj}) = D(V_a{}^1) + \cdots + D(V_e{}^m). \tag{15}$$

When no terminal distance is infinite, the general method of analysis has been independently discovered by a number of workers [1, 8, 12, 18]. We first recapitulate this solution. If $(x_1, x_2, \ldots, x_n)$ is any set of non-negative integers, define the function "comin $(x_1, x_2, \ldots, x_n)$" to be the smallest non-negative integer different from all the x_i (and hence to be 0 if the set is empty). Define the Sprague-Grundy function $G(.)$ for any one game inductively by the relation

$$G(V_i) = \text{comin}_j \; G(V_{ij})$$
$$\text{(so that in particular } G(V_t) = 0 \text{ for terminal } V_t). \tag{16a}$$

In other words, the G function must change its value at each move (16b)

and if $0 \leq g < G(V_i)$, there exists a follower V_{ij} of V_i with $G(V_{ij}) = g$. (16c)

If the G function is defined for all configurations of terminal distance less than d, then (16a) defines it for all configurations of terminal distance d; hence (assuming for the moment that no terminal distance is ∞), the G function is uniquely defined for all configurations. The following properties follow straightforwardly from (16a):

$$G(V_i) \leq D(V_i). \tag{17}$$

The P class contains all configurations V_i with $G(V_i) = 0$, and the N class all those with $G(V_i) > 0$. For, if the initial configuration V_s has $G(V_s) > 0$, and Abe moves to a configuration V_{si} with $G(V_{si}) = 0$, which is possible by (16c), then either this is terminal, and Abe wins, or Barbara must move to a configuration V_{sij} with $G(V_{sij}) > 0$. The argument then repeats. Sooner or later the process must terminate, and Abe must win.

Now take any finite set of non-negative integers, say $(x, y, ..., z)$, and write then in the scale of 2:

$$x = \Sigma\, 2^a\, \xi_a; \qquad y = \Sigma\, 2^a\, \eta_a; \;\; \cdots; \;\; z = \Sigma\, 2^a\, \zeta_a;$$

where ξ_a, η_a, ζ_a take the values 0, 1. Let

$$\tau_a = \xi_a + \eta_a + \cdots + \zeta_a.$$

Then

$$S = \Sigma\, 2^a\, \tau_a \tag{18}$$

is clearly the sum $x + y + \cdots + z$. Furthermore let ϱ_a be the remainder on dividing τ_a by 2. Then:

$$R = \Sigma\, 2^a\, \varrho_a \tag{19}$$

is the *nim-sum* of x, y, ..., z, and will be written $x +_2 y +_2 \cdots +_2 z$. Nim-addition is commutative and associative, and for all x

$$x +_2 0 = x; \qquad x +_2 x = 0. \tag{20}$$

It is also not difficult to prove the following.

LEMMA 3. *Let u, x, y, ..., z be non-negative integers such that $u < x +_2 y +_2 \cdots +_2 z$.* Then there exists either X with $0 \leq X < x$, such that

$$u = X +_2 y +_2 \cdots +_2 z$$

and/or Y, with $0 \leq Y < y$, such that

$$u = x +_2 Y +_2 \cdots +_2 z$$

and/or ... and/or Z, with $0 \leq Z < z$, such that

$$u = x +_2 y +_2 \cdots +_2 Z.$$

The analysis of a disjunctive compound game depends on the following result: Let

$$V_u^{dsj} = (V_a^1, V_b^2, ..., V_e^m).$$

Then

$$G(V_u^{dsj}) = G(V_a^1) +_2 G(V_b^2) +_2 \cdots +_2 G(V_e^m). \tag{21}$$

Denote the right-hand side of (21) by $G^*(V_u^{dsj})$. Then to prove (21) we have to show that the G^* function satisfies the conditions (16b), (16c). Now a move in the disjunctive compound consists in a move in one and only one of the component games. By (16b), this changes the G-value in this component, but not in any other component; it therefore changes the G^* value, which is the nim-sum of the G-values in the separate components. This verifies (16b) for G^*. Furthermore, suppose that $0 \leq g < G^*(V_u^{dsj})$. Applying Lemma 3, we find that there exists, say, $Y < G(V_b^2)$ such that

$$g = G(V_a^1) +_2 Y +_2 \cdots +_2 G(V_e^m).$$

Hence, by (16c), there is a V_{bj}^2 with $G(V_{bj}^2) = Y$, so that by the definition of $G^*(.)$,

$$g = G^*(V_a^1, V_{bj}^2, ..., V_e^m) = G^*(V_{uj}^{dsj}), \text{ say,} \tag{22}$$

where V_{uj}^{dsj} is a follower of V_u^{dsj}. This verifies (16c) for the $G^*(.)$ function, and completes the proof.

Numerical examples of $G(.)$ functions are given by Sprague [18, 19], Guy and Smith [10], Adams and Benson [1], and Holladay [12].

The analysis above can break down when any component configurations have terminal distance ∞. A function satisfying (16a) is called a *Sprague-Grundy function in the wide sense*. In some graphs (e.g., those containing loops), there does not exist such a function, while in others (e.g., in a cycle $V_1 \rightleftarrows V_2$ with two vertices) it is not unique. We therefore approach the problem somewhat differently.

We proceed to define pairs of functions, $G_u(.)$, $H_u(.)$. We restrict ourselves for the moment to non-negative integer values of these functions; later we also admit the value ∞. Such a pair of functions will be called a (G, H) pair if it satisfies the following conditions (23a) to (23e):

$$\text{The functions } G_u(V_i),\ H_u(V_i) \text{ need not be defined for all } V_i\text{; but if either } G_u(V_i) \text{ or } H_u(V_i) \text{ is a non-negative integer, so also is the other.} \tag{23a}$$

$$\text{If } V_t \text{ is terminal, } G_u(V_t) = H_u(V_t) = 0. \tag{23b}$$

$$\text{If } G_u(V_i) \text{ is a non-negative integer, and } 0 \leq y < G_u(V_i) \text{ (where } y \text{ is an integer), then there exists a follower } V_{ir} \text{ with } G_u(V_{ir}) = y \text{ and } H_u(V_{ir}) < H_u(V_i). \tag{23c}$$

$$\text{If } G_u(V_i) \text{ is a non-negative integer, and } H_u(V_{ij}) < H_u(V_i), \text{ then } G_u(V_{ij}) \neq G_u(V_i). \tag{23d}$$

$$\text{If } G_u(V_i) \text{ is a non-negative integer, and it is not true that } H_u(V_{ij}) < H_u(V_i) \text{ (i.e., if } H_u(V_{ij}) \geq H_u(V_i) \text{ or if } H_u(V_{ij}) \text{ is undefined), then there exists a follower } V_{ijk} \text{ of } V_{ij} \text{ with } G_u(V_{ijk}) = G_u(V_i) \text{ and } H_u(V_{ijk}) + 1 < H_u(V_i). \tag{23e}$$

Note that, as far as configurations with non-infinite terminal distance are concerned, these conditions are satisfied by putting $G_u(V_i) = G(V_i)$, the usual Sprague-Grundy function defined by (16a), and $H_u(V_i) = D(V_i)$. The function $G_u(.)$ is therefore a generalization of the Sprague-Grundy function, and will be called a *Sprague-Grundy function* (*in the strict sense*). The second function $H_u(.)$ is introduced merely to give a basis for induction.

We first show that, if V_{ij} is any follower of any configuration V_i such that both $G_u(V_{ij})$ and $G_u(V_i)$ are non-negative integers, then

$$G_u(V_{ij}) \neq G_u(V_i). \tag{24}$$

For if $H_u(V_{ij}) < H_u(V_i)$, this is (23d). Otherwise, by (23e) there exists V_{ijk} with $G_u(V_{ijk}) = G_u(V_i)$ and

$$H_u(V_{ijk}) < H_u(V_i) \leq H_u(V_{ij}),$$

whence by (23d) with "i" replaced by "ij"

$$G_u(V_{ij}) \neq G_u(V_{ijk}) = G_u(V_i),$$

completing the proof.

From (23c) and (23d) it follows that if $G_u(V_i)$ is a non-negative integer, it is comin$_j$ $G_u(V_{ij})$ taken over all V_{ij} for which $H_u(V_{ij}) < H_u(V_i)$; from (24) it further follows that

$$G_u(V_i) = g_u(V_i) \tag{25}$$

where $g_u(V_i) =$ comin$_j$ $G_u(V_{ij})$ taken over all V_{ij} for which $G_u(V_{ij})$ is defined.

Now if there are any vertices V_i for which $G_u(V_i)$ and $H_u(V_i)$ are not defined, it is natural to try to extend the functions $G_u(.)$ and $H_u(.)$ as far as possible by assigning values to the $G_u(V_i)$ and $H_u(V_i)$, subject to the conditions (23a) to (23e). Since the digraph of the game is at present supposed finite, this involves only a finite number of assignments. When no further values of $G_u(V_i)$ and $H_u(V_i)$ can be given without violating the conditions, we assign the value ∞ to all remaining undefined $G_u(V_i)$ and $H_u(V_i)$, much as we did also with the functions $D(.)$ and $R(.)$. From now on we suppose this done, so that $G_u(V_i)$ and $H_u(V_i)$ are defined, though possibly ∞, for every vertex V_i.

What is the condition that we should have $G_u(V_i) = H_u(V_i) = \infty$? V_i cannot be terminal, by (23b). Suppose that the following condition is fulfilled:

$$\begin{gathered}\text{There exists } V_{ij} \text{ such that } G_u(V_{ij}) = \infty, \text{ and no follower} \\ V_{ijk} \text{ of } V_{ij} \text{ has } G_u(V_{ijk}) = g_u(V_i).\end{gathered} \tag{26}$$

Then we must have $G_u(V_i) = \infty$. For, by (25), the only other possible value of $G_u(V_i)$ would be $g_u(V_i)$, and (26) would then violate (23e). Hence (26) is a sufficient condition for $G_u(V_i) = \infty$. Suppose on the other hand that (26) is untrue; and supposing $G_u(V_i)$, $H_u(V_i)$ undefined let us try assigning the values $G_u(V_i) = g_u(V_i)$, $H_u(V_i) =$ super$[H_u(V_{ij}), H_u(V_{ijk}) + 1]$, including only non-negative integral values of the functions inside the brackets. If these assigned values are compatible with the conditions (23a) to (23e), then we cannot have $G_u(V_i) = H_u(V_i) = \infty$, which happens by definition only when such assignment is impossible. It is straightforward to verify that (23a) to (23e) do in fact hold for V_i.

However, $V_i = V_{hi}$ could be the follower of some vertex V_h for which $G_u(V_h)$ is not ∞; it is then necessary to check that the assignments of the values of $G_u(V_i) = G_u(V_{hi})$ and $H_u(V_i) = H_u(V_{hi})$ do not violate the conditions on the values of $G_u(V_h)$ and $H_u(V_h)$. Now conditions (23a), (23b), (23c) on V_h are unaffected by the assignments, and (23e) is either unaffected or becomes void if $H_u(V_{hr}) < H_u(V_h)$. The only contradiction which could arise would therefore be to (23d), if we should have $G_u(V_{hi}) = G_u(V_h)$. But this is not possible, for, as $G_u(V_{hi})$ was supposed undefined before we made the assignment, there must exist by (23e) a V_{hij} with $G_u(V_h) = G_u(V_{hij}) = G_u(V_{ij})$, which by definition is different from $g_u(V_i) = G_u(V_i) = G_u(V_{hi})$. A similar argument shows that, if $V_i = V_{fhi}$, then the assignment will not violate any condition applied to V_f. Hence if (26) is false, $G_u(V_i) \neq \infty$; i.e., (26) is the necessary and sufficient condition for $G_u(V_i) = H_u(V_i) = \infty$.

We now assert that the Sprague-Grundy function (in the strict sense) is in fact unique; if $G_u(.)$, $H_u(.)$ and $G_w(.)$, $H_w(.)$ are two (G, H) pairs, then for all V_i

$$G_u(V_i) = G_w(V_i) \tag{27}$$

For suppose otherwise, say that we can find V_i such that $G_u(V_i) < G_w(V_i)$. Let h be the smallest value of $H_u(V_j)$ taken over all V_j with $G_u(V_j) \neq G_w(V_j)$, and let such a V_j be chosen with $H_u(V_j) = h$. We show that this leads to a contradiction.

By (23c), whenever $0 \leq y < G_u(V_j)$, there exists V_{jk} with $H_u(V_{jk}) < H_u(V_j) = h$ and $y = G_u(V_{jk})$. Hence, by the definition of h,

$$y = G_u(V_{jk}) = G_w(V_{jk}) \tag{28}$$

We now observe that no V_{jl} has $G_w(V_{jl}) = G_u(V_j)$. For suppose otherwise. Then if $H_u(V_{jl}) < h$, we have $G_u(V_{jl}) = G_w(V_{jl}) = G_u(V_j)$, contradicting (24); whereas if $H_u(V_{jl}) \geq h = H_u(V_j)$, then by (23e) there exists V_{jlm} with $H_u(V_{jlm}) < H_u(V_j) = h$, and so $G_w(V_{jlm}) = G_u(V_{jlm}) = G_u(V_j) = G_w(V_{jl})$, again contradicting (24). It follows from this and (28) that $g_w(V_j) = G_u(V_j)$. Now, if $G_w(V_j) \neq \infty$, (25) shows that $G_w(V_j) = g_w(V_j) = G_u(V_j)$, contradicting our definition of V_j; so we have $G_w(V_j) = \infty$. By (26) there exists V_{jk} with $G_w(V_{jk}) = \infty$, but

$$\text{no } V_{jkl} \text{ has } G_w(V_{jkl}) = g_w(V_j) = G_u(V_j) \tag{29}$$

We cannot have $H_u(V_{jk}) < h$, for that would imply that $\infty = H_w(V_{jk})$

$= H_u(V_{jk}) < h$, a contradiction. Hence $H_u(V_{jk}) \geq h = H_u(V_j)$, and so by (23e) there exists V_{jkl} with $H_u(V_{jkl}) < H_u(V_j) = h$, and so $G_u(V_{jkl}) = G_u(V_{jkl}) = G_u(V_j)$, contradicting (29). The uniqueness of the Sprague-Grundy function is therefore established, and we can drop suffixes, writting simply $G(.)$ instead of $G_u(.)$. In the argument which follows, it will not be important which H-function $H_u(.)$ we choose to use, and so it will also be convenient to write it simply as $H(.)$, leaving the suffix "u" to be understood. (If we wanted to pick out a unique H-function we could use $H_{\min}(.) = \min_u H_u(.)$; but as this is not needed, we leave the proof that this is in fact an H-function to the reader.)

If the starting configuration V_s has $G(V_s) = 0$, then the previous player (Barbara) can force a win. If V_s is terminal, this is trivial; if not, Abe moves to some position V_{si} with $G(V_{si}) \neq G(V_i) = 0$, by (24). If $H(V_{si}) < H(V_i)$, then $G(V_{si})$ must be a non-negative integer, and by (23c) there exists a follower V_{sij} with $G(V_{sij}) = 0$ and $H(V_{sij}) < H(V_{si}) < H(V_s)$. Let Barbara move to this. If, on the other hand, $H(V_{si}) \geq H(V_s)$, then by (23e) there exists V_{sij} with $G(V_{sij}) = 0$ and $H(V_{sij}) + 1 < H(V_s)$; let Barbara move to this. The play has thus returned to a position with zero G value, but the H value has been diminished by at least 2, and it is again Abe's turn to move. The argument is then repeated; sooner or later H will be reduced to zero and Barbara will win. Hence configurations V_i with $G(V_i) = 0$ are such that the previous player can force a win, and are therefore in class P. Also, since he can be sure of winning in not more than $H(V_s)$ moves, we have $H(V_s) \geq R(V_s)$.

If V_s has a follower V_{si} with $G(V_{si}) = 0$, then Abe can be sure of winning by moving to V_{si}. That is, the next player can force a win, and V_s is in N. By (23c) this is true whenever $G(V_s)$ is a positive integer; and in that case we can also make $H(V_{si}) < H(V_s)$. Thus Abe can be sure of winning in not more than $H(V_{si}) + 1 \leq H(V_s)$ moves, and hence $H(V_s) \geq R(V_s)$. It is also possible for there to be V_{si} with $G(V_{si}) = 0$ when $G(V_s) = \infty$; in that case, $H(V_s) > R(V_s)$ trivially.

The remaining case is that $G(V_s) = \infty$, but there is no V_{si} with $G(V_{si}) = 0$, and hence $g(V_s) = 0$. By (26) there does exist V_{si} with $G(V_{si}) = \infty$ and with no follower V_{sij} which has $G(V_{sij}) = 0$. Now Abe can certainly be defeated if he moves to a configuration V_{sk} with either $G(V_{sk})$ positive or $G(V_{sk}) = \infty$ and some $G(V_{skl}) = 0$. Hence his only hope of avoiding defeat is by moving to the position V_{si} defined above. From this, by the same argument, Barbara will either move to some position which enables Abe to force a win, or else she moves to a position V_{sij} such that $G(V_{sij}) = \infty$ and such that there exists no follower V_{sijm} with $G(V_{sijm}) = 0$. The argument now repeats; by moving in this way Abe can guarantee that he will not lose, and Barbara, by playing well, can also guarantee that she will not lose. Hence, if $G(V_s) = \infty$ and no $G(V_{sh}) = 0$, V_s lies in class O. In all cases we have

$$H(V_s) \geq R(V_s). \tag{30}$$

It is also not difficult to verify that for all V_i

$$H(V_i) \geq G(V_i), \tag{31}$$

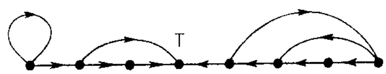
but we omit the proof.

In view of the classification we have given above of configurations with $G(V_i) = \infty$, it is convenient to define

$$J(V_i) = \text{the set of all } G(V_{ij}) \tag{32}$$

(taken over all followers V_{ij} of V_i). If $G(V_i) = \infty$, V_i belongs to N or O according as 0 is or is not a member of $J(V_i)$.

An example of a configuration digraph and its $G(.)$ and (minimum) $H(.)$ functions is shown in Figure 1. The $D(.)$ and $R(.)$ functions and the value class (N, O, P) are also shown for comparison.



Vertex V_i	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
$D(V_i)$	∞	2	1	0	∞	∞	∞	∞
$R(V_i)$	∞	1	1	0	1	2	3	3
$G(V_i)$	∞	2	1	0	∞	0	1	2
$H(V_i)$	∞	2	1	0	∞	2	3	4
Class	O	N	N	P	N	P	N	N

FIGURE 1

10. Disjunctive Compounds for Finite Digraphs

Consider a disjunctive compound of two games, and let $V_u^{dsj} = (V_a^1, V_b^2)$ be a typical configuration in it. We now assert that we can set

$$G(V_u^{dsj}) = G(V_a^1) +_2 G(V_b^2), \tag{33a}$$

$$H(V_u^{dsj}) = H(V_a^1) + H(V_1^2), \tag{33b}$$

where $x + y$ and $x +_2 y$ are both conventionally defined to be ∞ whenever either x or y is ∞. We have to verify that the (G, H) pair defined by (33a), (33b) satisfy the conditions (23a) to (23e) when they are integers, and (26) when they are ∞.

First, suppose that $G(V_a^1)$, $G(V_1^2)$ are both non-negative integers. Conditions (23a), (23b), and (23d) on the pair $G(V_u^{dsj})$, $H(V_u^{dsj})$ follow immediately from the same conditions on the component pairs; and (23c) for the compound follows from the same condition on the components by the use of Lemma 3, exactly as in the proof of Eq. (21) above. There remains condition (23e). Suppose, then, that V_{uv}^{dsj} is a follower of V_u^{dsj} with $H(V_{uv}^{dsj}) \geq H(V_u^{dsj})$. V_{uv}^{dsj} must have one of the forms (V_{aA}^1, V_b^2) or (V_a^1, V_{bB}^2); without loss of generality, we may suppose that the first of these forms holds. We must therefore have $H(V_{aA}^1) \geq H(V_a^1)$, and so by (23e) there exists V_{aAa}^1 with $G(V_{aAa}^1) = G(V_a^1)$ and $H(V_{aAa}^1) + 1 < H(V_a^1)$. Set $V_{uvw}^{dsj} = (V_{aAa}^1, V_b^2)$. Then $G(V_{uvw}^{dsj}) = G(V_u^{dsj})$ and $H(V_{uvw}^{dsj}) + 1 < H(V_u^{dsj})$, verifying (23e).

Now suppose that one of $G(V_a^1)$, $G(V_b^2)$ is ∞, say $G(V_a^1)$, but not $G(V_b^2)$. Let $G(V_{uv}^{dsj}) \neq \infty$; then by (33b) V_{uv}^{dsj} must be of the form (V_{aA}^1, V_b^2) with $G(V_{aA}^1) \neq \infty$, and so by (33a)

$$G(V_{uv}^{dsj}) = G(V_{aA}^1) +_2 G(V_b^2) \neq g(V_a^1) +_2 G(V_b^2),$$

whence

$$g(V_{uv}^{dsj}) = \operatorname{comin}_v G(V_{uv}^{dsj}) \leq g(V_a^1) +_2 G(V_b^2). \tag{34}$$

We now have two cases to consider:

CASE I. $g(V_u^{dsj}) = g(V_a^1) +_2 G(V_b^2)$.

Since $G(V_a^1) = \infty$, by (26) there exists V_{aj}^1 with $G(V_{aj}^1) = \infty$ and having no follower with $G(V_{ajk}^1) = g(V_a^1)$. Hence $V_{uw}^{dsj} = (V_{aj}^1, V_b^2)$ has also the properties that $G(V_{uw}^{dsj}) = \infty$ and no $G(V_{uwx}^{dsj}) = g(V_u^{dsj})$.

CASE II. $g(V_u^{dsj}) < g(V_a^1) +_2 G(V_b^2)$.

By Lemma 3 either there exists $y < g(V_a^1)$ such that

$$g(V_u^{dsj}) = y +_2 G(V_b^2) \tag{35a}$$

or there exists $z < G(V_b^2)$ such that

$$g(V_u^{dsj}) = g(V_a^1) +_2 z. \tag{35b}$$

Now if (35a) should hold, there would exist by (23c) V_{aA}^1 with $G(V_{aA}^1)=y$, and hence $G(V_{aA}^1, V_b^2) = g(V_u^{dsj})$, contrary to the definition of $g(V_u^{dsj})$. Hence (35b) must hold, and by (23c) there exists V_{bB}^2 with $G(V_{bB}^2) = z$. Thus $V_{uw}^{dsj} = (V_a^1, V_{bB}^2)$ is a follower of V_u^{dsj} with $G(V_{uw}^{dsj}) = \infty$. Any follower V_{uwx}^{dsj} with $G(_{uwx}^{dsj}) \neq \infty$ must be of the form (V_{aj}^1, V_{bB}^2), so that

$$G(V_{uwx}^{dsj}) = G(_{aj}^1) +_2 G(V_{bB}^2) \neq g(V_a^1) +_2 z = g(V_u^{dsj}),$$

using (35b). In either Case I or Case II (25) is accordingly verified.

Finally, suppose $G(V_a^1) = G(V_b^2) = \infty$. In this case, for every follower V_{uv}^{dsj} we have $G(V_{uv}^{dsj}) = \infty$; hence $g(V_u^{dsj}) = 0$. But there exists by (26) some follower $V_{uw}^{dsj} = (V_{aA}^1, V_b^2)$ with $G(V_{aA}^1) = G(V_b^2) = \infty$, and this can have no follower with $G(V_{uwx}^{dsj}) = 0 = g(V_u^{dsj})$; again (26) is true. Thus the functions defined by (33a), (33b) satisfy all the conditions for a (G, H) pair. Because the operations of disjunctive combination, ordinary and nim addition are all associative, these relations extend immediately to a disjunctive compound of any (finite) number of components:

$$G(V_a^1, V_b^2, V_c^3) = G(V_a^1) +_2 G(V_b^2) +_2 G(V_c^3), \tag{36a}$$

$$H(V_a^1, V_b^2, V_c^3) = H(V_a^1) + H(V_b^2) + H(V_c^3). \tag{36b}$$

If $G(V_a^1) = \infty$ it also follows directly from the definition of the $J(.)$ function that

$$x \in J(V_a^1) \Leftrightarrow [x +_2 G(V_b^2) +_2 G(V_c^3)] \in J(V_a^1, V_b^2, V_c^3). \tag{36c}$$

These formulae define the $G(.)$, $H(.)$, and $J(.)$ functions for disjunctive compounds, and hence allow the players to determine appropriate strategies for giving the best guaranteed outcomes.

Note that, if a disjunctive compound configuration has an infinite number of components, either all but a finite number are terminal, in which case the theory developed above applies, ignoring the terminal components, or else an infinite number are non-terminal, and the play must be infinitely long, resulting in an unterminated draw.

11. Games with Infinite Digraphs

In dealing with infinite graphs we will take for granted the classical theory of sets, including the axiom of choice (if only because this is not the most appropriate place to discuss possible criticisms of the theory). Most of the results we have given above then extend in a direct and straightforward way to infinite graphs, with the proviso that wherever we have referred to a "non-negative integral value" of the functions D, R, Q, G, H, J, this must now be replaced by an ordinal (finite or transfinite), and the proofs will proceed by transfinite induction. As a typical example we discuss the function $D(.)$; the arguments for the other functions follow analogous patterns.

We first construct a transfinite sequence of classes $\Delta(0)$, $\Delta(1)$, ... of vertices, where $\Delta(\alpha)$ will turn out to be the set of vertices V_i^X for which $D(V_i^X) = \alpha$. A vertex V_i^X is assigned to $\Delta(\alpha)$ if and only if every follower V_{ij}^Y belongs to some $\Delta(\beta_j)$ with $\beta_j < \alpha$, but V_i^X itself does not; in particular, $\Delta(0) = T$. Since this defines $\Delta(\alpha)$ in terms of the classes $\Delta(\gamma)$ with $\gamma < \alpha$, it follows that $\Delta(\alpha)$ is uniquely defined for every ordinal α. Also, V_i^X cannot belong to two $\Delta(\alpha)$, say $\Delta(\gamma)$ and $\Delta(\alpha)$ with $\gamma < \alpha$, for that would contradict the definition of $\Delta(\alpha)$. We set $\Delta(\infty) =$ the set of all V_i^X not in $\Delta(\alpha)$ for any ordinal α. The function $D(V_i^X) = \alpha$ where $V_i^X \in \Delta(\alpha)$ is then uniquely defined for all V_i^X, and it is not difficult to show that if all $D(V_{ij}^Y)$ are ordinal

$$D(V_i^X) = \text{super}_j\, D(V_{ij}^Y),$$

agreeing with (2), but with the right-hand side now meaning, of course, the "smallest ordinal greater than any $D(V_{ij}^Y)$ for varying j." If any $D(V_{ij}^Y) = \infty$, then $D(V_i^X) = \infty$. We look upon ∞ as standing for a conventional "number" "greater than all ordinals." Eq. (3) is therefore still satisfied, and the definition agrees with our former one for a finite graph. If $D(V_i^X)$ is finite, it is still the maximum length of play beginning at V_i^X, and conversely. If $D(V_i^X)$ is transfinite, the plays starting from V_i^X can no longer have bounded length, but we note that by (2)

$$D(V_i^X) > D(V_{ij}^Y) > D(V_{ijk}^X) > \cdots \tag{37}$$

and since any decreasing sequence of ordinals terminates, any play from V_i^X must have finite length. If $D(V_i^X) = \infty$, then, as before, there exists an unterminated play.

Any ordinal α has a unique expansion in the scale of 2:

$$\alpha = \Sigma_\lambda 2^\lambda a_\lambda \qquad (a_\lambda = 0,\ 1). \tag{38}$$

We say that α is odd or even according as $a_0 = 1$ or 0. If, in the same way, $\beta = \Sigma_\lambda 2^\lambda b_\lambda$, $\gamma = \Sigma_\lambda 2^\lambda c_\lambda$ then we define Hessenberg's [11] *natural sum* as

$$\alpha +_H \beta +_H \gamma = \Sigma_\lambda 2^\lambda (a_\lambda + b_\lambda + c_\lambda). \tag{39}$$

Here "$+_H$" is a commutative and associative operation different in general from the usual addition of ordinals. If t_λ is the remainder on dividing $a_\lambda + b_\lambda + c_\lambda$ by 2, the *nim-sum* is defined as

$$\alpha +_2 \beta +_2 \gamma = \Sigma_\lambda 2^\lambda t_\lambda. \tag{40}$$

These definitions allow the discussion of the R, G, H, and J functions to proceed substantially as before, except that every addition must now be understood in the sense of a natural addition. Thus, if the last player wins and $R(V_s^A)$ is finite odd, then as before Abe can force a win in at most $R(V_s^A)$ moves. If $R(V_s^A)$ is transfinite odd, Abe can still be sure of winning, but Barbara can choose to delay defeat for as long as she wishes (but not for an infinitely long time). Similar remarks apply to even values of $R(V_s^A)$, but with the two players interchanged. This use of transfinite ordinals was first introduced by Kalmár [13], although König [14] had already shown by another method that there exists a best strategy in any game which is sure to terminate. Other authors have rediscovered and extended the idea [1, 3–5, 12]. Note that, even if the D, R, G, or H functions are transfinite ordinal, the rules of the game may still be expressed in finite terms. Consider the following example: two players have between them a pool of d dollar coins and c cents. A move consists of removing one cent, except that if $c = 0$, and this is impossible a player must exchange one dollar for 100 cents, and then take his cent. The player who leaves no money wins. In this trivial game

$$D(\$d + c¢) = R(\$d + c¢) = 100d + c.$$

If, however, we modify the rules to say that he can exchange the dollar for any positive even number of cents he chooses, however large, before taking his cent, we have

$$D(\$d + c¢) = R(\$d + c¢) = \omega d + c,$$

where ω is as usual the smallest transfinite ordinal. The game becomes a little less trivial if we take a conjunctive compound of, say, two piles; Eq. (9) can still be shown to be true. In the finite case a good strategy in a conjunctive compound consists in following a good strategy independently in each component game. This will no longer be true in general when transfinite ordinal values occur. Suppose the starting position had two components of \$1.01¢ and \$1.02¢, respectively, and therefore with R values $\omega + 1$ and $\omega + 2$; the R value for the compound is the minimum of these, $\omega + 1$, which is odd, and hence Abe can force a win. His first move is necessarily to (\$1.00¢, \$1.01¢), and Barbara then moves to $((2m + 1)$¢, \$1.00¢). Abe will now move to $(2m$¢, $(2n - 1)$¢); if he chooses the strategies in the two components independently, he will choose n independently of the value of m. But if he chooses $n < m$ he will lose.

The simplest type of transfinite disjunctive compound is transfinite nim, due to R. Rado (personal communication). This is played with a set of ordinals α, β, ..., δ, with the rule that a player at his turn must diminish just one of the ordinals; a player who cannot move loses. Then

$$G(\alpha, \beta, \cdots, \delta) = \alpha +_2 \beta +_2 \cdots +_2 \delta.$$

12. Criticism of the Solution for Infinite Digraphs

The solution we have given above shows formally how to find a good strategy even for a game with an infinite digraph. But in several respects it is hardly realistic when applied to human players. Two of the more serious difficulties are that it may require an infinite amount of calculation to find the strategy, and that if a player is determined to win his opponent can sometimes prolong the play to an arbitrary length, e.g., to 10^{100} moves; a player might well prefer to lose rather than wait so long. However, if we allow that a quick loss may be preferable to a slow win, the interests of the two players are no longer completely opposed, and a solution, even for finite games, involves a difficult point in game theory [15]. We will therefore limit the discussion here to showing how to determine a player's *security level*, i.e., the best result he can guarantee however his opponent may play. We return to the general case, discussed in Sections 1 to 5, in which there are 3 classes of terminal position, T_A, T_B, and T_O, corresponding respectively to a win for

Abe, a win for Barbara, and a draw. We make the natural assumption that a player will prefer a draw for which he can guarantee an upper bound on the number of moves ("bounded draw" for brevity) to a draw which he can be sure to attain in a finite number of moves, without any guaranteed bound on the number ("terminated draw"), and that this in turn is preferable to a draw in possibily an infinite number of moves. Similarly he prefers a bounded win to a win without the guaranteed bound (which must be terminated), and the same for a loss. It is also natural to assume the preferences "bounded win to bounded draw to bounded loss," and "win to terminated draw to loss." These preferences impose a partial ordering on the outcomes, leaving the players free to choose for themselves whether they prefer, for instance, a guarantee only of a win (without bounded length) to a bounded draw, or vice versa.

In Section 5 we introduced two Q functions, $Q_1(.)$ (for which $S_A = T_A$) and $Q_0(.)$ (for which $S_A = T_A \cup T_0$), and corresponding strategies $\sigma_1{}^X(.)$ and $\sigma_0{}^X(.)$. If $Q_1(V_s{}^A)$ is odd at the starting position $V_s{}^A$, the strategy $\sigma_1{}^A(.)$ guarantees Abe a win; if moreover $Q_1(V_s{}^A)$ is *finite* odd,

TABLE 1

SECURITY LEVELS FOR ABE

(fin. = finite; bdd. = bounded; trs. = transfinite; ter. = terminated)

Values of Q_r functions			Security level under strategy		
$Q_1(V_s{}^A)$	$Q_0(V_s{}^A)$	$Q_{-1}(V_s{}^A)$	$\sigma_1{}^X(.)$	$\sigma_0{}^X(.)$	$\sigma_{-1}^X(.)$
fin. odd	fin. odd	fin. odd	bdd. win		
trs. odd	fin. odd	fin. odd	win	bdd. draw	
trs. odd	trs. odd	fin. odd	win		bdd. loss
trs. odd	trs. odd	trs. odd	win		
not odd	fin. odd	fin. odd		bdd. draw	
not odd	trs. odd	fin. odd		ter. draw	bdd. loss
not odd	trs. odd	trs. odd		ter. draw	
not odd	∞	fin. odd		unter. draw	bdd. loss
not odd	∞	trs. odd		draw	loss
not odd	not odd	∞		draw	
even	even	fin. odd			bdd. loss
even	even	trs. odd			loss
even	even	even			

the strategy guarantees a bounded win. The same applies to $Q_0(.)$ with the word "draw" substituted for "win," and where by "guaranteeing a draw" we include the possibility of doing better, i.e., winning, should the opponent not play as well as possible. We now introduce a third function, $Q_{-1}(.)$, defined by taking $S_A = T$; an odd value of $Q_{-1}(V_s^A)$ now guarantees Abe a loss (or better), i.e., a termination of the play, and a finite odd value of $Q_{-1}(V_s^A)$ guarantees a termination in not more than $Q_{-1}(V_s^A)$ moves by use of the corresponding strategy $\sigma_{-1}^V(.)$. Hence by considering the values of $Q_m(V_s^A)$ ($m = 1, 0, -1$) we can deduce the security levels for Abe shown in Table 1. A similar table of security levels for Barbara can be constructed by using instead of the functions $Q_1(.)$, $Q_0(.)$, $Q_{-1}(.)$, the functions $Q_0(.)$, $Q_1(.)$, $Q_2(.)$, respectively, where $Q_2(.)$ is obtained by taking $S_A = \emptyset$, $S_B = T$.

If, instead of the outcomes of the game being merely a win, draw, or loss, there are a number of possible outcomes such as money rewards in order of preference, a similar analysis can be performed by use of the functions $Q_v(.)$, where $Q_v(.)$ is defined by taking S_A to be the set of all terminal positions which result in a gain of v or more to Abe. If $Q_v(V_s^A)$ is odd, Abe can then be sure of winning at least v; and if $Q_v(V_s^A)$ is finite odd, Abe can win it in not more than $Q_v(V_s^A)$ moves.

I wish to acknowledge the kindness of Mr. A. J. Lee in preparing the diagram for Figure 1.

References

1. E. W. Adams and D. C. Benson, Nim-Type Games, *Tech. Rep. No. 13*, Carnegie Institute of Technology, Department of Mathematics, Pittsburgh, Pa, 1956.
2. C. Berge, Sur une Théorie ensembliste des Jeux alternatifs, *J. Math. Pures Appl.* **32** (1953), 129–184.
3. C. Berge, La Fonction de Grundy d'une Graphe infini, *Comp. Rend. Acad. Sci. Paris* **242** (1956), 1404.
4. C. Berge, *Théorie des Graphes et ses Applications*, Dunod, Paris, 1956.
5. C. Berge and M. P. Schützenberger, Jeux de Nim et Solutions, *Comp. Rend. Acad. Sci. Paris* **242** (1956), 1672.
6. C. L. Bouton, Nim a Game with a Complete Mathematical Theory, *Ann. of Math.* **3** (1902), 35–39.
7. D. Gale and F. M. Stewart, Infinite Games with Perfect Information, *Contributions to the Theory of Games* **2** (Annals of Mathematics Studies No. 28), Princeton, N. J., 1953, pp. 245–266.

8. P. M. GRUNDY, Mathematics and Games, *Eureka* **2** (1939), 6–8.

9. P. M. GRUNDY AND C. A. B. SMITH, Disjunctive Games with the Last Player Losing, *Proc. Cambridge Philos. Soc.* **52** (1956), 527–533.

10. R. K. GUY and C. A. B. SMITH, The *G*-Values of Various Games, *Proc. Cambridge Philos. Soc.* **52** (1956), 514–526.

11. G. HESSENBERG, Grundbegriffe der Mengenlehre, *Abh. Friesschen Schule* (*Göttingen*) **1**, Heft 4 1906.

12. J. C. HOLLADAY, Cartesian Products of termination games, *Contributions to the Theory of Games* 3 (Annals of Mathematics Studies No. 39), Princeton, N. J., 1957, pp. 189–199.

13. L. KALMÁR, Zur Theorie der Abstrakten Spiele, *Acta Sci. Math. Univ. Szeged.* **4** (1928), 65–85.

14. D. KÖNIG, Uber eine Schlussweise aus dem Endlichen ins Unendliche, *Acta Sci. Math. Univ. Szeged.* **3** (1927), 121–130.

15. R. D. LUCE and H. RAIFFA, *Games and Decisions*, Wiley, New York, 1957.

16. J. MILNOR, Sums of Positional Games, *Contributions to the Theory of Games* 2 (Annals of Mathematics Studies No. 28), Princeton, N. J., 1953, pp. 291–303.

17. W. W. ROUSE BALL and H. S. M. COXETER, *Mathematical Recreations and Essays*, 11th ed., Macmillan, London, 1949.

18. R. SPRAGUE, Über mathematische Kampfspiele, *Tôhoku Math. J.* **41** (1935), 438–444.

19. R. SPRAGUE, Über zwei Abarten von Nim, *Tôhoku Math. J.* **43** (1937), 351–359.

20. H. STEINHAUS, Definitions for a Theory of Games and Pursuit, *Myśl Akad. Lwów* **1**, No. 1 (1925), 13–14; reprinted in *Naval Res. Logist. Quart.* **7** (1960), 105–108.

21. J. VON NEUMANN and O. MORGENSTERN, *Theory of Games and Economic Behavior*, Princeton Univ. Press, Princeton, N. J., 1944.

22. E. ZERMELO, Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels, *Proc. 5th Internat. Cong. Math.* **2** (1912), 501–504.