Topology and its Applications 159 (2012) 3314–3318

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A proof of the Edwards–Walsh resolution theorem without Edwards–Walsh CW-complexes

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ARTICLE INFO

Article history:
Received 10 February 2011
Received in revised form 16 July 2012
Accepted 16 July 2012

MSC:
primary 54F45, 55M10, 55P20, 54C20

Keywords:
Bockstein basis
Cell-like map
Cohomological dimension
CW-complex
Dimension
Edwards–Walsh resolution
Inverse sequence
Simplicial complex

ABSTRACT

In the paper titled "Bockstein basis and resolution theorems in extension theory" (Tonić, 2010 [10]), we stated a theorem that we claimed to be a generalization of the Edwards–Walsh resolution theorem. The goal of this note is to show that the main theorem from Tonić (2010) [10] is in fact equivalent to the Edwards–Walsh resolution theorem, and also that it can be proven without using Edwards–Walsh complexes. We conclude that the Edwards–Walsh resolution theorem can be proven without using Edwards–Walsh complexes.

1. Introduction

In the paper titled "Bockstein basis and resolution theorems in extension theory" [10], the following theorem is proven.

Theorem 1.1. *Let G be an abelian group with $P_G = \mathbb{P}$, where $P_G = \{p \in \mathbb{P}\colon \mathbb{Z}_{(p)} \in$ Bockstein Basis $\sigma(G)\}$. Let $n \in \mathbb{N}$ and let K be a connected CW-complex with $\pi_n(K) \cong G$, $\pi_k(K) \cong 0$ for $0 \leqslant k < n$. Then for every compact metrizable space X with $X\tau K$ (i.e., with K an absolute extensor for X), there exist a compact metrizable space Z and a surjective map $\pi : Z \to X$ such that*

(a) *π is cell-like,*
(b) *$\dim Z \leqslant n$, and*
(c) *$Z\tau K$.*

This theorem turns out to be equivalent to the Edwards–Walsh resolution theorem, first stated by R. Edwards in [5], with proof published by J. Walsh in [11]:

Theorem 1.2 (R. Edwards, J. Walsh, 1981). *For every compact metrizable space X with $\dim_{\mathbb{Z}} X \leqslant n$, there exist a compact metrizable space Z and a surjective map $\pi : Z \to X$ such that π is cell-like, and $\dim Z \leqslant n$.*

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http://dx.doi.org/10.1016/j.topol.2012.07.013

We intend to explain this equivalence in Section 2.

However, the proof of Theorem 1.1 in [10] is interesting because it can be done without using Edwards–Walsh complexes, which were used in the original proof of Theorem 1.2. This requires changing the proof of Theorem 3.9 from [10], which will be done in Section 3 of this paper.

The definition and properties of Edwards–Walsh complexes can be found in [2,4] or [7]. Using Edwards–Walsh complexes, or CW-complexes built similarly to these, was the standard approach in proving resolution theorems, for example in [11,2,8]. But these complexes can become fairly complicated, which also complicates the algebraic topology machinery appearing in proofs using them. The proof of Theorem 1.1, after the adjustment of proof of Theorem 3.9 from [10], does not use Edwards–Walsh complexes – instead, it has a more involved point set topological part. Therefore the Edwards–Walsh resolution theorem can be proven without using Edwards–Walsh complexes.

2. The equivalence of the two theorems

We will use the following theorem by A. Dranishnikov, which can be found in [2] as Theorem 11.4, or in [3] as Theorem 9:

Theorem 2.1. *For any simple* CW-*complex M and any finite dimensional compactum X, the following are equivalent*:

1. $X\tau M$;
2. $X\tau SP^{\infty}M$;
3. $\dim_{H_i(M)} X \leqslant i$ *for all* $i \in \mathbb{N}$;
4. $\dim_{\pi_i(M)} X \leqslant i$ *for all* $i \in \mathbb{N}$.

A space M is called *simple* if the action of the fundamental group $\pi_1(M)$ on all homotopy groups is trivial. In particular, this implies that $\pi_1(M)$ is abelian. Also, $SP^{\infty}M$ is the infinite symmetric product of M, and for a CW-complex M, $SP^{\infty}M$ is homotopy equivalent to the weak cartesian product of Eilenberg–MacLane complexes $K(H_i(M), i)$, for all $i \in \mathbb{N}$.

In fact, Theorem 6 from [3] states that if X is a compact metrizable space, and M is any CW-complex, then $X\tau M$ implies $X\tau SP^{\infty}M$. Moreover, since $SP^{\infty}M$ is homotopy equivalent to the weak product of Eilenberg–MacLane complexes $K(H_i(M), i)$, then $X\tau SP^{\infty}M$ implies $X\tau K(H_i(M), i)$, for all $i \in \mathbb{N}$. This means that the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) from Theorem 2.1 are true for any compact metrizable space X, and not just for finite dimensional ones, as well as for any CW-complex M. So we can restate a part of the statement of Theorem 2.1 in the form we will need:

Theorem 2.2. *For any* CW-*complex M and any compact metrizable space X, we have $X\tau M \Rightarrow X\tau SP^{\infty}M \Rightarrow \dim_{H_i(M)} X \leqslant i$ for all $i \in \mathbb{N}$.*

Now X from Theorem 1.1 has property $X\tau K$, where K is a connected CW-complex with $\pi_n(K) \cong G$, $\pi_k(K) \cong 0$ for $0 \leqslant k < n$, and $n \in \mathbb{N}$. By the Hurewicz Theorem, if $n = 1$, since G is abelian we get $H_1(K) \cong \pi_1(K)$, and if $n \geqslant 2$ then $H_n(K) \cong \pi_n(K)$. Therefore, by Theorem 2.2, $X\tau K$ implies $\dim_{H_n(K)} X \leqslant n$, i.e., $\dim_G X \leqslant n$.

By the Bockstein Theorem and basic properties of Bockstein basis, as explained in Lemma 2.4 from [10], $P_G = \mathbb{P}$ implies that $\dim_G X = \dim_{\mathbb{Z}} X$. Now use the Edwards–Walsh resolution theorem to produce a compact metrizable space Z with $\dim Z \leqslant n$, and a cell-like map $\pi : Z \to X$. Since $\dim_A Z \leqslant \dim Z$ for any abelian group A, using $A = H_n(K) = G$ as well as other properties of K, and the fact that Z is finite dimensional, Lemma 3.10 from [10] shows $Z\tau K$.

3. How to avoid using Edwards–Walsh complexes

In the proof of Theorem 1.1 in [10], the following theorem is used – it appears in [10] as Theorem 3.9. This theorem is a known result, presented in a particular form that was adjusted to fit the needs of the proof of Theorem 1.1. This is why its proof was presented in [10].

Theorem 3.1 *(A variant of Edwards' Theorem). Let $n \in \mathbb{N}$ and let Y be a compact metrizable space such that $Y = \lim (|L_i|, f_i^{i+1})$, where $|L_i|$ are compact polyhedra with $\dim L_i \leqslant n+1$, and f_i^{i+1} are surjections. Then $\dim_{\mathbb{Z}} Y \leqslant n$ implies that there exists an $s \in \mathbb{N}$, $s > 1$, and there exists a map $g_1^s : |L_s| \to |L_1^{(n)}|$ which is an L_1-modification of f_1^s.*

The proof of this theorem in [10] had two parts, the first part for $n \geqslant 2$ and the second for $n = 1$. In the first part of the proof, Edwards–Walsh complexes were used. The proof is still correct, but it turns out that there was no need to use Edwards–Walsh complexes. In fact, the entire proof can be simplified, and done for any $n \in \mathbb{N}$ as it was done for the case when $n = 1$. Theorem 3.1 was the only place in [10] where Edwards–Walsh complexes were used, so the main result of [10] can be proven without ever using them. Consequently, the Edwards–Walsh resolution theorem can be proven without using Edwards–Walsh complexes.

The goal of this section is to give a simplified proof for Theorem 3.1. Here is a reminder of some facts from the original paper that are used in the new proof.

First of all, recall that a map $g : X \to |K|$ between a space X and a simplicial complex K is called a K-*modification* of f if whenever $x \in X$ and $f(x) \in \sigma$, for some $\sigma \in K$, then $g(x) \in \sigma$. This is equivalent to the following: whenever $x \in X$ and $f(x) \in \mathring{\sigma}$, for some $\sigma \in K$, then $g(x) \in \sigma$.

In the course of the simplified proof of Theorem 3.1, we will need the notion of *resolution in the sense of inverse sequences*. This usage of the word resolution is completely different from the notion from the title of this paper. The definition can be found in [9] for the more general case of inverse systems. We will give the definition for inverse sequences only.

Let X be a topological space. A *resolution* of X *in the sense of inverse sequences* consists of an inverse sequence of topological spaces $\mathbf{X} = (X_i, p_i^{i+1})$ and a family of maps $(p_i : X \to X_i)$ with the following two properties:

(R1) Let P be an ANR, $\mathcal{V}$ an open cover of P and $h : X \to P$ a map. Then there is an index $s \in \mathbb{N}$ and a map $f : X_s \to P$ such that the maps $f \circ p_s$ and h are $\mathcal{V}$-close.

(R2) Let P be an ANR and $\mathcal{V}$ an open cover of P. There exists an open cover $\mathcal{V}'$ of P with the following property: if $s \in \mathbb{N}$ and $f, f' : X_s \to P$ are maps such that the maps $f \circ p_s$ and $f' \circ p_s$ are $\mathcal{V}'$-close, then there exists an $s' \geqslant s$ such that the maps $f \circ p_s^{s'}$ and $f' \circ p_s^{s'}$ are $\mathcal{V}$-close.

By Theorem I.6.1.1 from [9], if all X_i in $\mathbf{X}$ are compact Hausdorff spaces, then $\mathbf{X} = (X_i, p_i^{i+1})$ with its usual projection maps $(p_i : \lim \mathbf{X} \to X_i)$ is a resolution of $\lim \mathbf{X}$ in the sense of inverse sequences. Moreover, since every compact metrizable space X is the inverse limit of an inverse sequence of compact polyhedra $\mathbf{X} = (P_i, p_i^{i+1})$ (see Corollary I.5.2.4 of [9]), this inverse sequence $\mathbf{X}$ will have the property (R1) mentioned above, and we will refer to this property as the *resolution property* (R1) *in the sense of inverse sequences*.

We will also use stability theory, about which more details can be found in §VI.1 of [6]. Namely, we will use the consequences of Theorem VI.1. from [6]: if X is a separable metrizable space with $\dim X \leqslant n$, then for any map $f : X \to I^{n+1}$, all values of f are unstable. A point $y \in f(X)$ is called an *unstable value* of f if for every $\delta > 0$ there exists a map $g : X \to I^{n+1}$ such that:

(1) $d(f(x), g(x)) < \delta$ for every $x \in X$, and
(2) $g(X) \subset I^{n+1} \setminus \{y\}$.

Moreover, this map g can be chosen so that $g = f$ on the complement of $f^{-1}(U)$, where U is an arbitrary open neighborhood of y, and so that g is homotopic to f (see Corollary I.3.2.1 of [9]).

Here is a technical result from [10], which is stated there as Lemma 3.7 and used in the proof of Theorem 3.1.

Lemma 3.2. *For any finite simplicial complex C, there are a map $r : |C| \to |C|$ and an open cover $\mathcal{V} = \{V_\sigma : \sigma \in C\}$ of $|C|$ such that for all $\sigma, \tau \in C$:*

(i) $\mathring{\sigma} \subset V_\sigma$,
(ii) *if $\sigma \neq \tau$ and $\dim \sigma = \dim \tau$, V_σ and V_τ are disjoint,*
(iii) *if $y \in \mathring{\tau}$, $\dim \sigma \geqslant \dim \tau$ and $\sigma \neq \tau$, then $y \notin V_\sigma$,*
(iv) *if $y \in \mathring{\tau} \cap V_\sigma$, where $\dim \sigma < \dim \tau$, then σ is a face of τ, and*
(v) $r(V_\sigma) \subset \sigma$.

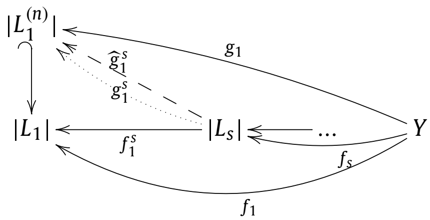
Proof of Theorem 3.1. Since $Y = \lim(|L_i|, f_i^{i+1})$, where $|L_i|$ are compact polyhedra with $\dim L_i \leqslant n+1$, we get that $\dim Y \leqslant n+1$. According to Aleksandrov's Theorem [1], $\dim Y$ being finite means $\dim_{\mathbb{Z}} Y = \dim Y$. Therefore, assuming $\dim_{\mathbb{Z}} Y \leqslant n$ really means that $\dim Y \leqslant n$, too.

Thus we can prove the theorem without using Edwards–Walsh complexes, but instead using the resolution property (R1) in the sense of inverse sequences.

We can construct a map $g_1 : Y \to |L_1^{(n)}|$ that equals f_1 on $f_1^{-1}(|L_1^{(n)}|)$. This can be done as follows. Let σ be an $(n+1)$-simplex of L_1 and $w \in \mathring{\sigma}$. Since $\dim \sigma = n+1$ and $\dim Y \leqslant n$, the point w is an unstable value for f_1 (f_1 is surjective, since all our bonding maps f_i^{i+1} are surjective). Therefore we can find a map $g_{1,\sigma} : Y \to |L_1|$ which agrees with f_1 on $Y \setminus (f_1^{-1}(\mathring{\sigma}))$, and $w \notin g_{1,\sigma}(Y)$. Then choose a map $r_\sigma : |L_1| \to |L_1|$ such that r_σ is the identity on $|L_1| \setminus \mathring{\sigma}$ and $r_\sigma(g_{1,\sigma}(Y)) \cap \mathring{\sigma} = \emptyset$. Finally, replace f_1 by $r_\sigma \circ g_{1,\sigma} : Y \to |L_1| \setminus \mathring{\sigma}$.

Continue the process with one $(n+1)$-simplex at a time. Since L_1 is finite, in finitely many steps we will reach the needed map $g_1 : Y \to |L_1^{(n)}|$. Note that from the construction of g_1, we get

(I) $g_1|_{f_1^{-1}(|L_1^{(n)}|)} = f_1|_{f_1^{-1}(|L_1^{(n)}|)}$, and for every $(n+1)$-simplex σ of L_1, $g_1(f_1^{-1}(\sigma)) \subset \partial\sigma$.



Let us choose an open cover $\mathcal{V}$ of $|L_1^{(n)}|$ by applying Lemma 3.2 to $C = L_1^{(n)}$. Now we can use resolution property (R1) in the sense of inverse sequences: there is an index $s > 1$ and a map $\widehat{g}_1^s : |L_s| \to |L_1^{(n)}|$ such that $\widehat{g}_1^s \circ f_s$ and g_1 are $\mathcal{V}$-close. Define $g_1^s := r \circ \widehat{g}_1^s : |L_s| \to |L_1^{(n)}|$, where $r : |L_1^{(n)}| \to |L_1^{(n)}|$ is the map from Lemma 3.2.

Notice that for any $y \in Y$, if $g_1(y) \in \mathring{\tau}$ for some $\tau \in L_1^{(n)}$, then $g_1(y) \in V_\tau$, and possibly also $g_1(y) \in V_{\gamma_j}$, where γ_j are some faces of τ (there can only be finitely many). Then either $\widehat{g}_1^s \circ f_s(y) \in V_\tau$, or $\widehat{g}_1^s \circ f_s(y) \in V_{\gamma_j}$, for some γ_j. In any case, $r \circ \widehat{g}_1^s \circ f_s(y) \in \tau$. Hence,

(II) for any $y \in Y$, $g_1(y) \in \mathring{\tau}$ for some $\tau \in L_1^{(n)}$ implies that $g_1^s(f_s(y)) \in \tau$.

Finally, for any $z \in |L_s|$, f_s is surjective implies that there is a $y \in Y$ such that $f_s(y) = z$. Then $f_1^s(z) = f_1^s(f_s(y)) = f_1(y)$. Now $f_1^s(z)$ is either in $\mathring{\sigma}$ for some $(n+1)$-simplex σ in L_1, or in $\mathring{\tau}$ for some $\tau \in L_1^{(n)}$.

If $f_1^s(z) \in \mathring{\sigma}$, that is $f_1(y) \in \mathring{\sigma}$ for some $(n+1)$-simplex σ, by (I) we get that $g_1(y) \in \partial\sigma$. Then by (II), $g_1^s(f_s(y)) \in \partial\sigma$, i.e., $g_1^s(z) \in \sigma$.

If $f_1^s(z) = f_1(y) \in \mathring{\tau}$ for some $\tau \in L_1^{(n)}$, then (I) implies that $g_1(y) = f_1(y) \in \mathring{\tau}$, so by (II), $g_1^s(f_s(y)) \in \tau$, i.e., $g_1^s(z) \in \tau$.

Therefore, g_1^s is indeed an L_1-modification of f_1^s. □

4. A note about the original proof of the Edwards–Walsh resolution theorem

In the original proof of Theorem 1.2 in [11], the following theorem is used. It is listed there as Theorem 4.2.

Theorem 4.1 *(R. Edwards). Let $n \in \mathbb{N}$ and let X be a compact metrizable space such that $X = \lim(P_i, f_i^{i+1})$, where P_i are compact polyhedra. The space X has cohomological dimension $\dim_{\mathbb{Z}} X \leqslant n$ if and only if for each integer k and each $\varepsilon > 0$ there are an integer $j > k$, and a triangulation L_k of P_k such that for any triangulation L_j of P_j there is a map $g_k^j : |L_j^{(n+1)}| \to |L_k^{(n)}|$ which is ε-close to the restriction of f_k^j.*

There were no additional assumptions made about dimension of polyhedra P_i, so in the proof of this theorem in [11], the usage of Edwards–Walsh complexes is indispensable. Therefore, the usage of Edwards–Walsh complexes was necessary in the original proof of Theorem 1.2 in [11].

Theorem 3.1 was modeled on Theorem 4.1, but with the additional assumption about dimension of polyhedra $\dim |L_i| \leqslant n+1$. This assumption, together with $\dim_{\mathbb{Z}} Y \leqslant n$ implies that $\dim Y \leqslant n$. Therefore the usage of Edwards–Walsh complexes in its proof can be avoided altogether. In fact, Theorem 3.1 becomes analogous to Theorem 4.1 from [11] – a weaker version of Edwards' Theorem:

Theorem 4.2. *Let $n \in \mathbb{N}$ and let X be a compact metrizable space such that $X = \lim\,(P_i, f_i^{i+1})$, where P_i are compact polyhedra. The space X has $\dim X \leqslant n$ if and only if for each integer k and each $\varepsilon > 0$ there are an integer $j > k$, a triangulation L_k of P_k, and a map $g_k^j : P_j \to |L_k^{(n)}|$ which is ε-close to f_k^j.*

Acknowledgement

The author cordially thanks the anonymous referee for their valuable comments which lead to significant improvement of this paper.

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