Lebesgue-type convergence theorems in Banach lattices
with applications to compact operators

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Dedicated to Professor Dr. A. C. Zaanen on the occasion of his 65th birthday

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ABSTRACT

The main result is a Lebesgue-type convergence theorem in the setting of Banach lattices, of which the classical Lebesgue dominated convergence theorem is the prime example. It is proved that whenever $E$ is a Banach lattice which is an ideal in a Riesz space $M$, $M$ having the principal projection property and the Egoroff property, and if a sequence in $E$ is order convergent in $M$, then the sequence is norm convergent in $E$ if it is contained in a set of uniformly absolutely continuous norm. This result enables one to derive compactness criteria for bounded linear operators on Banach lattices. If $T: E \to F$ ($E, F$ Banach lattices) is a norm bounded linear operator with $T[E] \subseteq F$ and if $U$ denotes the unit ball of $E$, then $T$ is compact if and only if

(a) $T[U]$ is of uniformly absolutely continuous norm;
(b) For every sequence $S \subseteq T[U]$ the band generated by $S$ in $F$ is an ideal in a Dedekind complete Riesz space $M$ with the Egoroff property and $S$ has a subsequence converging in order in $M$ to an element of $M$.

From this theorem all the known compactness criteria for order bounded linear operators on Banach lattices can be derived. In particular the well-known compactness conditions proved by W. A. J. Luxemburg and A. C. Zaanen (Math. Annalen 149, 150–180 (1963)) for kernel operators on Banach function spaces are generalized to Banach lattices.

1. INTRODUCTION

Let $E$ be a Banach lattice which is an ideal in a Riesz space $M$. By a Lebesgue-type theorem we have in mind a theorem providing conditions
for a sequence in $E$ which is order convergent in $M$ to be norm convergent in $E$. If $(X, \Sigma, \mu)$ is a $\sigma$-finite measure space we shall denote the Riesz space of $\mu$-measurable $\mu$-almost everywhere finite functions by $M(X, \mu)$ and the space of $\mu$-summable functions by $L^1(X, \mu)$. $L^1(X, \mu)$ is a norm ideal in $M(X, \mu)$ and, since pointwise $\mu$-almost everywhere convergence of a sequence on $X$ coincides with order convergence of the sequence in $M(X, \mu)$, the classical Lebesgue dominated convergence theorem is the prime example of this kind of result.

Our main theorem is a generalization of a result proved for Banach function spaces in [10] and [2]. We prove that if a sequence in $E$ is order convergent in $M$, $M$ having the principal projection property and the Egoroff-property, then the sequence is norm convergent in $E$ if it is contained in a set of uniformly absolutely continuous norm. (In the case of a Banach function space the above mentioned space $M(X, \mu)$ is super Dedekind complete and has the Egoroff-property ([6], theorem 71.6).) Conversely, if the norm on $E$ is order continuous, then the condition is also necessary. We also show that in Dedekind complete Banach lattices sets of uniformly absolutely continuous norm coincide with $L$-weakly compact sets as defined by P. Meyer-Nieberg [7].

The applicability of the main result is facilitated by the fact that if there exists a strictly positive order continuous linear functional (i.e. a strictly positive normal integral) on $E$, then $E$ is an ideal in an AL-space which then plays the rôle of $M$. In section 3 we illustrate the usefulness of the theorem by deriving compactness conditions for bounded linear operators on Banach lattices. From this result we obtain the compactness criteria of P. Dodds and D. H. Fremlin [1] and of R. J. Nagel and U. Schlotterbeck [8]. Our main theorem in this section is a generalization of the compactness conditions obtained by W. A. J. Luxemburg and A. C. Zaanen in [4] for kernel operators on Banach function spaces.

For notation and definitions concerning the general theory of Riesz spaces and Banach lattices we refer the reader to [5], [6] and [9].

2. CONVERGENCE THEOREMS

We recall the definition and a few elementary facts concerning order convergence.

**Definition:** Let $M$ be a Riesz space. The sequence $(f_n)$ in $M$ is said to converge in order to the element $f \in M$ whenever there exists a sequence $p_n \downarrow 0$ in $M^+$ such that $|f - f_n| < p_n$ holds for all $n$. We denote this by $f_n \to f$ in order.

If $E$ is an ideal in the Riesz space $M$ and if $(f_n)$ is a sequence in $E$ which is order convergent in $M$ to the element $f \in M$ and if $(f_n)$ is order bounded by an element $g \in E^+$, then $f \in E$ and $f_n \to f$ in order in $E$.

The next result exhibits the relationship between order convergence and norm convergence.

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2.1 Proposition: Let $E$ be a Banach lattice. Every sequence $(f_n)$ in $E$ which is norm convergent to an element $f \in E$ contains a subsequence which is order convergent to $f$ in $E$.

Proof: If $(f_n)$ is norm convergent it contains a subsequence $(f_{n_k})$ satisfying $||f - f_{n_k}|| < 2^{-k}$. The element $p = \sum_{k=1}^{\infty} |f - f_{n_k}|$ exists in $E$ since $E$ has the Riesz-Fischer property ([5], note VIII). Let $p_m = \sum_{k=1}^{m} |f - f_{n_k}|$, $m = 1, 2, \ldots$. Then we have $p - p_m \downarrow 0$ in $E$ and $|f - f_{n_k}| < p - p_{m-1}$. Hence $(f_{n_k})$ converges in order in $E$.

2.2 Theorem (Lebesgue's Dominated Convergence Theorem): Let $E$ be a Banach lattice which is an ideal in the Riesz space $M$ and suppose that $E$ has order continuous norm. If $(f_n)$ is a sequence in $E$ such that $f_n \rightarrow f$ in order in $M$ with $f \in M$ and if the sequence is order bounded in $E$, then $f \in E$ and $||f_n - f|| \rightarrow 0$ as $n \rightarrow \infty$.

Proof: From the remarks above we have $f \in E$ and $f_n \rightarrow f$ in order in $E$. Hence there exists a sequence $p_n \downarrow 0$ in $E^+$ such that $|f - f_n| < p_n$ holds for all $n \in \mathbb{N}$. But then $||f_n - f|| < ||p_n|| \downarrow 0$ by the order continuity of the norm of $E$.

Following Luxemburg and Zaanen we define

Definition: The sequence $(K_n)$ of bands in the Riesz space $E$ is called an exhausting sequence of bands whenever the sequence $(K_n)$ is increasing and the band generated by all the $K_n$ equals $E$.

If $(L_n)$ is a sequence of bands in $E$ we write $L_n \downarrow 0$ if and only if $(L_n^d)$ is an exhausting sequence of bands in $E$ with $L_n^d$ the disjoint complement of $L_n$ in $E$. We also denote the order projection of $E$ onto a projection band $L$ by $P[L]$. If $(K_n)$ is an exhausting sequence of projection bands, then it is not difficult to show that for every $f \in E^+$ we have $0 < P[K_n]f \uparrow f$.

2.3 Lemma: Let $E$ be an Archimedean Riesz space and $E_n \downarrow 0$ a sequence of projection bands in $E$. Then we have $E_n \downarrow 0$ if and only if $\bigcap_n E_n = \{0\}$.

Proof: We note that if $\{\bigcup_n E_n^d\}$ denotes the band generated by all the $E_n^d$, then

$$\{\bigcup_n E_n^d\} = \bigcup_n E_n^{dd} = (\bigcap_n E_n^d)^d = (\bigcap_n E_n^d)^d$$

([6], p. 107).

Hence if $\bigcap_n E_n = \{0\}$, we immediately have

$$\{\bigcup_n E_n^d\} = (\bigcap_n E_n)^d = E.$$ 

Conversely, let $E_n \downarrow 0$ and $f \in E_n$ for all $n$. Then we have $f \in (\bigcap_n E_n)^d$, which implies that $f \in (\bigcap_n E_n)^d = E$ and hence $\bigcap_n E_n = \{0\}$.

2.4 Proposition: Let $E$ be a Dedekind complete Banach lattice with continuous dual $E'$ and order continuous dual $E_{00}$. Then
(a) For every sequence of bands \( E_n \downarrow 0 \) in \( E \) there exists a sequence \( E_n^* \downarrow 0 \) of bands in \( E' \) such that \( P'[E_n]\phi = P[E_n^*]\phi \) for all \( \phi \in E_0 \).

(b) For every sequence \( E_n \downarrow 0 \) of bands in \( E' \) there exists a sequence \( E_n \downarrow 0 \) of bands in \( E \) such that \( P[E_n]\phi < P'[E_n]\phi \) for all \( \phi \in (E_0)_+ \).

PROOF: (a) Let \( E_n \downarrow 0 \) be a sequence of bands in \( E \). Then \( P'[E_n]\) is a projection in \( E' \) and \( 0 < P'[E_n]\phi < \phi \) for all \( \phi \in E_+ \). Hence there exists a sequence \( E_n \downarrow \) of bands in \( E' \) such that \( P'[E_n] = P[E_n^*] \) ([6], theorem 24.5). Let \( E_n = E_n \cap E_0 \) and suppose \( 0 < \phi \in \bigcap_n E_n^* \). Then we have for all \( f \in E_+ \) that \( 0 = \lim_n \langle P[E_n]f, \phi \rangle = \langle f, \phi \rangle \) and hence \( \phi = 0 \). Consequently, by lemma 2.3, \( E_n^* \downarrow 0 \) and clearly

\[ P'[E_n]\phi = P[E_n^*]\phi \] for all \( \phi \in E_0' \).

(b) Let \( E_n \downarrow 0 \) be a sequence of bands in \( E' \) and denote by \( E_n \) the carrier band of \( E_n^* = E_n \cap E_0 \) in \( E \). Since \( E_n^* \downarrow 0 \), i.e. \( \bigcap_n E_n^* = \{0\} \), we have \( \bigcap_n E_n = \{0\} \) ([5], theorem 27.15) and hence \( E_n \downarrow 0 \) is a sequence of bands in \( E \). Applying (a) there exists a sequence \( G_n^* \downarrow 0 \) of bands in \( E' \) such that \( P'[E_n]\phi = P[G_n]\phi \) for every \( \phi \in E_0 \).

We observe that \( E_n^* \subset G_n^* \) for every \( n \). Indeed, by [5], theorem 27.12, we have \( E_n = (0E_n^*)^d \) and hence \( \langle P[E_n^*]f, \phi \rangle = 0 \) for every \( \phi \in E_n^* \) and \( f \in E \). Consequently, if \( \phi \in E_n^* \), then

\[ \langle f, P'[E_n]\phi \rangle = \langle P[E_n]f, \phi \rangle + \langle P[E_n^*]f, \phi \rangle = \langle f, \phi \rangle \]

for every \( f \in E \), which shows that \( \phi \in G_n^* \). For every \( \phi \in (E_0)_+ \) we therefore have

\[ P[E_n]\phi = P[E_n^*]\phi < P[G_n]\phi = P'[E_n]\phi, \]

which completes the proof.

DEFINITION: Let \( E \) be a Banach lattice. A subset \( S \subset E \) is said to be of uniformly absolutely continuous norm (u.a.c. norm) whenever, given \( \varepsilon > 0 \) and a sequence \( (E_n) \) of projection bands with \( E_n \downarrow 0 \), there exists an index \( N \) such that \( ||P[E_n]f|| < \varepsilon \) holds for all \( n > N \) and for all \( f \in S \) simultaneously.

We now come to our main theorem. It generalizes theorem 2.2 and also lemma 4.1 of [2].

2.5 THEOREM: Let the Banach lattice \( E \) be an ideal in the Riesz space \( M \) and suppose that \( M \) has the principal projection property and the Egoroff property. If \( (f_n) \) is a sequence in \( E \) which is order convergent in \( M \) to an element \( f \in M \) and if the set \( S = \{f_n: n \in \mathbb{N}\} \) is of uniformly absolutely continuous norm, then we have \( f \in E \) and \( ||f_n - f|| \to 0 \) as \( n \to \infty \).

PROOF: (a) Assume that \( E \) has a weak order unit \( v \). Since \( f_n \to f \) in order in \( M \) there exists a sequence \( u_n \downarrow 0 \) in \( M^+ \) such that \( |f - f_n| < u_n \) holds for all \( n \). Applying the abstract Egoroff theorem ([6] theorem 74.3),
we find a sequence \( 0 < e_k \uparrow v \) with \( e_k \in M^+ \) for all \( k \), such that for \( k = 1, 2, \ldots \) the sequence \( (P[M_k]u_n) \) converges \( v \)-uniformly to zero as \( n \to \infty \). (Here \( M_k \) denotes the band generated by \( e_k \) in \( M \)). Now \( e_k \in E \) and hence the band \( B_k \) generated by \( e_k \) in \( E \) satisfies \( B_k = E \cap M_k \) and \( (B_k) \) is an exhausting sequence of bands in \( E \) ([6], lemma 74.1 and theorem 30.5).

Consequently, given \( \varepsilon > 0 \), there exists an index \( N \) such that \( \|P[B_k]f_n\| < \varepsilon/4 \) holds for all \( n \), and so also \( \|P[B_k](f_n - f_m)\| < \varepsilon/2 \) holds for all \( n \) and \( m \). Moreover we can find an index \( N_1 \) such that \( \|P[M_N](f_n - f_m)\| < \varepsilon/4 \) holds for all \( n \geq N_1 \). It follows that \( \|P[B_N](f_n - f_m)\| < \varepsilon/2 \) for \( n, m > N_1 \) and so

\[
\|f_n - f_m\| < \|P[B_N](f_n - f_m)\| + \|P[B_N](f_n - f_m)\| < \varepsilon \quad \text{for } n, m > N_1.
\]

The sequence \( (f_n) \) will therefore converge to some element \( g \in E \). By 2.1 it has a subsequence order convergent in \( E \) (and hence also in \( M \)) to \( g \). But this subsequence order converges in \( M \) to \( f \) and so \( f = g \in E \).

(b) In the general case, denote by \( B \) the band generated by \( S \). If \( v_n = \sup (|f_1|, |f_2|, \ldots, |f_n|), n \in \mathbb{N}, \) \( 0 < v_n \uparrow \) and the set \( \{v_n: n \in \mathbb{N}\} \) is a countable order basis for \( B \). If we denote the principal band generated by \( v_n \) by \( B_n \), we have that \( (B_n) \) is an exhausting sequence of projection bands for \( B \). Given \( \varepsilon > 0 \), there exists an index \( N \) such that

\[
\|P[B_n](f_n - f)\| < \varepsilon/2 \quad \text{for } n, l \in \mathbb{N}.
\]

Applying (a) to the sequence \( (P[B_N]f_n) \) in the band \( B_N \) and noting that the set \( \{P[B_N]f_n: n = 1, 2, \ldots\} \) is also of u.a.c. norm we get

\[
\|P[B_N](f_n - f)\| < \varepsilon/2 \quad \text{for } n, l > N_1.
\]

As in (a) we conclude that \( (f_n) \) is a Cauchy sequence which converges in norm to \( f \in E \).

We recall that the subset \( E^a \) of \( E \), consisting of all elements \( f \in E \) satisfying \( |f| \uparrow u_n \downarrow 0 \) in \( E \) implies \( \|u_n\| \downarrow 0 \), is a norm closed ideal in \( E \).

2.6 \textbf{THEOREM:} \textit{If \( (f_n) \) is a sequence in \( E^a \) which is norm convergent to the element \( f \in E^a \) then the set \( S = \{f_n: n \in \mathbb{N}\} \) is of uniformly absolutely continuous norm.}

\textbf{PROOF:} \textit{Let \( \varepsilon > 0 \) be given and let \( E_n \downarrow 0 \) be a sequence of projection bands. If \( N \) is such that \( \|f - f_n\| < \varepsilon/2, n > N \), if

\[
g = \sup (|f_1|, \ldots, |f_{N-1}|, |f|) \in E^a
\]

and if \( K \) is such that for \( k > K \) we have \( \|P[E_k]g\| < \varepsilon/4 \), then we have for every \( n \) and for every \( k > K \) that \( \|P[E_k]f_n\| < \varepsilon \) holds.}

2.7 \textbf{COROLLARY:} \textit{Let \( E \) and \( M \) be as in 2.5 and let \( E \) have order continuous norm. If \( f_n \in E \) for all \( n \) and \( f_n \to f \) in order in \( M \), then \( (f_n) \) is norm convergent to \( f \) if and only if the set \( S = \{f_n: n \in \mathbb{N}\} \) is of u.a.c. norm.}
If $E$ is a Dedekind complete Banach lattice and if $0 < \phi$ is an order continuous linear functional on $E$ we denote by $(E, \phi)$ the completion of the space $E/N_\phi$ with respect to the $L$-norm defined by $p_\phi(g) = \phi(|g|)$ for $g \in E$. ($N_\phi$ denotes the absolute kernel of $\phi$, i.e., $N_\phi = \{g \in E : \phi(|g|) = 0\}$). In the case which we consider the image of $E$ in $(E, \phi)$ is an ideal in the AL-space $(E, \phi)$ ([9], IV. 9.3). We denote the continuous canonical map from $E$ into $(E, \phi)$ by $j_*$. Moreover, if $\phi$ is strictly positive on $E$, $E$ may be regarded as an ideal in the AL-space $(E, \phi)$.

We recall that a subset $A \subset E$, $E$ a Banach lattice, is said to be $L$-weakly compact if it is norm bounded and if $\|f_n\| \to 0$ for every disjoint sequence $(f_n)$ in the positive part of the solid hull of $A$. (see [7]).

2.8 PROPOSITION: Let $E$ be a Dedekind complete Banach lattice. A norm bounded subset $A \subset E$ is $L$-weakly compact if and only if $A$ is of u.a.c. norm.

PROOF: Let $A$ be $L$-weakly compact and let $K_n \downarrow 0$ be a sequence of bands in $E$. Given $\varepsilon > 0$, there exists an element $g \in E_\phi$ such that $A \subset [-g, g] + V(\varepsilon/2)$, with $V(\varepsilon/2) = \{f \in E : \|f\| < \varepsilon/2\}$ ([7], Satz II.2). Hence, for every $f \in A$, we have $\|P[K_n]f\| < \|P[K_n]g\| + \varepsilon/2 < \varepsilon$ for all $n$ such that $\|P[K_n]g\| < \varepsilon/2$. This proves that $A$ is of u.a.c.norm.

Conversely, let $A$ be a norm bounded set of u.a.c. norm. The solid hull of $A$ has the same property and we therefore assume $A$ to be solid. Also, $A \subset E^\sigma$ and so we may assume that $E$ has order continuous norm. Let $(f_n)$ be a disjoint sequence in $A^+$. The band $B$ generated by $(f_n)$ is a Banach lattice with weak order unit and order continuous norm. Hence there exists a strictly positive order continuous linear functional $\phi \in B_\infty$ ([5], corollary 35.3) and $B$ is an ideal in the AL-space $(B, \phi)$. Now $(B, \phi) \cong L^1(X, \mu)$ and the disjoint sequence $(f_n)$ in $L^1(X, \mu)$ converges pointwise $\mu$-a.e. on $X$ to zero, i.e. it converges in $M(X, \mu)$ in order to zero. By assumption the set $\{f_n : n \in \mathbb{N}\} \subset A$ is of u.a.c. norm in $B$ and so by theorem 2.5, $\|f_n\| \to 0$ as $n \to \infty$. Hence $A$ is $L$-weakly compact.

For further reference we state the following result, a proof of which follows from 2.8 and [7], Satz II.6.

2.9 THEOREM. Let $E$ be a Dedekind complete Banach lattice. Then every norm bounded subset $S \subset E$ of u.a.c. norm is relatively weakly compact.

3. COMPACT OPERATORS

In this section $E$ will denote a Banach lattice, $F$ a Dedekind complete Banach lattice and $\mathcal{L}^b(E, F)$ the Dedekind complete Riesz space of order bounded linear operators from $E$ into $F$. Every positive operator $0 < T : E \to F$ belongs to $\mathcal{L}^b(E, F)$ and is continuous. On the other hand every operator in $\mathcal{L}^b(E, F)$ can be written as the difference of two positive linear operators and is therefore continuous.
We formulate the following compactness criterion for norm bounded linear operators on Banach lattices, thereby generalizing theorem 4.2 of [2].

3.1 THEOREM: Let $T: E \to F$ be a norm bounded linear operator with $T[E] \subseteq F^\sigma$. Let $U$ denote the unit ball of $E$. Then $T$ is compact if and only if

(a) $T[U]$ is of u.a.c. norm.
(b) For every sequence $S \subseteq T[U]$ the band generated by $S$ in $F^\sigma$ is an ideal in a Dedekind complete Riesz space $M$ with the Egoroff property and $S$ has a subsequence converging in order in $M$ to an element of $M$.

PROOF: By 2.5 the conditions are clearly sufficient. Conversely, let $T$ be compact but assume that (a) does not hold. Then there exist a sequence $K_n \uparrow 0$ of projection bands, a number $\varepsilon > 0$ and a sequence $(f_n)$ in $E$ with $\|f_n\| < 1$ such that $\|P[K_n]Tf_n\| > \varepsilon$ holds for all $n$. Since $T$ is compact, and $T[E] \subseteq F^\sigma$ we may assume that $\|Tf_n - g\| < c/2$ for some element $g \in F^\sigma$ and all $n > N_1 \in \mathbb{N}$. Also, $g \in F^\sigma$ and so we have $\|P[K_n]g\| < \varepsilon/2$ for all $n > N_2 \in \mathbb{N}$. Hence, for $n > N = \max(N_1, N_2)$, we have $\|P[K_n]Tf_n\| < \|P[K_n](Tf_n - g)\| + \|P[K_n]g\| < \varepsilon$. This contradiction shows that (a) must hold true.

To show that (b) holds, denote by $B$ the band generated in $F^\sigma$ by the sequence $S$. Then $B$ is a Banach lattice with order continuous norm and with weak order unit. Hence there exists a strictly positive linear functional $\phi \in B_0^\sigma$ and $B$ is an ideal in the AL-space $(B, \phi)$ which has the Egoroff property. Furthermore $S$ contains a subsequence converging in norm in $B$ and hence also in norm in $(B, \phi)$. By 2.1 this sequence again has a subsequence converging in order to an element of $(B, \phi)$. This completes the proof.

Following Dodds and Fremlin [1] we call a linear operator $T: E \to F$ $L$-weakly compact if $T[U]$ is an $L$-weakly compact set, $U$ the unit ball in $E$. Also we call a subset $A \subseteq E$ PL-compact if $S[A]$ is pre-compact in $L$ whenever $L$ is an AL-space and $S: E \to L$ is a positive linear operator. A linear operator $T: E \to F$ is PL-compact if $T[U]$ is a PL-compact set.

In [1] Dodds and Fremlin proved the following theorem.

THEOREM: If $E$ and $F$ are Banach lattices of which $F$ has order continuous norm, then $T: E \to F$ is compact if and only if $T$ is $L$-weakly compact and PL-compact.

Using proposition 2.8 it is clear that $T$ is $L$-weakly compact if and only if condition (a) of theorem 3.1 holds. The following proposition therefore shows that the Dodds-Fremlin result is a corollary to theorem 3.1.

3.2 PROPOSITION: Let $E$ and $F$ be Banach lattices of which $F$ has order continuous norm. If the linear operator $T: E \to F$ is PL-compact then condition (b) of 3.1 holds true.
PROOF: Let $S$ be a sequence in $T[U]$. $B$ the band generated in $F$ by $S$ and $\phi$ a strictly positive linear functional on $B$. As in the proof of 3.1 we need only show that $S$ contains a subsequence which converges in $(B, \phi)$. To do this, extend $\phi$ to a positive linear functional on $F$. Applying the definition of PL-compactness it follows that $S$ has a subsequence $(Tf_{n_k})$ satisfying $\phi \left( |Tf_{n_k} - Tf_{n_l}| \right) \to 0$ as $k,l \to \infty$. Hence $(Tf_{n_k})$ is a Cauchy sequence in $(B, \phi)$ and our proof is complete.

We now proceed by generalizing the main result of Luxemburg and Zaanen ([4], theorem 7.3). In order to do this, we assume henceforth that $E'$ and $F$ have order continuous norms. We need a few preliminary results.

3.3 LEMMA: For a sequence $(f_n)$ in $F$ the following statements are equivalent:

(a) For every sequence $F_\eta, \phi, \eta$ of bands in $F$ we have $\|P[F_\eta]f_n\| \to 0$ as $n \to \infty$.
(b) $S = \{f_n: n \in \mathbb{N}\}$ is a set of u.a.c. norm.

PROOF: We prove only (a) $\Rightarrow$ (b) since (b) $\Rightarrow$ (a) is obvious. Let us note first that if (a) is true for $(f_n)$ then also for any subsequence of $(f_n)$. Assume now that (b) does not hold. Then there exist an $\varepsilon > 0$ and a sequence of bands $F_k \downarrow 0$ such that for every $k$ there exists an element $z \in S$ with $\|P[F_k]z\| > \varepsilon$. Let $n(1)$ be the first index such that $\|P[F_1]f_{n(1)}\| > \varepsilon$. Since $F_k \downarrow 0$ and $F$ has order continuous norm there exists an index $k(2)$ such that $\|P[F_{k(2)}]f_{n(1)}\| < \varepsilon$. Let $n(2)$ be the first index such that $\|P[F_{k(2)}]f_{n(2)}\| > \varepsilon$ (necessarily $n(2) > n(1)$). Continuing in this way we obtain a subsequence $(f_{n(j)})$ and a sequence $F_{k(j)} \downarrow 0$ such that $\|P[F_{k(j)}]f_{n(j)}\| > \varepsilon$ for all $j$. This is a contradiction.

3.4 LEMMA: The sequence $(\phi_n)$ in $E'$ is norm convergent to zero if and only if

(a) $\lim_n \|\phi_n\| = 0$ for all $f \in E$.
(b) For every sequence $E_\eta, \phi, \eta$ of bands in $E'$ we have $\|P[E_\eta]\phi_n\| \to 0$ as $n \to \infty$.

PROOF: If $\phi_n \to 0$ in norm, then (a) and (b) follow trivially. Conversely, let the sequence $(\phi_n)$ in $E'$ satisfy (a) and (b) but suppose that $(\phi_n)$ does not converge in norm to zero. Then there exist an $\varepsilon > 0$, a subsequence $(\phi_{n_1})$ and a sequence $(f_{n_1})$ in $E$ with $\|f_{n_1}\| < 1$ such that $|\phi_{n_1}(f_{n_1})| > \varepsilon$ for all $n$. Denote by $G$ the band generated by $(f_{n_1})$ in $E$. Then $G$ has a weak order unit $0 < u = \sum_n 2^{-n}|f_{n_1}|$. Let $N_u$ denote the null ideal of $u$ in $E'$ and consider the quotient $E'/N_u$, which can be associated with the band $N_u^d$ in $E'$. The element $u$ defines a norm $p_u$ on $N_u^d$ by $p_u(\phi) = \langle u, |\phi| \rangle$. $N_u^d$ is also an ideal in the norm completion $(E', u)$ of $(E'/N_u, p_u)$.

Condition (a) implies that $(P[N_u^d]\phi_{n_1})$ is norm convergent to zero in
(E', u). Hence we may assume, by 2.1, that the sequence is order convergent to zero in (E', u). Condition (b) implies, via 3.3, that \{\phi_{in} : n \in \mathbb{N}\} is a set of u.a.c. norm in E', and hence the set \{P[N^d]\phi_{in} : n \in \mathbb{N}\} is also of u.a.c. norm. By 2.5 (\phi_{1n}) converges in norm to zero in the band N^d, i.e., \|P[N^d]\phi_{1n}\| \to 0 as n \to \infty. This contradicts \|\phi_{1n}(f_{1n})\| > \varepsilon for all n, since every f_{1n} belongs to the ideal generated by u.

3.5 Proposition: Let F_n \downarrow 0 be a sequence of bands in F. Then, for every norm bounded sequence \(\phi_n\) in F', the sequence \((P[F_n]\phi_n)\) is \(\sigma(F', F')\)-convergent to zero.

Proof: For every \(f \in F\) we have
\[
\langle f, P[F_n]\phi_n \rangle = \|P[F_n]f\| \cdot \|\phi_n\| < M \|P[F_n]f\| \to 0 \text{ as } n \to \infty.
\]
Using 3.4 and 3.5 we now obtain the following result.

3.6 Theorem: Let T \in \mathcal{L}(E, F). If \|P[E_n]T'P[F_n]\| \to 0 as n \to \infty for all sequences E_n \downarrow 0 of bands in E' and F_n \downarrow 0 of bands in F, then \|P[F_n]T\| \to 0 as n \to \infty.

Proof: We show that \|T'P[F_n]\| \to 0 as n \to \infty. In order to obtain this it is sufficient to prove that for every sequence \(\phi_n\) in F' with \|\phi_n\| < 1 we have \|T'P[F_n]\phi_n\| \to 0 as n \to \infty. By 3.5 the sequence \((P'[F_n]\phi_n)\) is \(\sigma(F', F')\)-convergent to zero. Since \|T'| is continuous, this implies that for every \(f \in E\)
\[
\langle T'P'[F_n]\phi_n, f \rangle = \langle T'|P'[F_n]\phi_n, |f| \rangle - \langle P'[F_n]\phi_n, |T'|(|f|) \rangle \to 0 \text{ as } n \to \infty,
\]
i.e., \((T'P'[F_n]\phi_n)\) is \(\sigma(E', E')\)-convergent to zero.

Furthermore, every sequence E_n \downarrow 0 of bands in E' satisfies
\[
\|P[E_n]T'P'[F_n]\phi_n\| \to \|P[E_n]T'P'[F_n]\| \to 0 \text{ as } n \to \infty.
\]
The required result follows from lemma 3.4.

The next result is analogous to lemma 3.4.

3.7 Lemma: The sequence \((f_n)\) in F converges in norm to zero if and only if
(a) \(\lim_n \phi(|f_n|) = 0\) for all \(\phi \in F'\).
(b) For every sequence F_n \downarrow 0 of bands in F we have \|P[F_n]f_n\| \to 0 as n \to \infty.

Proof: We need only prove that the conditions are sufficient. Let \((f_n)\) satisfy (a) and (b), but suppose that the sequence does not converge in norm to zero. Then there exist an \(\varepsilon > 0\) and a subsequence \((f_{1n})\) such
that \(|f_{n}| > \varepsilon\) for all \(n\). Denote by \(B\) the band generated by the sequence \((f_{n})\) in \(F\). Then \(B\) has a weak order unit and order continuous norm; hence there exists a strictly positive order continuous linear functional \(\phi\) on \(B\). Condition (a) now implies that \((f_{n})\) is norm convergent to zero in \((B, \phi)\) and by 2.1 we may assume that \((f_{n})\) is order convergent to zero in \((B, \phi)\). Condition (b) implies, via 3.3, that \(S = \{f_{n}: n \in \mathbb{N}\}\) is of u.a.c. norm and so theorem 2.5 implies that \(|f_{n}| \to 0\) as \(n \to \infty\). This contradiction completes the proof.

3.8 PROPOSITION: Let \(E_{n}' \downarrow 0\) be a sequence of bands in \(E'\). Then we have for every norm bounded sequence \((f_{n})\) in \(E\) that the sequence \((P'[E_{n}']f_{n})\) is \(\sigma(E'', E')\)-convergent to zero.

We omit the proof since it is identical to that of proposition 3.5.

Using 3.7 and 3.8 we obtain the following result.

3.9 THEOREM: Let \(T \in \mathcal{L}^{0}(E, F)\). If \(|P[E_{n}]T'P'[F_{n}]| \to 0\) as \(n \to \infty\) for every sequence \(E_{n}' \downarrow 0\) of bands in \(E'\) and \(F_{n} \downarrow 0\) of bands in \(F\), then we have that \(|P[E_{n}]T'| \to 0\) as \(n \to \infty\).

PROOF: To prove that \(|P[E_{n}]T'| \to 0\) as \(n \to \infty\), it is sufficient to show that \(|P[E_{n}]T'\phi_{n}| \to 0\) as \(n \to \infty\) for every sequence \((\phi_{n})\) in \(\overline{F}'\) with \(|\phi_{n}| < 1\). This statement is however equivalent to the statement that for every sequence \((f_{n})\) in \(E_{+}\) with \(|f_{n}| < 1\) we have that \(|\langle T'P'[E_{n}]f_{n}, \phi_{n} \rangle| = |\langle f_{n}, P[E_{n}]T'\phi_{n} \rangle| \to 0\) as \(n \to \infty\). Hence it is sufficient to show that \(|T'P'[E_{n}]f_{n}| \to 0\) as \(n \to \infty\) for every sequence \((f_{n})\) in \(E_{+}\) with \(|f_{n}| < 1\).

From theorem 3.6 we infer that \(T[U]\), \(U\) the unit ball in \(E\), is a set of u.a.c. norm and hence by theorem 2.9 \(T\) is a weakly compact operator. This implies that \((T'P'[E_{n}]f_{n})\) is a sequence in \(F\), and we can use 3.7 to show that \(|T'P'[E_{n}]f_{n}| \to 0\) as \(n \to \infty\). By 3.8 the sequence \((P'[E_{n}]f_{n})\) is \(\sigma(E'', E')\)-convergent to zero. Since \(|T'|\) is continuous, this implies that for every \(\phi \in F'\)

\[|\langle T'P'[E_{n}]f_{n}, \phi \rangle| \leq |\langle T'P'[E_{n}]f_{n}, |\phi| \rangle| = \langle P[E_{n}]f_{n}, |T'| |\phi| \rangle \to 0\] as \(n \to \infty\).

Hence \((|T'P'[E_{n}]f_{n}|)\) is \(\sigma(F, F')\)-convergent to zero. Furthermore we have for every sequence \(F_{n} \downarrow 0\) of bands in \(F\) that

\[|P[F_{n}]T'P'[E_{n}]f_{n}| \leq |P[F_{n}]T'P'[E_{n}]| = |P[E_{n}]T'P'[F_{n}]| \to 0\] as \(n \to \infty\).

The final result follows from 3.7.

Theorems 3.6 and 3.9 enable us to formulate the following result.

3.10 THEOREM: Let \(E\) and \(F\) have order continuous norms and let \(T \in \mathcal{L}^{0}(E, F)\). Then the following conditions are equivalent:

(a) \(T[U]\) is a set of u.a.c. norm, \(U\) the unit ball in \(E\).
(b) \( \|P[F_n]T\| \to 0 \) as \( n \to \infty \) for every sequence \( F_n \downarrow 0 \) of bands in \( F \).
(c) \( \|P[E_n]T'\| \to 0 \) as \( n \to \infty \) for every sequence \( E_n \downarrow 0 \) of bands in \( E' \).
(d) \( \|P[E_n]T'P[F_n]\| \to 0 \) as \( n \to \infty \) for all sequences \( E_n \downarrow 0 \) of bands in \( E' \) and \( F_n \downarrow 0 \) of bands in \( F \).

If \( T \) is positive we may add

(e) The norm of \( \mathcal{L}^b(E, F) \) induced on \([0, T] \subset \mathcal{L}^b(E, F) \) is order continuous.

**Proof:** (a) \( \Leftrightarrow \) (b) is evident from the definition.
(b) \( \Leftrightarrow \) (d) and (c) \( \Leftrightarrow \) (d) are the contents of theorems 3.6 and 3.9 respectively.
(e) \( \Rightarrow \) (b) is immediate, since \( P[F_n]T \downarrow 0 \) in \( \mathcal{L}^b(E, F) \) for every sequence \( F_n \downarrow 0 \) of bands in \( F \).
(b) \( \Rightarrow \) (e). Let \( 0 < T_n < T \) be a sequence of operators satisfying \( T_n \downarrow 0 \).
We have to prove that the sequence \( (T_n f_n) \) is norm convergent to zero for every sequence \( (f_n) \) in the band \( E_+ \) with \( \|f_n\| < 1 \). We use lemma 3.7: For any \( \phi \in E' \) it follows that \( \lim_n \|T_n \phi f_n\| = 0 \) since \( T_n \downarrow 0 \) and the norm on \( E' \) is order continuous. Also if \( F_n \downarrow 0 \) is a sequence of bands in \( F \) we have \( \|P[F_n]T_n f_n\| \leq \|P[F_n]T f_n\| \leq \|P[F_n]T\| \to 0 \) as \( n \to \infty \). By 3.7 we obtain the required result.

Luxemburg and Zaanen ([4], theorem 7.3) proved for kernel operators in Banach function spaces that conditions similar to the conditions (a) to (d) in 3.10 are equivalent to the fact that the kernel operator is compact.
Let \( (E' \otimes F)^{ad} \) denote the band generated by \( E' \otimes F \) in \( \mathcal{L}^b(E, F) \). In the special case that \( E \) and \( F \) are Banach function spaces every kernel operator belongs to this band. The following result therefore generalizes the main result of Luxemburg and Zaanen in [4]. The equivalence (a) \( \Leftrightarrow \) (f) is due to Nagel and Schlotebeck who obtained it assuming that \( E \) and \( F \) have quasi-interior positive elements [8].

3.11 **Theorem:** Let \( E' \) and \( F \) have order continuous norms and let \( T \in (E' \otimes F)^{ad} \). Then the following conditions are equivalent:
(a) \( T \) is compact.
(b) \( T[U] \) is a set of u.a.c. norm, \( U \) the unit ball in \( E \).
(c) \( \|P[F_n]T\| \to 0 \) as \( n \to \infty \) for every sequence \( F_n \downarrow 0 \) of bands in \( F \).
(d) \( \|P[E_n]T'\| \to 0 \) as \( n \to \infty \) for every sequence \( E_n \downarrow 0 \) of bands in \( E' \).
(e) \( \|P[E_n]T'P[F_n]\| \to 0 \) as \( n \to \infty \) for all sequences \( E_n \downarrow 0 \) of bands in \( E' \) and \( F_n \downarrow 0 \) of bands in \( F \).

If \( T \) is positive we may add

(f) The norm of \( \mathcal{L}^b(E, F) \) induced on \([0, T] \subset \mathcal{L}^b(E, F) \) is order continuous.

**Proof:** (a) \( \Rightarrow \) (b) follows from theorem 3.1.
(b) \( \Rightarrow \) (a). As we observed earlier, condition (b) implies that \( T \) is L-weakly compact. However for operators \( T \in (E' \otimes F)^{ad} \) Dodds and Fremlin [1] proved that L-weakly compactness of \( T \) implies compactness of \( T \). This completes the proof.
We can also obtain the equivalence of (a) and (b) by implementing theorem 3.1. This proof then avoids the disjointness arguments used by Dodds and Fremlin in their proof of this result.

We now show that if moreover $E$ is a Dedekind complete Banach lattice and $T$ a normal integral operator (order continuous operator), then the conditions (d) and (e) in 3.11 are equivalent to conditions which resemble closely the conditions given by Luxemburg and Zaanen for kernel operators in Banach function spaces. We shall use the fact that if $T : E \to F$ is a normal integral operator, then $T'$ maps $F_0 \to E_0$. Hence, since $F$ has order continuous norm, $T'$ maps $F'$ into $E_0$.

3.12 PROPOSITION: Let $E'$ and $F$ have order continuous norms, let $E$ be a Dedekind complete Banach lattice and assume $T$ to be a normal integral operator. Consider the following conditions:

(a) $\|P(E_n)T'P'[F_n]\| \to 0$ as $n \to \infty$ for all sequences $E_n \downarrow 0$ of bands in $E'$ and $F_n \downarrow 0$ of bands in $F$.

(b) $\|P[F_n]TP[E_n]\| \to 0$ as $n \to \infty$ for all sequences $E_n \downarrow 0$ of bands in $E$ and $F_n \downarrow 0$ of bands in $F$.

(c) $\|P[E_n]T'\| \to 0$ as $n \to \infty$ for every sequence $E_n \downarrow 0$ of bands in $E'$.

(d) $\|TP[E_n]\| \to 0$ as $n \to \infty$ for every sequence $E_n \downarrow 0$ of bands in $E$.

Then we have (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (d) (and hence all the statements are equivalent).

PROOF: (a) $\Rightarrow$ (b). Let $E_n \downarrow 0$ be a sequence of bands in $E$. By proposition 2.4(a) there exists a sequence $E'_n \downarrow 0$ of bands in $E'$ such that $P'[E'_n]T'P'=P[E'_n]T'$ for all $\phi \in F'$. By assumption we have for any sequence $F_n \downarrow 0$ of bands in $F$ that $\|P[E'_n]T'P'[F_n]\| \to 0$ as $n \to \infty$. Hence

$\|P[F_n]TP[E_n]\| = \|P'[E_n]T'P'[F_n]\| \to 0$ as $n \to \infty$.

(c) $\Rightarrow$ (d). This follows in exactly the same way.

(b) $\Rightarrow$ (a). Let $E'_n \downarrow 0$ be a sequence of bands in $E'$ and $F_n \downarrow 0$ a sequence of bands in $F$. By proposition 2.4(b) there exists a sequence $E_n \downarrow 0$ of bands in $E$ such that $P[E_n]\phi \leq P'[E_n]\phi$ for all $\phi \in (E_0)^\perp$. Let $(\phi_n)$ be a sequence in $F_+ \subseteq (E_0)^\perp$ satisfying $\|\phi_n\| < 1$. Then we have

$\|P[E'_n]T'P'[F_n]\phi_n\| - \|P[E'_n]T'P'[F_n]\phi_n\| < \|P'[E'_n]T'P'[F_n]\phi_n\| \leq \|P[E'_n]T'P'[F_n]\| = \|P[F_n]TP[E_n]\| \to 0$ as $n \to \infty$ by (b).

(d) $\Rightarrow$ (c). This follows in the same manner.

3.13 COROLLARY: Let $E'$ and $F$ have order continuous norms and assume that $E$ is a Dedekind complete Banach lattice. If $T \in (E_0 \otimes F)^{\text{ad}}$, then the following conditions are equivalent:
(a) $T$ is compact.
(b) $T[U]$ is a set of u.a.c. norm, $U$ the unit ball in $E$.
(c) $\|P[F_n]T\| \to 0$ as $n \to \infty$ for every sequence $F_n \downarrow 0$ of bands in $F$.
(d) $\|TP[E_n]\| \to 0$ as $n \to \infty$ for every sequence $E_n \downarrow 0$ of bands in $E$.
(e) $\|P[F_n]TP[E_n]\| \to 0$ as $n \to \infty$ for all sequences $F_n \downarrow 0$ of bands in $F$
and $E_n \downarrow 0$ of bands in $E$.
If $T$ is positive we may add
(f) The norm of $L^p(E,F)$ induced on $[0,T] \subset L^p(E,F)$ is order continuous.

For kernel operators on Banach function spaces the equivalence of (a) to (e) is given in [4], theorem 7.3.

REFERENCES