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# Convergence and Representation Theorems for Set Valued Random Processes

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In this paper we study set valued random processes in discrete time and with values in a separable Banach space. We start with set valued martingales and prove various convergence and regularity results. Then we turn our attention to larger classes of set valued processes. So we introduce and study set valued amarts and set valued martingales in the limit. Finally we prove a useful property of the set valued conditional expectation. © 1990 Academic Press, Inc

## 1. INTRODUCTION

In this paper we expand the work initiated in [25–27], where we studied the properties of the set valued conditional expectation and proved various convergence theorems for set valued martingales and martingale-like processes, with values in a separable Banach space.

Set valued random variables (random sets) have been studied recently by many authors. We refer to the interesting works of Alo–deKorvin–Roberts [1], Bagchi [5], Costé [10], Hiai–Umegaki [14], Hiai [15], and Luu [19], for details. Furthermore it was illustrated by the recent works of deKorvin–Kleyle [17] and the author [28], that the theory of set valued martingale-like processes is the natural tool in the study of certain problems in the theory of information systems (see [17]) and in mathematical economics (see [28]). Further applications can be found in the works Artstein–Hart [2] and Giné–Hahn–Zinn [13].

In this paper, starting from the notion of a set valued martingale, we then proceed and define broader classes of set valued random processes (set valued quasimartingales, set valued amarts, and set valued martingales–in the limit), for which we prove various convergence results. Briefly the structure of this paper is as follows. In the next section we establish our notation and recall some basic definitions and facts from the theory of measurable multifunctions (random sets) and set valued measures (multimeasures). In



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Section 3, we concentrate on set valued martingales and prove various convergence and regularity results for them. In Section 4, we study various real valued processes related to a set valued martingale. Sections 5 and 6 are devoted to extensions of the notion of a set valued martingale. So in Section 5, we introduce and study set valued amarts, while in Section 6, we study set valued martingales-in the limit. Finally in Section 7 we prove an interesting property of the set valued conditional expectation.

## 2. Preliminaries

Throughout this work $(\Omega, \Sigma, \mu)$ will be a complete probability space and $X$ a separable Banach space. Additional hypotheses will be introduced as needed. We will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X\text{: nonempty, closed, (convex)}\}$$

and

$$P_{(w)k(c)}(X) = \{A \subseteq X\text{: nonempty, } (w-)\text{compact, (convex)}\}.$$

Also if $A \in 2^X \setminus \{\varnothing\}$, by $|A|$ we will denote the "norm" of $A$, i.e., $|A| = \sup\{\|x\| : x \in A\}$, by $\sigma(\cdot, A)$ the "support function" of $A$, i.e., $\sigma(x^*, A) = \sup\{(x^*, x) : x \in A\}$, $x^* \in X^*$, and by $d(\cdot, A)$ the "distance function" from $A$, i.e., $d(z, A) = \inf\{\|z - x\| : x \in A\}$.

A multifunction $F\colon \Omega \to P_f(X)$ is said to be measurable, if one of the following equivalent conditions holds:

(a) for every $z \in X$, $\omega \to d(z, F(\omega))$ is measurable,

(b) there exist $f_n : \Omega \to X$ measurable functions s.t.

$$F(\omega) = \text{cl}\{f_n(\omega)\}_{n \geqslant 1}, \qquad \omega \in \Omega.$$

(c) $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, $B(X)$ being the Borel $\sigma$-field of $X$ (graph measurability).

More details on the measurability of multifunctions can be found in the survey paper of Wagner [34].

By $S_F^1$ we will denote the set of integrable selectors of $F(\cdot)$. So we have

$$S_F^1 = \{f(\cdot) \in L^1(X) : f(\omega) \in F(\omega)\, \mu\text{-a.e.}\}.$$

Having $S_F^1$ we can define a set valued integral for $F(\cdot)$, by setting $\int_\Omega F = \{\int_\Omega f : f \in S_F^1\}$. Note that $S_F^1$ (and so $\int_\Omega F$ too), may be empty. It is easy to show that $S_F^1$ is nonempty if and only if $\inf\{\|x\| : x \in F(\omega)\} \in L_+^1$.

We will say that a multifunction $F(\cdot)$ is integrably bounded if and only if $\omega \to |F(\omega)|$ is an $L^1_+$-function. In this case then $S^1_F \neq \emptyset$.

Let $\Sigma_0 \subseteq \Sigma$ be a sub-$\sigma$-field of $\Sigma$ and let $F: \Omega \to P_f(X)$ be a measurable multifunction s.t. $S^1_F \neq \emptyset$. Following Hiai–Umegaki [14], we define the set valued conditional expectation of $F(\cdot)$ with respect to $\Sigma_0$, to be the $\Sigma_0$-measurable multifunction $E^{\Sigma_0}F: \Omega \to P_f(X)$, for which we have $S^1_{E^{\Sigma_0}F} = \text{cl}\{E^{\Sigma_0}f : f \in S^1_F\}$ (the closure in the $L^1(X)$-norm). If $F(\cdot)$ is integrably bounded (resp. convex valued), then so is $E^{\Sigma_0}F(\cdot)$. Note that in Hiai–Umegaki [14], the definition was given for integrably bounded $F(\cdot)$. However, it is clear that it can be extended to the more general class of multifunctions $F(\cdot)$, used here.

Let $\{\Sigma_n\}_{n \geqslant 1}$ be an increasing sequence of sub-$\sigma$-fields of $\Sigma$ s.t. $\sigma(\bigcup_{n \geqslant 1} \Sigma_n) = \Sigma$. Let $F_n: \Omega \to P_f(X)$, $n \geqslant 1$, be measurable multifunctions adapted to $\{\Sigma_n\}_{n \geqslant 1}$. We say that $\{F_n, \Sigma_n\}_{n \geqslant 1}$ is a set valued martingale (resp. supermartingale, submartingale), if for every $n \geqslant 1$, we have

$$E^{\Sigma_n}F_{n+1}(\omega) = F_n(\omega)\, \mu\text{-a.e.}$$

(resp. $E^{\Sigma_n}F_{n+1}(\omega) \subseteq F_n(\omega)\, \mu$-a.e., $E^{\Sigma_n}F_{n+1}(\omega) \supseteq F_n(\omega)\, \mu$-a.e.).

On $P_f(X)$ we can define a (generalized) metric, known as the Hausdorff metric, by setting

$$h(A, B) = \max\{\sup(d(a, B) : a \in A), \sup(d(b, A) : b \in B)\}.$$

Recall that $(P_f(X), h)$ is a complete metric space. Similarly on the space of all $P_f(X)$-valued, integrably bounded multifunctions, we can define a metric $\Delta(\cdot, \cdot)$ by setting

$$\Delta(F, G) = \int_\Omega h(F(\omega), G(\omega))\, d\mu(\omega).$$

As usual we identify $F_1(\cdot)$ and $F_1(\cdot)$, if $F_1(\omega) = F_2(\omega)$ $\mu$-a.e. Again the space of $P_f(X)$-valued, integrably bounded multifunctions together with $\Delta(\cdot, \cdot)$ is a complete metric space.

Next, let us recall a few basic definitions and facts from the theory of set valued measures. A set valued measure (multimeasure) is a map $M: \Sigma \to 2^X \backslash \{\emptyset\}$ s.t. $M(\emptyset) = \{0\}$ and for $\{A_n\}_{n \geqslant 1} \subseteq \Sigma$ pairwise disjoint we have $M(\bigcup_{n \geqslant 1} A_n) = \sum_{n \geqslant 1} M(A_n)$. Depending on the way we interpret this last sum, we get different notions of multimeasures. However, when $M(\cdot)$ is $P_{wkc}(X)$-valued, then all of them are equivalent. This result is the set valued version of the Orlicz–Pettis theorem. Since we will be dealing only with $P_{wkc}(X)$-valued multimeasures, we can say that $M(\cdot)$ is a set valued measure, if for all $x^* \in X^* A \to \sigma(x^*, M(A))$ is a signed measure.

We will close this section, by recalling the notions of convergence of sets that we will be using in the sequel. So if $\{A_n, A\}_{n \geq 1} \subseteq 2^X \setminus \{\emptyset\}$, we set

$$s\text{-}\underline{\lim}\, A_n = \{x \in X : x_n \xrightarrow{s} x, x_n \in A_n, n \geq 1\}$$

and

$$w\text{-}\overline{\lim}\, A_n = \{x \in X : x_k \xrightarrow{w} x, x_k \in A_{n_k}, k \geq 1\}.$$

We say that the $A_n$'s converge to $A$ in the Kuratowski–Mosco sense, denoted by $A_n \to^{K-M} A$, if $w\text{-}\overline{\lim}\, A_n = A = s\text{-}\underline{\lim}\, A_n$. When $X$ is finite dimensional, the weak and strong topologies coincide and then the Kuratowski–Mosco convergence of sets is the well known Kuratowski convergence denoted by $A_n \to^K A$ (see Kuratowski [18] and Mosco [21]). We say that $A_n \to^h A$, if $h(A_n, A) \to 0$. Finally $A_n \to^w A$, if for all $x^* \in X^*$, $\sigma(x^*, A_n) \to \sigma(x^*, A)$.

## 3. Set Valued Martingales

We start with a regularity result for set valued martingales. Our result generalizes Theorem 6.5 of Hiai–Umegaki [14], since we drop the separability hypothesis on $X^*$.

In the sequel $\{\Sigma_n\}_{n \geq 1}$ will be an increasing sequence of complete sub-$\sigma$-fields of $\Sigma$ s.t. $\Sigma = \sigma(\bigcup_{n \geq 1} \Sigma_n)$. Recall that $X$ is always a separable Banach space.

A sequence $\{f_n\}_{n \geq 1} \subseteq L^1(X)$ s.t. $\{f_n, \Sigma_n\}_{n \geq 1}$ is a martingale and for each $n \geq 1$, $f_n(\cdot)$ is a selector of $F_n(\cdot)$, where $\{F_n(\cdot)\}_{n \geq 1}$ is a sequence of $P_{fc}(X)$-valued, integrably bounded multifunctions, is said to be a martingale selector of $\{F_n(\cdot)\}_{n \geq 1}$, and is denoted by $\langle f_n \rangle \in MS(F_n)$.

Theorem 3.1. *If $X$ has the R.N.P. and $F_n : \Omega \to P_{fc}(X)$ are $\Sigma_n$-measurable multifunctions s.t.*

(1) *$\{F_n, \Sigma_n\}_{n \geq 1}$ is a set valued martingale,*

(2) *$\{|F_n|\}_{n \geq 1}$ is uniformly integrable,*

*then there exists $F: \Omega \to P_{fc}(X)$ integrably bounded s.t. $E^{\Sigma_n} F(\omega) = F_n(\omega)\, \mu$-a.e. $n \geq 1$.*

*Proof.* Let $M \subseteq L^1(X)$ be defined by $M = \{f \in L^1(X) : E^{\Sigma_n} f \in S^1_{F_n}, n \geq 1\}$. As in the proof of Theorem 6.5 of Hiai–Umegaki [14], we can show that $M$ is a closed, convex, bounded, and decomposable subset of $L^1(X)$ (to get these properties no separability of $X^*$ is needed). Then combining Theorems 3.1, 3.2, and Corollary 1.6 of Hiai–Umegaki [14], we get

$F: \Omega \to P_{fc}(X)$ integrably bounded s.t. $M = S^1_F$. Our claim is that this is the desired $F(\cdot)$.

From Luu [19], we know that $S^1_{F_k}(\Sigma_k) = \text{cl}\{f_k : \langle f_n \rangle \in MS(F_n)\}$, $k \geq 1$. Let $f \in S^1_F$. Then $\langle E^{\Sigma_n} f \rangle \in MS(F_n) \Rightarrow \overline{E^{\Sigma_n} S^1_F} = S^1_{E^{\Sigma_n}F} \subseteq S^1_{F_n}$. On the other hand, given $\langle f_n \rangle \in MS(F_n)$, since $X$ has the R.N.P., there exists $f \in L^1(X)$ s.t. $E^{\Sigma_n} f = f_n \Rightarrow f \in M \Rightarrow S^1_{F_n} \subseteq S^1_{E^{\Sigma_n}F}$. Therefore we conclude that $S^1_{F_n} = S^1_{E^{\Sigma_n}F}$ $\Rightarrow F_n(\omega) = E^{\Sigma_n} F(\omega)\, \mu$-a.e. Q.E.D.

We can relax the R.N.P. assumption on $X$, by imposing additional hypotheses on the random sets $F_n(\cdot)$, $n \geq 1$.

THEOREM 3.2. *If $F_n: \Omega \to P_{fc}(X)$ are $\Sigma_n$-measurable multifunctions s.t.*

(1) *$\{F_n, \Sigma_n\}_{n \geq 1}$ is a set valued martingale,*

(2) *$F_n(\omega) \subseteq G(\omega)\, \mu$-a.e. with $G: \Omega \to P_{wkc}(X)$ integrably bounded,*

*then there exists $F: \Omega \to P_{fc}(X)$ measurable multifunction s.t. $F(\omega) \subseteq G(\omega)\, \mu$-a.e. and $E^{\Sigma_n} F(\omega) = F_n(\omega)\, \mu$-a.e. $n \geq 1$.*

*Proof.* Let $M \subseteq L^1(X)$ be as in the proof of Theorem 3.1. We saw that $M = S^1_F$ with $F: \Omega \to P_{fc}(X)$ integrably bounded. Also for all $f \in S^1_F$, from the definition of $M$, we have $E^{\Sigma_n} f(\omega) \in F_n(\omega) \subseteq G(\omega)\, \mu$-a.e. From Proposition V-2-6 of Neveu [23], we know that $E^{\Sigma_n} f(\omega) \to^s f(\omega)\, \mu$-a.e. $\Rightarrow$ $f(\omega) \in G(\omega)\, \mu$-a.e. $\Rightarrow F(\omega) \subseteq G(\omega)\, \mu$-a.e.

As in the proof of Theorem 3.1, through Luu's representation result [19], we get that $S^1_{E^{\Sigma_n}F} \subseteq S^1_{F_n}$. On the other hand, given $\{g_n\}_{n \geq 1} \in MS(F_n)$, from Proposition 4.4 of Chatterji [9], we know that $g_n(\omega) \to^s g(\omega)\, \mu$-a.e., $g \in L^1(X)$. Note that $g_n = E^{\Sigma_n} g$ (see Metivier [20, p. 62]). So since $S^1_{F_k}(\Sigma_k) = \text{cl}\{f_k : \langle f_n \rangle \in MS(F_n)\}$ (see Luu [19]), we have $S^1_{F_n} \subseteq S^1_{E^{\Sigma_n}F} \Rightarrow$ $S^1_{F_n} = S^1_{E^{\Sigma_n}F} \Rightarrow F_n(\omega) = E^{\Sigma_n} F(\omega)\, \mu$-a.e. Q.E.D.

Having those regularity results, we can now prove a convergence theorem for set valued martingales.

THEOREM 3.3. *If $X$ has the R.N.P., $X^*$ is separable and $F_n: \Omega \to P_{fc}(X)$ are $\Sigma_n$-measurable multifunctions s.t.*

(1) *$\{F_n, \Sigma_n\}_{n \geq 1}$ is a set valued martingale,*

(2) *$|F_n(\omega)| \leq \phi(\omega)\, \mu$-a.e. $\phi(\cdot) \in L^1_+$,*

*then $F_n(\omega) \to^{K-M} F(\omega)\, \mu$-a.e.*

*Proof.* From Theorem 3.1, we know that there exists $F: \Omega \to P_{fc}(X)$ $\Sigma$-measurable and integrably bounded by $\phi(\cdot)$ s.t. $E^{\Sigma_n} F(\omega) = F_n(\omega)$ $\mu$-a.e. Then for $f \in S^1_F$, we have $E^{\Sigma_n} f \in S^1_{F_n}$, $n \geq 1$. From Proposition V-2-6 of Neveu [23], we have that $E^{\Sigma_n} f(\omega) \to^s f(\omega)\, \mu$-a.e. Hence we get that

$$F(\omega) \subseteq s\text{-}\underline{\lim}\, F_n(\omega)\, \mu\text{-a.e.} \tag{1}$$

On the other hand, from Proposition 1.4 of Luu [19], we know that there exists $\langle f_n^k \rangle \in MS(F_n)$, $k \geq 1$, s.t. for all $n \geq 1$, $F_n(\omega) = \mathrm{cl}\{f_n^k(\omega)\}_{k \geq 1}$. Then given $x^* \in X^*$, we have that

$$\sigma(x^*, F_n(\omega)) = \sup_{k \geq 1} (x^*, f_n^k(\omega)).$$

But note that $\{(x^*, f_n^k(\cdot)), \Sigma_n\}_{n \geq 1}$ is an $\mathbb{R}$-valued martingale and $\sup_{n \geq 1} \int_\Omega \sup_{k \geq 1}(x^*, {}_n^k(\omega))^+ \, d\mu(\omega) \leq \|x^*\| \cdot \sup_{n \geq 1} \int_\Omega |F_n| < \infty$. Also from Corollary 11.8 of Metivier [20], we know that there exist $f^k \in L^1(X)$ s.t. $E^{\Sigma_n} f^k = f_n^k \Rightarrow f^k \in S_F^1$. Apply Lemma V-2-9 of Neveu [23], to get that

$$\sup_{k \geq 1} (x^*, f_n^k(\omega)) \to \sup_{k \geq 1} (x^*, f^k(\omega)) \omega \in \Omega \setminus N(x^*),\ \mu(N(x^*)) = 0 \text{ as } n \to \infty$$

$$\Rightarrow \overline{\lim}\, \sigma(x^*, F_n(\omega)) \leq \sigma(x^*, F(\omega)) \omega \in \Omega \setminus N(x^*),\ \mu(N(x^*)) = 0.$$

Given that $X^*$ is separable and $|F_n(\omega)| \leq \phi(\omega) \mu$-a.e. for all $n \geq 1$, a simple density argument gives us that

$$\overline{\lim}\, \sigma(x^*, F_n(\omega)) \leq \sigma(x^*, F(\omega)) \mu\text{-a.e.}$$

From Proposition 4.1 of [30], we deduce that

$$w\text{-}\overline{\lim}\, F_n(\omega) \subseteq F(\omega) \mu\text{-a.e.} \tag{2}$$

Combining (1) and (2) above, we conclude that

$$F_n(\omega) \xrightarrow{K-M} F(\omega) \mu\text{-a.e.} \qquad \text{Q.E.D.}$$

We can have the same convergence result, but with the hypotheses of Theorem 3.2.

THEOREM 3.4. *If the hypotheses of Theorem 3.2 hold then there exists $F: \Omega \to P_{fc}(X)$ integrably bounded s.t.*

$$F(\omega) \subseteq G(\omega) \mu\text{-a.e.} \quad \text{and} \quad F_n(\omega) \xrightarrow{K-M} F(\omega) \mu\text{-a.e.}$$

*Proof.* The proof is the same as that of Theorem 3.3, using this time Theorem 3.2. Also instead of Corollary 11.8 of Metivier [20] (which requires $X$ to have the R.N.P.), we use Proposition 4.4 of Chatterji [9] and Corollary 2, p. 126, of Diestel–Uhl [12]. Finally note that there exists $\{x_m^*\}_{m \geq 1} \subseteq X^*$ which is dense in $X^*$ for the Mackey topology $m(X^*, X)$ and recall that the support function of a weakly compact, convex set, is $m(X^*, X)$-continuous. Q.E.D.

*Remark.* Under stronger hypotheses, Daures [11] and Neveu [24] proved convergence in the metric $\Delta(\cdot, \cdot)$.

If $X$ is finite dimensional, then we have the following convergence result.

COROLLARY I. *If the hypotheses of Theorem* 3.3 *hold, then there exists* $F: \Omega \to P_{fc}(X)$ *integrably bounded s.t.*

$$F_n(\omega) \xrightarrow{h} F(\omega)\, \mu\text{-a.e.}$$

*Proof.* Follows from Theorem 3.3 above and Corollary 3A of Salinetti–Wets [32]. Q.E.D.

*Remark.* A more general finite dimensional convergence result can be found in Van Cutsem [33]. The result of Van Cutsem was extended to set valued quasi-martingales by the author in [27] (Theorem 2.3).

Another consequence of the convergence theorems is the following result.

COROLLARY II. *If the hypotheses of Theorem* 3.2 *hold, then there exists* $F: \Omega \to P_{fc}(X)$ *measurable s.t.*

$$F(\omega) \subseteq C(\omega)\, u\text{-a.e.} \qquad \text{and} \qquad S^1_{F_n} \xrightarrow{K-M} S^1_F.$$

*Proof.* Follows from Theorem 3.4 above and Theorem 4.4 of [30]. Q.E.D.

## 4. RELATED $\mathbb{R}$-VALUED PROCESSES

In this section we examine certain $\mathbb{R}$-valued processes associated with a set valued martingale.

THEOREM 4.1. *If* $F_n: \Omega \to P_{fc}(X)$ *are* $\Sigma_n$*-measurable multifunctions s.t.*

(1) $\{F_n, \Sigma_n\}_{n \geq 1}$ *is a set valued martingale,*

(2) $\sup_{n \geq 1} \| |F_n| \|_1 < \infty$,

*then there exists* $\phi(\cdot) \in L^1_+$ *s.t.* $|F_n(\omega)| \to \phi(\omega)\, \mu$*-a.e.*

*Proof.* From Proposition 1.4 of Luu [19], we know that there exists $\langle f^k_n \rangle \in MS(F_n)\; k \geq 1$ s.t. $F_n(\omega) = \text{cl}\{f^k_n(\omega)\}_{k \geq 1}\, \mu$-a.e. Then we have $|F_n(\omega)| = \sup_{k \geq 1} \|f^k_n(\omega)\|$ $\mu$-a.e. Note that for all $k \geq 1$, $E^{\Sigma_n} \|f^k_{n+1}(\omega)\| \geq \|E^{\Sigma_n} f^k_{n+1}(\omega)\| = \|f^k_n(\omega)\|$ $\mu$-a.e. So we see that for every $k \geq 1$, $\{\|f^k_n(\cdot)\|, \Sigma_n\}_{n \geq 1}$ is a submartingale and $\sup_{n \geq 1} \int_\Omega \sup_{k \geq 1} \|f^k_n(\omega)\|\, d\mu(\omega) = \sup_{n \geq 1} \int_\Omega |F_n(\omega)|\, d\mu(\omega) < \infty$. So we can apply Lemma V-2-9 of Neveu [23] and get that there exists $\phi(\cdot) \in L^1_+$ s.t. $\sup_{k \geq 1} \|f^k_n(\omega)\| = |F_n(\omega)| \to \phi(\omega)\, \mu$-a.e. Q.E.D.

Another $\mathbb{R}$-valued, martingale-like process asociated to $\{F_n(\cdot)\}_{n\geq 1}$ is that of the distance functions. Namely we have:

THEOREM 4.2. *If $F_n: \Omega \to P_f(X)$ are $\Sigma_n$-measurable multifunctions s.t.*

(1) $\{F_n, \Sigma_n\}_{n\geq 1}$ *is a set valued martingale,*

(2) $\sup_{n\geq 1} \| |F_n| \|_1 < \infty$,

*then given any $z \in X$, $\{d(z, F_n(\cdot)), \Sigma_n\}_{n\geq 1}$ is a submartingale which converges a.e. to a function $\psi(\cdot) \in L^1_+$.*

*Proof.* Let $g \in S^1_F$. Note that $E^{\Sigma_{n-1}} \|z - g(\omega)\| \geq \|z - E^{\Sigma_{n-1}} g(\omega)\|$ $\mu$-a.e. From the definition of the set valued conditional expectation, we have that $E^{\Sigma_{n-1}} g \in S^1_{E^{\Sigma_{n-1}} F_n}$. So we can write that

$$E^{\Sigma_{n-1}} \|z - g(\omega)\| \geq d(z, E^{\Sigma_{n-1}} F_n(\omega)) \, \mu\text{-a.e.}$$

Hence for all $A \in \Sigma_{n-1}$, we have

$$\int_A E^{\Sigma_{n-1}} \|z - g(\omega)\| \, d\mu(\omega)$$

$$= \int_A \|z - g(\omega)\| \, d\mu(\omega) \geq \int_A d(z, E^{\Sigma_{n-1}} F_n(\omega)) \, d\mu(\omega)$$

$$\Rightarrow \inf \left\{ \int_A \|z - g(\omega)\| \, d\mu(\omega) : g \in S^1_{F_n} \right\} \geq \int_A d(z, E^{\Sigma_{n-1}} F_n(\omega)) \, d\mu(\omega)$$

$$\Rightarrow \int_A \inf_{x \in F_n(\omega)} \|z - x\| \, d\mu(\omega) = \int_A d(z, F_n(\omega)) \, d\mu(\omega)$$

$$\geq \int_A d(z, E^{\Sigma_{n-1}} F_n(\omega)) \, d\mu(\omega) \qquad \text{(see Hiai–Umegaki [14])}$$

$$\Rightarrow E^{\Sigma_{n-1}} d(z, F_n(\omega)) \geq d(z, F_{n-1}(\omega)) \, \mu\text{-a.e.}$$

$$\Rightarrow \{d(z, F_n(\cdot)), \Sigma_n\}_{n\geq 1} \text{ is a submartingale.}$$

Finally, since $\sup_{n\geq 1} \int_\Omega d(z, F_n(\omega)) \, d\mu(\omega) \leq \|z\| + \sup_{n\geq 1} \int_\Omega |F_n(\omega)| < \infty$, Doob's convergence theorem tells us that there exists $\psi(\cdot) \in L^1_+$ s.t. $d(z, F_n(\omega)) \to \psi(\omega)$ $\mu$-a.e. Q.E.D.

We have an analogous result for the Hausdorff metric.

THEOREM 4.3. *If $X$ and $X^*$ are both separable Banach spaces and $F_n$, $G_n: \Omega \to P_{fc}(X)$ are $\Sigma_n$-measurable multifunctions s.t.*

(1) $\{F_n, \Sigma_n\}_{n\geq 1}$ *and* $\{G_n, \Sigma_n\}_{n\geq 1}$ *are set valued martingales,*

(2) $\sup_{n\geq 1} \| |F_n| \|_1 < \infty$ *and* $\sup_{n\geq 1} \| |G_n| \|_1 < \infty$,

*then* $h(F_n(\cdot), G_n(\cdot)), \Sigma_n\}_{n\geq 1}$ *is a submartingale that converges a.e. to a function* $\eta(\cdot) \in L^1_+$.

*Proof.* From Hörmander's formula [16], we know that

$$(h(F_n(\omega), G_n(\omega))) = \sup_{\|x^*\| \leq 1} |\sigma(x^*, F_n(\omega)) - \sigma(x^*, G_n(\omega))|.$$

Let $B^*$ be the closed unit ball in $X^*$. Using Theorem 2.2 of Hiai–Umegaki [14], for every $A \in \Sigma_{n-1}$, we have that

$$\int_A E^{\Sigma_{n-1}} h(F_n(\omega), G_n(\omega)\, d\mu(\omega) = \int_A h(F_n(\omega), G_n(\omega))\, d\mu(\omega)$$

$$= \int_A \sup_{\|x^*\| \leq 1} |\sigma(x^*, F_n(\omega)) - \sigma(x^*, G_n(\omega))|\, d\mu(\omega)$$

$$= \sup_{v(\ ) \in S^1_{B^*}(\Sigma_{n-1})} \int_A |\sigma(v(\omega), F_n(\omega)) - \sigma(v(\omega), G_n(\omega))|\, d\mu(\omega)$$

$$\geq \sup_{v(\ ) \in S^1_{B^*}(\Sigma_{n-1})} \int_A |E^{\Sigma_{n-1}}\sigma(v(\omega), F_n(\omega)) - E^{\Sigma_{n-1}}\sigma(v(\omega), G_n(\omega))|\, d\mu(\omega)$$

$$= \int_A \sup_{\|x^*\| \leq 1} |E^{\Sigma_{n-1}}\sigma(x^*, F_n(\omega)) - E^{\Sigma_{n-1}}\sigma(x^*, G_n(\omega))|\, d\mu(\omega).$$

From the lemma in [27], we get that

$$E^{\Sigma_{n-1}}\sigma(x^*, F_n(\omega)) = \sigma(x^*, E^{\Sigma_{n-1}}F_n(\omega))$$

and

$$E^{\Sigma_{n-1}}\sigma(x^*, G_n(\omega)) = \sigma(x^*, E^{\Sigma_{n-1}}G_n(\omega))$$

for all $x^* \in X^*$ and all $\omega \in \Omega \setminus N$, $\mu(N) = 0$. Hence finally we can write that

$$\int_A E^{\Sigma_{n-1}} h(F_n(\omega), G_n(\omega))\, d\mu(\omega)$$

$$\geq \int_A \sup_{\|x^*\| \leq 1} |\sigma(x^*, E^{\Sigma_{n-1}}F_n(\omega)) - \sigma(x^*, E^{\Sigma_{n-1}}G_n(\omega))|\, d\mu(\omega)$$

$$= \int_A h(E^{\Sigma_{n-1}}F_n(\omega), E^{\Sigma_{n-1}}G_n(\omega))\, d\mu(\omega)$$

$$= \int_A h(F_{n-1}(\omega), G_{n-1}(\omega))\, d\mu(\omega)$$

$\Rightarrow E^{\Sigma_{n-1}} h(F_n(\omega), G_n(\omega)) \geq h(F_{n-1}(\omega), G_{n-1}(\omega))\, \mu$-a.e.

$\Rightarrow \{h(F_n(\cdot), G_n(\cdot)), \Sigma_n\}_{n\geq 1}$ is a submartingale.

Since $\sup_{n\geq 1}\int_\Omega h(F_n(\omega), G_n(\omega))\, d\mu(\omega) \leq \sup_{n\geq 1}\int_\Omega |F_n(\omega)| + \sup_{n\geq 1}\int_\Omega |G_n(\omega)| < \infty$, from Doob's theorem we get that there exists $\eta(\cdot)\in L^1_+$ s.t. $h(F_n(\omega), G_n(\omega)) \to \eta(\omega)\,\mu$-a.e. Q.E.D.

*Remark.* Note that if for all $n\geq 1$ and all $\omega\in\Omega$, $G_n(\omega)=\{0\}$, then $h(F_n(\omega), G_n(\omega))=|F_n(\omega)|$ and so Theorem 4.3 produces Theorem 4.1 as a special case, with the additional hypothesis that $X^*$ is separable.

## 5. Set Valued Amarts

In this section, we turn our attention to a larger class of set valued processes, namely we examine set valued amarts.

Following Bagchi [5] and in the single valued case Bellow [6], we say that a sequence of multifunctions $F_n:\Omega\to P_{fc}(X)$ adapted to $\{\Sigma_n\}_{n\geq 1}$, is a "set valued amart," if there exists $K\in P_{fc}(X)$ s.t. $\lim_{\tau\in T} h(\int_\Omega F_\tau, K)=0$, where $T$ is the set of bounded stopping times. Note that $T$ with the usual pointwise ordering $\leq$ is a directed set filtering to the right. Clearly a set valued martingale is a set valued amart.

We start with a convergence theorem that partially extends Theorem 2.2 of Bagchi [5]. In that theorem, Bagchi considered a broader class of a set values processes, which he called $w^*$-amarts, which however take values in a separable, dual Banach space. Here we restrict ourselves to the smaller class of set values amarts, but we drop the requirement that they take their values in a dual Banach space.

THEOREM 5.1. *If both $X$ and $X^*$ are separable, $X$ has the R.N.P., and $F_n:\Omega\to P_{wkc}(X)$ are $\Sigma_n$-measurable multifunctions s.t.*

(1) *$\{F_n, \Sigma_n\}_{n\geq 1}$ is a set valued amart with $\Delta$-limit $K\in P_{wkc}(X)$,*

(2) *$\sup_{\tau\in T}\int_\Omega |F_\tau|<\infty$ (i.e., $\{F_n,\Sigma_n\}_{n\geq 1}$ is of class B),*

*then there exists $F:\Omega\to P_{fc}(X)$ integrable bounded s.t. $F_n(\omega)\to^{w} F(\omega)$ for all $\omega\in\Omega\setminus N$, $\mu(N)=0$.*

*Proof.* We claim that for fixed $k\geq 1$ and all $A\in\Sigma_k$, we have that $h$-$\lim_{\tau\in T}\int_A F_\tau$ exists in $P_{fc}(X)$. So let $\varepsilon>0$ be given. Then there exists $\sigma_0\in T$, $\sigma_0\geq k$ s.t. if $\sigma,\tau\in T(\sigma_0)=\{\sigma'\in T:\sigma_0\leq\sigma'\}$, then $h(\int_\Omega F_\sigma, \int_\Omega F_\tau)<\varepsilon$. Let $\sigma,\tau\in T(\sigma_0)$ and define $\hat\sigma,\hat\tau$ as follows: Let $n_1>\max(\sigma,\tau)$ and set $\hat\sigma=\sigma$, $\hat\tau=\tau$ on $A$, while $\hat\sigma=\hat\tau=n_1$ on $A^c$. It is easy to see that $\hat\sigma,\hat\tau\in T$ and $h(\int_A F_\sigma,\int_A F_\tau)=h(\int_\Omega F_{\hat\sigma},\int_\Omega F_{\hat\tau})<\varepsilon$. So $\lim_{\tau\in T}\int_A F_\tau$ exists in $(P_{fc}(X), h)$ and the convergence is uniform in $A\in\Sigma_k$. Since $k\geq 1$ was arbitrary, we deduce that the above $h$-limit exists for all $A\in\bigcup_{k\geq 1}\Sigma_k$. Recall that $\Sigma=\sigma(\bigcup_{k\geq 1}\Sigma_k)$, i.e., $\Sigma$ is generated by $\bigcup_{k\geq 1}\Sigma_k$. So given $A\in\Sigma$, there exists $A' \in \bigcup_{k\geq 1} \Sigma_k$ s.t. $\mu(A \Delta A') < \varepsilon$. Then $h(\int_A F_\tau, \int_{A'} F_\tau) < \varepsilon \int_\Omega u$, where as in Chacon–Sucheston [8, p. 57], we may assume, without any loss of generality, that $\sup_{n\geq 1} |F_n(\omega)| \leq u(\omega) \mu$-a.e. $u(\cdot) \in L^1_+$. Then, using the triangle inequality, it is easy to check that $h\text{-}\lim_{\tau\in T} \int_A F_\tau$ exists for all $A \in \Sigma$. Set $M_\tau(A) = \int_A F_\tau \in P_{wkc}(X)$ (see [25]) and $M(A) = h\text{-}\lim_{\tau\in T} \int_A F_\tau$. Then $\sigma(x^*, M_\tau(A)) \to \sigma(x^*, M(A))$ uniformly on $B^* =$ unit ball in $X^*$. But $x^* \to \sigma(x^*, M_\tau(A))$ is a signed measure. So by Nikodym's theorem $A \to \sigma(x^*, M(A))$ is a signed measure too. Also by hupothesis (2), $M(\Omega) \in P_{wkc}(X)$, while we saw that $M(A) \in P_{fc}(X)$ for all $A \in \Sigma$. Since $M(\Omega) = \overline{M(A) + M(\Omega \setminus A)}$, we deduce that $M(A) \in P_{wkc}(X)$ for all $A \in \Sigma$. Hence $M(\cdot)$ is a set valued measure with values in $P_{wkc}(X)$. Apply Theorem 2 of Costé [10], to get $F: \Omega \to P_{fc}(X)$ integrably bounded s.t. $M(A) = \int_A F$ for all $A \in \Sigma$. Now note that for fixed $x^* \in X^*$, the process $\{\sigma(x^*, F_n(\cdot)), \Sigma_n\}_{n\geq 1}$ is an $L^1$-bounded, real amart. From Theorem 2 of Austin–Edgar–Tulcea [4] and since $\sigma(x^*, M_n(A)) \to \sigma(x^*, M(A)) = \int_A \sigma(x^*, F(\omega))\, d\mu(\omega)$, we see that $\sigma(x^*, F_n(\omega)) \to \sigma(x^*, F(\omega))$ for all $\omega \in \Omega \setminus N(x^*)$, $\mu(N(x^*)) = 0$. Let $\{x_m^*\}_{m\geq 1}$ be dense in $X^*$ and set $N = \bigcup_{m\geq 1} N(x_m^*)$, for which clearly we have $\mu(N) = 0$. Given $x^* \in X^*$, we can find $\{x_k^*\}_{k\geq 1} \subseteq \{x_m^*\}_{m\geq 1}$ s.t. $x_k^* \to^s x^*$. Then for all $\omega \in \Omega \setminus N$, we have

$$\sigma(x_k^*, F_n(\omega)) \to \sigma(x_k^*, F(\omega)) \qquad \text{as} \quad n \to \infty.$$

Also from the continuity of $\sigma(\cdot, F(\omega))$, we have

$$\sigma(x_k^*, F(\omega)) \to \sigma(x^*, F(\omega)) \qquad \text{as} \quad k \to \infty.$$

Through a diagonalization lemma, we get

$$\sigma(x_{k(n)}^*, F_n(\omega)) \to \sigma(x^*, F(\omega)) \qquad \text{as} \quad n \to \infty.$$

Then for $\omega \in \Omega \setminus N$ and for any $x^* \in X^*$, we have

$$\begin{aligned} &|\sigma(x^*, F_n(\omega)) - \sigma(x^*, F(\omega))| \\ &\quad \leq |\sigma(x^*, F_n(\omega)) - \sigma(x_{k(n)}^*, F_n(\omega))| + |\sigma(x_{k(n)}^*, F_n(\omega)) \\ &\qquad - \sigma(x_{k(n)}^*, F(\omega))| + |\sigma(x_{k(n)}^*, F(\omega)) - \sigma(x^*, F(\omega))|. \end{aligned}$$

Note that the second and third terms of the sum on the right hand side of the inequality above tend to zero. Also

$$|\sigma(x^*, F_n(\omega)) - \sigma(x_{k(n)}^*, F_n(\omega))| \leq |F_n(\omega)| \cdot \|x^* - x_{k(n)}^*\| \to 0 \qquad \text{as} \quad n \to \infty.$$

Thus finally we have that

$$\sigma(x^*, F_n(\omega)) \to \sigma(x^*, F(\omega)) \qquad \text{as} \quad n \to \infty$$

for all $\omega \in \Omega \setminus N$, $\mu(N)=0$ and all $x^* \in X^*$. Therefore we conclude that $F_n(\omega) \to^{w} F(\omega)\, \mu$-a.e. Q.E.D.

As before, in the finite dimensional case, we can say more.

COROLLARY. *If* $\dim X < \infty$ *and the hypotheses of Theorem* 5.1 *hold, then there exists* $F: \Omega \to P_{kc}(X)$ *integrably bounded s.t.* $F_n(\omega) \to^h F(\omega)\, \mu$*-a.e.*

*Proof.* From Theorem 5.1, we know that there exists $F: \Omega \to P_{kc}(X)$ integrably bounded, s.t. for all $x^* \in X^*$ and all $\omega \in \Omega \setminus N$, $\mu(N)=0$, we have $\sigma(x^*, F_n(\omega)) \to \sigma(x^*, F(\omega))$. Then Corollary 2C of Salinetti–Wets [31] and Theorem 3.1 of Mosco [21], tell us that $F_n(\omega) \to^K F(\omega)\, \mu$-a.e. But since $F$ is compact and convex valued, we conclude that $F_n(\omega) \to^h F(\omega)\, \mu$-a.e.

Q.E.D.

## 6. Set Valued Martingales–In the Limit

In this section, we examine another large class of set valued stochastic processes that includes set valued martingales and is analogous to the family of single valued processes studied by Blake [7] and Mucci [22]. The results in this section, generalize those of Daures [11], Hiai–Umegaki [14], Hiai [15], Neveu [24], and Van Cutstem [33].

Let $F_n: \Omega \to P_{fc}(X)$ be measurable multifunctions adapted to $\{\Sigma_n\}_{n \geq 1}$. We will say $\{F_n, \Sigma_n\}_{n \geq 1}$ is a set valued martingale–in the limit (abbreviated as sv-mil), if for every $\varepsilon > 0$, we have

$$\mu\{\omega \in \Omega: h(E^{\Sigma_m} F_n(\omega), F_m(\omega)) > \varepsilon\} \to 0 \qquad \text{as} \quad n \geq m \to \infty.$$

Clearly every set valued martingale, or more generally every set valued quasi-martingale (see [27]) is a sv-mil.

We start with a "Riesz decomposition" type theorem for such set valued processes.

THEOREM 6.1. *If* $F_n: \Omega \to P_{fc}(X)$ *are* $\Sigma_n$*-measurable multifunctions s.t.*

(1) $\{F_n, \Sigma_n\}_{n \geq 1}$ *is a sv-mil,*

(2) $\{|F_n|\}_{n \geq 1}$ *is uniformly integrable,*

*then there exists a unique set valued martingale* $\{G_n, \Sigma_n\}_{n \geq 1}$ *with values in* $P_{fc}(X)$ *s.t.* $\Delta(F_n, G_n) \to 0$.

*Proof.* Note that for $n \geq m$, we have

$$h(E^{\Sigma_m} F_n, F_m) = h(E^{\Sigma_m} F_n, E^{\Sigma_m} F_m).$$

But from the proof of Theorem 4.3, we know that

$$h(E^{\Sigma_m}F_n, E^{\Sigma_m}F_m) \leqslant E^{\Sigma_m}h(F_n, F_m)\,\mu\text{-a.e.}$$

and

$$\int_\Omega E^{\Sigma_m}h(F_n, F_m) = \int_\Omega h(F_n, F_m) \leqslant \int_\Omega (|F_n| + |F_m|).$$

Therefore from hypothesis (2), we deduce that $\{h(E^{\Sigma_m}F_n, F_m)\}_{n \geqslant m}$ is uniformly integrable. Also, since by hypothesis (1), $h(E^{\Sigma_m}F_n, F_m) \to^{\mu} 0$ as $n \geqslant m \to \infty$, from the dominated convergence theorem (see Ash [3, p. 295]), we get that $\Delta(E^{\Sigma_m}F_n, F_m) \to 0$ as $n \geqslant m \to \infty$.

Now fix $m \geqslant 1$ and consider the sequence $\{E^{\Sigma_m}F_n\}_{n \geqslant m}$. From the triangle inequality for the metric $\Delta(\cdot, \cdot)$, we have for $n, k \geqslant m$

$$\Delta(E^{\Sigma_m}F_n, E^{\Sigma_m}F_k) \leqslant \Delta(E^{\Sigma_m}F_n, F_m) + \Delta(F_m, E^{\Sigma_m}F_k)$$

$\Rightarrow \{E^{\Sigma_m}F_n\}_{n \geqslant m}$ is a Cauchy sequence for the metric $\Delta(\cdot, \cdot)$.

Thus, Theorem 3.3 of Hiai–Umegaki [14] tells us that there exists $G_m: \Omega \to P_{fc}(X)$ integrably bounded multifunctions s.t. $E^{\Sigma_m}F_n \to^{\Delta} G_m$ as $n \to \infty$. We claim that $\{G_m, \Sigma_m\}_{m \geqslant 1}$ is a set valued martingale. So let $n \geqslant m$. We have

$$\begin{aligned} \Delta(E^{\Sigma_m}G_n, G_m) &\leqslant \Delta(E^{\Sigma_m}G_n, E^{\Sigma_m}E^{\Sigma_n}F_{n+k}) \\ &\quad + \Delta(E^{\Sigma_m}E^{\Sigma_n}F_{n+k}, G_m) \leqslant \Delta(G_n, E^{\Sigma_n}F_{n+k}) \\ &\quad + \Delta(E^{\Sigma_m}F_{n+k}, G_m) \to 0 \qquad \text{as} \quad k \to \infty \\ &\Rightarrow E^{\Sigma_m}G_n(\omega) = G_m(\omega)\,\mu\text{-a.e.} \\ &\Rightarrow \{G_m, \Sigma_m\}_{m \geqslant 1} \text{ is a set valued martingale.} \end{aligned}$$

Finally note that for $n \geqslant m$, we have

$$\Delta(F_m, G_m) \leqslant \Delta(F_m, E^{\Sigma_m}F_n) + \Delta(E^{\Sigma_m}F_n, G_m) \to 0 \qquad \text{as} \quad n \geqslant m \to \infty.$$

Now for the uniqueness of $\{G_n, \Sigma_n\}_{n \geqslant 1}$, suppose that there was another such set valued martingale $\{G'_n, \Sigma_n\}_{n \geqslant 1}$, for which we had $\Delta(F_n, G'_n) \to 0$ as $n \to \infty$. Then from Hiai–Umegaki [14], we have

$$\begin{aligned} \Delta(G_n, G'_n) &= \Delta(E^{\Sigma_n}G_{n+k}, E^{\Sigma_n}G'_{n+k}) \\ &\leqslant \Delta(G_{n+k}, G'_{n+k}) \\ &\leqslant \Delta(G_{n+k}, F_{n+k}) + \Delta(F_{n+k}, G'_{n+k}) \to 0 \qquad \text{as} \quad k \to \infty, \end{aligned}$$

$\Rightarrow \Delta(G_n, G'_n) = 0$ and so $G_n(\omega) = G'_n(\omega)\,\mu$-a.e. Q.E.D.

This leads us to the following regularity result for sv-mils.

THEOREM 6.2. *If $X$ has the R.N.P. and the hypotheses of Theorem 6.1 hold, then there exists $F: \Omega \to P_{fc}(X)$ integrably bounded multifunction s.t. $\Delta(F_n, E^{\Sigma_n}F) \to 0$ as $n \to \infty$.*

*Proof.* Apply Theorem 6.1 to get $G_n: \Omega \to P_{fc}(X)\Sigma_n$-measurable multifunctions s.t. $\{G_n, \Sigma_n\}_{n \geq 1}$ is a set valued martingale and $\Delta(F_n, G_n) \to 0$ as $n \to \infty$. Note that $|G_n| = h(G_n, 0) \leq h(G_n, F_n) + h(F_n, 0) = h(G_n, F_n) + |F_n| \Rightarrow \{|G_n|\}_{n \geq 1}$ is uniformly integrable. Use Theorem 3.1 to get $F: \Omega \to P_{fc}(X)$ integrably bounded s.t. $E^{\Sigma_n}F = G_n \mu$-a.e. Then $\Delta(F_n, E^{\Sigma_n}F) = \Delta(F_n, G_n) \to 0$ as $n \to \infty$. Q.E.D.

Again if $X$ is finite dimensional, we can say more.

COROLLARY. *If $\dim X < \infty$ and the hypotheses of Theorem 6.1 hold, then there exists $F: \Omega \to P_{kc}(X)$ integrably bounded s.t. $\Delta(F_n, F) \to 0$.*

*Proof.* Use Theorem 6.2 to get $F: \Omega \to P_{kc}(X)$ integrably bounded s.t. $\Delta(F_n, E^{\Sigma_n}F) \to 0$. Then note that $\Delta(F_n, F) \leq \Delta(F_n, E^{\Sigma_n}F) + \Delta(E^{\Sigma_n}F, F) \to 0$ as $n \to \infty$. Q.E.D.

## 7. SET VALUED CONDITIONAL EXPECTATION

In this section we present an interesting observation concerning set valued conditional expectations. Namely we show that the set valued conditional expectation of a $P_{wkc}(X)$-valued, integrably bounded multifunction is still a $P_{wkc}(X)$-valued multifunction (i.e., we have preservation of the weak compactness of the values).

THEOREM 7.1. *If $X$ and $X^*$ are separable, $\Sigma_0 \subseteq \Sigma$ is a sub-$\sigma$-field of $\Sigma$, $F: \Omega \to P_{wkc}(X)$ is integrably bounded, and every vector measure $m: \Sigma_0 \to X$ s.t. $m(A) \in M(A) = \int_A F(\omega)\, d\mu(\omega)$, has a Pettis integrable density, then $E^{\Sigma_0}F(\omega) \in P_{wkc}(X)\, \mu$-a.e.*

*Proof.* Let $M(A) = \int_A F(\omega)\, d\mu(\omega) = \{\int_A f(\omega)\, d\mu(\omega) : f \in S_F^1\}$, $A \in \Sigma_0$. Note that for every $x^* \in X^*$, we have (see [29])

$$\sigma(x^*, M(A)) = \int_A \sigma(x^*, F(\omega))\, d\mu(\omega)$$

$\Rightarrow A \to \sigma(x^*, M(A))$ is a signed measure on $\Sigma_0$, for every $x^* \in X^*$.

From the corollary to Proposition 3.1 in [25], we know that for all $A \in \Sigma_0$, $M(A) \in P_{wkc}(X)$. So $M(\cdot)$ is a set valued measure on $\Sigma_0$. Apply Theorem 3 of Costé [10] and get $G: \Omega \to P_{wkc}(X)$ integrably bounded s.t.

$$M(A) = \int_A G(\omega)\, d\mu(\omega).$$

From Theorem 5.4(i) of Hiai–Umegaki [14], we have that

$$\mathrm{cl} \int_A E^{\Sigma_0} F = \int_A F = \int_A G$$

$$\Rightarrow \int_A \sigma(x^*, E^{\Sigma_0} F) = \int_A \sigma(x^*, G)$$

$$\Rightarrow \sigma(x^*, E^{\Sigma_0} F(\omega)) = \sigma(x^*, G(\omega)), \qquad \text{for all } \omega \in \Omega \backslash N(x^*), \quad \mu(N(x^*)) = 0.$$

Let $\{x_m^*\}_{m \geq 1} \subseteq X^*$ be dense in $X^*$ and set $N = \bigcup_{m \geq 1} N(x_m^*)$. Then $\mu(N) = 0$. For every $x^* \in X^*$ and every $\omega \in \Omega \backslash N$, we have $\{x_k^*\}_{k \geq 1} \subseteq \{x_m^*\}_{m \geq 1} x_k^* \xrightarrow{s} x^*$ and

$$\begin{aligned} &|\sigma(x^*, E^{\Sigma_0} F(\omega)) - \sigma(x^*, G(\omega))| \\ &\quad \leqslant |\sigma(x^*, E^{\Sigma_0} F(\omega)) - \sigma(x_k^*, E^{\Sigma_0} F(\omega))| \\ &\qquad + |\sigma(x_k^*, E^{\Sigma_0} F(\omega)) - \sigma(x_k^*, G(\omega))| \\ &\qquad + |\sigma(x_k^*, G(\omega)) - \sigma(x^*, G(\omega))|. \end{aligned}$$

Note that since $\omega \in \Omega \backslash N$, $\sigma(x_k^*, E^{\Sigma_0} F(\omega)) = \sigma(x_k^*, G(\omega))$ for all $k \geqslant 1$. Also since $\sigma(\cdot, E^{\Sigma_0} F(\omega))$ and $\sigma(\cdot, G(\omega))$ are both strongly continuous, we have

$$\sigma(x_k^*, E^{\Sigma_0} F(\omega)) \to \sigma(x^*, E^{\Sigma_0} F(\omega))$$

and

$$\sigma(x_k^*, G(\omega)) \to \sigma(x^*, G(\omega)) \qquad \text{as} \quad k \to \infty,$$

Therefore for all $x^* \in X^*$ and all $\omega \in \Omega \backslash N$, we have

$$\begin{aligned} &\sigma(x^*, E^{\Sigma_0} F(\omega)) = \sigma(x^*, G(\omega)) \\ &\quad \Rightarrow E^{\Sigma_0} F(\omega) = G(\omega)\, \mu\text{-a.e.} \\ &\quad \Rightarrow E^{\Sigma_0} F(\omega) \in P_{wkc}(X)\, \mu\text{-a.e.} \end{aligned}$$

Q.E.D.

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