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On the existence of positive periodic solutions for neutral functional differential equation with multiple deviating arguments ☆

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Abstract

By means of the abstract continuation theory for k-contractions, some criteria are established for the existence and nonexistence of positive periodic solutions of the following neutral functional differential equation:

$$\frac{dN}{dt} = N(t)\left[a(t) - \beta(t)N(t) - \sum_{j=1}^{n} b_j(t)N\big(t - \sigma_j(t)\big) - \sum_{i=1}^{m} c_i(t)N'\big(t - \tau_i(t)\big)\right].$$

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1. Introduction

In this paper, we consider the following periodic neutral functional differential equation:

$$\frac{dN}{dt} = N(t)\Big[a(t) - \beta(t)N(t) - \sum_{j=1}^{n} b_j(t)N\big(t - \sigma_j(t)\big) - \sum_{i=1}^{m} c_i(t)N'\big(t - \tau_i(t)\big)\Big], \tag{1}$$

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where $\beta(t), a(t), b_j(t), c_i(t), \sigma_j(t), \tau_i(t)$ are continuous periodic functions of period $\omega > 0$ and $\beta(t) \geqslant 0$, $a(t) \geqslant 0$, $b_j(t) \geqslant 0$, $c_i(t) \geqslant 0$ $(i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$. For the ecological justification of Eq. (1), one can refer to [1–4].

In 1993, Kuang [5] proposed an open problem (Open problem 9.2) to obtain sufficient conditions for the existence of positive periodic solutions of the following equation:

$$\frac{dN}{dt} = N(t)\big[a(t) - \beta(t)N(t) - b(t)N\big(t - \tau(t)\big) - c(t)N'\big(t - \tau(t)\big)\big]. \tag{2}$$

In 1997, by using Mawhin's continuation theorem Li studied a kind of periodic neutral functional differential equation with constant multiple delays [6]:

$$\frac{dN}{dt} = N(t)\left[a(t) - \beta(t)N(t) - \sum_{j=1}^{n} b_j(t)N(t - \sigma_j) - \sum_{i=1}^{m} c_i(t)N'(t - \tau_i)\right]. \tag{3}$$

However, Li did not verify the important assumption that operator $N : \bar{\Omega} \to \Omega$ was L-compact. So the main results in [6] is not true. In fact, as the right side of Eq. (1) contains $N'(t - \tau_i)$ $(i = 1, 2, \ldots, m)$, it is not easy to verify the conclusion. Therefore, the result of [6] is far from clear and may be not even true.

Fang [7] studied Eq. (2) and gave an answer to Open problem 9.2 of [5]. But the paper [7] required that $c_0'(t) < b(t)$ or $c_0'(t) \leqslant b(t)$, $\beta(t) > 0$, $\forall t \in [0, \omega]$, where $c_0(t) = c(t)/(1 - \tau'(t))$.

It is easy to see that either Eq. (2) or Eq. (3) is a special case of Eq. (1). The purpose of this paper is to establish some criteria to guarantee the existence and nonexistence of positive periodic solutions of Eq. (1). By using the continuation theory for k-set contractions [8,9], we obtain some new results. If $m = n = 1$, $\tau_1(t) = \sigma_1(t) = \tau(t)$, $b_1(t) = b(t)$, and $c_1(t) = c(t)$, then our existence result is an answer to Open problem 9.2 due to Kuang.

Taking the transformation $N(t) = \exp x(t)$, Eq. (1) can be rewritten as

$$\begin{aligned} x'(t) &= a(t) - \beta(t)e^{x(t)} \\ &\quad - \sum_{j=1}^{n} b_j(t)e^{x(t-\sigma_j(t))} - \sum_{i=1}^{m} c_i(t)x'\big(t - \tau_i(t)\big)e^{x(t-\tau_i(t))}. \end{aligned} \tag{4}$$

In order to study Eq. (1), we should make some preparations.

Let E be a Banach space. For a bounded subset $A \subset E$, by

$$\alpha_E(A) = \inf\left\{\delta > 0 \mid \text{there is a finite number of subsets } A_i \subset A \text{ such that } A = \bigcup_{i=1} A_i \text{ and } \operatorname{diam}(A_i) \leqslant \delta\right\}$$

denote the (Kuratoski) measure of noncompactness, where $\operatorname{diam}(A_i)$ denotes the diameter of set A_i. Let X, Y be two Banach spaces and Ω be a bounded open subset of X. A continuous and bounded map $N : \bar{\Omega} \to Y$ is called k-set contractive if for any bounded set $A \subset \Omega$ we have

$$\alpha_Y\big(N(A)\big) \leqslant k\alpha_X(A).$$

Also, for a Fredholm operator $L : X \to Y$ with index zero, according to [8] we may define that

$$l(L) = \sup\{r \geqslant 0 \mid r\alpha_X(A) \leqslant \alpha_Y(L(A)), \text{ for all bounded subset } A \subset X\}.$$

Lemma 1 [10]. *Let $L : X \to Y$ be a Fredholm operator with index zero, and let $a \in Y$ be a fixed point. Suppose that $N : \Omega \to Y$ is a k-set contractive with $k < l(L)$, where $\Omega \subset X$ is bounded, open, and symmetric about $0 \in \Omega$. Further, we also assume that*

(1) *$Lx \neq \lambda Nx + \lambda a$, for $x \in \Omega$, $\lambda \in (0, 1)$, and*
(2) *$[QN(x) + Qa, x][QN(-x) + Qa, x] < 0$, for $x \in \ker L \cap \partial\Omega$, where $[\cdot, \cdot]$ is a bilinear form on $Y \times X$ and Q is the project of Y onto $\operatorname{coker}(L)$.*

Then there is $x \in \bar{\Omega}$ such that

$$Lx - Nx = a.$$

In order to use Lemma 1 for Eq. (1), we set

$$C_\omega^0 = \{x \mid x \in C^1(R, R),\ x(t + \omega) \equiv x(t)\}$$

with the norm defined by $|x|_0 = \max_{t\in[0,\omega]} |x(t)|$, and

$$C_\omega^1 = \{x \mid x \in C^1(R, R),\ x(t + \omega) \equiv x(t)\}$$

with the norm defined by $|x|_1 = \max\{|x|_0, |x'|_0\}$. Then C_ω^0, C_ω^1, are all Banach spaces. Let $L : C_\omega^1 \to C_\omega^0$ be defined by $Lx = dx/dt$, $N : C_\omega^1 \to C_\omega^0$ defined by

$$Nx = -\beta(t)e^{x(t)} - \sum_{j=1}^{n} b(t)e^{x(t-\sigma_j(t))} - \sum_{i=1}^{m} c_i(t)x'(t - \tau_i(t))e^{x(t-\tau_i(t))}. \tag{5}$$

It is easy to see that L is a Fredholm operator with index zero and is bounded with bound of 1. Now, Eq. (4) has an ω-periodic solution if and only if $Lx = Nx + a$ for some $x \in C_\omega^1$, where $a =: a(t)$.

In this paper, we denote $\bar{h} = (1/\omega)\int_0^\omega h(s)\,ds$, $h_m = \min_{t\in[0,\omega]} h(t)$, for $h \in C_\omega^0$.

Lemma 2 [9]. *The differential operator L is a Fredholm operator with index zero and satisfies $l(L) \geqslant 1$.*

Lemma 3. *Let r_1, r_2 be two positive constants and $\Omega = \{x \mid x \in C_\omega^1, |x|_0 < r_1, |x'|_0 < r_2\}$. If $k = (\sum_{i=1}^m |c_i|_0)e^{r_1} < 1$, then $N : \Omega \to C_\omega^0$ is a k-contractive map.*

Proof. Let $A \subset \bar{\Omega}$ be a bounded subset and let $\eta = \alpha_{C_\omega^1}(A)$. Then, for any $\varepsilon > 0$, there is a finite family of subsets A_i satisfying $A = \bigcup_{i=1} A_i$ with $\operatorname{diam}(A_i) \leqslant \eta + \varepsilon$. Now let

$$g(t, x, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_m, w_1, w_2, \ldots, w_m)$$
$$= \beta(t)e^x + \sum_{j=1}^{n} b_j(t)e^{y_j} + \sum_{i=1}^{m} c_i(t)w_i e^{z_i}.$$

Since $g(t, x, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_m, w_1, w_2, \ldots, w_m)$ is uniformly continuous on any compact subset of $R \times R^{2m+n+1}$, A and A_i are precompact in C_ω^0, it follows that there is a finite family of subsets A_{ij} of A_i such that $A_i = \bigcup_{j=1} A_{ij}$ with

$$\begin{aligned}
&\Big|g\Big(t, x(t), x\big(t-\sigma_1(t)\big), \ldots, x\big(t-\sigma_n(t)\big), x\big(t-\tau_1(t)\big), \ldots, x\big(t-\tau_m(t)\big),\\
&\quad u'\big(t-\tau_1(t)\big), \ldots, u'\big(t-\tau_m(t)\big)\Big)\\
&\quad - g\Big(t, u(t), u\big(t-\sigma_1(t)\big), \ldots, u\big(t-\sigma_n(t)\big), u\big(t-\tau_1(t)\big), \ldots, u\big(t-\tau_m(t)\big),\\
&\quad u'\big(t-\tau_1(t)\big), \ldots, u'\big(t-\tau_m(t)\big)\Big)\Big| \leqslant \varepsilon
\end{aligned}$$

for any $x, u \in A_{ij}$. Therefore, we have

$$\begin{aligned}
&|Nx - Nu|_0\\
&\quad = \sup_{t\in[0,T]}\Big|g\Big(t, x(t), x\big(t-\sigma_1(t)\big), \ldots, x\big(t-\sigma_n(t)\big), x\big(t-\tau_1(t)\big), \ldots,\\
&\qquad x\big(t-\tau_m(t)\big), x'\big(t-\tau_1(t)\big), \ldots, x'\big(t-\tau_n(t)\big)\Big)\\
&\qquad - g\Big(t, u(t), u\big(t-\sigma_1(t)\big), \ldots, u\big(t-\sigma_n(t)\big), u\big(t-\tau_1(t)\big), \ldots, u\big(t-\tau_m(t)\big),\\
&\qquad u'\big(t-\tau_1(t)\big), \ldots, u'\big(t-\tau_m(t)\big)\Big)\Big|\\
&\quad \leqslant \sup_{t\in[0,T]}\Big|g\Big(t, x(t), x\big(t-\sigma_1(t)\big), \ldots, x\big(t-\sigma_n(t)\big), x\big(t-\tau_1(t)\big), \ldots,\\
&\qquad x\big(t-\tau_m(t)\big), x'\big(t-\tau_1(t)\big), \ldots, x'\big(t-\tau_m(t)\big)\Big)\\
&\qquad - g\Big(t, x(t), x\big(t-\sigma_1(t)\big), \ldots, x\big(t-\sigma_n(t)\big), x\big(t-\tau_1(t)\big), \ldots, x\big(t-\tau_m(t)\big),\\
&\qquad u'\big(t-\tau_1(t)\big), \ldots, u'\big(t-\tau_m(t)\big)\Big)\Big|\\
&\qquad + \sup_{t\in[0,T]}\Big|g\Big(t, x(t), x\big(t-\sigma_1(t)\big), \ldots, x\big(t-\sigma_n(t)\big), x\big(t-\tau_1(t)\big), \ldots,\\
&\qquad x\big(t-\tau_m(t)\big), u'\big(t-\tau_1(t)\big), \ldots, u'\big(t-\tau_m(t)\big)\Big)\\
&\qquad - g\Big(t, u(t), u\big(t-\sigma_1(t)\big), \ldots, u\big(t-\sigma_n(t)\big), u\big(t-\tau_1(t)\big), \ldots, u\big(\tau_1(t)\big),\\
&\qquad u'\big(t-\tau_1(t)\big), \ldots, u'\big(\tau_m(t)\big)\Big)\Big|\\
&\quad \leqslant \sum_{i=1}^{m} |c_i|_0 \big|x'\big(t-\tau_i(t)\big) - u'\big(t-\tau_i(t)\big)\big| e^{r_1} + \varepsilon \leqslant k\eta + (k+1)\varepsilon.
\end{aligned}$$

As ε is arbitrary small, it is easy to see that

$$\alpha_{C_\omega^0}\big(N(A)\big) \leqslant k\alpha_{C_\omega^1}(A). \qquad \square$$

Lemma 4. *Let $g \in C^0_\omega$, $\tau \in C^1_\omega$, and $\tau' < 1$, $\forall t \in [0,\omega]$. Then $g(\mu(t)) \in C^0_\omega$, where $\mu(t)$ is the inverse function of $t - \tau(t)$.*

Proof. We need only to prove that $\mu(a+\omega) = \mu(a) + \omega$ for arbitrary $a \in R$. By the condition $\tau' < 1$, it is easy to see that the equation $t - \tau(t) = a$ and $t - \tau(t) = a + \omega$ has a unique solution t_0, t_1, respectively. That is

$$t_0 - \tau(t_0) = a, \qquad t_1 - \tau(t_1) = a + \omega,$$

i.e.,

$$\mu(a) = t_0 = a + \tau(t_0) \quad \text{and} \quad \mu(a+\omega) = t_1. \tag{6}$$

As

$$\begin{aligned} \omega + a + \tau(t_0) - \tau\big(\omega + a + \tau(t_0)\big) &= \omega + a + \tau(t_0) - \tau\big(a + \tau(t_0)\big) \\ &= \omega + a + \tau(t_0) - \tau(t_0) = \omega + a, \end{aligned}$$

it follows that $t_1 = \omega + a + \tau(t_0)$. So by (6) we have $\mu(a+\omega) = \mu(a) + \omega$ for $\forall a \in R$. □

Throughout this paper, we assume that $\tau_i \in C^2_\omega$, $\sigma_j \in C^2_\omega$, $\tau_i' < 1$, $\sigma_j' < 1$ ($i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$). So either $\tau_i(t)$ or $\sigma_j(t)$ has a unique inverse, and we set $\gamma_i(t), \mu_j(t)$ to represent the inverse of function $t - \tau_i(t), t - \sigma_j(t)$, respectively ($i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$).

2. Main results

Theorem 1. *Suppose $c_i \in C^1_\omega$, $\bar{a} > 0$, and*

$$\Gamma(t) =: \beta(t) + \sum_{j=1}^{n} \frac{b_j(\mu_j(t))}{1 - \sigma_j'(\mu_j(t))} - \sum_{i=1}^{m} \frac{c_{0,i}'(\gamma_i(t))}{1 - \tau_i'(\gamma_i(t))} > 0, \quad \forall t \in [0,\omega].$$

Let

$$\Gamma_1(t) =: \beta(t) + \sum_{j=1}^{n} \frac{b_j(\mu_j(t))}{1 - \sigma_j'(\mu_j(t))} + \sum_{i=1}^{m} \frac{|c_{0,i}'(\gamma_i(t))|}{1 - \tau_i'(\gamma_i(t))},$$

$$\begin{aligned} \Gamma_2(t) =: \theta_1 \Gamma(t) + \theta_2 \Bigg(& \beta(t) + \sum_{j=1}^{n} \frac{b_j(\mu_j(t))}{1 - \sigma_j'(\mu_j(t))} \\ & - \sum_{i=1}^{m} \frac{\beta(\gamma_i(t))}{1 - \tau_i'(\gamma_i(t))} - \sum_{i=1}^{m} \frac{b_i(\gamma_i(t))}{1 - \tau_i'(\gamma_i(t))} \Bigg), \end{aligned}$$

$$\Gamma_{3,i}(t) =: \beta(t) + b_i(t) - c_{0,i}'(t), \quad i = 1, 2, \ldots, m.$$

Further, we assume that

$$l = \min_{1 \leqslant i \leqslant m} \Big\{ \min_{t \in [0,\omega]} \Gamma_{3,i}(t) \Big\} > 0 \quad \textit{and} \quad k = \max\Bigg\{ \sum_{i=1}^{m} |c_{0,i}|_0, \sum_{i=1}^{m} |c_i|_0 \Bigg\} e^M < 1,$$

where

$$M=\max\left\{\left|\ln\left[\frac{\bar a}{\bar\beta+\sum_{j=1}^{n}\bar b}\right]\right|, R_1, M_1\right\},$$

$$R_1=\ln\left[\frac{(1-l_1)\bar a}{\theta_2 l}\right]+\max_{1\leqslant i\leqslant m}\left\{|c_{0,i}|_0\right\}\frac{(1-l_1)\bar a}{\theta_2 l}+\bar a\omega\left(1+\left(\frac{\Gamma_1}{\Gamma}\right)_0\right),$$

$$M_1=\left|\ln\frac{\bar a}{\bar\Gamma}\right|+\left(\bar a+\bar\beta e^{R_1}+\sum_{j=1}^{n}\bar b_j e^{R_1}\right)\omega\left(1-\sum_{i=1}^{m}|c_{0,i}|_0 e^{R_1}\right)^{-1},$$

$$l_1=\left(\frac{\Gamma_2(t)-\theta_2 l}{\Gamma(t)}\right)_m,\qquad c_{0,i}(t)=\frac{c_i(t)}{1-\tau_i'(t)},$$

θ_1,θ_2 are positive constants satisfying $\theta_1+\theta_2=1$, $\Gamma_2(t)>\theta_2 l$. Then Eq. (1) *has at least one positive ω-periodic solution.*

Proof. Corresponding to the operator equation

$$Lx=\lambda Nx+\lambda a,\quad \lambda\in(0,1),$$

we have

$$x'(t)=\lambda\Bigg[a(t)-\beta(t)e^{x(t)}-\sum_{j=1}^{n}b_j(t)e^{x(t-\sigma_j(t))}-\sum_{i=1}^{m}c_i(t)x'\big(t-\tau_i(t)\big)e^{x(t-\tau_i(t))}\Bigg].\tag{7}$$

Let $x(t)$ be an arbitrary ω-periodic solution of Eq. (7) for a certain $\lambda\in(0,1)$; then we have

$$x'(t)=\lambda\Bigg[a(t)-\beta(t)e^{x(t)}-\sum_{j=1}^{n}b_j(t)e^{x(t-\sigma_j(t))}-\sum_{i=1}^{m}c_i(t)x'\big(t-\tau_i(t)\big)e^{x(t-\tau_i(t))}\Bigg].\tag{8}$$

Integrating two sides of Eq. (8) on the interval $[0,\omega]$, we get that

$$\bar a\omega=\int_0^{\omega}\Bigg[\beta(t)e^{x(t)}+\sum_{j=1}^{n}b_j(t)e^{(t-\sigma_j(t))}-\sum_{i=1}^{m}c_{0,i}'(t)e^{x(t-\tau_i(t))}\Bigg]dt.\tag{9}$$

Let $t-\sigma_j(t)=s$, i.e., $t=\mu_j(s)$; then

$$\int_0^{\omega}b_j(t)e^{x(t-\sigma_j(t))}\,dt=\int_{-\sigma_j(0)}^{\omega-\sigma_j(\omega)}\frac{b_j(\mu_j(s))}{1-\sigma_j'(\mu_j(s))}e^{x(s)}\,ds.$$

According to Lemma 4, we have that

$$\int_0^\omega b_j(t)e^{x(t-\sigma_j(t))}\,dt = \int_0^\omega \frac{b_j(\mu_j(s))}{1-\sigma'(\mu_j(s))}e^{x(s)}\,ds \quad (j=1,2,\ldots,n).$$

Similarly,

$$\int_0^\omega c'_{0,i}(t)e^{x(t-\tau_i(t))}\,dt = \int_0^\omega \frac{c'_{0,i}(\gamma_i(s))}{1-\tau'(\gamma_i(s))}e^{x(s)}\,ds \quad (j=1,2,\ldots,m).$$

So from (9) we get that

$$\int_0^\omega \Gamma(s)e^{x(s)}\,ds = \bar{a}\omega. \tag{10}$$

It follows that there is $\xi \in [0,\omega]$ such that

$$e^{x(\xi)}\int_0^\omega \Gamma(s)\,ds = \bar{a}\omega,$$

i.e.,

$$e^{x(\xi)} = \frac{\bar{a}}{\bar{\Gamma}},$$

that is

$$x(\xi) = \ln\frac{\bar{a}}{\bar{\Gamma}}. \tag{11}$$

On the other hand,

$$\begin{aligned}
&\int_0^\omega \left|\left[x(t)+\lambda\sum_{i=1}^m c_{0,i}(t)e^{x(t-\tau_i(t))}\right]'\right|dt \\
&\quad = \lambda\int_0^\omega \left|a(t)-\beta(t)e^{x(t)}-\sum_{j=1}^n b_j(t)e^{x(t-\sigma_j(t))}+\sum_{i=1}^m c'_{0,i}(t)e^{x(t-\tau_i(t))}\right|dt \\
&\quad \leqslant \int_0^\omega \left|a(t)+\beta(t)e^{x(t)}+\sum_{j=1}^n b_j(t)e^{x(t-\sigma_j(t))}+\sum_{i=1}^m \left|c'_{0,i}(t)\right|e^{x(t-\tau_i(t))}\right|dt \\
&\quad \leqslant \int_0^\omega \left|a(t)+\beta(t)e^{x(t)}+\sum_{j=1}^n \frac{b_j(\mu_j(t))}{1-\sigma'(\mu_j(t))}e^{x(t)}+\sum_{i=1}^m \left|\frac{c'_{0,i}(\gamma_i(t))}{1-\tau'(\gamma_i(t))}\right|e^{x(t)}\right|dt \\
&\quad = \bar{a}\omega+\int_0^\omega \frac{\Gamma_1(t)}{\Gamma(t)}\Gamma(t)e^{x(t)}\,dt \leqslant \bar{a}\omega+\bar{a}\omega\left(\frac{\Gamma_1}{\Gamma}\right)_0 = \bar{a}\omega\left(1+\left(\frac{\Gamma_1}{\Gamma}\right)_0\right).
\end{aligned} \tag{12}$$

In addition to this, we have

$$
\begin{aligned}
\bar{a}\omega &= \theta_1 \int_0^\omega \left(\beta(t)e^{x(t)} + \sum_{j=1}^n b_j(t)e^{x(t-\sigma_j(t))} - \sum_{i=1}^m c'_{0,i}(t)e^{x(t-\tau_i(t))} \right) dt \\
&\quad + \theta_2 \int_0^\omega \left(\beta(t)e^{x(t)} + \sum_{j=1}^n b(t)e^{x(t-\sigma_j(t))} - \sum_{i=1}^m c'_0(t)e^{x(t-\tau_i(t))} \right) dt \\
&= \theta_1 \int_0^\omega \Gamma(t)e^{x(t)}\,dt + \theta_2 \int_0^\omega \left(\beta(t)e^{x(t)} + \sum_{j=1}^n b_j(t)e^{x(t-\sigma_j(t))} \right. \\
&\qquad \left. - \beta(t)\sum_{i=1}^m e^{x(t-\tau_i(t))} - \sum_{i=1}^m b_i(t)e^{x(t-\tau_i(t))} \right) dt \\
&\quad + \theta_2 \int_0^\omega \sum_{i=1}^m \bigl(b_i(t) + \beta(t) - c'_{0,i}(t)\bigr)e^{x(t-\tau_i(t))}\,dt \\
&= \int_0^\omega \left[\theta_1\Gamma(t) + \theta_2 \left(\beta(t) + \sum_{j=1}^n \frac{b_j(\mu(t))}{1-\sigma'(\mu(t))} \right. \right. \\
&\qquad \left. \left. - \sum_{i=1}^m \frac{\beta(\gamma_i(t))}{1-\tau_i'(\gamma_i(t))} - \sum_{i=1}^m \frac{b_i(\gamma_i(t))}{1-\tau_i'(\gamma_i(t))} \right) \right] e^{x(t)}\,dt \\
&\quad + \theta_2 \sum_{i=1}^m \int_0^\omega \Gamma_{3,i}(t)e^{x(t-\tau_i(t))}\,dt. \qquad (13)
\end{aligned}
$$

As $\Gamma(t) > 0$, it follows that there are two positive constants $\theta_1 > 0, \theta_2 > 0$ with $\theta_1 + \theta_2 = 1$ such that $\Gamma_2(t) > \theta_2 l$, $\forall t \in [0,\omega]$. Then from (13) we get that

$$
\begin{aligned}
\bar{a}\omega &= \int_0^\omega \bigl(\Gamma_2(t) - \theta_2 l\bigr)e^{x(t)}\,dt + \theta_2 \int_0^\omega \left(le^{x(t)} + \sum_{i=1}^m \Gamma_{3,i}(t)e^{x(t-\tau_i(t))} \right) dt \\
&\geqslant \left(\frac{\Gamma_2(t) - \theta_2 l}{\Gamma(t)} \right)_m \int_0^\omega \Gamma(t)e^{x(t)}\,dt + \theta_2 l \int_0^\omega \left(e^{x(t)} + \sum_{i=1}^m e^{x(t-\tau_i(t))} \right) dt. \qquad (14)
\end{aligned}
$$

From (14) we have that there is $\xi \in [0,\omega]$ such that

$$
\bar{a}\omega \geqslant \left(\frac{\Gamma_2(t) - \theta_2 l}{\Gamma(t)} \right)_m a\omega + \theta_2 l\omega \left(e^{x(\xi)} + \sum_{i=1}^m e^{x(\xi-\tau_i(\xi))} \right),
$$

i.e.,

$$e^{x(\xi)} + \sum_{i=1}^{m} e^{x(\xi-\tau_i(\xi))} \leqslant \frac{(1-l_1)\bar{a}}{\theta_2 l}, \tag{15}$$

where

$$l_1 = \left(\frac{\Gamma_2(t) - \theta_2 l}{\Gamma(t)}\right)_m.$$

Since

$$\Gamma_2(t) - \Gamma(t) = \theta_2 \sum_{i=1}^{m} \Gamma_{3,i}\bigl(\gamma_i(t)\bigr)\bigl[1 - \tau_i'\bigl(\gamma_i(t)\bigr)\bigr]^{-1} > 0,$$

we have $l_1 \in (0,1)$. It follows from (15) that

$$e^{x(\xi)} \leqslant \frac{(1-l_1)\bar{a}}{\theta_2 l},$$

i.e.,

$$x(\xi) \leqslant \ln \frac{(1-l_1)\bar{a}}{\theta_2 l} \tag{16}$$

and

$$\sum_{i=1}^{m} e^{x(\xi-\tau_i(\xi))} \leqslant \frac{(1-l_1)\bar{a}}{\theta_2 l}. \tag{17}$$

So by (16) and (17) we get that

$$\begin{aligned} x(t) &+ \lambda \sum_{i=1}^{m} c_{0,i}(t) e^{x(t-\tau_i(t))} \\ &\leqslant x(\xi) + \lambda \sum_{i=1}^{n} c_{0,i}(\xi) e^{x(\xi-\tau_i(\xi))} + \int_0^{\omega} \left| \left[x(t) + \lambda \sum_{i=1}^{m} c_{0,i}(t) e^{x(t-\tau_i(t))} \right]' \right| dt \\ &\leqslant \ln \frac{(1-l_1)\bar{a}}{\theta_2 l} + \max_{1\leqslant i\leqslant n} |c_{0,i}|_0 \frac{(1-l_1)\bar{a}}{\theta_2 l} + \bar{a}\omega\left(1 + \left(\frac{\Gamma_1}{\Gamma}\right)_0\right) = R_1. \end{aligned} \tag{18}$$

Thus

$$x(t) < R_1, \quad \forall t \in [0,\omega].$$

From (8) we have

$$\begin{aligned} \int_0^{\omega} \bigl|x'(t)\bigr|\, dt &\leqslant \bar{a}\omega + \bar{\beta}\omega e^{R_1} + \sum_{j=1}^{n} \bar{b}_j \omega e^{R_1} + e^{R_1} \sum_{i=1}^{m} \int_0^{\omega} c_i(t) \bigl|x'\bigl(t-\tau_i(t)\bigr)\bigr|\, dt \\ &\leqslant \bar{a}\omega + \bar{\beta}\omega e^{R_1} + \sum_{j=1}^{n} \bar{b}_j \omega e^{R_1} + e^{R_1} \sum_{i=1}^{m} \int_0^{\omega} \frac{c_i(\mu_i(t))}{1-\tau'(\mu_i(t))} \bigl|x'(t)\bigr|\, dt \end{aligned}$$

$$\leqslant \bar{a}\omega + \omega\bar{\beta}e^{R_1} + \omega\sum_{j=1}^{n}\bar{b}_j e^{R_1} + \sum_{i=1}^{m}|c_{0,i}|_0 e^{R_1}\int_0^{\omega}\left|x'(t)\right|dt,$$

i.e.,

$$\int_0^{\omega}\left|x'(t)\right|dt \leqslant \omega(\bar{a} + \bar{\beta}e^{R_1} + \bar{b}e^{R_1})\left(1 - \sum_{i=1}^{m}|c_{0,i}|_0 e^{R_1}\right)^{-1}.$$

It follows from (11) that

$$\begin{aligned}\left|x(t)\right| &\leqslant \left|\ln\left[\frac{\bar{a}}{\bar{\Gamma}}\right]\right| + \int_0^{\omega}\left|x'(t)\right|dt \\ &\leqslant \left|\ln\left[\frac{\bar{a}}{\bar{\Gamma}}\right]\right| + \omega(\bar{a} + \bar{\beta}e^{R_1} + \bar{b}e^{R_1})\left(1 - \sum_{i=1}^{m}|c_{0,i}|_0 e^{R_1}\right)^{-1} =: M_1, \qquad (19)\end{aligned}$$

and again, by (8), we get

$$|x'|_0 \leqslant |a|_0 + |b|_0 e^{R_1} + |\beta|_0 e^{R_1} + \sum_{i=1}^{m}|c_i|_0 e^{R_1}|x'|_0.$$

It follows that

$$|x'|_0 \leqslant \frac{|a|_0 + |b|_0 e^{R_1} + |\beta|_0 e^{R_1}}{1 - \sum_{i=1}^{m}|c_i|_0 e^{R_1}} = M_2. \qquad (20)$$

Let $\Omega = \{x \in C_\omega^1\colon |x|_0 < M_3,\ |x'|_0 < M_2\}$, and define a bounded bilinear form $[\cdot,\cdot]$ on $C_\omega^0 \times C_\omega^1$ by $[y,x] = \int_0^{\omega} y(t)x(t)\,dt$. Also we define $Q: y \to \operatorname{coker}(L)$ by $y \to \int_0^{\omega} y(t)\,dt$. Obviously,

$$\{x \mid x \in \ker L \cap \partial\Omega\} = \{x \mid x \equiv M_3 \text{ or } x \equiv -M_3\}.$$

Without loss of generality, we may assume that $x \equiv M_3$. Thus

$$\begin{aligned}&\left[QN(x) + Q(a), x\right]\left[QN(-x) + Q(a), x\right] \\ &\quad = M_3^2\omega^2\left[\int_0^{\omega} a(t)\,dt - e^{M_3}\int_0^{\omega}\left(\beta(t) + \sum_{j=1}^{n} b_j(t)\right)dt\right] \\ &\qquad \times \left[\int_0^{\omega} a(t)\,dt - e^{-M_3}\int_0^{\omega}\left(\beta(t) + \sum_{j=1}^{n} b_j(t)\right)dt\right] \\ &\quad = M_3^2\omega^2\left[\bar{a} - e^{M_3}\left(\bar{\beta} + \sum_{j=1}^{n}\bar{b}_j\right)\right]\left[\bar{a} - e^{-M_3}\left(\bar{\beta} + \sum_{j=1}^{n}\bar{b}_j\right)\right]. \qquad (21)\end{aligned}$$

If

$$M_3 > \left|\ln\frac{\bar{a}}{\bar{\beta} + \sum_{j=1}^{n}\bar{b}_j}\right|,$$

then

$$e^{M_3}\left(\bar{\beta}+\sum_{j=1}^{n}\bar{b}_j\right) > \frac{\bar{a}}{\bar{\beta}+\sum_{j=1}^{n}\bar{b}_j}\left(\bar{\beta}+\sum_{j=1}^{n}\bar{b}_j\right) = \bar{a}$$

and

$$e^{-M_3}\left(\bar{\beta}+\sum_{j=1}^{n}\bar{b}_j\right) < \frac{\bar{a}}{\bar{\beta}+\sum_{j=1}^{n}\bar{b}_j}\left(\bar{\beta}+\sum_{j=1}^{n}\bar{b}_j\right) = \bar{a}.$$

So from (21) we get

$$\big[QN(x)+Q(a),x\big]\big[QN(-x)+Q(a),x\big] < 0. \tag{22}$$

From the condition $\max\{\sum_{i=1}^{m}|c_i|_0, \sum_{i=1}^{m}|c_{i,0}|_0\}e^{M} < 1$ we have that there is a constant $\bar{M} > M$ such that $\sum_{i=1}^{m}|c_i|_0 e^{\bar{M}} < 1$. Applying Lemmas 1 and 3 with $\Omega = \{x \mid x \in C_\omega^1, |x|_0 < r_1, |x'|_0 < r_2\}$, we set $r_1 = \bar{M}$, $r_2 = M_2$. Then it follows from (18)–(20) and (22) that all the conditions in Lemma 1 are satisfied. Hence Eq. (1) has at least one positive ω-periodic solution. □

Theorem 2. *If $\bar{a} > 0$ and $\Gamma(t) \leqslant 0$, $\forall t \in [0,\omega]$, then Eq.* (1) *does not have any positive ω-periodic solution.*

Proof. We need only to prove that Eq. (4) does not have any ω-periodic solution. If Eq. (4) has a ω-periodic solution $x(t)$, then by integrating two sides of Eq. (4) on the interval $[0,\omega]$, we get that

$$\int_0^\omega a(t)\,dt = \int_0^\omega \left[\beta(t)e^{x(t)} + \sum_{j=1}^{n} b(t)e^{x(t-\sigma_j(t))} - \sum_{i=1}^{m} c'_{0,i}(t)e^{x(t-\tau_i(t))}\right]dt,$$

i.e.,

$$\bar{a}\omega = \int_0^\omega \Gamma(t)e^{x(t)}\,dt.$$

So there exists a number $\zeta \in [0,\omega]$ such that

$$\Gamma(\zeta)\int_0^\omega e^{x(t)}\,dt = \bar{a}\omega.$$

As $\bar{a} > 0$, it follows that $\Gamma(\zeta) > 0$ which contradicts to the condition $\Gamma(t) \leqslant 0$, $\forall t \in [0,\omega]$. This contradiction implies that Eq. (4) does not have any ω-periodic solution. □

Similarly, we have the following result:

Theorem 3. *If $\bar{a} = 0$, $\Gamma(t) \geqslant 0$ or $\Gamma(t) \leqslant 0$, and furthermore $\bar{\Gamma} \neq 0$, then Eq.* (1) *does not have any positive ω-periodic solution.*

For example, consider the following equation:

$$N'(t) = N(t)\left[a(t) - \delta N(t) - \delta N\left(t - \frac{1}{2}\sin t\right) - N'(t-1)\varepsilon \sin^2 t\right], \tag{23}$$

where $\delta > 0$, $\varepsilon > 0$ are two parameters, $a \in C_\omega^0$, $a(t) \geqslant 0$ for $t \in [0,\omega]$, and $\bar{a} > 0$. Let $N(t) = \exp x(t)$; then Eq. (23) can be written as

$$x'(t) = a(t) - \delta e^{x(t)} - \delta e^{x(t-(1/2)\sin t)} - \varepsilon \sin^2 t x'(t-1)e^{x(t-1)}. \tag{24}$$

Let $\mu(t)$ be the inverse of function of $t - (1/2)\sin t$; then

$$\Gamma(t) = \delta + \frac{\delta}{1-(1/2)\cos\mu(t)} - \varepsilon \sin 2(t+1),$$
$$\Gamma_1(t) = \delta + \frac{\delta}{1-(1/2)\cos\mu(t)} + \varepsilon\left|\sin 2(t+1)\right|,$$
$$\Gamma_2(t) = (\theta_1-\theta_2)\delta + \frac{\delta}{1-(1/2)\cos\mu(t)} - \theta_1\varepsilon \sin 2(t+1),$$
$$\Gamma_3(t) = 2\delta - \varepsilon\sin 2(t+1).$$

So $(5/3)\delta - \varepsilon \leqslant \Gamma(t) \leqslant 3\delta + \varepsilon$, $\bar{\Gamma} < 3\delta$, $\Gamma_1(t) \leqslant 3\delta + \varepsilon$, $\delta(\theta_1 - \theta_2 + 2/3) - \theta_1\varepsilon \leqslant \Gamma_2(t) \leqslant (\theta_1 - \theta_2 + 2)\delta + \theta_1\varepsilon$, $2\delta - \varepsilon \leqslant \Gamma_3(t) \leqslant 2\delta + \varepsilon$. Let $\theta_1 = 2/3$, $\theta_2 = 1/3$, $\varepsilon \in (0, \delta/13]$; then $l = (25/13)\delta$, $l_1 \in (13/120, 67/114)$, $(1-l_1)/(\theta_2 l) < 1.391\delta^{-1}$, and $(\Gamma_1/\Gamma)_0 < 34/31$, $\bar{\Gamma} < 3\delta$. Thus

$$R_1 < \ln 1.391\frac{\bar{a}}{\delta} + \varepsilon\frac{1391\bar{a}}{1000\delta} + 2\pi\bar{a}\left(1 + \frac{34}{31}\right) \leqslant \ln 1.391\frac{\bar{a}}{\delta} + \frac{1391\bar{a}}{13000\delta} + \frac{130}{31}\pi\bar{a}.$$

If $\varepsilon < \theta e^{-R_1}$, where $\theta \in (0,1)$ is a constant, then

$$M_1 = \left|\ln\frac{\bar{a}}{\bar{\Gamma}}\right| + (\bar{a} + 2\delta e^{R_1})\left(1 - \varepsilon|\sin 2t|e^{R_1}\right)^{-1} < \left|\ln\frac{\bar{a}}{3\delta}\right| + \frac{\bar{a} + 2\delta e^{R_1}}{1-\theta}.$$

Thus $\max\{|c|_0, |c_0|_0\}e^M < 1$, for $\varepsilon \in (0, \varepsilon_0)$, where

$$\varepsilon_0 = \min\left\{\frac{\delta}{13}, \theta e^{-R_1}, e^{-|\ln\bar{a}/(2\delta)|}, e^{-(|\ln\bar{a}/(3\delta)| + (\bar{a}+2\delta e^{R_1})/(1-\theta))}\right\}. \tag{25}$$

Therefore, by applying Theorem 1, we have that Eq. (23) has a positive 2π-periodic solution for $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is defined by (25).

Remark. It is easy to see that Eq. (2) is a special case of Eq. (1). So Theorem 1 for $m = n = 1$, $\sigma_1(t) = \tau_1(t) = \tau(t)$, $b_1(t) = b(t)$, $c_1(t) = c(t)$ is an answer to Open problem 9.2 due to Kuang [5]. For this case, we need

$$\Gamma(t) =: \beta(t) + \frac{b(\mu(t))}{1-\tau'(\mu(t))} - \frac{c_0'(\mu(t))}{1-\tau'(\mu(t))} > 0, \quad \forall t \in [0,\omega],$$

which is weaker than corresponding condition of [7] that $c_0'(t) < b(t)$ or $c_0'(t) \leqslant b(t)$, $\beta(t) > 0$, $\forall t \in [0,\omega]$.

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