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The ideal-valued index for a dihedral group action, and mass partition by two hyperplanes

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ABSTRACT

We compute the complete Fadell–Husseini index of the dihedral group $D_8 = (\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2$ acting on $S^d \times S^d$ for $\mathbb{F}_2$ and for $\mathbb{Z}$ coefficients, that is, the kernels of the maps in equivariant cohomology

$$H^*_{D_8}(\mathrm{pt}, \mathbb{F}_2) \longrightarrow H^*_{D_8}\left(S^d \times S^d, \mathbb{F}_2\right)$$

and

$$H^*_{D_8}(\mathrm{pt}, \mathbb{Z}) \longrightarrow H^*_{D_8}\left(S^d \times S^d, \mathbb{Z}\right).$$

This establishes the complete cohomological lower bounds, with $\mathbb{F}_2$ and with $\mathbb{Z}$ coefficients, for the two-hyperplane case of Grünbaum's 1960 mass partition problem: For which d and j can any j arbitrary measures be cut into four equal parts each by two suitably chosen hyperplanes in $\mathbb{R}^d$? In both cases, we find that the ideal bounds are not stronger than previously established bounds based on one of the maximal abelian subgroups of D_8.

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1. Introduction

1.1. The hyperplane mass partition problem

A *mass distribution* on $\mathbb{R}^d$ is a finite Borel measure $\mu(X) = \int_X f\,d\mu$ determined by an integrable density function $f : \mathbb{R}^d \to \mathbb{R}$.

Every affine hyperplane $H = \{x \in \mathbb{R}^d \mid \langle x, v\rangle = \alpha\}$ in $\mathbb{R}^d$ determines two open halfspaces

$$H^- = \left\{x \in \mathbb{R}^d \;\middle|\; \langle x, v\rangle < \alpha\right\} \quad \text{and} \quad H^+ = \left\{x \in \mathbb{R}^d \;\middle|\; \langle x, v\rangle > \alpha\right\}.$$

An *orthant* of an arrangement of k hyperplanes $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ in $\mathbb{R}^d$ is an intersection of halfspaces $\mathcal{O} = H_1^{\alpha_1} \cap \cdots \cap H_k^{\alpha_k}$, for some $\alpha_j \in \mathbb{Z}_2$. Thus there are 2^k orthants determined by $\mathcal{H}$ and they are naturally indexed by elements of the group $(\mathbb{Z}_2)^k$.

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An arrangement of hyperplanes $\mathcal{H}$ *equiparts* a collection of mass distributions $\mathcal{M}$ in $\mathbb{R}^d$ if for each orthant $\mathcal{O}$ and each measure $\mu \in \mathcal{M}$ we have

$$\mu(\mathcal{O}) = \frac{1}{2^k}\mu\left(\mathbb{R}^d\right).$$

A triple of integers (d, j, k) is *admissible* if for every collection $\mathcal{M}$ of j mass distributions in $\mathbb{R}^d$ there exists an arrangement of k hyperplanes $\mathcal{H}$ equiparting them.

The general problem formulated by Grünbaum [13] in 1960 can be stated as follows.

Problem 1.1. Determine the function $\Delta : \mathbb{N}^2 \to \mathbb{N}$ given by

$$\Delta(j,k) = \min\left\{d \;\middle|\; (d, j, k) \text{ is an admissible triple}\right\}.$$

The case of one hyperplane, $\Delta(j,1) = j$, is the famous ham sandwich theorem, which is equivalent to the Borsuk–Ulam theorem. The equality $\Delta(2,2) = 3$, and consequently $\Delta(1,3) = 3$, was proven by Hadwiger [14]. Ramos [25] gave a general lower bound for the function Δ,

$$\Delta(j,k) \geqslant \frac{2^k - 1}{k} j. \tag{1}$$

Recently, Mani-Levitska, Vrećica and Živaljević [21] applied Fadell–Husseini index theory for an elementary abelian subgroup $(\mathbb{Z}_2)^k$ of the Weyl group $W_k = (\mathbb{Z}_2)^k \rtimes S_k$ to obtain a new upper bound for the function Δ,

$$\Delta\left(2^q + r, k\right) \leqslant 2^{k+q-1} + r. \tag{2}$$

In the case of $j = 2^{l+1} - 1$ measures and $k = 2$ hyperplanes these bounds yield the equality

$$\Delta(j,2) = \left\lceil \frac{3}{2} j \right\rceil.$$

1.2. Statement of the main result ($k = 2$)

This paper addresses Problem 1.1 for $k = 2$ using two different but related Configuration Space/Test Map schemes (Section 2, Proposition 2.2).

- The **product scheme** is the classical one, already considered in [27] and [21]. The problem is translated to the problem of the existence of a W_k-equivariant map,

  $$Y_{d,k} := \left(S^d\right)^k \longrightarrow S\left((R_{2^k})^j\right),$$

  where $W_k = (\mathbb{Z}_2)^k \rtimes S_k$ is the Weyl group.
- The **join scheme** is a new one. It connects the problem with classical Borsuk–Ulam properties in the spirit of Marzantowicz [22]. It asks the question whether there exists a W_k-equivariant map

  $$X_{d,k} := \left(S^d\right)^{*k} \longrightarrow S\left(U_k \times (R_{2^k})^j\right).$$

  The W_k-representations R_{2^k} and U_k are introduced in Section 2.2.

Obstruction theory methods cannot be applied to either scheme directly for $k > 1$, since the W_k-actions on the respective configuration spaces $(S^d)^k$ and $(S^d)^{*k}$ are **not free** (compare [21, Section 2.3.3], assumptions on the manifold M^n). Therefore we analyze the associated equivariant question for $k = 2$ via the Fadell–Husseini ideal index theory method. We show that the join scheme considered from the Fadell–Husseini point of view, with either $\mathbb{F}_2$ or $\mathbb{Z}$ coefficients, yields no obstruction to the existence of the equivariant map in question (Remarks 5.3 and 6.2). In the case of the product scheme we give the ideal bounds obtained from the use of the full group of symmetries by proving the following theorem.

Theorem 1.2. *Let π_d, $d \geqslant 0$, be polynomials in $\mathbb{F}_2[y, w]$ given by*

$$\pi_d(y,w) = \sum_i \binom{d-1-i}{i}_{\text{mod } 2} w^i y^{d-2i}$$

and Π_d, $d \geqslant 0$, be polynomials in $\mathbb{Z}[\mathcal{Y}, \mathcal{M}, \mathcal{W}]/\langle 2\mathcal{Y}, 2\mathcal{M}, 4\mathcal{W}, \mathcal{M}^2 - \mathcal{W}\mathcal{Y}\rangle$ given by

$$\Pi_d(\mathcal{Y},\mathcal{W}) = \sum_i \binom{d-1-i}{i}_{\text{mod } 2} \mathcal{W}^i \mathcal{Y}^{d-2i}.$$

(A) $\mathbb{F}_2$-*bound*: *The triple* $(d, j, 2) \in \mathbb{N}^3$ *is admissible if*

$$y^j w^j \notin \langle \pi_{d+1}, \pi_{d+2} \rangle \subseteq \mathbb{F}_2[y, w].$$

(B) $\mathbb{Z}$-*bound*: *The triple* $(d, j, 2) \in \mathbb{N}^3$ *is admissible if*

$$\left\langle \begin{matrix} (j-1)_{\mathrm{mod}\ 2}\mathcal{Y}^{\frac{j}{2}}\mathcal{W}^{\frac{j}{2}}, & \\ j_{\mathrm{mod}\ 2}\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j-1}{2}}\mathcal{M}, & j_{\mathrm{mod}\ 2}\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}} \end{matrix} \right\rangle \subseteq \left\langle \begin{matrix} (d-1)_{\mathrm{mod}\ 2}\Pi_{\frac{d+2}{2}}, & (d-1)_{\mathrm{mod}\ 2}\Pi_{\frac{d+4}{2}}, \\ (d-1)_{\mathrm{mod}\ 2}\mathcal{M}\Pi_{\frac{d}{2}}, & \\ d_{\mathrm{mod}\ 2}\Pi_{\frac{d+1}{2}}, & d_{\mathrm{mod}\ 2}\Pi_{\frac{d+3}{2}} \end{matrix} \right\rangle$$

in the ring $\mathbb{Z}[\mathcal{Y}, \mathcal{M}, \mathcal{W}]/\langle 2\mathcal{Y}, 2\mathcal{M}, 4\mathcal{W}, \mathcal{M}^2 - \mathcal{W}\mathcal{Y}\rangle$.

Remark 1.3. Let $\hat{\Pi}_d$, $d \geqslant 0$, be the sequence of polynomials in $\mathbb{Z}[Y, W]$ defined by $\hat{\Pi}_0 = 0$, $\hat{\Pi}_1 = Y$ and $\hat{\Pi}_{d+1} = Y\hat{\Pi}_d + W\hat{\Pi}_{d-1}$ for $d \geqslant 2$. Then the sequences of polynomials Π_d and π_d are reductions of the polynomials $\hat{\Pi}_d$. The polynomials $\hat{\Pi}_d$ can be also described by the generating function (formal power series)

$$\sum_{d \geqslant 0} \hat{\Pi}_d = \frac{Y}{1 - Y - W}$$

where $\hat{\Pi}_d$ is homogeneous of degree $2d$ if we set $\deg(Y) = 2$ and $\deg(W) = 4$.

Theorem 1.2 is a consequence of a topological result, the complete and explicit computation of the relevant Fadell–Husseini indexes of the D_8-space $S^d \times S^d$ and the D_8-sphere $S(R_4^{\oplus j})$.

Theorem 1.4.

(A) $\mathrm{Index}^{3j}_{D_8,\mathbb{F}_2} S(R_4^{\oplus j}) = \mathrm{Index}_{D_8,\mathbb{F}_2} S(R_4^{\oplus j}) = \langle y^j w^j \rangle$.

(B) $\mathrm{Index}^{d+2}_{D_8,\mathbb{F}_2}(S^d \times S^d) = \langle \pi_{d+1}, \pi_{d+2} \rangle$.

(C) $\mathrm{Index}^{3j+1}_{D_8,\mathbb{Z}} S(R_4^{\oplus j}) = \begin{cases} \langle \mathcal{Y}^{\frac{j}{2}}\mathcal{W}^{\frac{j}{2}} \rangle, & \textit{for } j \textit{ even}, \\ \langle \mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j-1}{2}}\mathcal{M}, \mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}} \rangle, & \textit{for } j \textit{ odd}. \end{cases}$

(D) $\mathrm{Index}^{d+2}_{D_8,\mathbb{Z}} S^d \times S^d = \begin{cases} \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}}, \mathcal{M}\Pi_{\frac{d}{2}} \rangle, & \textit{for } d \textit{ even}, \\ \langle \Pi_{\frac{d+1}{2}}, \Pi_{\frac{d+3}{2}} \rangle, & \textit{for } d \textit{ odd}. \end{cases}$

The sequence of Fadell–Husseini indexes will be introduced in Section 3. The actions of the dihedral group D_8 and the definition of the representation space $R_4^{\oplus j}$ are given in Section 2. Even though it does not seem to have any relevance to our study of Problem 1.1, the complete index $\mathrm{Index}_{D_8,\mathbb{F}_2}(S^d \times S^d)$ will also be computed in the case of $\mathbb{F}_2$ coefficients,

$$\mathrm{Index}_{D_8,\mathbb{F}_2}\big(S^d \times S^d\big) = \big\langle \pi_{d+1}, \pi_{d+2}, w^{d+1} \big\rangle. \tag{3}$$

Final remark 1.5. The preprint versions of this paper, posted on the arXiv in April 2007 and July 2008, arXiv0704.1943v1–v2, have been referenced in diverse applications: see Gonzalez and Landweber [12], Adem and Reichstein [2], as well as [5].

1.3. Proof overview

Problem 1.1 about mass partitions by hyperplanes can be connected with the problem of the existence of equivariant maps as discussed in Section 2, Proposition 2.2. The topological problems we face, about the existence of $W_k = (\mathbb{Z}_2)^k \rtimes S_k$-equivariant maps, for the product/join schemes,

$$\big(S^d\big)^k \longrightarrow S\big(R_{2^k}^{\oplus j}\big), \qquad \big(S^d\big)^{*k} \longrightarrow S\big(U_k \times R_{2^k}^{\oplus j}\big),$$

have to be treated with care because the actions of the Weyl groups W_k are not free. Note that there is no naive Borsuk–Ulam theorem for fixed point free actions. Indeed, in the case $k = 2$ when $W_2 = D_8$ there exists a W_2-equivariant map [4, Theorem 3.22, p. 49]

$$S\big((V_{+-} \oplus V_{-+})^{10}\big) \longrightarrow S\big((U_2 \oplus V_{--})^8\big)$$

even though $\dim(V_{+-} \oplus V_{-+})^{10} > \dim(U_2 \oplus V_{--})^8$. The $W_2 = D_8$-representations $V_{+-} \oplus V_{-+}$, V_{--} and U_2 are introduced in Section 2.2.

In this paper we focus on the case of $k=2$ hyperplanes. Theorem 1.2 gives the best possible answer to the question about the existence of $W_2 = D_8$-equivariant maps

$$S^d \times S^d \longrightarrow S\big(R_4^{\oplus j}\big)$$

from the point of view of Fadell–Husseini index theory (Section 3). We explicitly compute the relevant Fadell–Husseini indexes with $\mathbb{F}_2$ and $\mathbb{Z}$ coefficients (Theorem 1.4, Sections 5, 6, 7 and 8). Then Theorem 1.2 is a consequence of the basic index property, Proposition 3.2.

The index of the sphere $S(R_4^{\oplus j})$, with $\mathbb{F}_2$ coefficients, is computed in Section 5 by

- decomposition of the D_8-representation $R_4^{\oplus j}$ into a sum of irreducible ones, and
- computation of indexes of spheres of all irreducible D_8-representations.

The main technical tool is the restriction diagram derived in Section 4.2.2, which connects the indexes of the subgroups of D_8.

The index with $\mathbb{Z}$ coefficients is computed in Section 6 using

- (for j even) the results for $\mathbb{F}_2$ coefficients and comparison of Serre spectral sequences, and
- (for j odd) the Bockstein spectral sequence combined with known results for $\mathbb{F}_2$ coefficients and comparison of Serre spectral sequences.

The index of the product $S^d \times S^d$ is computed in Sections 7 and 8 by an explicit study of the Serre spectral sequence associated with the fibration

$$S^d \times S^d \to \mathrm{E}D_8 \times_{D_8} \big(S^d \times S^d\big) \to \mathrm{B}D_8.$$

The major difficulty comes from non-triviality of the local coefficients in the Serre spectral sequence. The computation of the spectral sequence with non-trivial local coefficients is done by an independent study of $H^*(D_8,\mathbb{F}_2)$-module and $H^*(D_8,\mathbb{Z})$-module structures of relevant rows in the Serre spectral sequence (Sections 7.1 and 8.1).

1.4. Evaluation of the index bounds

1.4.1. $\mathbb{F}_2$-evaluation

It was pointed out to us by Siniša Vrećica that, with $\mathbb{F}_2$-coefficients, the D_8 index bound gives the same bounds as the $H_1 = (\mathbb{Z}_2)^2$ index bound. This observation follows from the implication

$$a^j b^j (a+b)^j \in \big\langle a^{d+1}, (a+b)^{d+1}\big\rangle \quad\Rightarrow\quad a^j b^j (a+b)^j \in \big\langle a^{d+1} + (a+b)^{d+1}, a^{d+2} + (a+b)^{d+2}\big\rangle.$$

By introducing a new variable $c := a+b$, it is enough to prove the implication

$$a^j c^j (a+c)^j \in \big\langle a^{d+1}, c^{d+1}\big\rangle \quad\Rightarrow\quad a^j c^j (a+c)^j \in \big\langle a^{d+1} + c^{d+1}, a^{d+2} + c^{d+2}\big\rangle. \tag{4}$$

Let us assume that $a^j c^j (a+c)^j \in \langle a^{d+1}, c^{d+1}\rangle$. The monomials in the expansion of $a^j c^j (a+c)^j$ always come in pairs

$$a^{d+k} c^{3j-d-k} + c^{d+k} a^{3j-d-k}.$$

This is also true when j is even since $\binom{j}{j/2} =_{\mathrm{mod}\ 2} 0$ implies there are no middle terms. The sequence of equations

$$a^{d+1}c^{3j-d-1} + c^{d+1}a^{3j-d-1} = \big(a^{d+1}+c^{d+1}\big)\big(c^{3j-d-1}+a^{3j-d-1}\big) + a^{3j} + c^{3j}$$
$$a^{d+2}c^{3j-d-2} + c^{d+2}a^{3j-d-2} = \big(a^{d+1}+c^{d+1}\big)\big(ac^{3j-d-2}+a^{3j-d-2}c\big) + a^{3j-1}c + ac^{3j-1}$$
$$\ldots$$
$$a^{3j} + c^{3j} = \big(a^{d+2}+c^{d+2}\big)\big(a^{3j-d-2}+c^{3j-d-2}\big) + a^{d+2}c^{3j-d-2} + c^{d+2}a^{3j-d-2}$$

shows that all the binomials

$$a^{d+1}c^{3j-d-1} + c^{d+1}a^{3j-d-1}, \quad a^{d+2}c^{3j-d-2} + c^{d+2}a^{3j-d-2}, \quad \ldots, \quad a^{3j} + c^{3j}$$

belong to the ideal $\langle a^{d+1} + c^{d+1}, a^{d+2} + c^{d+2}\rangle$ or none of them do.

Since for $3j-d-1$ even

$$a^{d+1+\frac{3j-d-1}{2}} c^{\frac{3j-d-1}{2}} + c^{d+1+\frac{3j-d-1}{2}} a^{\frac{3j-d-1}{2}} = \big(a^{d+1}+c^{d+1}\big) a^{\frac{3j-d-1}{2}} c^{\frac{3j-d-1}{2}} \in \big\langle a^{d+1}+c^{d+1}, a^{d+2}+c^{d+2}\big\rangle,$$

and for $3j-d-1$ odd

$$a^{d+2+\frac{3j-d-2}{2}} c^{\frac{3j-d-2}{2}} + c^{d+2+\frac{3j-d-2}{2}} a^{\frac{3j-d-2}{2}} = \big(a^{d+2}+c^{d+2}\big) a^{\frac{3j-d-2}{2}} c^{\frac{3j-d-2}{2}} \in \big\langle a^{d+1}+c^{d+1}, a^{d+2}+c^{d+2}\big\rangle,$$

the implication (4) is proved.

1.4.2. $\mathbb{Z}$-evaluation

More is true, even the complete D_8 index bound, now with $\mathbb{Z}$-coefficients, implies the same bounds as does the subgroup $H_1 = (\mathbb{Z}_2)^2$ for the $k = 2$ hyperplanes mass partition problem.

Lemma 1.6. *Let $a = \sum_{i=1}^{k} a_i 2^i$ and $b = \sum_{i=1}^{k} b_i 2^i$ be the dyadic expansions. Then*

$$\binom{b}{a}_{\mathrm{mod}\ 2} = \prod_{i=1}^{k} \binom{b_i}{a_i}_{\mathrm{mod}\ ,2}.$$

This classical fact [20] about binomial coefficients mod 2 yields the following property for the sequence of polynomials Π_d, $d \geqslant 0$.

Lemma 1.7. *Let $q > 0$ and i be integers. Then*

(A) $\binom{2^q-1-i}{i} = \begin{cases} 0, & i \neq 0, \\ 1, & i = 0, \end{cases}$

(B) $\Pi_{2^q} = \mathcal{Y}^{2^q}$.

Proof. The statement (B) is a direct consequence of the fact (A) and the definition of polynomials Π_d. For $i \notin \{1, \ldots, 2^{q-1}\}$ the statement (A) is true from boundary conditions on binomial coefficients. Let $i \in \{1, \ldots, 2^{q-1}\}$ and $i = \sum_{k \in I \subseteq \{0,\ldots,q-1\}} 2^k$. Then

$$2^q - 1 - i = 2^0 + 2^1 + 2^2 + \cdots + 2^{q-1} - \sum_{k \in I \subseteq \{0,\ldots,q-1\}} 2^k = \sum_{k \in I^c \subseteq \{0,\ldots,q-1\}} 2^k$$

where I^c is the complementary index set in $\{0, \ldots, q-1\}$. The statement (A) follows from Lemma 1.6 $\square$

Let j be an integer such that $j = 2^q + r$ where $0 \leqslant r < 2^q$ and $d = 2^{q+1} + r - 1$. Let us introduce the following ideals

$$A_j = \begin{cases} \langle \mathcal{Y}^{\frac{j}{2}} \mathcal{W}^{\frac{j}{2}} \rangle, & \text{for } j \text{ even}, \\ \langle \mathcal{Y}^{\frac{j+1}{2}} \mathcal{W}^{\frac{j-1}{2}} \mathcal{M}, \mathcal{Y}^{\frac{j+1}{2}} \mathcal{W}^{\frac{j+1}{2}} \rangle, & \text{for } j \text{ odd}, \end{cases} \quad \text{and} \quad B_d = \begin{cases} \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}}, \mathcal{M}\Pi_{\frac{d}{2}} \rangle, & \text{for } d \text{ even}, \\ \langle \Pi_{\frac{d+1}{2}}, \Pi_{\frac{d+3}{2}} \rangle, & \text{for } d \text{ odd}. \end{cases}$$

The fact that the D_8 index bound with $\mathbb{Z}$-coefficients does not improve the mass partition bounds obtained by using the subgroup $H_1 = (\mathbb{Z}_2)^2$ is a consequence of the following facts:

- $r = 0 \Rightarrow A_j \subseteq B_d$,
- $(r \neq 2^q - 1$ and $A_j \subseteq B_d) \Rightarrow A_{j+1} \subseteq B_{d+1}$,

that are proved in Lemma 1.8 and Lemma 1.9, respectively.

Lemma 1.8. $\langle \mathcal{Y}^{2^{q-1}} \mathcal{W}^{2^{q-1}} \rangle = A_{2^q} \subseteq B_{2^{q+1}-1} = \langle \Pi_{2^q}, \Pi_{2^q+1} \rangle$.

Proof. Since $\mathcal{Y}^{2^{q-1}} = \Pi_{2^{q-1}}$ by Lemma 1.7,

$$\mathcal{Y}^{2^{q-1}} \mathcal{W} = \Pi_{2^{q-1}} \mathcal{W} = \Pi_{2^{q-1}+2} + \mathcal{Y}\Pi_{2^{q-1}+1} \in \langle \Pi_{2^{q-1}+1}, \Pi_{2^{q-1}+2} \rangle.$$

By induction on the power i of $\mathcal{W}$ in $\mathcal{Y}^{2^{q-1}} \mathcal{W}^{2i}$,

$$\mathcal{Y}^{2^{q-1}} \mathcal{W}^i \in \langle \Pi_{2^{q-1}+i}, \Pi_{2^{q-1}+i+1} \rangle,$$

and consequently

$$\mathcal{Y}^{2^{q-1}} \mathcal{W}^{2^{q-1}} \in \langle \Pi_{2^q}, \Pi_{2^q+1} \rangle. \qquad \square$$

Lemma 1.9. *If $r \neq 2^q - 1$ and $A_j \subseteq B_d$ then $A_{j+1} \subseteq B_{d+1}$.*

Proof. We distinguish two cases depending on the parity of j.

(A) Let j be even and $\mathcal{Y}^{\frac{j}{2}} \mathcal{W}^{\frac{j}{2}} \in \langle \Pi_{\frac{d+1}{2}}, \Pi_{\frac{d+3}{2}} \rangle$. There are polynomials α and β such that

$$\mathcal{Y}^{\frac{j}{2}} \mathcal{W}^{\frac{j}{2}} = \alpha \Pi_{\frac{d+1}{2}} + \beta \Pi_{\frac{d+3}{2}}.$$

Then

$$\mathcal{Y}^{\frac{(j+1)+1}{2}}\mathcal{W}^{\frac{(j+1)-1}{2}}\mathcal{M} = \mathcal{Y}^{\frac{j+2}{2}}\mathcal{W}^{\frac{j}{2}}\mathcal{M} = \mathcal{Y}\mathcal{M}(\alpha\Pi_{\frac{d+1}{2}} + \beta\Pi_{\frac{d+3}{2}})$$
$$\in \langle \Pi_{\frac{(d+1)+2}{2}}, \mathcal{M}\Pi_{\frac{d+1}{2}}\rangle \subseteq \langle \Pi_{\frac{d+3}{2}}, \Pi_{\frac{d+5}{2}}, \mathcal{M}\Pi_{\frac{d+1}{2}}\rangle = B_{d+1},$$

and

$$\mathcal{Y}^{\frac{(j+1)+1}{2}}\mathcal{W}^{\frac{(j+1)+1}{2}} = \mathcal{Y}\mathcal{W}\big(\mathcal{Y}^{\frac{j}{2}}\mathcal{W}^{\frac{j}{2}}\big) = \mathcal{Y}\mathcal{W}(\alpha\Pi_{\frac{d+1}{2}} + \beta\Pi_{\frac{d+3}{2}}) = \alpha\mathcal{M}^2\Pi_{\frac{d+1}{2}} + \beta\mathcal{Y}\mathcal{W}\Pi_{\frac{d+3}{2}}$$
$$\in \langle \mathcal{M}\Pi_{\frac{d+1}{2}}, \Pi_{\frac{d+3}{2}}\rangle \subseteq \langle \Pi_{\frac{d+3}{2}}, \Pi_{\frac{d+5}{2}}, \mathcal{M}\Pi_{\frac{d+1}{2}}\rangle = B_{d+1}.$$

Thus $A_{j+1} \subseteq B_{d+1}$.

(B) Let j be odd and

$$\big\langle \mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j-1}{2}}\mathcal{M}, \mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}}\big\rangle = A_j \subseteq B_d = \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}}, \mathcal{M}\Pi_{\frac{d}{2}}\rangle.$$

There are polynomials α, β and γ such that

$$\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}} = \alpha\Pi_{\frac{d+2}{2}} + \beta\Pi_{\frac{d+4}{2}} + \gamma\mathcal{M}\Pi_{\frac{d}{2}}$$

and no occurrence of the defining relation $\Pi_{\frac{d+4}{2}} = \mathcal{Y}\Pi_{\frac{d+2}{2}} + \mathcal{W}\Pi_{\frac{d}{2}}$, Remark 1.3, can be subtracted from the presentation. Then $\gamma\mathcal{M}\Pi_{\frac{d}{2}} \in \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}}\rangle$, and since $\mathcal{M}$ is of odd degree $\gamma = \mathcal{M}\gamma'$. In the first case the inclusion $A_{j+1} \subseteq B_{d+1}$ follows directly. Consider $\gamma = \mathcal{M}\gamma'$. Since $(\mathcal{Y}+\mathcal{X})\mathcal{W}\Pi_i = \mathcal{Y}\mathcal{W}\Pi_i$ for every $i > 0$, we have that

$$\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}} = \alpha\Pi_{\frac{d+2}{2}} + \beta\Pi_{\frac{d+4}{2}} + \gamma'\mathcal{M}^2\Pi_{\frac{d}{2}} = \alpha\Pi_{\frac{d+2}{2}} + \beta\Pi_{\frac{d+4}{2}} + \gamma'\mathcal{Y}\mathcal{W}\Pi_{\frac{d}{2}}$$
$$= \alpha\Pi_{\frac{d+2}{2}} + \beta\Pi_{\frac{d+4}{2}} + \gamma'\mathcal{Y}(\mathcal{Y}\Pi_{\frac{d}{2}+1} + \Pi_{\frac{d}{2}+2}) \in \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}}\rangle = B_{d+1}.$$

Thus $A_{j+1} \subseteq B_{d+1}$. □

2. Configuration Space/Test Map scheme

The Configuration Space/Test Map (CS/TM) paradigm (formalized by Živaljević in [26], and also beautifully exposited by Matoušek in [23]) has been very powerful in the systematic derivation of topological lower bounds for problems of Combinatorics and of Discrete Geometry.

In many instances, the problem suggests natural configuration spaces X, Y, a finite symmetry group G, and a test set $Y_0 \subset Y$, where one would try to show that every G-equivariant map $f : X \to Y$ must hit Y_0. The canonical tool is then Dold's theorem, which says that if the group actions are free, then the map f must hit the test set $Y_0 \subset Y$ if the connectivity of X is higher than the dimension of $Y \setminus Y_0$.

For the success of this "canonical approach" one crucially needs that a result such as Dold's theorem is applicable. Thus the group action must be free, so one often reduces the group action to a prime order cyclic subgroup of the full symmetry group, and results may follow only in "the prime case", or with more effort and deeper tools in the prime power case. The main example for this is the Topological Tverberg Problem, which is still not resolved for (d, q) if $d > 1$ and q is not a prime power [23, Section 6.4, p. 165]. So in general one has to work much harder when the "canonical" approach fails.

In the following, we present configuration spaces and test maps for the mass partition problem.

2.1. Configuration space

The space of all oriented affine hyperplanes in $\mathbb{R}^d$ can be naturally identified with the subspace of the sphere S^d obtained by removing two points, namely the "oriented hyperplanes at infinity". Indeed, let $\mathbb{R}^d$ be embedded in $\mathbb{R}^{d+1}$ by $(x_1, \ldots, x_d) \longmapsto (x_1, \ldots, x_d, 1)$. Then every oriented affine hyperplane H in $\mathbb{R}^d$ determines a unique oriented hyperplane $\tilde{H}$ through the origin in $\mathbb{R}^{d+1}$ such that $\tilde{H} \cap \mathbb{R}^d = H$, and conversely if the hyperplane at infinity is included. The oriented hyperplane uniquely determined by the unit vector $v \in S^d$ is denoted by H_v and the assumed orientation is determined by the half-space $H_v^{\pm}$. Then $H_{-v}^{-} = H_v^{+}$. The obvious and classically used candidate for the configuration space associated with the problem of testing admissibility of (d, j, k) is

$$Y_{d,k} = \big(S^d\big)^k.$$

The relevant group acting on this space is the Weyl group $W_k = (\mathbb{Z}_2)^k \rtimes S_k$. Each $\mathbb{Z}_2 = (\{+1, -1\}, \cdot)$ acts antipodally on the appropriate copy of S^d (changing the orientation of hyperplanes), while S_k acts by permuting copies. The second configuration space that we can use is

$$X_{d,k} = \underbrace{S^d * \cdots * S^d}_{k \text{ copies}} \cong S^{dk+k-1}.$$

The elements of $X_{d,k}$ are denoted by $t_1v_1+\cdots+t_kv_k$, with $t_i\geqslant 0$, $\sum t_1=1$, $v_i\in S^d$. The Weyl group W_k acts on $X_{d,k}$ by

$$\varepsilon_i\cdot(t_1v_1+\cdots+t_iv_i+\cdots+t_kv_k)=t_1v_1+\cdots+t_i(-v_i)+\cdots+t_kv_k,$$
$$\pi\cdot(t_1v_1+\cdots+t_iv_i+\cdots+t_kv_k)=t_{\pi^{-1}(1)}v_{\pi^{-1}(1)}+\cdots+t_{\pi^{-1}(i)}v_{\pi^{-1}(i)}+\cdots+t_{\pi^{-1}(k)}v_{\pi^{-1}(k)},$$

where ε_i is the generator of the i-th copy of $\mathbb{Z}_2$ and $\pi\in S_k$ is an arbitrary permutation.

2.2. Test map

Let $\mathcal{M}=\{\mu_1,\ldots,\mu_j\}$ be a collection of mass distributions in $\mathbb{R}^d$. Let the coordinates of $\mathbb{R}^{2^k}$ be indexed by the elements of the group $(\mathbb{Z}_2)^k$. The Weyl group W_k acts on $\mathbb{R}^{2^k}$ by acting on its coordinate index set $(\mathbb{Z}_2)^k$ in the following way:

$$\big((\beta_1,\ldots,\beta_k)\rtimes\pi\big)\cdot(\alpha_1,\ldots,\alpha_k)=(\beta_1\alpha_{\pi^{-1}(1)},\ldots,\beta_k\alpha_{\pi^{-1}(k)}).$$

The test map $\phi:Y_{d,k}\to(\mathbb{R}^{2^k})^j$ used with the configuration space $Y_{d,k}$ is a W_k-equivariant map given by

$$\phi(v_1,\ldots,v_k)=\left(\left(\mu_i\big(H_{v_1}^{\alpha_1}\cap\cdots\cap H_{v_k}^{\alpha_k}\big)-\frac{1}{2^k}\mu_i\big(\mathbb{R}^d\big)\right)_{(\alpha_1,\ldots,\alpha_k)\in(\mathbb{Z}_2)^k}\right)_{i\in\{1,\ldots,j\}}.$$

Denote the i-th component of ϕ by ϕ_i, $i=1,\ldots,j$.

To define a test map associated with the configuration space $X_{d,k}$, we discuss the $(\mathbb{Z}_2)^k$- and W_k-module structures on $\mathbb{R}^{2^k}$.

All irreducible representations of the group $(\mathbb{Z}_2)^k$ are 1-dimensional. They are in bijection with the homomorphisms (characters) $\chi:(\mathbb{Z}_2)^k\to\mathbb{Z}_2$. These homomorphisms are completely determined by the values on generators $\varepsilon_1,\ldots,\varepsilon_k$ of $(\mathbb{Z}_2)^k$, i.e. by the vector $(\chi(\varepsilon_1),\ldots,\chi(\varepsilon_k))$. For $(\alpha_1,\ldots,\alpha_k)\in(\mathbb{Z}_2)^k$ let $V_{\alpha_1\ldots\alpha_k}=\operatorname{span}\{v_{\alpha_1\ldots\alpha_k}\}\subset\mathbb{R}^{2^k}$ denote the 1-dimensional representation given by

$$\varepsilon_i\cdot v_{\alpha_1\ldots\alpha_k}=\alpha_iv_{\alpha_1\ldots\alpha_k}.$$

The vector $v_{\alpha_1\ldots\alpha_k}\in\{+1,-1\}^{2^k}$ is uniquely determined up to a scalar multiplication by -1. Note that

$$\langle v_{\alpha_1\ldots\alpha_k},v_{\beta_1\ldots\beta_k}\rangle=0$$

for $\alpha_1\ldots\alpha_k\neq\beta_1\ldots\beta_k$. For $k=2$, with the abbreviation $+$ for $+1$, $-$ for -1, the coordinate index set for $\mathbb{R}^4$ is $\{++,+-,-+,--\}$. Then

$$v_{++}=(1,1,1,1),\qquad v_{+-}=(1,-1,1,-1),$$
$$v_{-+}=(1,1,-1,-1),\qquad v_{--}=(1,-1,-1,1).$$

The following decomposition of $(\mathbb{Z}_2)^k$-modules holds, with the index identification $(\mathbb{Z}_2)^k=\{+,-\}^k$,

$$\mathbb{R}^{2^k}\cong V_{+\cdots+}\oplus\sum_{\alpha_1\ldots\alpha_k\in(\mathbb{Z}_2)^k\setminus\{+\cdots+\}}V_{\alpha_1\ldots\alpha_k}$$

where $V_{+\cdots+}$ is the trivial $(\mathbb{Z}_2)^k$-representation. Let R_{2^k} denote the orthogonal complement of $V_{+\cdots+}$ and $\pi:\mathbb{R}^{2^k}\to R_{2^k}$ the associated (equivariant) projection. Explicitly

$$R_{2^k}=\left\{(x_1,\ldots,x_{2^k})\in\mathbb{R}^{2^k}\;\Big|\;\sum x_i=0\right\}=\sum_{\alpha_1\ldots\alpha_k\in(\mathbb{Z}_2)^k\setminus\{+\cdots+)\}}V_{\alpha_1\ldots\alpha_k},\tag{5}$$

and

$$\mathbf{x}=(x_1,\ldots,x_{2^k})\overset{\pi}{\longmapsto}\frac{1}{2^{k-1}}\big(\langle\mathbf{x},v_{\alpha_1\ldots\alpha_k)}\rangle\big)_{\alpha_1\ldots\alpha_k\in(\mathbb{Z}_2)^k\setminus\{+\cdots+\}},$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product of $\mathbb{R}^{2^k}$. Observe that

$$\operatorname{im}\phi=\phi(Y_{d,k})\subseteq(R_{2^k})^j.$$

Let $\alpha_1\ldots\alpha_k\in(\mathbb{Z}_2)^k$ and let $\eta(\alpha_1\ldots\alpha_k)=\frac{1}{2}(k-\sum\alpha_i)$. The following decomposition of W_k-modules holds

$$\mathbb{R}^{2^k}\cong V_{+\cdots+}\oplus\sum_{n=1}^{k}\sum_{n=\eta(\alpha_1,\ldots,\alpha_k)}V_{\alpha_1\ldots\alpha_k}\cong V_{+\cdots+}\oplus R_{2^k}.\tag{6}$$

The test map $\tau : X_{d,k} \to U_k \times (R_{2^k})^j$ is defined by

$$\tau(t_1 v_1 + \cdots + t_k v_k) = \left(t_1 - \frac{1}{k}, \ldots, t_k - \frac{1}{k}\right) \times \left(\left(t_1^{\frac{1-\alpha_1}{2}} \cdots t_k^{\frac{1-\alpha_k}{2}} \langle \phi_i(v_1, \ldots, v_k), v_{\alpha_1 \ldots \alpha_k}\rangle\right)_{\alpha_1 \ldots \alpha_k \in (\mathbb{Z}_2)^k \setminus \{+\cdots+\}}\right)_{i=1}^{j}.$$

Here $U_k = \{(\xi_1, \ldots, \xi_k) \in \mathbb{R}^k \mid \sum \xi_i = 0\}$ is a W_k-module with an action given by

$$\big((\beta_1, \ldots, \beta_k) \rtimes \pi\big) \cdot (\xi_1, \ldots, \xi_k) := (\xi_{\pi^{-1}(1)}, \ldots, \xi_{\pi^{-1}(k)}).$$

The subgroup $(\mathbb{Z}_2)^k$ acts trivially on U_k. The action on $U_k \times (R_{2^k})^j$ is assumed to be the diagonal action. The test map τ is well defined, continuous and W_k-equivariant.

Example 2.1. The test map $\tau : X_{d,k} \to U_k \times (R_{2^k})^j$ is in the case of $k = 2$ hyperplanes and $j = 1$ measure given by $\tau : X_{d,2} \to U_2 \times R_4 = U_2 \times ((V_{+-} \oplus V_{-+}) \oplus V_{--})$ and

$$\tau(t_1 v_1 + t_2 v_2) = \left(t_1 - \frac{1}{2}, t_2 - \frac{1}{2}, t_1\big\langle\phi(v_1, v_2), v_{-+}\big\rangle, t_2\big\langle\phi(v_1, v_2), v_{+-}\big\rangle, t_1 t_2\big\langle\phi(v_1, v_2), v_{--}\big\rangle\right)$$

where

$$\phi(v_1, v_2) = \left(\mu_i\big(H_{v_1}^{\alpha_1} \cap H_{v_2}^{\alpha_2}\big) - \frac{1}{4}\mu\big(\mathbb{R}^d\big)\right)_{\alpha_1\alpha_2 \in (\mathbb{Z}_2)^2} \in \mathbb{R}^4.$$

2.3. The test space

The test spaces for the maps ϕ and τ are the origins of $(R_{2^k})^j$ and $U_k \times (R_{2^k})^j$, respectively. The constructions that we perform in this section satisfy the usual hypotheses for the CS/TM scheme.

Proposition 2.2.

(i) *For a collection of mass distributions $\mathcal{M} = \{\mu_1, \ldots, \mu_j\}$ let $\phi : Y_{d,k} \to (R_{2^k})^j$ and $\tau : X_{d,k} \to U_k \times (R_{2^k})^j$ be the corresponding test maps. If*

$$(0, \ldots, 0) \in \phi(Y_{d,k}) \quad \textit{or} \quad (0, \ldots, 0) \in \tau(X_{d,k})$$

then there exists an arrangement of k hyperplanes $\mathcal{H}$ in $\mathbb{R}^d$ equiparting the collection $\mathcal{M}$.

(ii) *If there is no W_k-equivariant map with respect to the actions defined above,*

$$Y_{d,k} \to (R_{2^k})^j \setminus \big\{(0, \ldots, 0)\big\}, \qquad \textit{or} \quad Y_{d,k} \to S\big((R_{2^k})^j\big) \approx S^{j(2^k-1)-1}, \quad \textit{or}$$
$$X_{d,k} \to U_k \times (R_{2^k})^j \setminus \big\{(0, \ldots, 0)\big\}, \quad \textit{or} \quad X_{d,k} \to S\big(U_k \times (R_{2^k})^j\big) \approx S^{j(2^k-1)+k-2},$$

then the triple (d, j, k) is admissible.

(iii) *Specifically, for $k = 2$, if there is no $D_8 \cong W_2$ equivariant map, with the already defined actions,*

$$Y_{d,2} \to (R_4)^j \setminus \big\{(0, \ldots, 0)\big\}, \qquad \textit{or} \quad Y_{d,2} \to S\big((R_4)^j\big) \approx S^{3j-1}, \quad \textit{or}$$
$$X_{d,2} \to U_2 \times (R_4)^j \setminus \big\{(0, \ldots, 0)\big\}, \quad \textit{or} \quad S^{2d+1} \approx X_{d,2} \to S\big(U_2 \times (R_4)^j\big) \approx S^{3j},$$

then the triple $(d, j, 2)$ is admissible.

Remark 2.3. The action of W_k on the sphere $S(U_2 \times (R_4)^j)$ is *fixed point free, but not free*. For $k = 2$, the action of the unique $\mathbb{Z}_4$ subgroup of $W_2 = D_8$ on the sphere $S(U_2 \times (R_4)^j)$ is fixed point free.

The necessary condition for the non-existence of an equivariant W_k-map

$$X_{d,k} \to S\big(U_k \times (R_{2^k})^j\big)$$

implied by the equivariant Kuratowski–Dugundji theorem [3, Theorem 1.3, p. 25] is

$$dk + k - 1 > j\big(2^k - 1\big) + k - 2 \iff d \geqslant \frac{2^k - 1}{k} j. \tag{7}$$

For $k = 2$ the condition (7) becomes

$$d \geqslant \left\lceil \frac{3}{2} j \right\rceil. \tag{8}$$

3. The Fadell–Husseini index theory

3.1. Equivariant cohomology

Let X be a G-space and $X \to \mathrm{E}G \times_G X \xrightarrow{\pi_X} \mathrm{B}G$ the associated universal bundle, with X as a typical fibre. $\mathrm{E}G$ is a contractible cellular space on which G acts freely, and $\mathrm{B}G := \mathrm{E}G/G$. The space $\mathrm{E}G \times_G X = (\mathrm{E}G \times X)/G$ is called the *Borel construction* of X with respect to the action of G. The *equivariant cohomology* of X is the ordinary cohomology of the Borel construction $\mathrm{E}G \times_G X$,

$$H^*_G(X) := H^*(\mathrm{E}G \times_G X).$$

The equivariant cohomology is a module over the ring $H^*_G(\mathrm{pt}) = H^*(\mathrm{B}G)$. When X is a free G-space the homotopy equivalence $\mathrm{E}G \times_G X \simeq X/G$ induces a natural isomorphism

$$H^*_G(X) \cong H^*(X/G).$$

The universal bundle $X \to \mathrm{E}G \times_G X \xrightarrow{\pi_X} \mathrm{B}G$, for coefficients in the ring R, induces a Serre spectral sequence converging to the graded group $\mathrm{Gr}(H^*_G(X,R))$ associated with $H^*_G(X,R)$ appropriately filtered. In this paper "ring" means commutative ring with a unit element. The E_2-term is given by

$$E_2^{p,q} \cong H^p\big(\mathrm{B}G, \mathcal{H}^q(X,R)\big), \tag{9}$$

where $\mathcal{H}^q(X,R)$ is a system of local coefficients. For a discrete group G, the E_2-term of the spectral sequence can be interpreted as the cohomology of the group G with coefficients in the G-module $H^*(X,R)$,

$$E_2^{p,q} \cong H^p\big(G, H^q(X,R)\big). \tag{10}$$

3.2. $\mathrm{Index}_{G,R}$ and $\mathrm{Index}^k_{G,R}$

Let X be a G-space, R a ring and π_X^* the ring homomorphism in cohomology

$$\pi_X^* : H^*(\mathrm{B}G, R) \to H^*(\mathrm{E}G \times_G X, R)$$

induced by the projection $\mathrm{E}G \times_G X \to \mathrm{E}G \times_G \mathrm{pt} \approx \mathrm{B}G$.

The *Fadell–Husseini* (*ideal-valued*) *index* of a G-space X is the kernel ideal of π_X^*,

$$\mathrm{Index}_{G,R} X := \ker \pi_X^* \subseteq H^*(\mathrm{B}G, R).$$

The Serre spectral sequence (9) yields a representation of the homomorphism π_X^* as the composition

$$H^*(\mathrm{B}G,R) \to E_2^{*,0} \to E_3^{*,0} \to E_4^{*,0} \to \cdots \to E_\infty^{*,0} \subseteq H^*(\mathrm{E}G \times_G X, R).$$

The k-th *Fadell–Husseini index* is defined by

$$\mathrm{Index}^k_{G,R} X = \ker\big(H^*(\mathrm{B}G,R) \to E_k^{*,0}\big), \quad k \geqslant 2,$$
$$\mathrm{Index}^1_{G,R} X = \{0\}.$$

From the definitions the following properties of indexes can be derived.

Proposition 3.1. *Let X, Y be G-spaces.*

(1) $\mathrm{Index}^k_{G,R} X \subseteq H^*(\mathrm{B}G,R)$ *is an ideal, for every* $k \in \mathbb{N}$;
(2) $\mathrm{Index}^1_{G,R} X \subseteq \mathrm{Index}^2_{G,R} X \subseteq \mathrm{Index}^3_{G,R} X \subseteq \cdots \subseteq \mathrm{Index}_{G,R} X$;
(3) $\bigcup_{k\in\mathbb{N}} \mathrm{Index}^k_{G,R} X = \mathrm{Index}_{G,R} X$.

Proposition 3.2. *Let X and Y be G-spaces and $f : X \to Y$ a G-map. Then*

$$\mathrm{Index}_{G,R}(X) \supseteq \mathrm{Index}_{G,R}(Y)$$

and for every $k \in \mathbb{N}$

$$\mathrm{Index}^k_{G,R}(X) \supseteq \mathrm{Index}^k_{G,R}(Y).$$

Proof. Functoriality of all constructions implies that the following diagrams commute:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{E}G \times_G X \arrow[rr, "\hat{f}"] \arrow[dr, "\pi_X"] & & \mathrm{E}G \times_G Y \arrow[dl, "\pi_Y"'] \\ & \mathrm{B}G & \end{tikzcd}$$

and consequently applying cohomology functor

$$\begin{tikzcd} H^*(\mathrm{E}G \times_G X, R) & & H^*(\mathrm{E}G \times_G Y, R) \arrow[ll, "f^*"'] \\ & H^*(\mathrm{B}G, R) \arrow[ul, "\pi_X^*"] \arrow[ur, "\pi_Y^*"'] & \end{tikzcd}$$

$\pi_X = \pi_Y \circ \hat{f}$ and $\pi_X^* = f^* \circ \pi_Y^*$. Thus $\ker \pi_X^* \supseteq \ker \pi_Y^*$. $\square$

Example 3.3. S^n is a $\mathbb{Z}_2$-space with the antipodal action. The action is free and therefore

$$\mathrm{E}\mathbb{Z}_2 \times_{\mathbb{Z}_2} S^n \simeq S^n/\mathbb{Z}_2 \approx \mathbb{R}\mathrm{P}^n \quad \Rightarrow \quad H^*_{\mathbb{Z}_2}\big(S^n, R\big) \cong H^*\big(\mathbb{R}\mathrm{P}^n, R\big).$$

1. $R = \mathbb{F}_2$: The cohomology ring $H^*(\mathrm{B}\mathbb{Z}_2, \mathbb{F}_2) = H^*(\mathbb{R}\mathrm{P}^\infty, \mathbb{F}_2)$ is the polynomial ring $\mathbb{F}_2[t]$ where $\deg(t) = 1$. The $\mathbb{Z}_2$-index of S^n is the principal ideal generated by t^{n+1}:
$$\mathrm{Index}_{\mathbb{Z}_2,\mathbb{F}_2} S^n = \mathrm{Index}^{n+2}_{\mathbb{Z}_2,\mathbb{F}_2} S^n = \big\langle t^{n+1}\big\rangle \subseteq \mathbb{F}_2[t].$$
2. $R = \mathbb{Z}$: The cohomology ring $H^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}) = H^*(\mathbb{R}\mathrm{P}^\infty, \mathbb{Z})$ is the quotient polynomial ring $\mathbb{Z}[\tau]/\langle 2\tau\rangle$ where $\deg(\tau) = 2$. The $\mathbb{Z}_2$-index of S^n is the principal ideal
$$\mathrm{Index}_{\mathbb{Z}_2,\mathbb{Z}} S^n = \mathrm{Index}^{n+2}_{\mathbb{Z}_2,\mathbb{Z}} S^n = \begin{cases} \langle \tau^{\frac{n+1}{2}}\rangle, & \text{for } n \text{ odd}, \\ \langle \tau^{\frac{n+2}{2}}\rangle, & \text{for } n \text{ even}. \end{cases}$$

Example 3.4. Let G be a finite group and H a subgroup of index 2. Then $H \lhd G$ and $G/H \cong \mathbb{Z}_2$. Let V be the 1-dimensional real representation of G defined for $v \in V$ by

$$g \cdot v = \begin{cases} v, & \text{for } g \in H, \\ -v, & \text{for } g \notin H. \end{cases}$$

There is a G-homeomorphism $S(V) \approx \mathbb{Z}_2$. Therefore by [17, last equation on p. 34]:

$$\mathrm{E}G \times_G S(V) \approx \mathrm{E}G \times_G (G/H) \approx (\mathrm{E}G \times_G G)/H \approx \mathrm{E}G/H \approx \mathrm{B}H$$

and

$$\mathrm{Index}_{G,R} S(V) = \ker\big(\mathrm{res}^G_H : H^*(G, R) \to H^*(H, R)\big). \tag{11}$$

3.3. The restriction map and the index

Let X be a G-space and $K \subseteq G$ a subgroup. Then there is a commutative diagram of fibrations [9, pp. 179–180]:

$$\begin{tikzcd} \mathrm{E}G \times_G X \arrow[d] & \mathrm{E}G \times_K X \arrow[l, "f"'] \arrow[d] \\ \mathrm{B}G = \mathrm{E}G/G & \mathrm{E}G/K = \mathrm{B}K \arrow[l, "\mathrm{B}i"'] \end{tikzcd} \tag{12}$$

induced by inclusion $i : K \subset G$. Here $\mathrm{E}G$ in the lower right corner is understood as a K-space and consequently a model for $\mathrm{E}K$. The map $\mathrm{B}i$ is a map between classifying spaces induced by inclusion i. Now with coefficients in the ring R we define

$$\mathrm{res}^G_K := H^*(f) : H^*(\mathrm{E}G \times_G X, R) \to H^*(\mathrm{E}G \times_K X, R).$$

If G is a finite group, then the induced map on the cohomology of the classifying spaces

$$\mathrm{res}^G_K = (\mathrm{B}i)^* : H^*(\mathrm{B}G, R) \to H^*(\mathrm{B}K, R)$$

coincides with the restriction homomorphism between group cohomologies

$$\mathrm{res}^G_K : H^*(G, R) \to H^*(K, R).$$

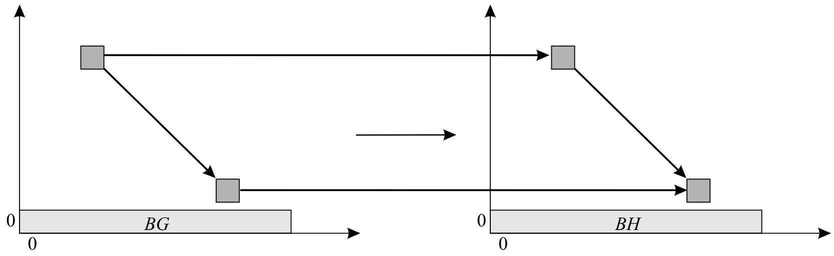

Fig. 1. Illustration of Proposition 3.5 (D) and (E).

Proposition 3.5. *Let X be a G-space, and K and L subgroups of G.*

(A) *The morphism of fibrations* (12) *provides the following commutative diagram in cohomology*:

$$\begin{tikzcd} H^*(\mathrm{E}G\times_G X,R) \arrow[r, "\mathrm{res}^G_K"] & H^*(\mathrm{E}G\times_K X,R) \\ H^*(\mathrm{B}G,R) \arrow[r, "\mathrm{res}^G_K"] \arrow[u, "\pi^*_X"] & H^*(\mathrm{B}K,R) \arrow[u, "\pi^*_X"'] \end{tikzcd} \tag{13}$$

(B) *For every $x\in H^*(\mathrm{B}G,R)$ and $y\in H^*(\mathrm{E}G\times_G X,R)$,*

$$\mathrm{res}^G_K(x\cdot y)=\mathrm{res}^G_K(x)\cdot\mathrm{res}^G_K(y).$$

(C) $L\subset K\subset G\Rightarrow \mathrm{res}^G_L=\mathrm{res}^K_L\circ\mathrm{res}^G_K$.

(D) *The map of fibrations* (12) *induces a morphism of Serre spectral sequences* (*see Fig.* 1)

$$\Gamma^{*,*}_i : E^{*,*}_i(\mathrm{E}G\times_G X,R)\to E^{*,*}_i(\mathrm{E}K\times_K X,R)$$

such that
(1) $\Gamma^{*,*}_\infty=\mathrm{res}^G_K : H^{*+*}(\mathrm{E}G\times_G X,R)\to H^{*+*}(\mathrm{E}G\times_K X,R)$,
(2) $\Gamma^{*,0}_2=\mathrm{res}^G_K : H^*(\mathrm{B}G,R)\to H^*(\mathrm{B}K,R)$.

(E) *Let R and S be commutative rings and $\phi : R\to S$ a ring homomorphism. There are morphisms:*
(1) *in equivariant cohomology $\Phi^* : H^*(\mathrm{E}G\times_G X,R)\to H^*(\mathrm{E}G\times_G X,S)$,*
(2) *in group cohomology $\Phi^* : H^*(G,R)\to H^*(G,S)$, and*
(3) *between Serre spectral sequences $\Phi^{*,*}_i : E^{*,*}_i(\mathrm{E}G\times_G X,R)\to E^{*,*}_i(\mathrm{E}G\times_G X,S)$,*
induced by ϕ such that the following diagram commutes:

$$\begin{tikzcd} H^*(\mathrm{E}G\times_G X,R) \arrow[rr] \arrow[dr, "\phi"'] & & H^*(\mathrm{E}G\times_K X,R) \arrow[dr, "\phi"'] & \\ & H^*(\mathrm{E}G\times_G X,S) \arrow[rr] & & H^*(\mathrm{E}G\times_K X,S) \\ H^*(\mathrm{B}G,R) \arrow[rr] \arrow[uu] \arrow[dr, "\phi"'] & & H^*(\mathrm{B}K,R) \arrow[uu] \arrow[dr, "\phi"'] & \\ & H^*(\mathrm{B}G,S) \arrow[rr] \arrow[uu] & & H^*(\mathrm{B}K,S) \arrow[uu] \end{tikzcd} \tag{14}$$

Remark 3.6. By a morphism of spectral sequences in properties (D) and (E) we mean that

$$\Gamma^{*,*}_i\circ\partial_i=\partial_i\circ\Gamma^{*,*}_i \quad \text{and} \quad \Phi^{*,*}_i\circ\partial_i=\partial_i\circ\Phi^{*,*}_i.$$

These relations are applied in the situations where the right-hand side is $\neq 0$ for a particular element x, to imply that the left-hand side $\Gamma^{*,*}_i\circ\partial_i(x)$ or $\Phi^{*,*}_i\circ\partial_i(x)$ is also $\neq 0$. In particular, then $\partial_i(x)\neq 0$.

Proposition 3.7. *Let X be a G-space and K a subgroup of G. Let R and S be rings and $\phi : R\to S$ a ring homomorphism. Then*

(1) $\mathrm{res}^G_K(\mathrm{Index}_{G,R}X)\subseteq\mathrm{Index}_{K,R}X$,
(2) $\mathrm{res}^G_K(\mathrm{Index}^r_{G,R}X)\subseteq\mathrm{Index}^r_{K,R}X$ *for every $r\in\mathbb{N}$*,
(3) $\Phi^*(\mathrm{Index}_{G,R}X)\subseteq\mathrm{Index}_{G,S}X$,
(4) $\Phi^*(\mathrm{Index}^r_{G,R}X)\subseteq\mathrm{Index}^r_{G,S}X$.

Proof. The assertions about the $\mathrm{Index}_{G,R}$ follow from diagrams (13) and (14). The commutative diagrams

$$\begin{tikzcd} E_r^{*,0}(\mathrm{E}G\times_G X,R) \arrow[r, "\Gamma_r^{*,0}"] & E_r^{*,0}(\mathrm{E}K\times_K X,R) \\ H^*(\mathrm{B}G,R) \arrow[u] \arrow[r, "\mathrm{res}_K^G"] & H^*(\mathrm{B}K,R) \arrow[u] \end{tikzcd}$$

and

$$\begin{tikzcd} E_r^{*,0}(\mathrm{E}G\times_G X,R) \arrow[r, "\Phi_r^{*,0}"] & E_r^{*,0}(\mathrm{E}G\times_G X,S) \\ H^*(\mathrm{B}G,R) \arrow[u] \arrow[r, "\Phi^*"] & H^*(\mathrm{B}G,S) \arrow[u] \end{tikzcd}$$

imply the partial index assertions. □

3.4. Basic calculations of the index

3.4.1. The index of a product

Let X be a G-space and Y an H-space. Then $X\times Y$ has the natural structure of a $G\times H$-space. What is the relation between the three indexes $\mathrm{Index}_{G\times H}(X\times Y)$, $\mathrm{Index}_G(X)$, and $\mathrm{Index}_H(Y)$? Using the Künneth formula one can prove the following proposition [11, Corollary 3.2], [27, Proposition 2.7] when the coefficient ring is a field.

Proposition 3.8. *Let X be a G-space and Y an H-space and*

$$H^*(\mathrm{B}G,\Bbbk)\cong\Bbbk[x_1,\dots,x_n],\qquad H^*(\mathrm{B}H,\Bbbk)\cong\Bbbk[y_1,\dots,y_m]$$

the cohomology rings of the associated classifying spaces with coefficients in the field $\Bbbk$. If

$$\mathrm{Index}_{G,\Bbbk}X=\langle f_1,\dots,f_i\rangle\quad \textit{and}\quad \mathrm{Index}_{H,\Bbbk}(Y)=\langle g_1,\dots,g_j\rangle,$$

then

$$\mathrm{Index}_{G\times H,\Bbbk}X=\langle f_1,\dots,f_i,g_1,\dots,g_j\rangle\subseteq\Bbbk[x_1,\dots,x_n,y_1,\dots,y_m].$$

The $(\mathbb{Z}_2)^k$-index of a product of spheres can be computed using this proposition and Example 3.3.

Corollary 3.9. *Let $S^{n_1}\times\cdots\times S^{n_k}$ be a $(\mathbb{Z}_2)^k$-space with the product action. Then*

$$\mathrm{Index}_{(\mathbb{Z}_2)^k,\mathbb{F}_2}S^{n_1}\times\cdots\times S^{n_k}=\langle t_1^{n_1+1},\dots,t_k^{n_k+1}\rangle\subseteq\mathbb{F}_2[t_1,\dots,t_k].$$

Unfortunately when the coefficient ring is not a field the claim of Proposition 3.8 does not hold.

Example 3.10. Let $S^n\times S^n$ be a $(\mathbb{Z}_2)^2$-space with the product action. From the previous corollary

$$\mathrm{Index}_{(\mathbb{Z}_2)^2,\mathbb{F}_2}S^n\times S^n=\langle t_1^{n+1},t_2^{n+1}\rangle\subseteq\mathbb{F}_2[t_1,t_2]=H^*\big((\mathbb{Z}_2)^2,\mathbb{F}_2\big).\tag{15}$$

The cohomology ring $H^*((\mathbb{Z}_2)^2,\mathbb{Z})$, as in [19, Proposition 4.1, p. 508] or [16, Example 3E.5, pp. 306–307], can be described as the quotient:

$$H^*\big((\mathbb{Z}_2)^2,\mathbb{Z}\big)\cong\big(\mathbb{Z}[\tau_1,\tau_2]\otimes Z[\mu]\big)/\mathcal{I}\tag{16}$$

where $\deg\tau_1=\deg\tau_2=2$, $\deg\mu=3$ and $\mathcal{I}$ is the ideal generated by the relations:

$$2\tau_1=2\tau_2=2\mu=0\quad\text{and}\quad \mu^2=\tau_1\tau_2(\tau_1+\tau_2).$$

The ring morphism $c:\mathbb{Z}\to\mathbb{F}_2$ induces a morphism $c_*:H^*((\mathbb{Z}_2)^2,\mathbb{Z})\to H^*((\mathbb{Z}_2)^2,\mathbb{F}_2)$ given by:

$$\tau_1\longmapsto t_1^2,\qquad \tau_2\longmapsto t_2^2,\qquad \mu\longmapsto t_1t_2(t_1+t_2).\tag{17}$$

The $(\mathbb{Z}_2)^2$-action on $S^n\times S^n$, as a product of antipodal actions, is free and therefore

$$\mathrm{E}(\mathbb{Z}_2)^2\times_{(\mathbb{Z}_2)^2}\big(S^n\times S^n\big)\simeq\big(S^n\times S^n\big)/(\mathbb{Z}_2)^2\approx\mathbb{R}P^n\times\mathbb{R}P^n.$$

Using equality (15), Proposition 3.5.E.3 on the coefficient morphism $c:\mathbb{Z}\to\mathbb{F}_2$, the isomorphism

$$H^*_{(\mathbb{Z}_2)^2}\big(S^n\times S^n,\mathbb{Z}\big)\cong H^*\big(\mathbb{R}P^n\times\mathbb{R}P^n,\mathbb{Z}\big)$$

and the existence of the $(\mathbb{Z}_2)^2$-inclusions

$$S^{n-1} \times S^{n-1} \subset S^n \times S^n \subset S^{n+1} \times S^{n+1},$$

it can be concluded that

$$\operatorname{Index}_{(\mathbb{Z}_2)^2,\mathbb{Z}} S^n \times S^n = \begin{cases} \langle \tau_1^{\frac{n+1}{2}}, \tau_2^{\frac{n+1}{2}} \rangle, & \text{for } n \text{ odd} \\ \langle \tau_1^{\frac{n+2}{2}}, \tau_2^{\frac{n+2}{2}}, \tau_1^{\frac{n}{2}}\mu, \tau_2^{\frac{n}{2}}\mu \rangle, & \text{for } n \text{ even} \end{cases} \subseteq H^*\big((\mathbb{Z}_2)^2, \mathbb{Z}\big). \tag{18}$$

3.4.2. The index of a sphere

We need to know how to compute the index of a sphere admitting an action of a finite group different from the antipodal $\mathbb{Z}_2$-action. The following three propositions will be of some help [11, Proposition 3.13], [27, Proposition 2.9].

Proposition 3.11. *Let G be a finite group and V an n-dimensional complex representation of G. Then*

$$\operatorname{Index}_{G,\mathbb{Z}} S(V) = \big\langle c_n(V_G) \big\rangle \subset H^*(G, \mathbb{Z})$$

where $c_n(V_G)$ is the n-th Chern class of the bundle $V \to \mathrm{E}G \times_G V \to \mathrm{B}G$.

Proof. The fact that the index is generated by the Euler class $e(V_G)$ of the orientable vector bundle V_G,

$$V \to \mathrm{E}G \times_G V \to \mathrm{B}G,$$

follows from the Gysin exact sequence, [24, Theorem 12.2, p. 143]. In the particular case of the complex representation V the Euler class $e(V_G)$ coincides with the top Chern class $c_n(V_G)$, [18, Exercise 3, p. 261]. □

Proposition 3.12. *Let U, V be two G-representations and let $S(U)$, $S(V)$ be the associated G-spheres. Let R be a ring and assume that $H^*(S(U), R)$, $H^*(S(V), R)$ are trivial G-modules. If $\operatorname{Index}_{G,R}(S(U)) = \langle f \rangle \subseteq H^*(\mathrm{B}G, R)$ and $\operatorname{Index}_{G,R}(S(V)) = \langle g \rangle \subseteq H^*(\mathrm{B}G, R)$, then*

$$\operatorname{Index}_{G,R} S(U \oplus V) = \langle f \cdot g \rangle \subseteq H^*(\mathrm{B}G, R).$$

Proposition 3.13.

(A) *Let V be the 1-dimensional $(\mathbb{Z}_2)^k$ -representation with the associated ± 1 vector $(\alpha_1, \dots, \alpha_k) \in (\mathbb{Z}_2)^k$ (as defined in Section 2). Then*

$$\operatorname{Index}_{(\mathbb{Z}_2)^k,\mathbb{F}_2} S(V) = \langle \bar{\alpha}_1 t_1 + \cdots + \bar{\alpha}_k t_k \rangle \subseteq \mathbb{F}_2[t_1, \dots, t_k],$$

where $\bar{\alpha}_i = 0$ if $\alpha_i = 1$, and $\bar{\alpha}_i = 1$ if $\alpha_i = -1$.

(B) *Let U be an n-dimensional $(\mathbb{Z}_2)^k$ -representation with a decomposition $U \cong V_1 \oplus \cdots \oplus V_n$ into 1-dimensional $(\mathbb{Z}_2)^k$-representations $V_1, \dots, V_n$. If $(\alpha_{1i}, \dots, \alpha_{ki}) \in (\mathbb{Z}_2)^k$ is the associated ± 1 vector of V_i, then*

$$\operatorname{Index}_{(\mathbb{Z}_2)^k,\mathbb{F}_2} S(U) = \left\langle \prod_{i=1}^{n} (\bar{\alpha}_{1i} t_1 + \cdots + \bar{\alpha}_{ki} t_k) \right\rangle \subseteq \mathbb{F}_2[t_1, \dots, t_k].$$

Example 3.14. Let V_{-+}, V_{+-} and V_{--} be 1-dimensional real $(\mathbb{Z}_2)^2$-representations introduced in Section 2.2. Then by Proposition 3.13

$$\operatorname{Index}_{(\mathbb{Z}_2)^2,\mathbb{F}_2} S(V_{-+}) = \langle t_1 \rangle, \qquad \operatorname{Index}_{(\mathbb{Z}_2)^2,\mathbb{F}_2} S(V_{+-}) = \langle t_2 \rangle, \qquad \operatorname{Index}_{(\mathbb{Z}_2)^2,\mathbb{F}_2} S(V_{--}) = \langle t_1 + t_2 \rangle.$$

On the other hand, Example 3.4 and the restriction diagram (23) imply that

$$\operatorname{Index}_{(\mathbb{Z}_2)^2,\mathbb{Z}} S(V_{-+}) = \langle \tau_1, \mu \rangle, \qquad \operatorname{Index}_{(\mathbb{Z}_2)^2,\mathbb{Z}} S(V_{+-}) = \langle \tau_2, \mu \rangle, \qquad \operatorname{Index}_{(\mathbb{Z}_2)^2,\mathbb{Z}} S(V_{--}) = \langle \tau_1 + \tau_2, \mu \rangle.$$

4. The cohomology of D_8 and the restriction diagram

The dihedral group $W_2 = D_8 = (\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2 = (\langle \varepsilon_1 \rangle \times \langle \varepsilon_2 \rangle) \rtimes \langle \sigma \rangle$ can be presented by

$$D_8 = \big\langle \varepsilon_1, \sigma \;\big|\; \varepsilon_1^2 = \sigma^2 = (\varepsilon_1 \sigma)^4 = 1 \big\rangle.$$

Then $\langle \varepsilon_1 \sigma \rangle \cong \mathbb{Z}_4$ and $\varepsilon_2 = \sigma \varepsilon_1 \sigma$.

4.1. *The poset of subgroups of D_8*

The poset $\mathrm{Sub}(G)$ denotes the collection of all nontrivial subgroups of a given group G ordered by inclusion. The poset $\mathrm{Sub}(G)$ can be interpreted as a small category $\mathfrak{G}$ in the usual way:

- $\mathrm{Ob}(\mathfrak{G}) = \mathrm{Sub}(G)$,
- for every two objects H and K, subgroups of G, there is a unique morphism $f_{H,K} : H \to K$ if $H \supseteq K$, and no morphism if $H \not\supseteq K$, i.e.

$$\mathrm{Mor}(H,K) = \begin{cases} \{f_{H,K}\}, & H \supseteq K, \\ \emptyset, & H \not\supseteq K. \end{cases}$$

The Hasse diagram of the poset $\mathrm{Sub}(D_8)$ is presented in the following diagram.

$$\begin{tikzcd}[column sep=small]
 & & & D_8 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & \begin{array}{|c|}\hline H_1 \\ \hline \langle \varepsilon_1, \varepsilon_2 \rangle \\ \hline \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \hline \end{array} \arrow[dl] \arrow[d] \arrow[drr] & & \begin{array}{|c|}\hline H_2 \\ \hline \langle \varepsilon_1 \sigma \rangle \\ \hline \mathbb{Z}_4 \\ \hline \end{array} \arrow[d] & & \begin{array}{|c|}\hline H_3 \\ \hline \langle \varepsilon_1 \varepsilon_2, \sigma \rangle \\ \hline \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \hline \end{array} \arrow[dll] \arrow[d] \arrow[dr] & \\
\begin{array}{|c|}\hline K_1 \\ \hline \langle \varepsilon_1 \rangle \\ \hline \mathbb{Z}_2 \\ \hline \end{array} & \begin{array}{|c|}\hline K_2 \\ \hline \langle \varepsilon_2 \rangle \\ \hline \mathbb{Z}_2 \\ \hline \end{array} & & \begin{array}{|c|}\hline K_3 \\ \hline \langle \varepsilon_1 \varepsilon_2 \rangle \\ \hline \mathbb{Z}_2 \\ \hline \end{array} & & \begin{array}{|c|}\hline K_4 \\ \hline \langle \sigma \rangle \\ \hline \mathbb{Z}_2 \\ \hline \end{array} & \begin{array}{|c|}\hline K_5 \\ \hline \langle \varepsilon_1 \varepsilon_2 \sigma \rangle \\ \hline \mathbb{Z}_2 \\ \hline \end{array}
\end{tikzcd}$$

4.2. *The cohomology diagram of subgroups with coefficients in $\mathbb{F}_2$*

Let G be a finite group and R an arbitrary ring. Then the diagram $\mathrm{Res}_{(R)} : \mathfrak{G} \to \mathfrak{Ring}$ (covariant functor) defined by

$$\mathrm{Ob}(\mathfrak{G}) \ni H \longmapsto H^*(H,R),$$
$$(H \supseteq K) \longmapsto \big(\mathrm{res}^H_K : H^*(H,R) \to H^*(K,R)\big)$$

is the *cohomology diagram of subgroups* of G with coefficients in the ring R. In this section we assume that $R = \mathbb{F}_2$.

4.2.1. *The $\mathbb{Z}_2 \times \mathbb{Z}_2$-diagram*

The cohomology of any elementary abelian 2-group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a polynomial ring $\mathbb{F}_2[x,y]$, $\deg(x) = \deg(y) = 1$. The restrictions to the three subgroups of order 2 are given by all possible projections $\mathbb{F}_2[x,y] \to \mathbb{F}_2[t]$, $\deg(t) = 1$:

$$(x \mapsto t, y \mapsto 0) \quad \text{or} \quad (x \mapsto 0, y \mapsto t) \quad \text{or} \quad (x \mapsto t, y \mapsto t).$$

Thus the cohomology diagram of the subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is

$$\begin{tikzcd}
 & \begin{array}{|c|}\hline \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \hline \mathbb{F}_2\,[x,y] \\ \hline \end{array} \arrow[dl, "x\mapsto 0"', "y\mapsto t_1"] \arrow[d, "x\mapsto t_2"', "y\mapsto 0"] \arrow[dr, "x\mapsto t_3", "y\mapsto t_3"'] & \\
\begin{array}{|c|}\hline \mathbb{Z}_2 \\ \hline \mathbb{F}_2[t_1] \\ \hline \end{array} & \begin{array}{|c|}\hline \mathbb{Z}_2 \\ \hline \mathbb{F}_2[t_2] \\ \hline \end{array} & \begin{array}{|c|}\hline \mathbb{Z}_2 \\ \hline \mathbb{F}_2[t_3] \\ \hline \end{array}
\end{tikzcd} \tag{19}$$

4.2.2. *The D_8-diagram*

The cohomology of D_8 can be presented, as in [1, p. 119] and [7,8], by

$$H^*(D_8, \mathbb{F}_2) = \mathbb{F}_2[x,y,w]/\langle xy \rangle$$

where $\deg(x) = \mathbf{deg}(y) = 1$ and $\deg(w) = 2$. Following [7,8] the two top levels of the diagram can be presented by:

$$\begin{tikzcd}
 & \begin{array}{|c|}\hline D_8 \\ \hline \mathbb{F}_2[x,y;w]/\langle xy\rangle \\ \hline \deg:1,1,2 \\ \hline \end{array} \arrow[dl, "x\mapsto 0,\, y\mapsto b"', "w\mapsto a(a+b)"] \arrow[d, "x,y\mapsto e"', "w\mapsto u"] \arrow[dr, "x\mapsto d,\, y\mapsto 0", "w\mapsto c(c+d)"'] & \\
\begin{array}{|c|}\hline H_1 \\ \hline \mathbb{F}_2[a,b] \\ \hline \deg:1,1 \\ \hline \end{array} & \begin{array}{|c|}\hline H_2 \\ \hline \mathbb{F}_2[e,u]/\langle e^2\rangle \\ \hline \deg:1,2 \\ \hline \end{array} & \begin{array}{|c|}\hline H_3 \\ \hline \mathbb{F}_2[c,d] \\ \hline \deg:1,1 \\ \hline \end{array}
\end{tikzcd} \tag{20}$$

Let $H^*(K_i, \mathbb{F}_2) = \mathbb{F}_2[t_i]$, $\deg(t_i) = 1$. From [1, Corollary II.5.7, p. 69] the restriction

$$\mathrm{res}^{H_2}_{K_3} : \big(H^*(H_2, \mathbb{F}_2) = \mathbb{F}_2[e,u]\langle e^2\rangle\big) \longrightarrow \big(H^*(K_3, \mathbb{F}_2) = \mathbb{F}_2[t_3]\big)$$

is given by $e \mapsto 0$, $u \mapsto t_3^2$. Thus, the restriction $\mathrm{res}^{D_8}_{K_3}$ is given by $x \mapsto 0$, $y \mapsto 0$, $w \mapsto t_3^2$. Using diagrams (19), (20) with the property (C) from Proposition 3.5 we almost completely reveal the cohomology diagram of subgroups of D_8. The equalities

$$\mathrm{res}^{D_8}_{K_3} = \mathrm{res}^{H_2}_{K_3} \circ \mathrm{res}^{D_8}_{H_2} = \mathrm{res}^{H_1}_{K_3} \circ \mathrm{res}^{D_8}_{H_1} = \mathrm{res}^{H_3}_{K_3} \circ \mathrm{res}^{D_8}_{H_3}$$

imply that

- $\mathrm{res}^{H_1}_{K_3} : (H^*(H_1, \mathbb{F}_2) = \mathbb{F}_2[a,b]) \longrightarrow (H^*(K_3, \mathbb{F}_2) = \mathbb{F}_2[t_3])$ is given by $a \mapsto t_3$, $b \mapsto 0$,
- $\mathrm{res}^{H_3}_{K_3} : (H^*(H_3, \mathbb{F}_2) = \mathbb{F}_2[c,d]) \longrightarrow (H^*(K_3, \mathbb{F}_2) = \mathbb{F}_2[t_3])$ is given by $c \mapsto t_3$, $d \mapsto 0$.

$$\begin{tikzcd}
\begin{array}{|c|}\hline H_1 \\ \hline \mathbb{F}_2[a,b] \\ \hline \deg:1,1 \\ \hline \end{array} \arrow[dr, "b\mapsto 0", "a\mapsto t_3"'] & \begin{array}{|c|}\hline H_2 \\ \hline \mathbb{F}_2[e,u]/\langle e^2\rangle \\ \hline \deg:1,2 \\ \hline \end{array} \arrow[d, "u\mapsto t_3^2"', "e\mapsto 0"] & \begin{array}{|c|}\hline H_3 \\ \hline \mathbb{F}_2[c,d] \\ \hline \deg:1,1 \\ \hline \end{array} \arrow[dl, "d\mapsto 0"', "c\mapsto t_3"] \\
 & \begin{array}{|c|}\hline K_3 \\ \hline \mathbb{F}_2[t_3] \\ \hline \deg:1 \\ \hline \end{array} &
\end{tikzcd} \tag{21}$$

The cohomology diagram (19) of subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the part (21) of the D_8 diagram imply that

- $\mathrm{res}^{H_1}_{K_1} : \mathbb{F}_2[a,b] \longrightarrow \mathbb{F}_2[t_1]$ and $\mathrm{res}^{H_1}_{K_2} : \mathbb{F}_2[a,b] \longrightarrow \mathbb{F}_2[t_2]$ are given by
$$(a \mapsto t_1, b \mapsto t_1 \text{ and } a \mapsto 0, b \mapsto t_2) \quad \text{or} \quad (a \mapsto 0, b \mapsto t_1 \text{ and } a \mapsto t_2, b \mapsto t_2),$$
- $\mathrm{res}^{H_3}_{K_4} : \mathbb{F}_2[c,d] \longrightarrow \mathbb{F}_2[t_4]$ and $\mathrm{res}^{H_3}_{K_5} : \mathbb{F}_2[a,b] \longrightarrow \mathbb{F}_2[t_5]$ are given by
$$(c \mapsto t_4, d \mapsto t_4 \text{ and } c \mapsto 0, d \mapsto t_5) \quad \text{or} \quad (c \mapsto 0, d \mapsto t_4 \text{ and } c \mapsto t_5, d \mapsto t_5).$$

Proposition 4.1. *For all $i \neq 3$, $\mathrm{res}^{D_8}_{K_i}(w) = 0$, while $\mathrm{res}^{D_8}_{K_3}(w) \neq 0$.*

Proof. The result follows from the diagram (20) in the following way:

(a) For $i \in \{1,2\}$:
$$\mathrm{res}^{D_8}_{K_i}(w) = \mathrm{res}^{H_1}_{K_i} \circ \mathrm{res}^{D_8}_{H_1}(w) = \mathrm{res}^{H_1}_{K_i}\big(a(a+b)\big) = 0$$
since either $a \mapsto t_i$, $b \mapsto t_i$ or $a \mapsto 0$, $b \mapsto t_i$.

(b) For $i \in \{4,5\}$:
$$\mathrm{res}^{D_8}_{K_i}(w) = \mathrm{res}^{H_3}_{K_i} \circ \mathrm{res}^{D_8}_{H_3}(w) = \mathrm{res}^{H_3}_{K_i}\big(c(c+d)\big) = 0$$
since either $c \mapsto t_i$, $d \mapsto t_i$ or $c \mapsto 0$, $d \mapsto t_i$. $\square$

Corollary 4.2. *The cohomology of the dihedral group D_8 is*

$$H^*(D_8, \mathbb{F}_2) = \mathbb{F}_2[x, y, w]/\langle xy\rangle$$

where

(a) $x \in H^1(D_8, \mathbb{F}_2)$ *and* $\mathrm{res}^{D_8}_{H_1}(x) = 0$,
(b) $y \in H^1(D_8, \mathbb{F}_2)$ *and* $\mathrm{res}^{D_8}_{H_3}(y) = 0$,
(c) $w \in H^1(D_8, \mathbb{F}_2)$ *and* $\mathrm{res}^{D_8}_{K_1}(w) = \mathrm{res}^{D_8}_{K_2}(w) = \mathrm{res}^{D_8}_{K_4}(w) = \mathrm{res}^{D_8}_{K_5}(w) = 0$ *and* $\mathrm{res}^{D_8}_{K_3}(w) \neq 0$.

Assumption. Without lose of generality we can assume that

$$\mathrm{res}^{H_1}_{K_1}(a) = t_1, \qquad \mathrm{res}^{H_1}_{K_1}(b) = t_1, \qquad \mathrm{res}^{H_1}_{K_2}(a) = 0, \qquad \mathrm{res}^{H_1}_{K_2}(b) = t_2. \tag{22}$$

4.3. *The D_8-diagram with coefficients in $\mathbb{Z}$*

Let G be a finite group and R and S rings. A ring homomorphism $\phi : R \to S$ induces a morphism of diagrams (natural transformation of covariant functors) $\Phi : \mathrm{Res}_{(R)} \to \mathrm{Res}_{(S)}$. The morphism Φ on each object $H \in \mathrm{Ob}(\mathfrak{G})$ is defined by the coefficient reduction $\Phi(H) : H^*(H, R) \to H^*(H, S)$ induced by ϕ. Particularly in this section, as a tool for the reconstruction of the diagram $\mathrm{Res}_{(\mathbb{Z})}$, we use the diagram morphism $C : \mathrm{Res}_{(\mathbb{Z})} \to \mathrm{Res}_{(\mathbb{F}_2)}$ induced by the coefficient reduction homomorphism $c : \mathbb{Z} \to \mathbb{F}_2$.

4.3.1. *The $\mathbb{Z}_2 \times \mathbb{Z}_2$-diagram*

The cohomology restriction diagram $\mathrm{Res}_{(\mathbb{F}_2)}$ of the elementary abelian 2-group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is given in the diagram (19). Using the presentation of cohomology $H^*(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z})$ and the homomorphism $H^*(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}) \to H^*(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{F}_2)$ given in Example 3.10 we can reconstruct the restriction diagram $\mathrm{Res}_{(\mathbb{Z})}$:

$$\begin{tikzcd}
& \boxed{\begin{array}{l|l} \mathbb{Z}_2 \times \mathbb{Z}_2 & \mathbb{Z}[\tau_1, \tau_2] \otimes \mathbb{Z}[\mu] \\ \hline \multicolumn{2}{l}{\deg \tau_1 = \deg \tau_2 = 2, \deg \mu = 3} \\ \hline \multicolumn{2}{l}{2\tau_1 = 2\tau_2 = 2\mu = 0,} \\ \hline \multicolumn{2}{l}{\mu^2 = \tau_1 \tau_2 (\tau_1 + \tau_2)} \end{array}} \arrow[dl, "{\tau_1 \mapsto 0, \tau_2 \mapsto \theta_1, \mu \mapsto 0}"'] \arrow[d, "{\tau_1 \mapsto \theta_2, \tau_2 \mapsto 0, \mu \mapsto 0}"'] \arrow[dr, "{\tau_1 \mapsto \theta_3, \tau_2 \mapsto \theta_3, \mu \mapsto 0}"] & \\
\boxed{\begin{array}{l|l} \mathbb{Z}_2 & \mathbb{Z}[\theta_1] \\ \hline \multicolumn{2}{l}{\deg \theta_1 = 2} \\ \hline \multicolumn{2}{l}{2\theta_1 = 0} \end{array}} & \boxed{\begin{array}{l|l} \mathbb{Z}_2 & \mathbb{Z}[\theta_2] \\ \hline \multicolumn{2}{l}{\deg \theta_2 = 2} \\ \hline \multicolumn{2}{l}{2\theta_2 = 0} \end{array}} & \boxed{\begin{array}{l|l} \mathbb{Z}_2 & \mathbb{Z}[\theta_3] \\ \hline \multicolumn{2}{l}{\deg \theta_3 = 2} \\ \hline \multicolumn{2}{l}{2\theta_3 = 0} \end{array}}
\end{tikzcd} \tag{23}$$

4.3.2. *The D_8-diagram*

The cohomology ring $H^*(D_8, \mathbb{Z})$ can be presented by

$$H^*(D_8, \mathbb{Z}) = \mathbb{Z}[\mathcal{X}, \mathcal{Y}, \mathcal{M}, \mathcal{W}]/\mathcal{I} \tag{24}$$

where $\deg \mathcal{X} = \deg \mathcal{Y} = 2$, $\deg \mathcal{M} = 3$, $\deg \mathcal{W} = 4$, and the ideal $\mathcal{I}$ is generated by the relations

$$2\mathcal{X} = 2\mathcal{Y} = 2\mathcal{M} = 4\mathcal{W} = 0, \qquad \mathcal{X}\mathcal{Y} = 0, \qquad \mathcal{M}^2 = \mathcal{W}(\mathcal{X} + \mathcal{Y}). \tag{25}$$

The map $c_* : H^*(D_8, \mathbb{Z}) \longrightarrow H^*(D_8, \mathbb{F}_2)$, induced by the reduction of coefficients $\mathbb{Z} \to \mathbb{F}_2$, is given by

$$\mathcal{X} \mapsto x^2, \qquad \mathcal{Y} \mapsto y^2, \qquad \mathcal{M} \mapsto w(x + y), \qquad \mathcal{W} \mapsto w^2. \tag{26}$$

For the details consult [15, Theorem 5.2, p. 27].

Now using:

- the D_8 restriction diagram (20) and (21) with $\mathbb{F}_2$ coefficients,
- the $\mathbb{Z}_2 \times \mathbb{Z}_2$ restriction diagrams (23) with $\mathbb{Z}$ coefficients,
- the presentation of $H^*(D_8, \mathbb{Z})$ given in (24),
- the homomorphism $c_* : H^*(D_8, \mathbb{Z}) \to H^*(D_8, \mathbb{F}_2)$ described in (26),

we can reconstruct the restriction diagram of D_8 with $\mathbb{Z}$ coefficients.

$$
\begin{tikzcd}
& \begin{array}{|c|c|}\hline D_8 & \mathbb{Z}[\mathcal{X},\mathcal{Y},\mathcal{M},\mathcal{W}] \\ \hline \multicolumn{2}{|l|}{\deg: 2,2,3,4} \\ \hline \multicolumn{2}{|l|}{2\mathcal{X}=2\mathcal{Y}=2\mathcal{M}=4\mathcal{W}=0,} \\ \hline \multicolumn{2}{|l|}{\mathcal{X}\mathcal{Y}=0,\ \mathcal{M}^2=\mathcal{W}(\mathcal{X}+\mathcal{Y})} \\ \hline \end{array} \arrow[dl, "{\mathcal{X}\mapsto 0,\ \mathcal{Y}\mapsto\beta,\ \mathcal{M}\mapsto\mu,\ \mathcal{W}\mapsto\alpha(\alpha+\beta)}"'] \arrow[d, "{\mathcal{Y}\mapsto 2U,\ \mathcal{W}\mapsto U^2}"', "{\mathcal{X}\mapsto 2U,\ \mathcal{M}\mapsto 0}"] \arrow[dr, "{\mathcal{X}\mapsto\delta,\ \mathcal{Y}\mapsto 0,\ \mathcal{M}\mapsto\eta,\ \mathcal{W}\mapsto\gamma(\gamma+\delta)}"] & \\
\begin{array}{|c|c|}\hline H_1 & \mathbb{Z}[\alpha,\alpha+\beta,\mu] \\ \hline \multicolumn{2}{|l|}{\deg: 2,2,3,} \\ \hline \multicolumn{2}{|l|}{2\alpha=2\beta=2\mu=0,} \\ \hline \multicolumn{2}{|l|}{\mu^2=\alpha\beta(\alpha+\beta)} \\ \hline \end{array} \arrow[dr, "{\alpha\mapsto\theta_3,\ \beta\mapsto 0,\ \mu\mapsto 0}"] & \begin{array}{|c|c|}\hline H_2 & \mathbb{Z}[U] \\ \hline \multicolumn{2}{|l|}{\deg: 2} \\ \hline \multicolumn{2}{|l|}{4U=0,} \\ \hline \end{array} \arrow[d, "U\mapsto\theta_3"'] & \begin{array}{|c|c|}\hline H_3 & \mathbb{Z}[\gamma,\gamma+\delta,\eta] \\ \hline \multicolumn{2}{|l|}{\deg: 2,2,3,} \\ \hline \multicolumn{2}{|l|}{2\gamma=2\delta=2\eta=0,} \\ \hline \multicolumn{2}{|l|}{\eta^2=\gamma\delta(\gamma+\delta)} \\ \hline \end{array} \arrow[dl, "{\gamma\mapsto\theta_3,\ \delta\mapsto 0,\ \eta\mapsto 0}"'] \\
& \begin{array}{|c|c|c|}\hline K_3 & \mathbb{Z}[\theta_3] & \deg\theta_3=2 \\ \hline \end{array} &
\end{tikzcd}
\tag{27}
$$

Now the determination of the diagram morphism $C:\mathrm{Res}_{(\mathbb{Z})}\to\mathrm{Res}_{(\mathbb{F}_2)}$ induced by the coefficient reduction homomorphism $c:\mathbb{Z}\to\mathbb{F}_2$ is just a routine exercise.

5. $\mathbf{Index}_{D_8,\mathbb{F}_2} S(R_4^{\oplus j})$

In this section we show the following equality:

$$\mathrm{Index}_{D_8,\mathbb{F}_2} S\big(R_4^{\oplus j}\big)=\mathrm{Index}^{3j+1}_{D_8,\mathbb{F}_2} S\big(R_4^{\oplus j}\big)=\big\langle w^j y^j\big\rangle.$$

The D_8-representation $R_4^{\oplus j}$ can be decomposed into a sum of irreducibles in the following way

$$R_4=(V_{-+}\oplus V_{+-})\oplus V_{--}\quad\Rightarrow\quad R_4^{\oplus j}=(V_{-+}\oplus V_{+-})^{\oplus j}\oplus V_{--}^{\oplus j}$$

where $V_{-+}\oplus V_{+-}$ is a 2-dimensional irreducible D_8 -representation. Since in this section $\mathbb{F}_2$ coefficients are assumed, Proposition 3.12 implies that computing the indexes of the spheres $S(V_{-+}\oplus V_{+-})$ and $S(V_{--})$ suffices. The strategy employed uses Proposition 3.7 and the following particular facts.

A. Let $X=S(T)$ for some D_8-representation T. Then the E_2-term of the Serre spectral sequence associated to $\mathrm{E}D_8\times_{D_8}X$ is

$$E_2^{p,q}=H^p(D_8,\mathbb{F}_2)\otimes H^q(X,\mathbb{F}_2).\tag{28}$$

The local coefficients are trivial since X is a sphere and the coefficients are $\mathbb{F}_2$. Since only $\partial_{\dim T,\mathbb{F}_2}$ may be $\neq 0$, from the multiplicative property of the spectral sequence it follows that

$$\mathrm{Index}_{D_8,\mathbb{F}_2}X=\big\langle\partial^{0,\dim V-1}_{\dim V,\mathbb{F}_2}(1\otimes l)\big\rangle$$

where $l\in H^{\dim V-1}(X,\mathbb{F}_2)$ is the generator. Therefore, $\mathrm{Index}_{D_8,\mathbb{F}_2}(X)=\mathrm{Index}^{\dim V+1}_{D_8,\mathbb{F}_2}(X)$.

B. For any subgroup H of D_8, with some abuse of notation,

$$\Gamma^{\dim V,0}_{\dim V}\circ\partial^{0,\dim V-1}_{\dim V,\mathbb{F}_2}(1\otimes l)=\partial^{0,\dim V-1}_{\dim V,\mathbb{F}_2}\circ\Gamma^{0,\dim V-1}_{\dim V}(1\otimes l),\tag{29}$$

where Γ denotes the restriction morphism of Serre spectral sequences introduced in Proposition 3.5(D). Therefore, for every subgroup H of D_8 we get

$$\mathrm{Index}_{D_8,\mathbb{F}_2}X=\langle a\rangle,\qquad\mathrm{Index}_{H,\mathbb{F}_2}X=\langle a_H\rangle\quad\Longrightarrow\quad\mathrm{res}^G_K(a)=a_H.$$

In particular, if $a_H\neq 0$ then $a\neq 0$.

Our computation of $\mathrm{Index}_{D_8,\mathbb{F}_2}X$ for $X=S(V_{-+}\oplus V_{+-})$ and $X=S(V_{--})$ has two steps:

- compute $\mathrm{Index}_{H,\mathbb{F}_2}X=\langle a_H\rangle$ for all proper subgroups H of D_8,
- search for an element $a\in H^*(D_8,\mathbb{F}_2)$ such that for every computed a_H

$$\mathrm{res}^G_K(a)=a_H.$$

5.1. $\mathrm{Index}_{D_8,\mathbb{F}_2}S(V_{-+}\oplus V_{+-})=\langle w\rangle$

Proposition 3.13 and the properties of the action of D_8 on $V_{-+}\oplus V_{+-}$ provide the following information:

$$\mathrm{Index}_{H_1,\mathbb{F}_2}S(V_{-+}\oplus V_{+-})=\begin{cases}\langle a(a+b)\rangle & \text{or}\\ \langle b(a+b)\rangle & \text{or}\\ \langle ab\rangle.\end{cases}$$

Since initially we do not know which of the possible generators a, b, $a+b$ of $\mathbb{F}_2[a,b]$ correspond to the generators ε_1, ε_2, $\varepsilon_1\varepsilon_2$, we have to take all three possibilities into account. Similarly:

$$\mathrm{Index}_{H_3,\mathbb{F}_2} S(V_{-+}\oplus V_{+-}) = \begin{cases} \langle c(c+d)\rangle & \text{or} \\ \langle d(c+d)\rangle & \text{or} \\ \langle cd\rangle. \end{cases}$$

Furthermore,

$$\begin{array}{lll} \varepsilon_1 \text{ acts trivially on } V_{+-} & \Rightarrow & \mathrm{Index}_{K_1,\mathbb{F}_2} S(V_{-+}\oplus V_{+-}) = 0, \\ \varepsilon_2 \text{ acts trivially on } V_{-+} & \Rightarrow & \mathrm{Index}_{K_2,\mathbb{F}_2} S(V_{-+}\oplus V_{+-}) = 0, \\ \sigma \text{ acts trivially on } \{(x,x)\in V_{-+}\oplus V_{+-}\} & \Rightarrow & \mathrm{Index}_{K_4,\mathbb{F}_2} S(V_{-+}\oplus V_{+-}) = 0, \\ \varepsilon_1\varepsilon_2\sigma \text{ acts trivially on } \{(x,-x)\in V_{-+}\oplus V_{+-}\} & \Rightarrow & \mathrm{Index}_{K_5,\mathbb{F}_2} S(V_{-+}\oplus V_{+-}) = 0. \end{array}$$

The only nonzero element of $H^2(D_8,\mathbb{F}_2)$ satisfying all requirements of commutativity with restrictions is w. Hence,

$$\mathrm{Index}_{D_8,\mathbb{F}_2} S(V_{-+}\oplus V_{+-}) = \langle w\rangle. \tag{30}$$

Remark 5.1. The side information coming from this computation is that generators ε_1 and ε_2 of the group H_1 correspond to generators a and $a+b$ in the cohomology ring $H^*(H_1,\mathbb{F}_2)$.

5.2. $\mathrm{Index}_{D_8,\mathbb{F}_2} S(V_{--}) = \langle y\rangle$

Again, V_{--} is a concrete D_8-representation, and from Proposition 3.13:

$$\mathrm{Index}_{H_1,\mathbb{F}_2} S(V_{--}) = \begin{cases} \langle a+b\rangle, & \text{or} \\ \langle a+(a+b)\rangle, & \text{or} \\ \langle b+(a+b)\rangle. \end{cases}$$

Again, we allow all three possibilities since we do not know the correspondence between generators of H_1 and the chosen generators of $H^*(H_q,\mathbb{F}_2)$. Furthermore, since K_1 and K_2 act nontrivially on V_{--},

$$\mathrm{Index}_{K_1,\mathbb{F}_2} S(V_{--}) = \langle t_1\rangle, \qquad \mathrm{Index}_{K_2,\mathbb{F}_2} S(V_{--}) = \langle t_2\rangle.$$

On the other hand, H_3 acts trivially on $S(V_{--})$ and so

$$\mathrm{Index}_{H_3,\mathbb{F}_2} S(V_{--}) = 0.$$

By commutativity of the restriction diagram, or since the groups K_3, K_4 and K_5 act trivially on $V_{(1,1)}$, it follows that

$$\mathrm{Index}_{K_3,\mathbb{F}_2} S(V_{--}) = \mathrm{Index}_{K_4,\mathbb{F}_2} S(V_{--}) = \mathrm{Index}_{K_5,\mathbb{F}_2} S(V_{--}) = 0.$$

The only element satisfying the commutativity requirements is $y\in H^1(D_8,\mathbb{F}_2)$, so

$$\mathrm{Index}_{D_8,\mathbb{F}_2} S(V_{--}) = \langle y\rangle. \tag{31}$$

Remark 5.2. From the previous remark the fact that $\mathrm{Index}_{H_1,\mathbb{F}_2} S(V_{--}) = \langle b\rangle = \langle a+(a+b)\rangle$ follows directly. Alternatively, Eq. (31) is a consequence of (11) and (20).

5.3. $\mathrm{Index}_{D_8,\mathbb{F}_2} S(R_4^{\oplus j}) = \langle y^j w^j\rangle$

From Proposition 3.12 we get that

$$\mathrm{Index}_{D_8,\mathbb{F}_2} S\big(R_4^{\oplus j}\big) = \mathrm{Index}_{D_8,\mathbb{F}_2} S\big((V_{-+}\oplus V_{+-})^{\oplus j}\oplus V_{--}^{\oplus j}\big) = \big\langle y^j w^j\big\rangle.$$

Remark 5.3. In the same way we can compute that

$$\mathrm{Index}_{D_8,\mathbb{F}_2}(U_2) = \langle x\rangle. \tag{32}$$

Therefore $\mathrm{Index}_{D_8,\mathbb{F}_2}(U_2\oplus R_4^{\oplus j}) = 0$. This means that on the join CS/TM scheme the Fadell–Husseini index theory with $\mathbb{F}_2$ coefficients yields no obstruction to the existence of the equivariant map in question.

6. $\mathbf{Index}_{D_8,\mathbb{Z}} S(R_4^{\oplus j})$

In this section we show that

$$\mathrm{Index}_{D_8,\mathbb{Z}} S\big(R_4^{\oplus j}\big) = \mathrm{Index}^{3j+1}_{D_8,\mathbb{Z}} S\big(R_4^{\oplus j}\big) = \begin{cases} \langle \mathcal{Y}^{\frac{j}{2}}\mathcal{W}^{\frac{j}{2}}\rangle, & \text{for } j \text{ even}, \\ \langle \mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j-1}{2}}\mathcal{M}, \mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}}\rangle, & \text{for } j \text{ odd}. \end{cases} \tag{33}$$

6.1. The case when j is even

In the case when j is even the group D_8 acts trivially on the cohomology $H^*(S(R_4^{\oplus j}),\mathbb{Z})$. Then the E_2 -term of the Serre spectral sequence associated to $\mathrm{E}D_8\times_{D_8} S(R_4^{\oplus j})$ is a tensor product

$$E_2^{p,q}=H^p(D_8,\mathbb{Z})\otimes H^q\big(S\big(R_4^{\oplus j}\big),\mathbb{Z}\big).$$

Since only $\partial_{3j,\mathbb{Z}}$ may be $\neq 0$, the multiplicative property of the spectral sequence implies that

$$\mathrm{Index}_{D_8,\mathbb{Z}}S\big(R_4^{\oplus j}\big)=\mathrm{Index}_{D_8,\mathbb{Z}}^{\dim V+1}S\big(R_4^{\oplus j}\big)=\big\langle\partial_{3j,\mathbb{Z}}^{0,3j-1}(1\otimes l)\big\rangle$$

where $l\in H^{3j-1}(S(R_4^{\oplus j}),\mathbb{Z})$ is a generator. The coefficient reduction morphism $c:\mathbb{Z}\to\mathbb{F}_2$ induces a morphism of Serre spectral sequences (Proposition 3.5(E)(3)) associated with the Borel construction of the sphere $S(R_4^{\oplus j})$. Thus,

$$c_*\big(\partial_{3j,\mathbb{Z}}^{0,3j-1}(1\otimes l)\big)=\partial_{3j,\mathbb{F}_2}^{0,3j-1}\big(c_*(1\otimes l)\big)\in H^{3j}(D_8,\mathbb{F}_2)$$

and according to the result of the previous section

$$c_*\big(\partial_{3j,\mathbb{Z}}^{0,3j-1}(1\otimes l)\big)=y^jw^j.$$

Now, from the description of the map $c_*:H^*(D_8,\mathbb{Z})\longrightarrow H^*(D_8,\mathbb{F}_2)$ in (26) follows the statement for j even.

6.2. The case when j is odd

The group D_8 acts nontrivially on the cohomology $H^*(S(R_4^{\oplus j}),\mathbb{Z})$. Precisely, the D_8-module $\mathcal{Z}=H^{3j-1}(S(R_4^{\oplus j}),\mathbb{Z})$ is a nontrivial D_8-module and for $z\in\mathcal{Z}$:

$$\varepsilon_1\cdot z=z,\qquad \varepsilon_2\cdot z=z,\qquad \sigma\cdot z=-z.$$

Then the E_2-term of the Serre spectral sequence associated to $\mathrm{E}D_8\times_{D_8} S(R_4^{\oplus j})$ is not a tensor product and

$$E_2^{p,q}=H^p\big(D_8,H^q\big(S\big(R_4^{\oplus j}\big),\mathbb{Z}\big)\big)=\begin{cases}H^p(D_8,\mathbb{Z}), & q=0,\\ H^p(D_8,\mathcal{Z}), & q=3j-1,\\ 0, & q\neq 0,3j-1.\end{cases}\tag{34}$$

To compute the index in this case we have to study the $H^*(D_8,\mathbb{Z})$-module structure of $H^*(D_8,\mathcal{Z})$. This module structure is completely described in [15, Theorem 5.11(a)]. The necessary information relevant for the computation of the index is summarized in the following proposition.

Proposition 6.1.

(A) $2\cdot H^*(D_8,\mathcal{Z})=0$,
(B) *$H^*(D_8,\mathcal{Z})$ is generated as a $H^*(D_8,\mathbb{Z})$-module by three elements ρ_1, ρ_2, ρ_3 of degree* 1, 2, 3 *such that*

$$\rho_1\cdot\mathcal{Y}=0,\qquad \rho_2\cdot\mathcal{X}=0,\qquad \rho_3\cdot\mathcal{X}=0$$

and

$$c_*(\rho_1)=x,\qquad c_*(\rho_2)=y^2,\qquad c_*(\rho_3)=yw$$

where c_* is the map induced by the D_8-modulo map $\mathcal{Z}\to\mathcal{Z}/2\mathcal{Z}\cong\mathbb{F}_2$.

Thus, the index is given by

$$\mathrm{Index}_{D_8,\mathbb{Z}}S\big(R_4^{\oplus j}\big)=\big\langle\partial_{3j,\mathbb{Z}}^{1,3j-1}(\rho_1),\partial_{3j,\mathbb{Z}}^{2,3j-1}(\rho_2),\partial_{3j,\mathbb{Z}}^{3,3j-1}(\rho_3)\big\rangle.$$

The morphism C from spectral sequence (34) to spectral sequence (28) induced by the reduction homomorphism $\mathbb{Z}\to\mathbb{F}_2$ implies that:

$$\begin{aligned}
&C\big(\partial_{3j,\mathbb{Z}}^{1,3j-1}(\rho_1)\big)=\partial_{3j,\mathbb{F}_2}^{1,3j-1}\big(c_*(\rho_1)\big)=\partial_{3j,\mathbb{F}_2}^{1,3j-1}(x)=0,\\
&C\big(\partial_{3j,\mathbb{Z}}^{2,3j-1}(\rho_2)\big)=\partial_{3j,\mathbb{F}_2}^{2,3j-1}\big(c_*(\rho_2)\big)=\partial_{3j,\mathbb{F}_2}^{2,3j-1}\big(y^2\big)=y^{j+2}w^j=y^{j+1}w^{j-1}(y+x)w,\\
&C\big(\partial_{3j,\mathbb{Z}}^{3,3j-1}(\rho_3)\big)=\partial_{3j,\mathbb{F}_2}^{3,3j-1}\big(c_*(\rho_3)\big)=\partial_{3j,\mathbb{F}_2}^{3,3j-1}(yw)=y^{j+1}w^{j+1}.
\end{aligned}\tag{35}$$

The sequence of D_8 inclusion maps

$$S\big(R_4^{\oplus(j-1)}\big)\subset S\big(R_4^{\oplus j}\big)\subset S\big(R_4^{\oplus(j+1)}\big)$$

provides (Proposition 3.2) a sequence of inclusions:

$$\big\langle\mathcal{Y}^{\frac{j-1}{2}}\mathcal{W}^{\frac{j-1}{2}}\big\rangle=\mathrm{Index}_{D_8,\mathbb{Z}}S\big(R_4^{\oplus(j-1)}\big)\supseteq\mathrm{Index}_{D_8,\mathbb{Z}}S\big(R_4^{\oplus j}\big)\supseteq\mathrm{Index}_{D_8,\mathbb{Z}}S\big(R_4^{\oplus(j+1)}\big)=\big\langle\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}}\big\rangle.\tag{36}$$

The relations (35), (36) and (26), along with Proposition 6.1 imply that for j odd:

$$\mathrm{Index}_{D_8,\mathbb{Z}}S\big(R_4^{\oplus j}\big)=\big\langle\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j-1}{2}}\mathcal{M},\mathcal{Y}^{\frac{j+1}{2}}\mathcal{W}^{\frac{j+1}{2}}\big\rangle.$$

Remark 6.2. The index $\mathrm{Index}_{D_8,\mathbb{Z}} S(U_k \times R_4^{\oplus j})$ appearing in the join test map scheme can now be computed. From Example 3.4 and the restriction diagram (27) it follows that

$$\mathrm{Index}_{D_8,\mathbb{Z}} S(U_k) = \mathrm{Index}_{D_8,\mathbb{Z}} D_8/H_1 = \ker\big(\mathrm{res}^{D_8}_{H_1} : H^*(D_8,\mathbb{Z}) \to H^*(H_1,\mathbb{Z})\big) = \langle \mathcal{X} \rangle.$$

The inclusions

$$\mathrm{Index}_{D_8,\mathbb{Z}} S\big(U_k \times R_4^{\oplus j}\big) \subseteq \mathrm{Index}_{D_8,\mathbb{Z}} S\big(R_4^{\oplus j}\big) \quad \text{and} \quad \mathrm{Index}_{D_8,\mathbb{Z}} S\big(U_k \times R_4^{\oplus j}\big) \subseteq \mathrm{Index}_{D_8,\mathbb{Z}} S(U_k)$$

imply that

$$\mathrm{Index}_{D_8,\mathbb{Z}} S\big(U_k \times R_4^{\oplus j}\big) \subseteq \mathrm{Index}_{D_8,\mathbb{Z}} S\big(R_4^{\oplus j}\big) \cap \mathrm{Index}_{D_8,\mathbb{Z}} S(U_k) = \{0\}.$$

Thus, as in the case of $\mathbb{F}_2$ coefficients, the Fadell–Husseini index theory with $\mathbb{Z}$ coefficients on the join CS/TM scheme does not lead to any obstruction to the existence of the equivariant map in question.

7. $\mathrm{Index}_{D_8,\mathbb{F}_2} S^d \times S^d$

This section is devoted to the proof of the equality

$$\mathrm{Index}_{D_8,\mathbb{F}_2} S^d \times S^d = \big\langle \pi_{d+1}, \pi_{d+2}, w^{d+1} \big\rangle. \tag{37}$$

The index will be determined by the explicit computation of the Serre spectral sequence associated with the Borel construction

$$S^d \times S^d \to \mathrm{E}D_8 \times_{D_8} \big(S^d \times S^d\big) \to \mathrm{B}D_8.$$

The group D_8 acts nontrivially on the cohomology of the fibre, and therefore the spectral sequence has nontrivial local coefficients. The E_2-term is given by

$$E_2^{p,q} = H^p\big(\mathrm{B}D_8, \mathcal{H}^q\big(S^d \times S^d, \mathbb{F}_2\big)\big) = H^p\big(D_8, H^q\big(S^d \times S^d, \mathbb{F}_2\big)\big) = \begin{cases} H^p(D_8, \mathbb{F}_2), & q = 0, 2d, \\ H^p(D_8, \mathbb{F}_2[D_8/H_1]), & q = d, \\ 0, & q \neq 0, d, 2d. \end{cases} \tag{38}$$

The nontriviality of local coefficients appears in the d-th row of the spectral sequence.

7.1. *The d-th row as an $H^*(D_8, \mathbb{F}_2)$-module*

Since the spectral sequence is an $H^*(D_8,\mathbb{F}_2)$-module and the differentials are module maps we need to understand the $H^*(D_8,\mathbb{F}_2)$-module structure of the E_2-term.

Proposition 7.1. $H^*(D_8, \mathbb{F}_2[D_8/H_1]) \cong_{\mathrm{ring}} H^*(H_1, \mathbb{F}_2)$.

Proof. Here $H_1 = \langle \varepsilon_1, \varepsilon_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is a maximal (normal) subgroup of index 2 in D_8. The statement follows from Shapiro's lemma [6, Proposition 6.2, p. 73] and the fact that when $[G : H] < \infty$, then there is an isomorphism of G-modules $\mathrm{Coind}_H^G M \cong \mathrm{Ind}_H^G M$. □

The first information about the $H^*(D_8,\mathbb{F}_2)$-module structure on $H^*(D_8, \mathbb{F}_2[D_8/H_1])$, as well as the method for revealing the complete structure, comes from the following proposition.

Proposition 7.2. *We have $x \cdot H^*(D_8, \mathbb{F}_2[D_8/H_1]) = 0$ for the nonzero element $x \in H^1(D_8,\mathbb{F}_2)$ that is characterized by $\mathrm{res}^{D_8}_{H_1}(x) = 0$.*

Proof. The isomorphism $H^*(D_8, \mathbb{F}_2[D_8/H_1]) \cong_{\mathrm{ring}} H^*(H_1,\mathbb{F}_2)$ induced by Shapiro's lemma [6, Proposition 6.2, p. 73] carries the $H^*(D_8,\mathbb{F}_2)$-module structure to $H^*(H_1,\mathbb{F}_2)$ via the restriction homomorphism $\mathrm{res}^{D_8}_{H_1} : H^*(D_8,\mathbb{F}_2) \to H^*(H_1,\mathbb{F}_2)$. In this way the complete $H^*(D_8,\mathbb{F}_2)$-module structure is given on $H^*(D_8, \mathbb{F}_2[D_8/H_1])$. In particular, since $\mathrm{res}^{D_8}_{H_1}(x) = 0$, the proposition is proved. □

Corollary 7.3. $\mathrm{Index}^{d+2}_{D_8,\mathbb{F}_2} S^d \times S^d = \mathrm{im}(\partial_{d+1} : E_{d+1}^{*,d} \to E_{d+1}^{*+d+1,0}) \subseteq y \cdot H^*(D_8,\mathbb{F}_2)$.

Proof. Let $\alpha \in E_{d+1}^{*,d}$ and $\partial_{d+1}(\alpha) \notin y \cdot H^*(D_8,\mathbb{F}_2)$. Then $x \cdot \partial_{d+1}(\alpha) \neq 0$. Since ∂_{d+1} is a $H^*(D_8,\mathbb{F}_2)$-module map and x acts trivially on $H^*(D_8, \mathbb{F}_2[D_8/H_1])$, as indicated by Proposition 7.2, there is a contradiction

$$0 = \partial_{d+1}(x \cdot \alpha) = x \cdot \partial_{d+1}(\alpha) \neq 0. \qquad □$$

Proposition 7.4. *$H^*(D_8, \mathbb{F}_2[D_8/H_1])$ is generated as an $H^*(D_8,\mathbb{F}_2)$-module by*

$$H^0\big(D_8, \mathbb{F}_2[D_8/H_1]\big) \quad \textit{and} \quad H^1\big(D_8, \mathbb{F}_2[D_8/H_1]\big).$$

Proof. We already observed that Shapiro's lemma $H^*(D_8, \mathbb{F}_2[D_8/H_1]) \cong_{\mathrm{ring}} H^*(H_1,\mathbb{F}_2)$ carries the $H^*(D_8,\mathbb{F}_2)$-module structure to $H^*(H_1,\mathbb{F}_2)$ via the restriction homomorphism $\mathrm{res}^{D_8}_{H_1} : H^*(D_8,\mathbb{F}_2) \to H^*(H_1,\mathbb{F}_2)$. Thus $H^*(H_1,\mathbb{F}_2)$ as an $H^*(D_8,\mathbb{F}_2)$-module is generated by $1 \in H^0(H_1,\mathbb{F}_2)$ together with $a \in H^1(H_1,\mathbb{F}_2)$. □

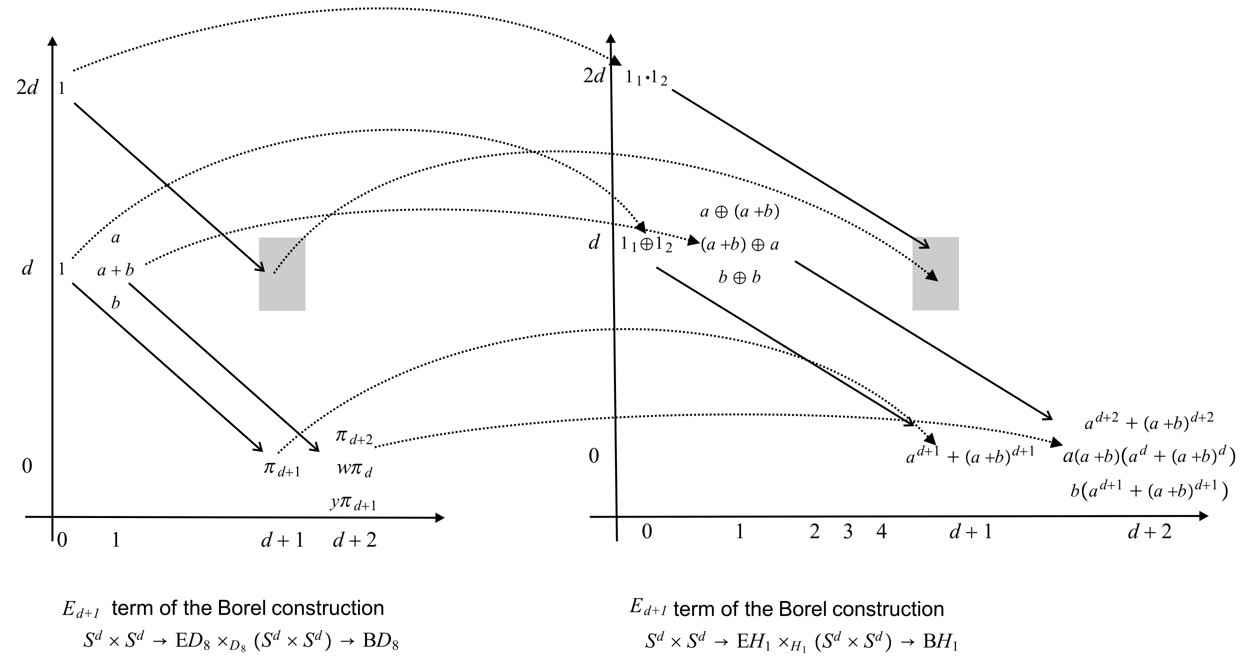

Fig. 2. The morphism of spectral sequences.

7.2. $\mathrm{Index}^{d+2}_{D_8,\mathbb{F}_2} S^d \times S^d = \langle \pi_{d+1}, \pi_{d+2} \rangle$

The index by definition is

$$\mathrm{Index}^{d+2}_{D_8,\mathbb{F}_2} S^d \times S^d = \mathrm{im}\big(\partial_{d+1} : E^{*,d}_{d+1} \to E^{*+d+1,0}_{d+1}\big) = \mathrm{im}\big(\partial_{d+1} : H^*\big(D_8, \mathbb{F}_2[D_8/H_1]\big) \to H^{*+d+1}(D_8, \mathbb{F}_2)\big).$$

From Proposition 7.4 this image is generated as a module by the ∂_{d+1}-images of $H^0(D_8, \mathbb{F}_2[D_8/H_1])$ and of $H^1(D_8, \mathbb{F}_2[D_8/H_1])$. The ∂_{d+1} image is computed by applying restriction properties given in Proposition 3.5 to the subgroup H_1. With the identification of $H^*(D_8, \mathbb{F}_2[D_8/H_1])$ given by Shapiro's lemma the morphism of spectral sequences of Borel constructions induced by restriction is specified in Fig. 2. Also,

$$\mathrm{Index}^{d+2}_{D_8,\mathbb{F}_2} S^d \times S^d = \big\langle \partial^{D_8}_{d+1}(1), \partial^{D_8}_{d+1}(a), \partial^{D_8}_{d+1}(b), \partial^{D_8}_{d+1}(a+b) \big\rangle.$$

To simplify notation let $\rho_d := a^d + (a+b)^{d+1}$. Then from

$$1 \quad \xmapsto{\mathrm{res}^{D_8}_{H_1}} \quad 1_1 \oplus 1_2 \quad \xmapsto{\partial^{H_1}_{d+1}} \quad \rho_{d+1}$$

$$\{a, a+b, b\} \xmapsto{\mathrm{res}^{D_8}_{H_1}} \left\{ \begin{array}{c} a \oplus (a+b) \\ (a+b) \oplus a \\ b \oplus b \end{array} \right\} \xmapsto{\partial^{H_1}_{d+1}} \big\{\rho_{d+2}, a(a+b)\rho_d, b\rho_{d+1}\big\}$$

it follows that

$$\mathrm{res}^{D_8}_{H_1}\big(\big\{\partial^{D_8}_{d+1}(1), \partial^{D_8}_{d+1}(a), \partial^{D_8}_{d+1}(b), \partial^{D_8}_{d+1}(a+b)\big\}\big) = \big\{\rho_{d+2}, a(a+b)\rho_d, b\rho_{d+1}\big\}.$$

The formula

$$\rho_{d+2} = a^{d+2} + (a+b)^{d+2} = (a+a+b)\left(\rho_{d+1} + a(a+b)\sum_{i=0}^{d-1} a^i (a+b)^{d-1-i}\right)$$

$$= b\rho_{d+1} + a(a+b)(a+a+b)\sum_{i=0}^{d-1} a^i (a+b)^{d-1-i} = b\rho_{d+1} + a(a+b)\rho_d$$

together with Remark 1.3 and the knowledge of the restriction $\mathrm{res}^{D_8}_{H_1}$ implies that

$$\mathrm{res}^{D_8}_{H_1}(\pi_d) = \rho_d.$$

Therefore, there exist $x\alpha, x\beta, x\gamma, x\delta \in \ker(\mathrm{res}^{D_8}_{H_1})$ such that

$$\partial^{D_8}_{d+1}(1) = \pi_{d+1} + x\alpha$$

and

$$\left\{\partial^{D_8}_{d+1}(a), \partial^{D_8}_{d+1}(b), \partial^{D_8}_{d+1}(a+b)\right\} = \{\pi_{d+2} + x\beta, y\pi_{d+1} + x\gamma, w\pi_d + x\delta\}.$$

Since y divides π_d, Proposition 7.2 implies that $\alpha = \beta = \gamma = \delta = 0$, and

$$\begin{aligned}\mathrm{Index}^{d+2}_{D_8,\mathbb{F}_2} S^d \times S^d &= \left\langle \partial^{D_8}_{d+1}(1), \partial^{D_8}_{d+1}(a), \partial^{D_8}_{d+1}(b), \partial^{D_8}_{d+1}(a+b)\right\rangle \\ &= \langle \pi_{d+1}, \pi_{d+2},\ y\pi_{d+1}, w\pi_d\rangle \\ &= \langle \pi_{d+1}, \pi_{d+2}\rangle.\end{aligned}$$

Remark 7.5. The property that the concretely described homomorphism

$$\mathrm{res}^{D_8}_{H_1} : H^*\big(D_8, \mathbb{F}_2[D_8/H_1]\big) \to H^*\big(H_1, \mathbb{F}_2[D_8/H_1]\big)$$

is injective holds more generally [10, Lemma on p. 187].

7.3. $\mathrm{Index}_{D_8,\mathbb{F}_2} S^d \times S^d = \langle \pi_{d+1}, \pi_{d+2}, w^{d+1}\rangle$

In the previous section we described the differential $\partial^{D_8}_{d+1}$ of the Serre spectral sequence associated with the Borel construction

$$S^d \times S^d \to \mathrm{E}D_8 \times_{D_8} \big(S^d \times S^d\big) \to \mathrm{B}D_8.$$

The only remaining, possibly non-trivial, differential is $\partial^{D_8}_{2d+1}$.

The following proposition describing $E^{*,2d}_{2d+1}$ can be obtained from Fig. 2.

Proposition 7.6. $E^{*,2d}_{2d+1} = \ker(\partial^{D_8}_{d+1} : E^{*,2d}_{d+1} \to E^{*+d+1,d}_{d+1}) = x \cdot H^*(D_8, \mathbb{F}_2)$.

Proof. The restriction property from Proposition 3.5(D), applied to the element $1 \in E^{0,2d}_{d+1} = H^*(D_8, \mathbb{F}_2)$ implies that $\partial^{D_8}_{d+1}(1) \neq 0$. Proposition 7.2, together with the fact that multiplication by y and by w in $H^*(D_8, \mathbb{F}_2[D_8/H_1])$ is injective, implies that $\ker(\partial^{D_8}_{d+1} : E^{*,2d}_{d+1} \to E^{*+d+1,d}_{d+1}) = xH^*(D_8, \mathbb{F}_2)$. $\square$

The description of the differential $\partial^{D_8}_{2d+1} : E^{*,2d}_{2d+1} \to E^{*+2d+1,0}_{2d+1}$ comes in an indirect way. There is a D_8-equivariant map

$$S^d \times S^d \to S^d * S^d \approx S\big((V_{+-} \oplus V_{-+})^{\oplus(d+1)}\big)$$

given by $S^d \times S^d \ni (t_1, t_2) \mapsto \frac{1}{2}t_1 + \frac{1}{2}t_2 \in S^d * S^d$. The result of Section 5.1 and the basic property of the index (Proposition 3.2) imply that

$$\mathrm{Index}_{D_8,\mathbb{F}_2} S^d \times S^d \supseteq \mathrm{Index}_{D_8,\mathbb{F}_2} S\big((V_{+-} \oplus V_{-+})^{\oplus(d+1)}\big) = \big\langle w^{d+1}\big\rangle.$$

Thus $w^{d+1} \in \mathrm{Index}_{D_8,\mathbb{F}_2} S^d \times S^d$. Since by Corollary 7.3 $w^{d+1} \notin \mathrm{Index}^{d+1}_{D_8,\mathbb{F}_2} S^d \times S^d$ it follows that

$$w^{d+1} \in \mathrm{im}\big(\partial^{D_8}_{2d+1} : E^{1,2d}_{2d+1} \to E^{2d+2,0}_{2d+1}\big).$$

But the only nonzero element in $E^{1,2d}_{2d+1}$ is x, therefore

$$\partial^{D_8}_{2d+1}(x) = w^{d+1}.$$

This concludes the proof of Eq. (37).

8. $\mathrm{Index}_{D_8,\mathbb{Z}} S^d \times S^d$

Let $\Pi_0 = 0$, $\Pi_1 = \mathcal{Y}$ and $\Pi_{n+2} = \mathcal{Y}\Pi_{n+1} + \mathcal{W}\Pi_n$, for $n \geqslant 0$, be a sequence of polynomials in $H^*(D_8, \mathbb{Z})$. This section is devoted to the proof of the equality

$$\mathrm{Index}^{d+2}_{D_8,\mathbb{Z}} S^d \times S^d = \begin{cases} \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}}, \mathcal{M}\Pi_{\frac{d}{2}}\rangle, & \text{for } d \text{ even}, \\ \langle \Pi_{\frac{d+1}{2}}, \Pi_{\frac{d+3}{2}}\rangle, & \text{for } d \text{ odd}. \end{cases} \tag{39}$$

The index is determined by the explicit computation of the E_{d+2}-term of the Serre spectral sequence associated with the Borel construction

$$S^d \times S^d \to \mathrm{E}D_8 \times_{D_8} \big(S^d \times S^d\big) \to \mathrm{B}D_8.$$

As in the previous section, the group D_8 acts nontrivially on the cohomology of the fibre and thus the coefficients in the spectral sequence are local. The E_2-term is given by

$$E_2^{p,q} = H^p\big(\mathrm{B}D_8, \mathcal{H}^q\big(S^d \times S^d, \mathbb{Z}\big)\big) = H^p\big(D_8, H^q(S^d \times S^d, \mathbb{Z})\big) = \begin{cases} H^p(D_8, \mathbb{Z}), & q = 0, 2d, \\ H^p(D_8, H^d(S^d \times S^d, \mathbb{Z})), & q = d, \\ 0, & q \neq 0, d, 2d. \end{cases} \tag{40}$$

The local coefficients are nontrivial in the d-th row of the spectral sequence.

8.1. The d-th row as an $H^*(D_8, \mathbb{Z})$-module

The D_8-module $M := H^d(S^d \times S^d, \mathbb{Z})$, as an abelian group, is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Since the action of D_8 on M depends on d we distinguish two cases.

8.1.1. The case when d is odd

The action on M is given by

$$\varepsilon_1 \cdot (x, y) = (x, y), \qquad \varepsilon_2 \cdot (x, y) = (x, y), \qquad \sigma \cdot (x, y) = (y, x).$$

Thus, there is an isomorphism of D_8-modules $M \cong \mathbb{Z}[D_8/H_1]$. The situation resembles the one in Section 7.1, and therefore the following propositions hold.

Proposition 8.1. $H^*(D_8, \mathbb{Z}[D_8/H_1]) \cong_{\mathrm{ring}} H^*(H_1, \mathbb{Z})$.

Proof. The claim follows from Shapiro's lemma [6, Proposition 6.2, p. 73] and the fact that when $[G : H] < \infty$ there is an isomorphism of G-modules $\mathrm{Coind}_H^G M \cong \mathrm{Ind}_H^G M$. □

Proposition 8.2. *Let $\mathcal{T} \in H^*(D_8, \mathbb{Z})$ and $P \in H^*(H_1, \mathbb{Z}) \cong H^*(D_8, \mathbb{Z}[D_8/H_1])$.*

(A) *The action of $H^*(D_8, \mathbb{Z})$ on $H^*(D_8, \mathbb{Z}[D_8/H_1])$ is given by*

$$\mathcal{T} \cdot P := \mathrm{res}_{H_1}^{D_8}(\mathcal{T}) \cdot P.$$

Here P on the right-hand side is an element of $H^*(H_1, \mathbb{Z})$ and on the left-hand side is its isomorphic image under the isomorphism from the previous proposition. In particular,

$$\mathcal{X} \cdot H^*\big(D_8, \mathbb{Z}[D_8/H_1]\big) = 0.$$

(B) *$H^*(D_8, \mathbb{Z})$-module $H^*(D_8, \mathbb{Z}[D_8/H_1])$ is generated by the two elements*

$$1, \alpha \in H^*(H_1, \mathbb{Z}) \cong H^*\big(D_8, \mathbb{Z}[D_8/H_1]\big)$$

of degree 0 *and* 2.

(C) *The map $H^*(D_8, \mathbb{Z}[D_8/H_1]) \to H^*(D_8, \mathbb{F}_2[D_8/H_1])$, induced by the coefficient map $\mathbb{Z} \to \mathbb{F}_2$, is given by $1, \alpha \longmapsto 1, a^2$.*

Proof. The isomorphism $H^*(D_8, \mathbb{Z}[D_8/H_1]) \cong_{\mathrm{ring}} H^*(H_1, \mathbb{Z})$ induced by Shapiro's lemma [6, Proposition 6.2, p. 73] carries the $H^*(D_8, \mathbb{Z})$-module structure to $H^*(H_1, \mathbb{Z})$ via $\mathrm{res}_{H_1}^{D_8} : H^*(D_8, \mathbb{Z}) \to H^*(H_1, \mathbb{Z})$. In this way the complete $H^*(D_8, \mathbb{Z})$-module structure is given on $H^*(D_8, \mathbb{Z}[D_8/H_1])$. The claim (B) follows from the restriction diagram (27). The morphism of restriction diagrams induced by the coefficient reduction homomorphism $c : \mathbb{Z} \to \mathbb{F}_2$ implies the last statement. □

8.1.2. The case when d is even

The action on M is given by

$$\varepsilon_1 \cdot (x, y) = (-x, y), \qquad \varepsilon_2 \cdot (x, y) = (x, -y), \qquad \sigma \cdot (x, y) = (y, x).$$

In this case we are forced to analyze the Bockstein spectral sequence associated with the exact sequence of D_8-modules

$$0 \to M \xrightarrow{\times 2} M \to \mathbb{F}_2[D_8/H_1] \to 0, \tag{41}$$

i.e. with the exact couple

$$\begin{tikzcd} H^*(D_8, M) \arrow[rr, "\times 2"] & & H^*(D_8, M) \arrow[dl, "c"] \\ & H^*\big(D_8, \mathbb{F}_2[D_8/H_1]\big) \arrow[ul, "\delta"] & \end{tikzcd} \tag{42}$$

First we study the Bockstein spectral sequence

$$\begin{tikzcd}
H^*(H_1, M) \arrow[rr, "\times 2"] & & H^*(H_1, M) \arrow[dl, "c"] \\
& H^*(H_1, \mathbb{F}_2[D_8/H_1]) \arrow[ul, "\delta"] &
\end{tikzcd} \tag{43}$$

As in Section 7.2, we have that $H^*(H_1, \mathbb{F}_2[D_8/H_1]) = \mathbb{F}_2[a, a+b] \oplus \mathbb{F}_2[a, a+b]$. The module M as an H_1-module can be decomposed into the sum of two H_1-modules Z_1 and Z_2. The modules $Z_1 \cong_{Ab} \mathbb{Z}$ and $Z_2 \cong_{Ab} \mathbb{Z}$ are given by

$$\varepsilon_1 \cdot x = -x, \qquad \varepsilon_2 \cdot x = x \quad \text{and} \quad \varepsilon_1 \cdot y = y, \qquad \varepsilon_2 \cdot y = -y$$

for $x \in Z_1$ and $y \in Z_2$. This decomposition also induces a decomposition of H_1-modules $\mathbb{F}_2[D_8/H_1] \cong \mathbb{F}_2 \oplus \mathbb{F}_2$. Thus, the exact couple (43) decomposes into the direct sum of two exact couples

$$\begin{tikzcd}
H^*(H_1, Z_1) \arrow[rr, "\times 2"] & & H^*(H_1, Z_1) \arrow[dl, "c"] & H^*(H_1, Z_2) \arrow[rr, "\times 2"] & & H^*(H_1, Z_2) \arrow[dl, "c"] \\
& H^*(H_1, \mathbb{F}_2) \arrow[ul, "\delta"] & & & H^*(H_1, \mathbb{F}_2) \arrow[ul, "\delta"] &
\end{tikzcd} \tag{44}$$

Since all the maps in these exact couples are $H^*(H_1, \mathbb{Z})$-module maps, the following proposition completely determines both exact couples.

Proposition 8.3. *In the exact couples* (44) *differentials $d_1 = c \circ \delta$ are determined, respectively, by*

$$d_1(1) = a, \qquad d_1(b) = b(b+a) \quad \textit{and} \quad d_1(1) = a+b, \qquad d_1(a) = d_1(b) = ab. \tag{45}$$

Proof. In both claims we use the following diagram of exact couples induced by restrictions, where $i \in \{1, 2\}$:

$$\begin{tikzcd}[column sep=small]
& & & H^*(H_1, Z_i) \arrow[rr, "\times 2"] & & H^*(H_1, Z_i) \arrow[dl, "c"] & & & \\
& & & & H^*(H_1, \mathbb{F}_2) \arrow[ul, "\delta"] \arrow[dlll] \arrow[d] \arrow[drrr] & & & & \\
& \, & & & \, & & & \, & \\
H^*(K_1, Z_i) \arrow[rr, "\times 2"] & & H^*(K_1, Z_i) \arrow[dl, "c"] & H^*(K_2, Z_i) \arrow[rr, "\times 2"] & & H^*(K_2, Z_i) \arrow[dl, "c"] & H^*(K_3, Z_i) \arrow[rr, "\times 2"] & & H^*(K_3, Z_i) \arrow[dl, "c"] \\
& H^*(K_1, \mathbb{F}_2) \arrow[ul, "\delta"] & & & H^*(K_2, \mathbb{F}_2) \arrow[ul, "\delta"] & & & H^*(K_3, \mathbb{F}_2) \arrow[ul, "\delta"] &
\end{tikzcd}$$

The first exact couple. The module Z_1 is a non-trivial K_1 and K_3-module, but a trivial K_2-module. Therefore by the long exact sequence in the group cohomology associated to the exact sequence of $\mathbb{Z}_2$-modules $0 \to Z_1 \overset{\times 2}{\to} Z_1 \to \mathbb{F}_2 \to 0$, properties of Steenrod squares and the assumption at the end of Section 4.2.2:

(A) K_1-exact couple: $d_1(1) = t_1$ and $d_1(t_1) = 0$;
(B) K_2-exact couple: $d_1(1) = 0$ and $d_1(t_2) = t_2^2$;
(C) K_3-exact couple: $d_1(1) = t_3$ and $d_1(t_3) = 0$.

Now

$$\left.\begin{array}{l} \mathrm{res}^{H_1}_{K_1}(d_1(1)) = t_1 \\ \mathrm{res}^{H_1}_{K_2}(d_1(1)) = 0 \\ \mathrm{res}^{H_1}_{K_3}(d_1(1)) = t_3 \end{array}\right\} \quad \Rightarrow \quad d_1(1) = a \qquad \left.\begin{array}{l} \mathrm{res}^{H_1}_{K_1}(d_1(b)) = 0 \\ \mathrm{res}^{H_1}_{K_2}(d_1(b)) = t_2^2 \\ \mathrm{res}^{H_1}_{K_3}(d_1(b)) = 0 \end{array}\right\} \quad \Rightarrow \quad d_1(b) = b(b+a).$$

The second exact couple. The module Z_2 is a non-trivial K_2 and K_3-module, while it is a trivial K_1-module. Therefore by the long exact sequence in the group cohomology associated to the exact sequence of $\mathbb{Z}_2$-modules $0 \to Z_2 \overset{\times 2}{\to} Z_2 \to \mathbb{F}_2 \to 0$, properties of Steenrod squares and the assumption at the end of Section 4.2.2:

(A) K_1-exact couple: $d_1(1) = 0$ and $d_1(t_1) = t_1^2$;
(B) K_2-exact couple: $d_1(1) = t_2$ and $d_1(t_2) = 0$;
(C) K_3-exact couple: $d_1(1) = t_3$ and $d_1(t_3) = 0$.

Now

$$\left.\begin{array}{l} \mathrm{res}^{H_1}_{K_1}(d_1(1)) = 0 \\ \mathrm{res}^{H_1}_{K_2}(d_1(1)) = t_2 \\ \mathrm{res}^{H_1}_{K_3}(d_1(1)) = t_3 \end{array}\right\} \quad \Rightarrow \quad d_1(1) = a+b \qquad \left.\begin{array}{l} \mathrm{res}^{H_1}_{K_1}(d_1(b)) = t_1^2 \\ \mathrm{res}^{H_1}_{K_2}(d_1(b)) = 0 \\ \mathrm{res}^{H_1}_{K_3}(d_1(b)) = 0 \end{array}\right\} \quad \Rightarrow \quad d_1(b) = ab. \qquad \square$$

Remark 8.4. The result of the previous proposition can be seen as a key step in an alternative proof of Eq. (18).

Proposition 8.5. *In the exact couple* (42), *with identification* $H^*(D_8, \mathbb{F}_2[D_8/H_1]) = \mathbb{F}_2[a, a+b]$, *the differential* $d_1 = s \circ \delta$ *satisfies*

$$d_1(1) = a, \qquad d_1(a+b) = d_1(b) = b(b+a), \qquad d_1\big(a^2\big) = a^3. \tag{46}$$

(*This determines* d_1 *completely since* c *and* δ *are* $H^*(D_8, \mathbb{Z})$*-module maps.*)

Proof. Recall from Remark 7.5 that the restriction map

$$\mathrm{res}_{H_1}^{D_8} : H^*\big(D_8, \mathbb{F}_2[D_8/H_1]\big) \to H^*\big(H_1, \mathbb{F}_2[D_8/H_1]\big)$$

is injective. Then Eqs. (46) are obtained by filling the empty places in the following diagrams

$$\begin{tikzcd}
1 \arrow[r, "d_1"] \arrow[d] & \square \arrow[d] & a+b \arrow[r, "d_1"] \arrow[d] & \square \arrow[d] & a^2 \arrow[r, "d_1"] \arrow[d] & \square \arrow[d] \\
1 \oplus 1 \arrow[r, "d_1"] & a \oplus (a+b) & (a+b) \oplus a \arrow[r, "d_1"] & b(b+a) \oplus ab & a^2 \oplus (a+b)^2 \arrow[r, "d_1"] & a^3 \oplus (a+b)^3
\end{tikzcd}$$

where all vertical maps are $\mathrm{res}_{H_1}^{D_8}$. $\square$

Corollary 8.6. $H^*(D_8, M)$ *is generated as a* $H^*(D_8, \mathbb{Z})$*-module by three elements* ζ_1, ζ_2, ζ_3 *of degree* 1, 2, 3 *such that*

$$c(\zeta_1) = a, \qquad c(\zeta_2) = b(a+b), \qquad c(\zeta_3) = a^3$$

where c *is the map* $H^*(D_8, M) \to H^*(D_8, \mathbb{F}_2[D_8/H_1])$ *from the exact couple* (42).

8.2. $\mathrm{Index}_{D_8,\mathbb{Z}}^{d+2} S^d \times S^d$

The relation between the sequences of polynomials $\pi_d \in H^*(D_8, \mathbb{F}_2)$ and $\Pi_d \in H^*(D_8, \mathbb{Z})$ is described by the following lemma.

Lemma 8.7. *Let* $c_* : H^*(D_8, \mathbb{Z}) \to H^*(D_8, \mathbb{F}_2)$ *be the map induced by the coefficient morphism* $\mathbb{Z} \to \mathbb{F}_2$ (*explicitly given by* (26)). *Then for every* $d \geqslant 0$,

$$c_*(\Pi_d) = \pi_{2d}.$$

Proof. Induction on $d \geqslant 0$. For $d = 0$ and $d = 1$ the claim is obvious. Let $d \geqslant 2$ and let us assume that claim holds for every $d \leqslant k+1$. Then

$$\begin{aligned}
c_*(\Pi_{k+2}) &= c_*(\mathcal{Y}\Pi_{k+1} + \mathcal{W}\Pi_k) \overset{\text{hypo.}}{=} y^2\pi_{2k+2} + w^2\pi_{2k} = y^2\pi_{2k+2} + yw\pi_{2d+1} + yw\pi_{2d+1} + w^2\pi_{2k} \\
&= y(y\pi_{2k+2} + w\pi_{2d+1}) + w(y\pi_{2d+1} + w\pi_{2k}) = y\pi_{2k+3} + w\pi_{2k+2} \\
&= \pi_{2k+4}. \qquad \square
\end{aligned}$$

There is a sequence of D_8-inclusions

$$S^1 \times S^1 \subset S^2 \times S^2 \subset \cdots \subset S^{d-1} \times S^{d-1} \subset S^d \times S^d \subset S^{d+1} \times S^{d+1} \subset \cdots$$

implying a sequence of ideal inclusions

$$\mathrm{Index}_{D_8,\mathbb{Z}}^3 S^1 \times S^1 \supseteq \mathrm{Index}_{D_8,\mathbb{Z}}^4 S^2 \times S^2 \supseteq \cdots \supseteq \mathrm{Index}_{D_8,\mathbb{Z}}^{d+1} S^{d-1} \times S^{d-1} \supseteq \mathrm{Index}_{D_8,\mathbb{Z}}^{d+2} S^d \times S^d \supseteq \cdots \tag{47}$$

8.2.1. The case when d is odd

In this section we prove that

$$\mathrm{Index}_{D_8,\mathbb{Z}}^{d+2} S^d \times S^d = \langle \Pi_{\frac{d+1}{2}}, \Pi_{\frac{d+3}{2}} \rangle. \tag{48}$$

The proof can be conducted as in the case of $\mathbb{F}_2$ coefficients (Section 7.2). The results of Section 7.2 can also be used to simplify the proof of Eq. (48). The morphism $c_* : H^*(D_8, \mathbb{Z}) \to H^*(D_8, \mathbb{F}_2)$ induced by the coefficient morphism $\mathbb{Z} \to \mathbb{F}_2$ is a part of the morphism C of Serre spectral sequences (40) and (38). Thus, for $1 \in E_{d+1}^{0,d} = H^0(D_8, H^d(S^d \times S^d, \mathbb{Z}))$, $\hat{1} \in E_{d+1}^{0,d} = H^0(D_8, H^d(S^d \times S^d, \mathbb{F}_2))$, $\alpha \in E_{d+1}^{2,d} = H^2(D_8, H^d(S^d \times S^d, \mathbb{Z}))$ and $a \in E_{d+1}^{1,d} = H^1(D_8, H^d(S^d \times S^d, \mathbb{Z}))$,

$$C\big(\partial_{d+1}(1)\big) = \partial_{d+1}\big(C(1)\big) = \partial_{d+1}(\hat{1}) = \pi_{d+1} = C(\Pi_{\frac{d+1}{2}}),$$

$$C\big(\partial_{d+1}(\alpha)\big) = \partial_{d+1}\big(C(\alpha)\big) = \partial_{d+1}\big(a^2\big) = \partial_{d+1}(w \cdot \hat{1} + y \cdot a) = w\pi_{d+1} + y\pi_{d+2} = \pi_{d+3} = C(\Pi_{\frac{d+3}{2}}).$$

From Proposition 8.2 and the sequence of inclusions (47) it follows that

$$\partial_{d+1}(1) = \Pi_{\frac{d+1}{2}} \quad \text{and} \quad \partial_{d+1}(\alpha) = \Pi_{\frac{d+3}{2}}.$$

Finally, the statement (B) of Proposition 8.2 implies Eq. (48).

8.2.2. The case when d is even

In this section we prove that

$$\mathrm{Index}^{d+2}_{D_8,\mathbb{Z}} S^d \times S^d = \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}}, \mathcal{M}\Pi_{\frac{d}{2}} \rangle. \tag{49}$$

The previous section implies that

$$\langle \Pi_{\frac{d}{2}}, \Pi_{\frac{d+2}{2}} \rangle \supseteq \mathrm{Index}^{d+2}_{D_8,\mathbb{Z}} S^d \times S^d \supseteq \langle \Pi_{\frac{d+2}{2}}, \Pi_{\frac{d+4}{2}} \rangle. \tag{50}$$

From Corollary 8.6 we know that $\mathrm{Index}^{d+2}_{D_8,\mathbb{Z}} S^d \times S^d$ is generated by three elements $\partial_{d+1}(\zeta_1)$, $\partial_{d+1}(\zeta_2)$, $\partial_{d+1}(\zeta_3)$ of degrees $d+2$, $d+3$, $d+4$. Thus, $\partial_{d+1}(\zeta_1) = \Pi_{\frac{d+2}{2}}$ and $\partial_{d+1}(\zeta_2) = \mathcal{M}\Pi_{\frac{d}{2}}$. Since $\Pi_{\frac{d+4}{2}} \notin \langle \Pi_{\frac{d+2}{2}}, \mathcal{M}\Pi_{\frac{d}{2}} \rangle$, then $\partial_{d+1}(\zeta_3) = \Pi_{\frac{d+4}{2}}$. The proof of Eq. (49) is concluded.

Alternatively, the proof can be obtained with the help of the morphism C of Serre spectral sequences (40) and (38).

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References

[1] A. Adem, R.J. Milgram, Cohomology of Finite Groups, second ed., Grundlehren Math. Wiss., vol. 309, Springer-Verlag, Berlin, 2004.
[2] A. Adem, Z. Reichstein, Cohomology and truncated symmetric polynomials, arXiv:0906.4799, 2009.
[3] Z. Balanov, A. Kushkuley, Geometric Methods in Degree Theory for Equivariant Maps, Lecture Notes in Math., vol. 1632, Springer-Verlag, Berlin, 1996.
[4] T. Bartsch, Topological Methods for Variational Problems with Symmetries, Lecture Notes in Math., vol. 1560, Springer-Verlag, Berlin, 1993.
[5] P. Blagojević, G.M. Ziegler, Tetrahedra on deformed spheres and integral group cohomology, Electron. J. Combin. 16 (2) (2009) R16, 1-11.
[6] K.S. Brown, Cohomology of Groups, Grad. Texts in Math., vol. 87, Springer-Verlag, New York, Berlin, 1982.
[7] J. Carlson, Group 4: Dihedral(8), results published on the web page http://www.math.uga.edu/~lvalero/cohohtml/groups_8_4_frames.htm.
[8] J. Carlson, L. Townsley, L. Valero-Elizondo, M. Zhang, Cohomology Rings of Finite Groups. With an Appendix: Calculations of Cohomology Rings of Groups of Order Dividing 64, Kluwer Academic, 2003.
[9] T. Tom Dieck, Transformation Groups, de Gruyter Stud. Math., vol. 8, de Gruyter, Berlin, 1987.
[10] L. Evens, On the Chern classes of representations of finite groups, Trans. Amer. Math. Soc. 115 (1965) 180–193.
[11] E. Fadell, S. Husseini, An ideal-valued cohomological index, theory with applications to Borsuk–Ulam and Bourgin–Yang theorems, Ergodic Theory Dynam. Systems 8 (1988) 73–85.
[12] J. González, P. Landweber, The integral cohomology groups of configuration spaces of pairs of points in real projective spaces, arXiv:1004.0746, 2010.
[13] B. Grünbaum, Partition of mass-distributions and convex bodies by hyperplanes, Pacific J. Math. 10 (1960) 1257–1261.
[14] H. Hadwiger, Simultane Vierteilung zweier Körper, Arch. Math. (Basel) 17 (1966) 274–278.
[15] D. Handel, On products in the cohomology of the dihedral groups, Tohoku Math. J. (2) 45 (1993) 13–42.
[16] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, 2002, xii+544 pp.
[17] W.Y. Hsiang, Cohomology Theory of Topological Transformation Groups, Springer-Verlag, 1975.
[18] D. Husemoller, Fibre Bundles, third ed., Springer-Verlag, 1993.
[19] G. Lewis, The integral cohomology rings of groups of order p^3, Trans. Amer. Math. Soc. 132 (1968) 501–529.
[20] E. Lucas, Sur les congruences des nombres eulériens et les coefficients différentiels des functions trigonométriques suivant un module premier, Bull. Soc. Math. France 6 (1878) 49–54.
[21] P. Mani-Levitska, S. Vrećica, R. Živaljević, Topology and combinatorics of partition masses by hyperplanes, Adv. Math. 207 (2006) 266–296.
[22] W. Marzantowicz, An almost classification of compact Lie groups with Borsuk–Ulam properties, Pacific J. Math. 144 (1990) 299–311.
[23] J. Matoušek, Using the Borsuk–Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry, second, corrected printing, Universitext, Springer-Verlag, Heidelberg, 2008.
[24] J. Milnor, J. Stasheff, Characteristic Classes, Ann. of Math. Stud., vol. 76, Princeton University Press, University of Tokyo Press, Princeton, NJ, Tokyo, 1974, vii+331 pp.
[25] E. Ramos, Equipartitions of mass distributions by hyperplanes, Discrete Comput. Geom. 15 (1996) 147–167.
[26] R.T. Živaljević, Topological methods, in: J.E. Goodman, J. O'Rourke (Eds.), CRC Handbook on Discrete and Computational Geometry, CRC Press, Boca Raton, FL, 2004, pp. 305–329 (Chapter 14).
[27] R. Živaljević, User's guide to equivariant methods in combinatorics II, Publ. Inst. Math. Belgrade 64 (78) (1998) 107–132.