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Singular Perturbations of Linear Systems with Multiparameters and Multiple Time Scales*

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In this paper, an alternate approach to the method of asymptotic expansions for the study of a singularly perturbed, linear system with multiparameters and multiple time scales is developed. The method consists of developing a linear, nonsingular transformation that enables one to transform the original system into an upper triangular form. This process of upper triangularization will enable us to investigate (i) stability and (ii) approximation of solutions of the original system in terms of the overall reduced system and the corresponding boundary layer systems. © 1988 Academic Press, Inc.

INTRODUCTION

Singular perturbation models with multiple time scales do arise in many of the dynamical models of physical as well as biochemical processes [1, 2, 4, 9, 15, 16]. In previous years, the method of asymptotic expansions has been widely used for the study of such systems [5, 6, 13, 14]. In recent years, an alternative approach has been investigated in which one develops a suitable, nonsingular, linear transformation that enables one to partially or totally decouple the original system. This transformed system will then enable one to study the stability properties of the original system with relative ease. Moreover, one gets a closed form expression for first-order approximations of solutions to the original system. This idea was initiated by Khalil and Kokotovic [7, 8] for a two-time scale problem, and by Chang [3] for a general boundary value problem. Later, Ladde and Siljak [11] and Ladde and Rajalakshmi [10] have used the idea for a three-time scale problem.

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In this paper, we have used this idea for the case of an arbitrary n-time scale problem. The method consists of developing a linear, nonsingular transformation that transforms the original system into an upper triangular form. This transformed system is then used to investigate (i) stability and (ii) approximation of solutions of the original system in terms of the overall reduced system and the corresponding boundary layer systems.

The paper is organized as follows: In Section 2, by following [10, 11], a joint n-time scale multiparameter, singular perturbation problem is formulated. Also, a hierarchical scheme for aggregating and arranging the groups of small parameters according to their order and an order reduction scheme are presented. In Section 3, an n-fold version of the transformation in [11] which transforms the given system into an upper triangular form is formulated. In Section 4, validity of the transformation developed in Section 3 is established. Section 5, the main stability result and the approximation to the solution of the original relative to the overall reduced and the various boundary layer systems are given.

Although the results obtained here are only first-order approximations, this method of analysis is simple and straightforward. Moreover, the forms of these solutions indicate a possibility of developing an algorithm for the process, thereby enabling one to employ a computer. In addition, it is expected to have great potential in the study of control theory. Work in this direction is in progress and will be reported elsewhere.

2. Formulation of the Problem

Let us consider a linear system described by

$$\begin{aligned} \dot{x}_1 &= A^1_{11}(t)x_{11} + \sum_{k=1}^{\gamma_2} A^2_{1k}(t)x_{2k} + \cdots + \sum_{k=1}^{\gamma_n} A^n_{1k}(t)x_{nk} \\ \varepsilon^i_j \dot{x}_{ji} &= \sum_{l=1}^{n}\left(\sum_{k=1}^{\gamma_l} A^{li}_{jk}(t)x_{lk}\right), \qquad i \in I(1,\gamma_j),\ j \in I(2,n), \end{aligned} \tag{2.1}$$

where $I(a,b)=\{a,\ a+1,\ a+2, ..., b\}$, $a \leqslant b$, $I(a,b)=\phi$, $a>b$, $a,b \in Z^+$, $\gamma_1 = 1$, $x_1 = x_{11} \in R^{l_1}$, $x_{ji} \in R^{l_{ji}}$, $l_j = \sum_{i=1}^{\gamma_j} l_{ji}$ and the dimension of the system (2.1) is $N=\sum_{j=1}^{n} l_j$. In (2.1), all matrix functions are continuous on R and have appropriate dimensions. The parameters ε^i_j are small, positive real numbers. The crucial assumption is that ε^i_j's have different orders for fixed i and have the same order for fixed j. This implies that the ratio of ε^i_j with ε^k_j for a fixed j is bounded, that is,

$$\underline{\varepsilon_j} \leqslant \varepsilon^i_j/\varepsilon^k_j \leqslant \overline{\varepsilon_j}, \qquad \forall i, k \in I(1,\gamma_j),\ j \in I(2,n), \tag{2.2}$$

where $\underline{\varepsilon}_j$ and $\bar{\varepsilon}_j$ are some positive, real numbers. Now, we make the following important assumption:

(A1) $\lim_{\varepsilon_{i-1} \to 0^+} (\varepsilon_i/\varepsilon_{i-1}) = 0$, $\forall i \in I(2, n)$, where ε_i is defined by

$$\varepsilon_i = (\varepsilon_i^1, \varepsilon_i^2, ..., \varepsilon_i^{\gamma_i})^{1/\gamma_i}, \qquad i \in I(2, n) \text{ and } \varepsilon_1 = 1. \tag{2.3}$$

Using (A1) and (2.3), we can rewrite the system (2.1) as

$$\varepsilon_m \dot{x}_m = \sum_{k=1}^{n} D_m A_{mk}(t) x_k, \qquad x_m(t_0) = x_{m0}, m \in I(1, n), \tag{2.4}$$

where $x_k = (x_{k1}^T, x_{k2}^T, ..., x_{k\gamma_k}^T)^T \in R^{l_k}$ and the block matrices of (2.4) are formed in an obvious way from the matrices of (2.1) with D_m's given by

$$D_m = \text{diag}\left\{\frac{\varepsilon_m}{\varepsilon_m^1} I_{m1}, \frac{\varepsilon_m}{\varepsilon_m^2} I_{m2}, ..., \frac{\varepsilon_m}{\varepsilon_m^{\gamma_m}} I_{m,\gamma_m}\right\}, \quad m \in I(2, n).$$
$$D_1 = I_{11}. \tag{2.5}$$

Here, I_{mk}'s are identity matrices of appropriate dimensions and the elements of the D_m matrices are also bounded, that is,

$$\underline{\varepsilon}_m \leqslant \varepsilon_m/\varepsilon_m^j \leqslant \tilde{\varepsilon}_m, \qquad j \in I(1, \gamma_m), m \in I(2, n). \tag{2.6}$$

Thus, system (2.4) gives us an n-time scale, multiparameter problem.

Before proceeding further, let us introduce the following notation:

$$A_{ij}^{(k+1)}(t) = A_{ij}^{(k)}(t) - A_{i,n-k}^{(k)}(t)$$
$$\times (E_{n-k}^{(k)})(t)^{-1} A_{n-k,j}^{(k)}(t), \qquad K \in I(0, n-1) \tag{2.7}$$
$$A_{ij}^{(0)}(t) = A_{ij}(t),$$

where

$$E_{n-k}^{(k)}(t) = A_{n-k,n-k}^{(k)}(t).$$

(Note: (2.7) is a recursive relation.)

To perform an n-stage reduction of system (2.4), we make the following assumption:

(A2) The block matrices $A_{n-j,n-j}^{(j)}(t)$ are nonsingular for all $t \geqslant t_0$, where $j \in I(0, n-2)$.

Using (A2), we can get the ε_{n-j}-reduced system by setting $\varepsilon_{n-j} = 0$ as

$$\varepsilon_l \dot{x}_l = \sum_{k=1}^{n-(j+1)} D_l A_{lk}^{(j+1)}(t) x_k,$$
$$x_l(t_0) = x_{l0}, l \in I(1, n-j-1) \tag{2.8}$$
$$x_{n-j}(t) = -\sum_{k=1}^{n-(j+1)} A_{n-j,n-j}^{(j)}(t))^{-1} A_{n-j,k}^{(j)}(t) x_k, \quad j \in I(0, n-2).$$

We now get the various boundary layer systems as follows. By using the transformation $\tau_n = (t - t_0)/\varepsilon_n$ and setting $\varepsilon_n = 0$ in (2.4), we first get the ε_n-boundary layer system. The other $\varepsilon_{n-(j+1)}$-boundary layer systems for $j \in I(0, n-3)$ are then obtained by using the transformation $\tau_{j+1} = (t - t_0)/\varepsilon_{n-(j+1)}$ and setting $\varepsilon_{n-(j+1)} = 0$ in the corresponding ε_{n-j}-reduced subsystem (2.8). They are given by

$$\frac{d\hat{x}_l}{d\tau_{n-(j+1)}} = 0, \qquad l \in I(1, n-j-2)$$

$$\frac{d\hat{x}_{n-(j+1)}}{d\tau_{j+1}} = A^{(j+1)}_{n-(j+1),n-(j+1)}(t)\, \hat{x}_{n-(j+1)}(\tau_{j+1})$$

$$\begin{aligned} \hat{x}_{n-(j+1)}(0) = x_{n-(j+1),0} & \\ & + \sum_{k=1}^{j} A^{(j+1)}_{n-(j+1),n-(j+1)}(t_0) \\ & \times A^{(j+1)}_{n-(j+1)} x_{n-(j+1),0}, \qquad j \in I(1, n-3). \end{aligned} \tag{2.9}$$

Two of our main objectives here are (i) to study the stability of system (2.4) and (ii) to get approximate solutions to (2.4) in terms of the solutions to the overall reduced system (obtained by setting $j = n-2$ in (2.8)) and the various boundary layer systems (2.9).

3. Triangularization

In this section, we develop an n-fold version of the transformation in [11] which results in transforming the system (2.4) into an upper triangular form. This form of partial decoupling will then enable us to achieve our objectives with relative ease.

We can rewrite system (2.4) in a matrix form as

$$\dot{x} = A(t)x, \tag{3.1}$$

where

$$x = (x_1, x_2, ..., x_n)^T, \qquad A(t) = (a_{ij}(t))_{n \times n}$$

and

$$a_{ij}(t) = \varepsilon_i^{-1} D_i A_{ij}(t).$$

Now, we introduce a similarity transformation from $R^{l_1} \times R^{l_2} \times \cdots \times R^{l_n}$ into itself represented by the matrix $T = (T_{pq})_{n \times n}$. The elements of T are submatrix functions of appropriate dimensions. These submatrix functions are given by

$$T_{pq} = \begin{cases} 0 & \text{for} \quad q \in I(p+1, n) \\ I_p & \text{for} \quad p = q \\ L_{pq} & \text{for} \quad q \in I(1, p-1), \end{cases} \tag{3.2}$$

where L_{pq} are determined by the initial value problem

$$\begin{aligned} \varepsilon_p \dot{L}_{p,p-1} &= -\sum_{j=1}^{p} \frac{\varepsilon_p}{\varepsilon_j} L_{pj} \left(\sum_{k=p-1}^{n} D_j A_{jk}(t) R_{k,p-1}(t) \right), \quad p \in I(2, n) \\ \varepsilon_p \dot{L}_{pq} &= -\left[\sum_{j=1}^{p} \frac{\varepsilon_p}{\varepsilon_j} L_{pj} D_j A_{jq}(t) R_{qq} - Q_{pq} \right], \quad p \in I(3, n), q \in I(1, p-2) \end{aligned} \tag{3.3}$$

with initial conditions

$$\begin{aligned} &L_{pq}(t_0) = (A_{pp}^{(n-p)}(t_0))^{-1} A_{pq}^{(n-p)}(t_0), \\ &p \in I(2, n), q \in I(1, p-1), \end{aligned} \tag{3.4}$$

where

$$Q_{pq}(t) = \sum_{m=p}^{n} \left[\sum_{j=1}^{p} \frac{\varepsilon_p}{\varepsilon_j} L_{pj}(t) D_j A_{jm}(t) \left(\sum_{k=p}^{m} R_{mk}(t) L_{kq}(t) \right) \right], \tag{3.5}$$

and for all $j \in I(1, n-1)$ we have

$$R_{ij}(t) = \begin{cases} 0, & i \in I(1, j-1) \\ I_i, & i = j \\ -\sum_{k=j}^{i-1} L_{ik}(t) R_{kj}(t), & i \in I(j+1, n). \end{cases} \tag{3.6}$$

We make a note that, for a fixed j, the elements $R_{ij}(t)$ given by (3.6) are obtained in a recursive fashion. Moreover, it is easy to verify that $R = (R_{ij})_{n \times n}$ is the matrix representing the inverse of the transformation T.

The similarity transformation (3.2) determined by (3.3) and (3.4) transforms (3.1) into an upper triangular form as

$$\dot{u} = B(t) u, \tag{3.7}$$

where $u = (u_1, u_2, ..., u_n)^T$, $B = (b_{pq}(t))_{n \times n}$, and for all $p \in I(1, n)$, b_{pq}'s are given by

$$b_{pq} = \begin{cases} \sum_{k=q}^{n} \left(\sum_{j=1}^{p} \frac{\varepsilon_p}{\varepsilon_j} L_{pj}(t) D_j A_{jk}(t) \right) R_{kq}(t), & q \in I(p, n) \\ 0 \quad \text{for} \quad q \in I(1, p-1). \end{cases} \tag{3.8}$$

The upper triangular form (3.7) of (3.1) will make the study of stability properties of multi-time scale problems relatively easy. In the next section, we will proceed to verify the validity of the transformation (3.2).

4. Verification of the Transformation

In this section, our objective is to verify the validity of the transformation (3.2) which is necessary in establishing the upper triangular form (3.7) of (3.1). For the transformation to be valid, we need to establish the existence, uniqueness, and boundedness of solutions to the initial value problems (3.3) and (3.4).

The existence and uniqueness of these solutions follow from continuity of the coefficient matrices in (2.4) and continuous differentiability of the right-hand side of the system of Eq. (3.3) with respect to L_{ij}. Thus, it is enough to show that these solutions are bounded. In addition, we develop a certain type of convergence results which enable us to obtain approximations for the matrix functions $L_{pq}(t)$. The usefulness of these approximations will become obvious in the next section.

To establish the boundedness and approximations of these solutions, we use the following recursive procedure. First, we establish the boundedness of the matrix functions in the last row of the transformation and show that these functions have good approximations. We then proceed to show the boundedness of matrix elements in the previous row and obtain the corresponding approximations. We repeat this procedure till we come to the second row of the transformation. Once we have established the boundedness of all these solutions, it is easy to see that the matrix function $L_{21}(t)$ is bounded. Thus, we only need to get an approximation for $L_{21}(t)$.

We now actually establish these results. First, we make the following assumptions.

(A3) All matrix functions $A_{ij}(t)$ in (2.4) are bounded on R.

(A4) For all D_j satisfying (2.6), there exist positive numbers $\alpha_{jj}^{(n-j)} > 0$ such that

$$L\{D_j A_{jj}^{(n-j)}(t)\} \leqslant -\alpha_{jj}^{(n-j)} \qquad \forall t \geqslant t_0,\ j \in I(2, n). \tag{4.1}$$

(A5) The matrix functions $(A_{ii}^{(n-i)}(t))^{-1} A_{ij}^{(n-i)}(t)$ for $i \in I(2, n)$, $j \in I(2, i-1)$ and their first derivatives are bounded on R.

Consider the differential equations for $L_{nj}(t)$, $j \in I(1, n-1)$. Using (3.3)–(3.6), we can rewrite these equations in the form

$$\begin{aligned} \varepsilon_n \dot{L}_{nj} &= D_n A_{nn}(t) L_{nj} - \frac{\varepsilon_n}{\varepsilon_j} L_{jn} D_j A_{jj}(t) + S_{nj}(t) \\ L_{nj}(t_0) &= A_{nn}^{-1}(t_0) A_{nj}(t_0), \end{aligned} \tag{4.2}$$

where

$$S_{nj}(t) = \sum_{k=1}^{n} \frac{\varepsilon_n}{\varepsilon_k} L_{nk}\{(1-\delta_{kn}) D_k A_{kn}(t) L_{nj}(t) - (1-\delta_{kj}) D_k A_{kj}(t)\}. \tag{4.3}$$

Let $\phi_{nn}(t, \tau; \varepsilon_n)$ denote the state transition matrix of

$$\varepsilon_n \dot{z} = D_n A_{nn}(t) z.$$

Then, we have the following lemma.

LEMMA 4.1. *Under* (A4), *we have*

$$\phi_{nn}(t, \tau; \varepsilon_n) \leqslant \exp\left[-\frac{\alpha_{nn}}{\varepsilon_n}(t-\tau)\right], \qquad t \geqslant \tau. \tag{4.4}$$

Proof. It is analogous to the proof of Lemma 4.12 in [10].

Let $\phi_{jj}(t, \tau; \varepsilon_j)$ be the state transition matrix of

$$\varepsilon_j \dot{y}_j = D_j A_{jj}(t) y_j, \qquad j \in I(1, n-1). \tag{4.5}$$

Then the estimates on their norms are of the form

$$\phi_{jj}(t, \tau; \varepsilon_j) \leqslant \exp[\alpha_{jj}(t-\tau)], \qquad t \geqslant \tau, \tag{4.6}$$

where α_{jj} are some positive real numbers. We now have the following result.

Result 4.1. Under (A1), (A3), and (A4), there exist $\varepsilon_j^+ > 0$, $k \in I(2, n)$, such that for all $\varepsilon_k/\varepsilon_{k-1} < \varepsilon_k^+/\varepsilon_{k-1}^+$, $k \in I(2, n)$, the solutions $L_{nj}(t)$ of (4.2) are bounded for all $t \geqslant t_0$.

Proof. $L_{nj}(t)$ satisfy the initial value problem (4.2). We observe that $Y_j = \phi_{nn}(t, t_0; \varepsilon_n) L_{nj}(t_0) \phi_{jj}(t_0, t; \varepsilon_j)$ satisfy the homogeneous part of (4.2), where $\phi_{jj}(t_0, t; \varepsilon_j)$ represents the state transition matrix of the adjoint system of (4.5). Using variation of parameters formula [12], we can write

$$L_{nj}(t) = \phi_{nn}(t, t_0; \varepsilon_n) L_{nj}(t_0) \phi_{jj}(t_0, t; \varepsilon_j) + \int_{t_0}^{t} \phi_{nn}(t, \tau; \varepsilon_n) \varepsilon_n^{-1} S_{nj}(\tau) \phi_{jj}(\tau, t; \varepsilon_j)\, d\tau, \tag{4.7}$$

where $S_{nj}(t)$ is given by (4.3).

Using (A1), we can choose $\varepsilon_2^+, \varepsilon_3^+, ..., \varepsilon_{n-1}^+$ and ε_n^+ such that

$$2\varepsilon_n^+ \max_{m \in I(1, n-1)} \{\alpha_{mm}\} \leqslant \alpha_{nn}$$

$$\frac{8(n-1)}{\alpha_{nn}} \frac{\varepsilon_n^+}{\varepsilon_m^+} \left[\|D_m A_{mn}\| \rho + \max_{\substack{1 \leqslant k \leqslant n-1 \\ m \neq k}} \{\|D_m A_{mk}\|\}\right] \leqslant 1,$$

for $m \in I(1, n-1)$,

where ρ is defined by

$$\rho = 2\left\{\frac{2}{\alpha_{nn}} \max_{1 \leqslant k \leqslant n-1} \{\|D_n A_{nk}\|\} + \max_{1 \leqslant k \leqslant n-1} \{\|L_{nk}(t_0)\|\}\right\}.$$

We now show that $\|L_{nj}\| < \rho$ for all $t \geqslant t_0$ and $j \in I(1, n-1)$.

If it is not true, then there exists $j_0 \in I(1, n-1)$ and $t^* \in (t_0, t]$ such that $\|L_{nj_0}(t^*)\| = \rho$ and $\|L_{nj_0}(t)\| < \rho$ $\forall j \neq j_0$ and $t < t^*$. Taking the norms on both sides of (4.7) and after a few simple steps, we arrive at the contradiction $\rho > 3/4\rho$. Hence $\|L_{nj}\| < \rho$ for all $t \geqslant t_0$ and $j \in I(1, n-1)$. Thus, we have Result 4.1.

Next, we establish the convergence results for $L_{nj}(t)$.

Result 4.2. Under Result 4.1 and assumption (A5), we have

$$L_{nj}(t) = \tilde{L}_{nj}(t) + O(\varepsilon_n/\varepsilon_{n-1}),$$

where (4.8)

$$\tilde{L}_{nj}(t) = [A_{nn}(t)]^{-1} A_{nj}(t), \qquad j \in I(1, n-1).$$

Proof. Let $L_{nj}(t) = \tilde{L}_{nj}(t) + \Delta L_{nj}(t)$. Then $\Delta L_{nj}(t)$ satisfies the differential equation

$$\varepsilon_n \dot{\Delta L}_{nj} = \left[D_n A_{nn}(t) + \sum_{k=1}^{n-1} \frac{\varepsilon_n}{\varepsilon_k} \tilde{L}_{nk}(t) D_k A_{kn}(t)\right] \Delta L_{nj}$$

$$-\frac{\varepsilon_n}{\varepsilon_j} \Delta L_{nj}[D_j(A_{jj}(t) - A_{jn}(t)\tilde{L}_{nj}(t))] + P_{nj}(t) \qquad (4.9)$$

$$\Delta L_{nj}(t_0) = 0,$$

where

$$P_{nj}(t) = \sum_{k=1}^{n-1} \frac{\varepsilon_n}{\varepsilon_k} \{[\tilde{L}_{nk}(t) + (1-\delta_{kj}) \Delta L_{nk}] D_k,$$
$$\cdot [A_{kn}(t)\tilde{L}_{nj}(t) - (1-\delta_{kj}) A_{kj}(t)]$$
$$+ \Delta L_{nk} D_k A_{kn}(t) \Delta L_{nj}\} - \varepsilon_n \dot{\tilde{L}}_{nj}. \qquad (4.10a)$$

For the sake of clarity, we introduce the following notation: $\varepsilon_{kj} = (\varepsilon_j, \varepsilon_{j+1}, ..., \varepsilon_k)$, $j < k$. Let $\phi_{Mn}(t, \tau; \varepsilon_{n2})$ and $\phi_{Aj}(t, \tau; \varepsilon_j)$ be the state transition matrices of

$$\varepsilon_n \dot{m} = \left[D_n A_{nn}(t) + \sum_{k=1}^{n-1} \frac{\varepsilon_n}{\varepsilon_k} \tilde{L}_{nk}(t) D_k A_{kn}(t)\right] m$$

$$\varepsilon_j \dot{P}_j = D_j[A_{jj}(t) - A_{jn}(t)[A_{nn}(t)]^{-1} A_{nj}(t)] P_j, \qquad j \in I(1, n-1).$$

Now, using the methods of differential inequalities and properties of the logarithmic norm, it is easy to prove that

$$\|\phi_{Mn}(t, \tau; \varepsilon_{n2})\| \leqslant \exp\left[\frac{-\alpha_{Mn}}{\varepsilon_n}(t-\tau)\right], \quad t \geqslant \tau, \alpha_{Mn} < \alpha_{nn},$$

$$\|\phi_{Aj}(t, \tau; \varepsilon_j)\| \leqslant \exp[\alpha_{Aj}(t-\tau)],$$

$$\alpha_{Aj} > 0, t > \tau, \text{ and } j \in I(1, n-1).$$

Then $y_j = \phi_{Mn}(t, t_0; \varepsilon_{n2})$ Ⓐ $\phi_{Aj}(t_0, t; \varepsilon_j)$, $y_j(t_0) =$ Ⓐ is the solution for the unperturbed part of (4.9). Using variation of parameters formula, we get

$$\Delta L_{nj} = \int_{t_0}^{t} \phi_{Mn}(t, \tau; \varepsilon_{n2})\, \varepsilon_n^{-1} P_{nj}(\tau)\, \phi_{Aj}(\tau, t; \varepsilon_j)\, d\tau.$$

Taking norms on both sides and doing some calculations, we arrive at

$$\|\Delta L_{nj}\| \leqslant \frac{\varepsilon_n}{\varepsilon_{n-1}} \frac{k_{nj}}{\alpha_{Mn} - \varepsilon_n \alpha_{Aj}} \left[1 - \exp\left(\frac{-(\alpha_{mn} - \varepsilon_n \alpha_{Aj})}{\varepsilon_n}(t - t_0)\right)\right],$$

where

$$k_{nj} = \sum_{k=1}^{n-1} \frac{\varepsilon_{n-1}}{\varepsilon_k} b_{kj} + \varepsilon_{n-1} \|\dot{\tilde{L}}_{nj}\|$$

$$\begin{aligned} b_{kj} = {} & [(1-\delta_{kj}) \|L_{nk}\| + \delta_{kj} \|\tilde{L}_{nj}\|] \\ & \times \|D_k(A_{kn}(t)\tilde{L}_{nj} - (1-\delta_{kj}) A_{kj}(t))\| + \|D_k A_{kn}(t)\| \qquad (4.10\text{b}) \\ & \times (\|L_{nk}\| + \|\tilde{L}_{nk}\|)(\|\tilde{L}_{nj}\| + \|L_{nj}\|). \end{aligned}$$

As a consequence of assumption (A1), we can now choose ε_n small enough such that $\alpha_{Mn} - \varepsilon_n \alpha_{Aj} > 0$. Then, we have

$$\|\Delta L_{nj}\| \leqslant \frac{\varepsilon_n}{\varepsilon_{n-1}} \frac{k_{nj}}{\alpha_{Mn} - \varepsilon_n \alpha_{Aj}} \simeq O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right).$$

Thus, we have

$$L_{nj}(t) = \tilde{L}_{nj}(t) + O(\varepsilon_n / \varepsilon_{n-1}).$$

Next, we proceed to show the boundedness of the elements in the $(n-1)$th row of the transformation (3.2), i.e., $L_{n-1,j}$'s. Let us denote $n-1 = i$. L_{ij}'s satisfy the differential equation given by (3.3) with the corresponding initial condition given by (3.4). Using (4.8) and rearranging the terms appropriately, we can rewrite the equations for L_{ij}'s as follows:

$$\varepsilon_i \dot{L}_{ij} = D_i \left[A_{ii}^{(1)}(t) + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right] L_{ij}(t) - \frac{\varepsilon_i}{\varepsilon_j} L_{ij}(t) D_j$$
$$\cdot \left[A_{jj}^{(1)}(t) + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right] + S_{ij}(t) \tag{4.11}$$
$$L_{ij}(t_0) = [A_{ii}^{(1)}(t_0)]\, A_{ii}^{(1)}(t_0),$$

where

$$S_{ij}(t) = \sum_{k=1}^{i} \frac{\varepsilon_i}{\varepsilon_k} L_{ik} D_k \{(1-\delta_{ki}) A_{ki}^{(1)}(t) L_{ij}(t)$$
$$-(1-\delta_{kj}) A_{kj}^{(1)}(t)\} + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right).$$

We now state and prove the following result.

Result 4.3. Under (A1), (A3), (A4), and (A5), we have

(i) there exist $\varepsilon_k^{(1)} > 0$, $k \in I(2, n-1)$, such that for all $\varepsilon_k/\varepsilon_{k-1} < \varepsilon_k^{(1)}/\varepsilon_{k-1}^{(1)}$, $k \in I(2, n-1)$, the solutions $L_{ij}(t)$ of (4.11) are bounded for all $t \geqslant t_0$.

(ii) The forms of $L_{ij}(t)$ are given by

$$L_{ij}(t) = \tilde{L}_{ij}(t) + \sum_{k=i}^{n} O(\varepsilon_k/\varepsilon_{k-1}),$$

where

$$\tilde{L}_{ij}(t) = (A_{ii}^{(1)}(t))^{-1} A_{ij}^{(1)}(t), \qquad j \in I(1, n-1). \tag{4.12}$$

Proof. First, let us consider the proof of (i). L_{ij}'s satisfy (4.11). Let $\phi_{ii}^{(1)}(t, \tau; \varepsilon_i)$, $\phi_i^{(1)}(t, \tau; \varepsilon_{ni})$, and $\phi_j^{(1)}(t, \tau; \varepsilon_j, \varepsilon_{ni})$ be the state transition matrices of

$$\varepsilon_i \dot{y} = D_i A_{ii}^{(1)}(t)\, y,$$
$$\varepsilon_i \dot{y} = D_i [A_{ii}^{(1)}(t) + O(\varepsilon_n/\varepsilon_i)]\, y,$$

and

$$\varepsilon_j \dot{y} = D_j [A_{jj}^{(1)}(t) + O(\varepsilon_n/\varepsilon_i)]\, y,$$

respectively, for $j \in I(1, n-1)$. Using (A4), it is easy to see that these state transition matrices have estimates of the form

$$\begin{aligned}
&\|\phi_{ii}^{(1)}(t,\tau;\varepsilon_i)\| \leqslant \exp\left[\frac{-\alpha_{ii}^{(1)}}{\varepsilon_i}(t-\tau)\right],\\
&\|\phi_{i}^{(1)}(t,\tau;\varepsilon_{ni})\| \leqslant \exp\left[\frac{-\alpha_{i}^{(1)}}{\varepsilon_i}(t-\tau)\right], \qquad \alpha_i^{(1)}<\alpha_{ii}^{(1)},\\
&\|\phi_{j}^{(1)}(t,\tau;\varepsilon_j,\varepsilon_{ni})\| \leqslant \exp[\alpha_j^{(1)}(t-\tau)],\\
&\qquad\qquad \alpha_j^{(1)}>0,\ j\in I(1,i-1), \text{ for all } t\geqslant\tau.
\end{aligned} \tag{4.13}$$

Then, we observe that $Y_j=\phi_i^{(1)}(t,t_0;\varepsilon_{ni})\,L_{ij}(t_0),\ \phi_j^{(1)}(t_0,t;\varepsilon_j,\varepsilon_{ni})$ satisfy the homogeneous part of (4.11). Once again, making use of the variation of parameters formula, we can write

$$\begin{aligned}
L_{ij}(t)&=\phi_i^{(1)}(t,t_0;\varepsilon_{ni})\,L_{ij}(t_0)\,\phi_j^{(1)}(t_0,t;\varepsilon_j,\varepsilon_{ni})\\
&\quad+\int_{t_0}^{t}\phi_i^{(1)}(t,\tau;\varepsilon_{ni})\,\varepsilon_{i-1}^{-1}S_{ij}(\tau)\,\phi_j^{(1)}(\tau,t;\varepsilon_j,\varepsilon_{ni})\,d\tau.
\end{aligned}$$

Assumption (A1) allows us to choose $\varepsilon_2^{(1)}, \varepsilon_3^{(1)}, ..., \varepsilon_i^{(1)}$ such that

$$2\varepsilon_i^{(1)}\max\{\alpha_1^{(1)},\alpha_2^{(1)},...,\alpha_{i-1}^{(1)}\}\leqslant\alpha_i^{(1)}$$

$$\frac{8(i-1)}{\alpha_i^{(1)}}\frac{\varepsilon_i^{(1)}}{\varepsilon_m^{(1)}}\Big[\|D_mA_{mi}^{(1)}\|\,\rho+\max_{\substack{1\leqslant k\leqslant i-1\\ k\neq m}}\{\|D_mA_{mk}^{(1)}\|\}\Big]\leqslant 1, \qquad m\in I(1,i-1),$$

where ρ is defined by

$$\rho=2\left\{\frac{2}{\alpha_i^{(1)}}\max_{1\leqslant k\leqslant i-1}\left\{\|D_iA_{ik}^{(1)}\|+\left\|O\left(\frac{\varepsilon_n}{\varepsilon_i}\right)\right\|\right\}+\max_{1\leqslant k\leqslant i-1}\{\|L_{ik}(t_0)\|\right\}.$$

We now show that $\|L_{ij}(t)\|<\rho$ for $j\in I(1,i-1)$ and for all $t\geqslant t_0$. The procedure is exactly the same as in the proof of Result 4.1. Thus, we have now established the boundedness of L_{ij}'s. Now, let us consider the proof of (ii).

Let $L_{ij}(t)=\tilde{L}_{ij}(t)+\Delta L_{ij}$. Then ΔL_{ij} satisfies the differential equation

$$\begin{aligned}
\varepsilon_i\,\Delta\dot{L}_{ij}&=\left[D_iA_{ii}^{(1)}(t)+\sum_{k=1}^{i-1}\frac{\varepsilon_i}{\varepsilon_k}\tilde{L}_{ik}\,D_kA_{ki}^{(1)}(t)+O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right]\\
&\quad-\frac{\varepsilon_i}{\varepsilon_j}\Delta L_{ij}\left[D_j\left(A_{jj}^{(1)}(t)-A_{ji}^{(1)}(t)\tilde{L}_{ij}+O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right)\right]+P_{ij}(t),
\end{aligned}$$

where

$$P_{ij}(t)=\sum_{k=1}^{i-1}\frac{\varepsilon_i}{\varepsilon_k}\{[\tilde{L}_{ik}+(1-\delta_{kj})\,\Delta L_{ik}]$$
$$\times D_k[A_{Ki}^{(1)}(t)\tilde{L}_{ij}-(1-\delta_{kj})\,A_{kj}^{(1)}(t)]$$
$$+\Delta L_{ik}\,D_k A_{ki}^{(1)}(t)\,\Delta L_{ij}\}-\varepsilon_i\dot{\tilde{L}}_{ij}+O\left(\frac{\varepsilon_n}{n-1}\right).$$

Let $\phi_{Mi}^{(1)}(t,\tau;\varepsilon_{n2})$ and $\phi_{Aj}^{(1)}(t,\tau;\varepsilon_j,\varepsilon_{ni})$ be the state transition matrices of

$$\varepsilon_i\dot{m}=\left[D_iA_{ii}^{(1)}(t)+\sum_{k=1}^{i-1}\frac{\varepsilon_i}{\varepsilon_k}\tilde{L}_{ik}\,D_kA_{ki}^{(1)}(t)+O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right]m$$

and

$$\varepsilon_j\dot{p}_j=D_j\left[A_{jj}^{(1)}(t)-A_{ji}^{(1)}(t)\tilde{L}_{ij}+O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right]p_j,\qquad j\in I(1,i-1).$$

It is easy to see that

$$\|\phi_{Mi}^{(1)}(t,\tau;\varepsilon_{n2})\|\leqslant\exp\left[\frac{-\alpha_{Mi}^{(1)}}{\varepsilon_i}(t-\tau)\right],\qquad t\geqslant\tau,\quad \alpha_{Mi}^{(1)}<\alpha_{ii}^{(1)},$$

and

$$\|\phi_{Aj}^{(1)}(t,\tau;\varepsilon_j,\varepsilon_{ni})\|\leqslant\exp[\alpha_{Aj}^{(1)}(t-\tau)],\qquad t\geqslant\tau,\,\alpha_{Aj}^{(1)}>0,\,j\in I(1,n-1).$$

Using the same procedures as in the proof of Result 4.2, we get

$$\|\Delta L_{ij}\|\leqslant\left[\varepsilon_{i-1}^{-1}\left(\sum_{k=1}^{i-1}\frac{\varepsilon_{i-1}}{\varepsilon_k}b_{kj}^{(1)}+\varepsilon_{i-1}\|\dot{\tilde{L}}_{ij}\|\right)\right.$$
$$\left.+\varepsilon_i^{-1}\left\|O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right\|\right]\frac{\varepsilon_i}{\alpha_{Mi}^{(1)}-\varepsilon_i\alpha_{Aj}^{(1)}},$$

where

$$b_{kj}^{(1)}=[(1-\delta_{kj})\,\|L_{ik}\|+\delta_{kj}\,\|\tilde{L}_{ik}\|]$$
$$\cdot\|D_k(A_{ki}^{(1)}\tilde{L}_{ij}-(1-\delta_{kj})\,A_{kj}^{(1)})\|$$
$$+\|D_kA_{ki}^{(1)}\|\,(\|L_{ik}\|+\|\tilde{L}_{ik}\|)(\|L_{ij}\|+\|\tilde{L}_{ij}\|),$$

that is,

$$\|\Delta L_{ij}\|\leqslant\frac{\varepsilon_i}{\varepsilon_{i-1}}\frac{k_{ij}^{(1)}}{\alpha_{Mi}^{(1)}-\varepsilon_i\alpha_{Aj}^{(1)}}+\frac{1}{\alpha_{Mi}^{(1)}-\varepsilon_i\alpha_{Aj}^{(1)}}\left\|O\left(\frac{\varepsilon_i}{\varepsilon_{n-1}}\right)\right\|,$$

where

$$K_{ij}^{(1)} = \sum_{k=1}^{i-1} \frac{\varepsilon_{i-1}}{\varepsilon_k} b_{kj}^{(1)} + \varepsilon_{i-1} \|\dot{\tilde{L}}_{ij}\|.$$

This implies

$$\Delta L_{ij} \simeq O\left(\frac{\varepsilon_i}{\varepsilon_{i-1}}\right) + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right) = \sum_{k=i}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)$$

and hence the result (ii).

Using this process in a recessive fashion, we establish the boundedness and convergence results for the elements in the nth, $(n-1)$th, ..., $(p-1)$th rows of the transformation (3.2) and we get the corresponding approximations for these matrix functions as

$$L_{kj}(t) = \left(A_{kk}^{(n-k)}(t)\right)^{-1} A_{kj}^{(n-k)}(t) + \sum_{l=k}^{n} O\left(\frac{\varepsilon_l}{\varepsilon_{l-1}}\right),$$
$$\text{for} \quad k = n, n-1, ..., n-(n-p+1). \qquad (4.14)$$

Let us now consider the elements in the pth row of the transformation. Using (4.14), we can write the differential equation for L_{pq}'s as

$$\varepsilon_p \dot{L}_{pq} = D_p \left[A_{pp}^{(d)}(t) + \sum_{k=p+1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right] L_{pq}$$
$$+ \frac{-\varepsilon_p}{\varepsilon_q} L_{pq} D_q \left[A_{qq}^{(d)}(t) + \sum_{k=p}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right] + S_{pq}(t) \qquad (4.15)$$
$$L_{pq}(t_0) = (A_{pp}^{(d)}(t_0))^{-1} A_{pq}^{(d)}(t_0), \qquad q \in I(1, p-1),$$

where

$$S_{pq}(t) = \sum_{k=1}^{p} \frac{\varepsilon_p}{\varepsilon_k} L_{pk} D_k \{(1-\delta_{kj}) A_{kp}^{(d)}(t) L_{pq}$$
$$-(1-\delta_{kq}) \cdot A_{kq}^{(d)}(t)\} + \sum_{k=p+1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)$$

and $d = n - p$.

We now state and prove the following result.

Result 4.4. Under assumptions of Result 4.3, we have

(i) There exist $\varepsilon_k^{(d)} > 0$, $k \in I(2, d)$, such that for all $\varepsilon_k/\varepsilon_{k-1} < \varepsilon_k^{(d)}/\varepsilon_{k-1}^{(d)}$, $k \in I(2, d)$, the solutions $L_{pq}(t)$ of (4.15) are bounded for all $t \geq t_0$.

(ii) The L_{pq}'s have the form

$$L_{pq}(t) = \tilde{L}_{pq}(t) + \sum_{k=p}^{n} \left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right),$$

where (4.16)

$$\tilde{L}_{pq}(t) = (A_{pp}^{(d)}(t))^{-1} A_{pq}^{(d)}(t), \qquad q \in I(1, p-1).$$

Proof. As before, let $\phi_{pp}^{(d)}(t, \tau; \varepsilon_p)$, $\phi_p^{(d)}(t, \tau; \varepsilon_{np})$, and $\phi_q^{(d)}(t, \tau; \varepsilon_q, \varepsilon_{nd})$ be the state transition matrices of

$$\varepsilon_p \dot{y} = D_p A_{pp}^{(d)}(t)\, y,$$

$$\varepsilon_p \dot{y} = D_p \left[A_{pp}^{(d)}(t) + \sum_{k=p+1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right] y,$$

and

$$\varepsilon_q \dot{y} = D_q \left[A_{qq}^{(d)}(t) + \sum_{k=p+1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right] y, \; g \in I(1, p-1).$$

These transition matrices have the estimates on the norms similar to (4.13) with positive constants $\alpha_{pp}^{(d)}$, $\alpha_p^{(d)}$, and $\alpha_q^{(d)}$, where $\alpha_p^{(d)} < \alpha_{pp}^{(d)}$. The proof of (i) is now similar to the proof of part (i) of Result 4.3, where the choice of ε_k's is such that

$$2\varepsilon_p^{(d)} \max\{\alpha_1^{(d)}, \alpha_2^{(d)}, ..., \alpha_{p-1}^{(d)}\} < \alpha_p^{(d)}$$

$$\frac{8(p-1)}{\alpha_p^{(d)}} \frac{\varepsilon_p^{(d)}}{\varepsilon_k^{(d)}} \left[\|D_k A_{kp}^{(d)}\|\, \rho + \max_{\substack{1 \leqslant j \leqslant p-1 \\ j \neq k}} \{\|D_k A_{kj}^{(d)}\|\}\right]$$

$$\leqslant 1 \qquad \text{for} \quad k \in I(1, p-1),$$

where ρ is defined by

$$\rho = 2\left\{\frac{2}{\alpha_p^{(d)}} \max_{1 \leqslant j \leqslant p-1} \left\{\|D_p A_{pj}^{(d)}\| + \sum_{k-p+1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right\} + \max_{1 \leqslant j \leqslant p-1} \{\|L_{pj}(t_0)\|\}\right\}.$$

Consider the proof of (ii). We let $L_{pq}(t) = \tilde{L}_{pq}(t) + \Delta L_{pq}$. Then ΔL_{pq} satisfies the differential equation

$$\varepsilon_p \Delta \dot{L}_{pq} = \left[D_p A_{pp}^{(d)}(t) + \sum_{k=1}^{p-1} \frac{\varepsilon_p}{\varepsilon_k} \tilde{L}_{pk} D_k A_{kp}^{(d)}(t) + \sum_{k=p+1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right] \Delta L_{pq}$$

$$+ \frac{-\varepsilon_p}{\varepsilon_q} \Delta L_{pq} \left[D_q (A_{qq}^{(d)}(t) - A_{qp}^{(d)}(t)\, \tilde{L}_{pq}) + \sum_{k=p+1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right] + P_{pq}(t),$$

where

$$P_{pq}(t) = \sum_{k=1}^{p-1} \frac{\varepsilon_p}{\varepsilon_k} \{[\tilde{L}_{pk} + (1-\delta_{kq})\, \Delta L_{pk}] \times D_k[A_{kp}^{(d)}(t)\, \tilde{L}_{pq} - (1-\delta_{kq})\, A_{kq}^{(d)}(t)] + \Delta L_{pk}\, D_k A_{kp}^{(d)}(t)\, \Delta L_{pq}\} - \varepsilon_p \dot{\tilde{L}}_{pq} + \sum_{k=p+1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).$$

The rest of the proof is analogous to the proof of (ii) of Result 4.3.

Finally, the matrix function $L_{21}(t)$ satisfies the differential equation

$$\varepsilon_2 \dot{L}_{21} = D_2\left[A_{22}^{(n-2)}(t) + \sum_{k=3}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right] L_{21} - \varepsilon_2 L_{21}\left[A_{11}^{(n-2)}(t) + \sum_{k=3}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right] + S_{21}(t), \tag{4.17}$$

where

$$S_{21}(t) = \sum_{k=1}^{2} \frac{\varepsilon_2}{\varepsilon_k} L_{2k}\, D_k\{(1-\delta_{k2})\, A_{k2}^{(n-2)}(t) L_{21} - (1-\delta_{k1}) A_{k1}^{(n-2)}(t)\} + \sum_{k=3}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).$$

From the boundedness of L_{pq}'s, $p \in I(3, n)$, $q \in I(1, p-1)$, assumption (A5), and noting that the coefficient matrices in (4.17) are bounded, it follows that the solution $L_{21}(t)$ of (4.17) is bounded. The convergence of $L_{21}(t)$ is established in the next result.

Result 4.5. Under (A4) and (A5), we have

$$L_{21}(t) = \tilde{L}_{21}(t) + \sum_{k=2}^{n} O(\varepsilon_k/\varepsilon_{k-1}),$$

where (4.18)

$$\tilde{L}_{21}(t) = [A_{22}^{(n-2)}(t)]^{-1}\, A_{21}^{(n-2)}(t).$$

Proof. Let $L_{21}(t) = \tilde{L}_{21}(t) + \Delta L_{21}$. Then ΔL_{21} satisfies

$$\varepsilon_2\, \Delta\dot{L}_{21} = \left[D_2 A_{22}^{(n-2)}(t) + \varepsilon_2 \tilde{L}_{21} A_{12}^{(n-2)}(t) + \sum_{k=3}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right] \Delta L_{21} - \varepsilon_2\, \Delta L_{21}\left[A_{11}^{(n-2)}(t) + A_{12}^{(n-2)}(t) \tilde{L}_{21} + \sum_{k=3}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right] + P_{21}(t),$$

where

$$P_{21}(t)=\varepsilon_2[\tilde{L}_{21}(t)(A_{12}^{(n-2)}(t)\tilde{L}_{21}-A_{11}^{(n-2)}(t))+\Delta L_{21}A_{12}^{(n-2)}(t)\,\Delta L_{21}]$$
$$-\varepsilon_2\dot{\tilde{L}}_{21}+\sum_{k=3}^{n}O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).$$

The rest of the proof is analogous to the proof of (ii) of Result 4.3.

By establishing the boundedness and convergence of the solutions $L_{pq}(t)$, $p\in I(2,n)$, $q\in I(1,P-1)$, of the auxiliary system (3.3) with initial conditions (3.4), we have verified the validity of the transformation (3.2). Therefore, the transformed system (3.7) can be rewritten in terms of the original matrices of (3.1) as

$$\dot{u}=\bar{B}u, \qquad \text{where the elements of } \bar{B} \text{ are given by} \tag{4.19}$$

$$\varepsilon_p b_{pq}=D_pA_{pp}^{(n-q)}(t)+(1-\delta_{pn})(1-\delta_{qn})\sum_{k=q+1}^{n}O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)$$
$$+(1-\delta_{p1})\,O\left(\frac{\varepsilon_p}{\varepsilon_{p-1}}\right),\qquad q\geqslant p \tag{4.20}$$
$$b_{pq}=0,\qquad q<p,$$

where

$$O\left(\frac{\varepsilon_p}{\varepsilon_{p-1}}\right)=\sum_{j=1}^{p-1}\left[O\left(\frac{\varepsilon_p}{\varepsilon_j}\right)+\left(\frac{\varepsilon_p}{\varepsilon_j}\right)\sum_{k=q}^{n}O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right] \tag{4.21}$$

and

$$O\left(\frac{\varepsilon_p}{\varepsilon_j}\right)=\frac{\varepsilon_p}{\varepsilon_j}\,[D_p(A_{pp}^{(n-p)}(t))^{-1}\,A_{pj}^{(n-p)}(t)\,A_{jq}^{(n-q)}(t)], \tag{4.22}$$

with initial conditions given by

$$u_1(t_0)=x_{10}$$
$$u_i(t_0)=x_{i0}+\sum_{j=1}^{i-1}(A_{ii}^{(n-i)}(t_0))^{-1}\,A_{ij}^{(n-i)}(t_0)x_{j0}$$
$$+\sum_{k=i}^{n}O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right),\qquad i=I(2,n).$$

In the next section, using the transformed system (4.19), we obtain our main result concerning the qualitative properties of the original system (3.1).

5. Main Result

We are now in a position to use the triangular form of the transformed system (4.19) and establish stability of the orginal system (3.1). For this, we need to make the assumption concerning the stability of the reduced subsystem (2.8) (corresponding to $j=n-2$) with the system matrix $A_{11}^{(n-1)}(t)$.

(A6) There exists a positive number α_R such that

$$\limsup_{t\to\infty}\left(\frac{1}{t-t_0}\int_{t_0}^{t} L\{A_{11}^{(n-1)}(\tau)\}\,d\tau\right)\leqslant -\alpha_R. \tag{5.1}$$

Assumption (A6) implies that the reduced subsystem is globally, exponentially stable. We now have the following main result.

Result 5.1. Under (A1)–(A6), there exist positive numbers $\hat{\varepsilon}_k$, $k=1, 2, ..., n$, such that if $\varepsilon_n<\hat{\varepsilon}_n$ and $\varepsilon_k/\varepsilon_{k-1}<\hat{\varepsilon}_k/\hat{\varepsilon}_{k-1}$ for $k=2, 3, ..., n$, then the equilibrium of the system (3.1) is globally, exponentially stable.

Proof. The inverse of the transformation T is given by the matrix $R=(R_{ij})$ in (3.6). From the boundedness of the submatrix functions in (3.2), it is easy to see that $T^{-1}=R$ is also bounded. From the boundedness of T^{-1}, it follows that the stability of (4.19) implies the stability of (3.1). To show stability of (4.19), it is necessary to verify the stability of each decoupled subsystem by considering the smallness of their interactions as regular perturbation terms. This is done as follows:

First, consider the u_n subsystem (or ε_n-subsystem). u_n satisfies the differential equation

$$\varepsilon_n\dot{u}_n=\left[D_nA_{nn}(t)+O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right]u_n. \tag{5.2}$$

Let $\phi_n(t, \tau; \varepsilon_n, \varepsilon_{n-1})$ be the state transition matrix of (5.2). Then, using Lemma 4.1, it is easy to see that

$$\|\phi_n(t, \tau; \varepsilon_n, \varepsilon_{n-1})\|\leqslant \exp\left[\frac{-\alpha_n}{\varepsilon_n}(t-\tau)\right], \qquad t\geqslant\tau,\ \alpha_n<\alpha_{nn}. \tag{5.3}$$

This implies that the ε_n-subsystem of (4.19) is stable.

Next, consider the differential equation satisfied by u_{n-1}, that is, the ε_{n-1} subsystem of (4.19):

$$\varepsilon_{n-1}\dot{u}_{n-1}=\left[D_{n-1}A_{n-1,n-1}^{(1)}(t)+O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)+O\left(\frac{\varepsilon_{n-1}}{\varepsilon_{n-2}}\right)\right]u_{n-1}(t)$$
$$+\left[D_{n-1}A_{n-1,n}(t)+O\left(\frac{\varepsilon_{n-1}}{\varepsilon_{n-2}}\right)\right]u_n(t).$$

If $\tilde{\phi}_{n-1}(t, \tau; \varepsilon_{nn-2})$ is the state transition matrix of

$$\varepsilon_{n-1}\dot{y} = \left[D_{n-1}A^{(1)}_{n-1,n-1}(t) + \sum_{k=n-1}^{n} O(\varepsilon_k/\varepsilon_{k-1})\right] y,$$

then $\|\tilde{\phi}_{n-1}(t, \tau; \varepsilon_{nn-2})\|$ has an estimate similar to that of $\|\phi^{(1)}_{n-1}(t, \tau; \varepsilon_{nn-1})\|$ given by (4.13) with a constant $\tilde{\alpha}_{n-1}$, that is,

$$\tilde{\phi}_{n-1}(t, \tau; \varepsilon_{nn-1}) \leqslant \exp\left[\frac{-\tilde{\alpha}_{n-1}}{\varepsilon_{n-1}}(t-\tau)\right], \qquad t \geqslant \tau. \tag{5.4}$$

Using the variation of parameters formula, we have

$$u_{n-1}(t) = \tilde{\phi}_{n-1}(t, t_0)\, u_{n-1}(t_0) + \int_{t_0}^{t} \tilde{\phi}_{n-1}(t, \tau)\, \tilde{S}_{n-1,n}(\tau)\, u_n(\tau)\, d\tau, \tag{5.5}$$

where

$$\tilde{S}_{n-1,n}(t) = \varepsilon_{n-1}^{-1} S_{n-1,n}(t),$$

$$S_{n-1,n}(t) = \left[D_{n-1}A_{n-1,n}(t) + O\left(\frac{\varepsilon_{n-1}}{\varepsilon_{n-1}}\right)\right],$$

and

$$u_n(t) = \phi_n(t, t_0, \varepsilon_n, \varepsilon_{n-1})\, u_n(t_0).$$

Taking norms on both sides of (5.5) and after a few simple calculations, we get

$$\|u_{n-1}(t)\| \leqslant k_{n-1,n} \exp\left[\frac{-\alpha_{n-1}}{{}_{n-1}}(t-t_0)\right], \tag{5.6}$$

where

$$k_{n-1,n} = \|u_{n-1}(t_0)\| + \frac{\varepsilon_n}{\varepsilon_{n-1}} \|S_{n-1,n}\|\, \|u_n(t_0)\|\, k_n^{-1}[1 - \exp(-k_n(t-t_0))] \tag{5.7}$$

and

$$k_n = \left(\alpha_n - \frac{\varepsilon_n}{\varepsilon_{n-1}}\tilde{\alpha}_{n-1}\right).$$

From (A1), we can choose $\tilde{\varepsilon}_n^{(1)}$, $\tilde{\varepsilon}_{n-1}^{(1)}$ sufficiently small such that for all $\varepsilon_n/\varepsilon_{n-1} < \tilde{\varepsilon}_n^{(1)}/\tilde{\varepsilon}_{n-1}^{(1)}$, we have $k_n >$. Then (5.6) implies the stability of the ε_{n-1}-subsystem. Consider the ε_{n-2}-subsystem. u_{n-2} satisfies

$$\varepsilon_{n-2}\dot{u}_{n-2} = \varepsilon_{n-2}[b_{n-2,n-2}(t)u_{n-2}(t) + \tilde{S}_{n-2,n}(t)],$$
$$\tilde{S}_{n-2,n}(t) = \varepsilon_{n-2}^{-1}(S_{n-2,n-1}(t)\,u_{n-1}(t) + S_{n-2,n}(t)\,u_n(t)), \tag{5.8}$$
$$S_{n-2,n-1}(t) = \left[D_{n-2}A_{n-2,n-1}^{(1)}(t) + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right) + O\left(\frac{\varepsilon_{n-2}}{\varepsilon_{n-3}}\right)\right],$$

and

$$S_{n-2,n}(t) = \left[D_{n-2}A_{n-2,n}(t) + O\left(\frac{\varepsilon_{n-2}}{\varepsilon_{n-3}}\right)\right].$$

From (5.3), (5.6), and (5.8), we see that

$$\|\tilde{S}_{n-2,n}(t)\| \leqslant \exp\left[\frac{-\tilde{\alpha}_{n-1}}{\varepsilon_{n-1}}(t-t_0)\right]$$
$$\times \{\|S_{n-2,n-1}\|\, K_{n-1,n} + \|S_{n-2,n}\|\, \|u_n(t_0)\|\}\, \varepsilon_{n-2}^{-1}.$$

Let $\tilde{\Phi}_{n-2}(t, \tau; \varepsilon_{nn-3})$ be the state transition matrix of

$$\varepsilon_{n-2}\,\dot{y} = \left[D_{n-2}A_{n-2,n-2}^{(2)}(t) + \sum_{k=n-2}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right]y.$$

Then, we can see that

$$\|\tilde{\Phi}_{n-2}(t, \tau; \varepsilon_{nn-3})\| \leqslant \exp\left[\frac{-\tilde{\alpha}_{n-2}}{\varepsilon_{n-2}}(t-\tau)\right], \qquad \tilde{\alpha}_{n-2} < \alpha_{n-2}^{(2)}. \tag{5.9}$$

Using variation of parameters formula, we can write the expression for $u_{n-2}(t)$ as

$$u_{n-2}(t) = \tilde{\Phi}_{n-2}(t, t_0)\, u_{n-2}(t_0) + \int_{t_0}^{t} \tilde{\Phi}_{n-2}(t, \tau)\, \tilde{S}_{n-2,n}(\tau)\, d\tau.$$

After some simple computations, we have

$$\|u_{n-2}(t)\| \leqslant \exp\left[\frac{-\tilde{\alpha}_{n-2}}{\varepsilon_{n-2}}(t-t_0)\right]$$
$$\times \left\{\|u_{n-2}(t_0)\| + \frac{\varepsilon_{n-1}}{\varepsilon_{n-2}} k_{n-1}^{-1}[\|S_{n-2,n-1}\|\, k_{n-1,n}(t_0)\right.$$
$$\left. + \|S_{n-2,n}\|\, \|u_n(t_0)\|\, \{1 - \exp[-k_{n-1}(t-t_0)]\}]\right\},$$

where

$$k_{n-1}=\left[\tilde{\alpha}_{n-1}-\frac{\varepsilon_{n-1}}{\varepsilon_{n-2}}\tilde{\alpha}_{n-2}\right].$$

Now, choose $\tilde{\varepsilon}^{(2)}_{n-1}/\tilde{\varepsilon}^{(2)}_{n-2}$ small enough such that for all $\varepsilon_{n-1}/\varepsilon_{n-2}<\tilde{\varepsilon}^{(2)}_{n-1}/\tilde{\varepsilon}^{(2)}_{n-2}$, we have $k_{n-1}>0$. Thus, for all $\varepsilon_n/\varepsilon_{n-1}<\tilde{\varepsilon}^{(1)}_n/\tilde{\varepsilon}^{(1)}_{n-1}$ and $\varepsilon_{n-1}/\varepsilon_{n-2}<\tilde{\varepsilon}^{(2)}_{n-1}/\tilde{\varepsilon}^{(2)}_{n-2}$, we have

$$\|u_{n-2}\|\leqslant k_{n-2,n}\exp\left[\frac{-\tilde{\alpha}_{n-2}}{\varepsilon_{n-2}}(t-t_0)\right], \tag{5.10}$$

where

$$k_{n-2,n}=\|u_{n-2}(t_0)\|+\frac{\varepsilon_{n-1}}{\varepsilon_{n-2}}k^{-1}_{n-1}$$
$$\times[\|S_{n-2,n-1}\|\,k_{n-1,n}(t_0)+\|S_{n-2,n}\|\,\|u_n(t_0)\|].$$

Equation (5.10) implies that the ε_{n-2}-subsystem is stable.

Using a similar procedure, we show that there exist $\hat{\varepsilon}_k>0$, $k\in I(1,n)$, such that for all $\varepsilon_k/\varepsilon_{k-1}<\bar{\varepsilon}^{(n-2)}_k/\bar{\varepsilon}^{(n-2)}_{k-1}$ and $\varepsilon_n<\hat{\varepsilon}_n$, we have the stability of $u_k(t)$, $k\in I(2,n)$.

Finally, consider the differential equation for $u_1(t)$:

$$\dot{u}_1=b_{11}u_1=\left[A^{(n-1)}_{11}(t)+\sum_{k=2}^{n}O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)\right]u_1. \tag{5.11}$$

Let $\phi_R(t,\tau;\varepsilon_{n2})$ be the state transition matrix of (5.11) and $m_R(t)=\|\phi_R(t,\tau;\varepsilon_{n2})\|$. Solving the differential inequality

$$D_+m_R(t)\leqslant\left[L\{A^{(n-1)}_{11}(t)\}+\sum_{j=2}^{n}k_j\frac{\varepsilon_j}{\varepsilon_{j-1}}\right]m_R(t),$$

where k_j is a positive number such that $\|O(\varepsilon_j/\varepsilon_{j-1})\|\leqslant k_j(\varepsilon_j/\varepsilon_{j-1})$, we get

$$m_R(t)\leqslant\exp\left[\left\{\frac{1}{t-\tau}\int_\tau^t L\{A^{(n-1)}_{11}(\tau)\}\,d\tau+k_p\right\}(t-\tau)\right], \tag{5.12}$$

where $k_p=\sum_{j=2}^{n}k_j(\varepsilon_j/\varepsilon_{j-1})$. From (A6), we have

$$m_R(t)\leqslant\exp[-(\alpha_R-k_p)(t-\tau)],\qquad t\geqslant\tau+T, \tag{5.13}$$

and from (A1), we can choose $\varepsilon_{kr}/\varepsilon_{k-1,r}$, $k\in I(2,n)$, such that for all $\varepsilon_k/\varepsilon_{k-1}<\varepsilon_{kr}/\varepsilon_{k-1,r}$, we have $(\alpha_R-k_p)>0$. Thus, for $t\in[\tau,\tau+T]$, we can write (5.12) as

$$m_R(t)\leqslant k\exp[-(\alpha_R-k_p)(t-\tau)],\qquad t\in[\tau,\tau+T], \tag{5.14}$$

where k is given by

$$k = \max_{t \in [\tau, \tau + T]} \left[\exp \left[\left\{ \frac{1}{t - \tau} \int_\tau^t L\{A_{11}^{(n-1)}(\tau)\} \, d\tau + k_p \right\} (t - \tau) \right] \right].$$

From (5.13) and (5.14), we have exponential stability for (5.11). We now choose

$$\frac{\hat{\varepsilon}_k}{\hat{\varepsilon}_{k-1}} = \min \left\{ \frac{\bar{\varepsilon}_k^{(n-2)}}{\bar{\varepsilon}_{k-1}^{(n-2)}}, \frac{\varepsilon_{kr}}{\varepsilon_{k-1,r}} \right\}, \qquad k \in I(2, n).$$

Then, for $\varepsilon_k/\varepsilon_{k-1} < \hat{\varepsilon}_k/\hat{\varepsilon}_{k-1}$, $k \in I(2, n)$, we have the overall stability of (4.19) and thus the stability of (3.1).

In the remaining part of this section, we get the approximate solutions for $u_i(t)$, $i \in I(1, n)$, in terms of the solutions for the boundary layer system (2.9) and the overall reduced problem (2.8) (corresponding to the value of $j = n - 2$). Then, we make use of these solutions and the inverse of the transformation T to get the approximate solutions for $x_i(t)$ in terms of $u_i(t)$.

To get the approximate solutions for $u_i(t)$, we need to make one last assumption.

(A7) The matrix functions $A_{jj}^{(n-j)}(t)$ are Lipschitzian on R, that is, there exist constants $\tilde{k}_j$ such that

$$|A_{jj}^{(n-j)}(t_1) - A_{jj}^{(n-j)}(t_2)| \leqslant \tilde{k}_j \, |t_1 - t_2|, \qquad \forall t_1, t_2 \in R, \; j \in I(2, n).$$

First, consider the ε_n-subsystem. We have

$$\dot{u}_n = \left[D_n \varepsilon_n^{-1} A_{nn}(t) + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right) \right] u_n \tag{5.15}$$

and the corresponding ε_n-boundary layer system is given by

$$\frac{d\hat{x}_n}{d\tau_n} = D_n A_{nn}(t_0) \, \hat{x}_n(\tau_n). \tag{5.16}$$

Let $\phi_n(t, \tau; \varepsilon_{nn-1})$ and $\hat{\phi}_n(t, \tau; \varepsilon_n)$ be the state transition matrices of (5.15) and (5.16), respectively. Denote

$$\psi_n(t, \tau; \varepsilon_{nn-1}) = \phi_n(t, \tau; \varepsilon_{nn-1}) - \hat{\phi}_n(t, \tau; \varepsilon_n). \tag{5.17}$$

Consider the differential equation satisfied by ψ_n:

$$\varepsilon_n \dot{p} = \left[D_n A_{nn}(t) + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right) \right] p$$
$$+ \left[D_n(A_{nn}(t) - A_{nn}(\tau)) + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right) \right] \hat{\phi}_n(t, \tau; \varepsilon_n).$$

Now, $\|\hat{\phi}_n(t, \tau; \varepsilon_n)\| \leqslant \exp[(-\alpha_{nn}/\varepsilon_n)(t-\tau)]$, $t \geqslant \tau$. Let $m(t) = \|\psi_n(t, \tau; \varepsilon_{nn-1})\|$. Computing $\lim_{h \to 0^+} ((m(t+h)-m(t))/h)$, we get

$$D_+ m(t) \leqslant \left[L\left\{\frac{1}{\varepsilon_n} D_n A_{nn}(t)\right\} + L\left\{\frac{1}{\varepsilon_n} O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right\}\right] m(t) + \frac{1}{\varepsilon_n} \|q_n(t, \tau, \varepsilon_{nn-1})\|,$$

where

$$q_n(t, \tau; \varepsilon_{nn-1}) = \left[D_n(A_{nn}(t) - A_{nn}(\tau)) + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right] \hat{\phi}_n$$

and

$$m(\tau) = 0.$$

Using the comparison principle, we get $m(t) \leqslant r(t, \tau, 0)$, $t \geqslant \tau$, where $r(t, \tau, 0)$ is the solution of the differential equation

$$\dot{r} = \left[L\left\{\frac{1}{\varepsilon_n} D_n A_{nn}(t)\right\} + \frac{1}{\varepsilon_n}\left\| O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right\|\right] r + S(t),$$

where

$$S(t) = \varepsilon_n^{-1}\left[\left\| D_n(A_{nn}(t) - A_{nn}(\tau)) + O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right\|\right] \times \exp\left[-\frac{1}{\varepsilon_n}\alpha_{nn}(t-\tau)\right].$$

The solution $r(t)$ is given by

$$r(t; \tau, 0) = \int_\tau^t \exp\left\{\int_\theta^t \left[L\left\{\frac{1}{\varepsilon_n} D_n A_{nn}(\eta)\right\} + \frac{1}{\varepsilon_n}\left\| O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)\right\|\right] d\eta\right\} S(\theta)\, d\theta.$$

Using assumption (A7) and doing some simple computations, we get

$$r(t; \tau, 0) \leqslant \frac{\varepsilon_n}{\varepsilon_{n-1}}\left[\frac{k_n}{e\alpha_n} + \frac{\tilde{k}_n}{2\alpha_n^2 e^2}\varepsilon_{n-1}\right] \sim O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right).$$

Thus, we have

$$m(t) = \|\psi_n(t, \tau; \varepsilon_n^{n-1})\| \sim O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right). \tag{5.18}$$

Now, consider the expression

$$\|u_n - \hat{x}_n\| = \|\phi_n(t, t_0; \varepsilon_{nn-1})\, u_n(t_0) - \hat{\phi}_n(t, t_0; \varepsilon_{nn-1})\, \hat{x}_n(0)\|.$$

Substituting for $u_n(t_0)$, $\hat{x}_n(0)$ and rewriting, we get

$$\|u_n - \hat{x}_n\| \leqslant \|\phi_n - \hat{\phi}_n\| \left\| x_{n0} + \sum_{j=1}^{n-1} [A_{nn}(t_0)]^{-1} A_{nj}(t_0)\, x_{j0} \right\| + \left\| O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right) \right\| \|\phi_n(t, t_0, \varepsilon_{nn-1})\|.$$

From (5.18), we have $\|u_n - \hat{x}_n\| \simeq O(\varepsilon_n/\varepsilon_{n-1})$, that is, $u_n(t) \simeq \hat{x}_n((t-t_0)/\varepsilon_n) + O(\varepsilon_n/\varepsilon_{n-})$. Let us now consider $\|u_{n-1} - \hat{x}_{n-1}\|$:

$$\begin{aligned}\|u_{n-1} - \hat{x}_{n-1}\| &= \Big\| \tilde{\phi}_{n-1}(t, t_0)\, u_{n-1}(t_0) \\ &\quad + \int_{t_0}^{t} \tilde{\phi}_{n-1}(t, \tau)\, \tilde{S}_{n-1,n}(\tau)\, u_n(\tau)\, dt - \hat{\phi}_{n-1}(t, t_0)\, \hat{x}_{b-1}(0) \Big\| \\ &\leqslant c_{n-1} \|\tilde{\phi}_{n-1}(t, t_0) - \hat{\phi}_{n-1}(t, t_0)\| + \|\tilde{\phi}_{n-1}(t, t_0)\| \\ &\quad \times \left\| \sum_{k=n-1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right) \right\| + \int_{t_0}^{t} \|\tilde{\phi}_{n-1}\|\, \|\tilde{S}_{n-1,n}(\tau)\|\, \|u_n(\tau)\|\, d\tau,\end{aligned}$$

where

$$c_{n-1} = \left\| x_{n-1,0} + \sum_{j=1}^{n-2} [A^{(1)}_{n-1,n-1}(t_0)]^{-1} A^{(1)}_{n-1,j}(t_0) x_{j0} \right\|.$$

Letting $m(t) = \|\psi_{n-1}(t, \tau; \varepsilon_{nn-2})\| = \|\tilde{\phi}_{n-1}(t, \tau) - \hat{\phi}_{n-1}(t, \tau)\|$ and repeating the same procedure as before, we can show that

$$\|\psi_n(t; \tau, 0)\| \simeq O\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right) + O\left(\frac{\varepsilon_{n-1}}{\varepsilon_{n-2}}\right) \simeq \sum_{k=n-1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).$$

Then, we have the estimate for $\|u_{n-1} - \hat{x}_{n-1}\|$ as

$$\begin{aligned}\|u_{n-1} - \hat{x}_{n-1}\| &\leqslant \sum_{k=n-1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right) \left[c_{n-1} + \exp\left[\frac{-\alpha_{n-1}}{\varepsilon_{n-1}} (t - t_0) \right] \right] \\ &\quad + \varepsilon_n k_n^{-1} \|\tilde{S}_{n-1,n}\|\, \|u_n(t_0)\| \exp\left[\frac{-\tilde{\alpha}_{n-1}}{\varepsilon_{n-1}} (t - t_0) \right].\end{aligned}$$

Now $\varepsilon_n k_n^{-1} \|\tilde{S}_{n-1,n}\| \simeq O(\varepsilon_n/\varepsilon_{n-1})$. Then

$$\|u_{n-1} - \hat{x}_{n-1}\| \simeq \sum_{k=n-1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).$$

Thus, we have

$$u_{n-1}(t) \simeq \hat{x}_{n-1}\left(\frac{t-t_0}{\varepsilon_{n-1}}\right) + \sum_{k=n-1}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right).$$

Using this process repeatedly, we show that

$$\begin{aligned} u_i(t) &= \hat{x}_i\left(\frac{t-t_0}{\varepsilon_i}\right) + \sum_{k=i}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right), \qquad i \in I(2, n), \\ u_1(t) &= \tilde{x}_1(t) + \sum_{k=2}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right), \end{aligned} \tag{5.19}$$

where $\tilde{x}_1(t)$ is the solution of the overall reduced system (2.8) and $\hat{x}_i(t)$ are the solutions of the corresponding ε_{n-i}-boundary layer systems (2.9).

Using the approximations for $L_{ij}(t)$, $i \in I(2, n)$, $j \in I(1, i-1)$, the inverse of transformation T (given by the matrix elements R_{ij}), and (5.19), we can write the approximate solutions for $x_i(t)$ as

$$\begin{aligned} x_i(t) &= \hat{x}_i(t) - \sum_{k=1}^{i-1} c_{ik}(t)\, \hat{x}_k(t) + \sum_{k=2}^{n} O\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right) \\ \hat{x}_1(t) &= \hat{x}_1(t), \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} c_{ij}(t) &= [A_{ii}^{(n-i)}(t)]^{-1} A_{ij}^{(n-i)}(t) \\ &\quad - (1 - \delta_{i-1,j}) \sum_{k=j+1}^{i-1} [A_{ii}^{(n-i)}(t)]^{-1} A_{ik}^{(n-i)}(t)\, c_{kj}(t), \\ &\quad \text{for} \quad i \in I(2, n) \text{ and } j \in I(1, i-1). \end{aligned}$$

Note that $c_{ij}(t)$ are obtained in a recursive fashion.

Thus (5.20) and (5.21) together give the approximate solutions of system (3.1) in terms of the reduced system and the various boundary layer systems.

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