JOURNAL OF COMBINATORIAL THEORY 5, 153–156 (1968)

On a Conjecture of Milner on k-Graphs with Non-Disjoint Edges

DANIEL KLEITMAN*

Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

ABSTRACT

The following theorem is proven. It is a slight generalization of a conjecture of Eric Milner.

Consider two families, one consisting of k and the other of l element subsets of an n element set. Let each member of one have nonempty intersection with each member of the other and let $k + l$ be less than or equal to n.

Then either there are no more than $\binom{n-1}{k-1}$ members of the first family or there are fewer than $\binom{n-1}{l-1}$ members of the second.

Let S be an n-element set. Suppose we have a collection of k-element subsets of S with the property that no two of the sets are disjoint. Erdös, Ko, and Rado [1] showed that, if $2k \leqslant n$, there can be no more than $\binom{n-1}{k-1}$ sets in the collection. E. C. Milner has raised the following related question. Suppose we have two collections of k-element subsets (which we will call k-edges below) such that each member of one has non-empty intersection with each member of the other. Milner conjectured that the number of members of the smaller collection would, if $2k \leqslant n$, have to have $\binom{n-1}{k-1}$ or fewer members.

We prove below a generalization of Milner's conjecture; it is interesting that it is much easier to prove the more general result then to prove the special case directly. Our result is as follows: If we have two collections of subsets of S, one of k-edges, the other of l-edges, with the restriction that each member of one has non-empty intersection with each member of the other, then if $k + l \leqslant n$ either the first has $\binom{n-1}{k-1}$ or fewer members, or the second has $\binom{n-1}{l-1}$ or fewer members.

A direct proof based on complementation can be given for $k + l = n$. For $k + l < n$, we proceed by induction on n, making use of the fact that we can always find maximal pairs of collections for which there is an element of S which can never be the intersection of a member of one with a member of the other.

* This research was supported in part by NSF Contract GP6165.

582/5/2-4

We first present the direct argument for the case $k + l = n$. Then we verify the last remark in the paragraph immediately above. Finally we apply this remark to yield our desired result.

Let F_k and G_l be collections of k-edges and l-edges of S (i.e., a k-graph and l-graph, respectively, of S), and let each k-edge of F_k intersect each l-edge of G_l. Let the number of members of F_k be f_k, and of G_k, g_k.

Suppose that $k + l = n$. Then the complement in S of the l-edges in G_l form a k-graph $\bar{G}_k$ no member of which can lie in F_k. We can immediately deduce then that

$$f_k + g_l \leqslant \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{l-1},$$

from which it follows that either $f_k \leqslant \binom{n-1}{k-1}$ or $g_l < \binom{n-1}{l-1}$.

Let us order the n-elements of S as $s_1, \ldots, s_n$ and let us write each subset A of S as an ordered sequence of zeros and ones; thus we write A as $\{A_j\}$ with $A_j = 1$ when $s_j \in A$.

We define the following set of mappings m_i, for $1 \leqslant i \leqslant n-1$, which take k-edges into k-edges for each $k \leqslant n$:

$$\begin{aligned} m_i(A_i) &= A_i + A_{i+1} - A_i A_{i+1} = \max(A_i, A_{i+1}), \\ m_i(A_{i+1}) &= A_i A_{i+1} = \min(A_i, A_{i+1}), \\ m_i(A_j) &= A_j \qquad \text{for} \quad i \neq j \neq i+1. \end{aligned}$$

Notice that the mapping m_i acting on a subset A which contains one of s_i and s_{i+1} yields the subset, otherwise identical to A, which contains s_i and not s_{i+1}. All other subsets are unchanged by the action of m_i.

Further, for any collection F of subsets of S we define $m_i(A; F)$ according to

$$\begin{aligned} m_i(A; F) &= m_i(A) \qquad \text{if} \quad m_i(A) \notin F, \\ m_i(A; F) &= A \qquad \text{if} \quad m_i(A) \in F. \end{aligned}$$

Suppose now, with F_k and G_l satisfying the conditions imposed on them above, we examine the collections $m_i(F_k; F_k)$ and $m_i(G_l; G_l)$. These will have the same number of members as F_k and G_l, respectively, and will again have the property that each member of one intersects each member of the other. For suppose some member C of $m_i(F_k; F_k)$ fails to intersect a member D of $m_i(G_l; G_l)$. Suppose that $C = m_i(A; F_k)$, $D = m_i(B; G_l)$. We then have the following situation: $A \cap B \neq \emptyset$ by hypothesis, hence $m_i(A) \cap m_i(B) \neq \emptyset$. But $C \cap D = m_i(A, F_k) \cap m_i(B, G_l) = \emptyset$; hence either $m_i(A) \neq (A, F_k)$ or $m_i(B) \neq m_i(B, G_l)$. Suppose $m_i(A) \neq m_i(A, F_k)$; then $m_i(A) \in F_k$ and $m_i(A) \cap B = A \cap m_i(B) = C \cap D = \emptyset$, which violates our hypotheses. The argument that applies for $m_i(B) \neq m_i(B, G_l)$ is identical to this one.

We define a stable pair of collections F_k, G_l, to be a pair which is invariant under the transformations $F_k \to m_i(F_k; F_k)$ $G_l \to m_i(G_l; G_l)$ for all i, $l \leqslant i \leqslant n - 1$. The argument above tells us that, starting with any pair (F_k, G_l) satisfying our conditions, we can, by repeated application of the m_i transformations, obtain new pairs (F_k, G_l) which have the same number of edges in each component as have F_k and G_l, and which again satisfy our intersection property.

For any collection F of subsets let

$$\alpha(F) = \sum_{A \in F} \sum_{j=1}^{n} A_j .$$

For each i, $\alpha(m_i(F, F)) < \alpha(F)$ unless $m_i(F, F) = F$; also for all F $\alpha(F) \geqslant 0$. Consequently repeated applications of the m-transformations must eventually yield a stable pair $(\bar{F}_k, \bar{G}_l)$ starting from any pair (F_k, G_l).

A stable pair $(\bar{F}_k, \bar{G}_l)$ have the property that, if one takes any member of $\bar{F}_k$ (or $\bar{G}_l$) and replaces any element S_l in it by a "smaller element" (S_r for $r < l$), the resulting subset is again in $\bar{F}_k$ (or $\bar{G}_l$). We may therefore conclude that no member A of $\bar{F}_k$ can intersect a member B of $\bar{G}_l$ in s_n only, if $k + l \leqslant n$. Otherwise we could pick an element in neither A nor B (one must exist since $A \cup B$ can contain at most $n - 1$ elements) and consider the set A' obtained from A by replacing s_n by it. Then $A' \in \bar{F}_k$, and also if $A \cap B = \{s_n\}$, we would have $A' \cap B = \emptyset$, violating our assumptions about $\bar{F}_k$ and $\bar{G}_l$.

We are now in a position to prove our theorem. Suppose $k + l < n$, and let $\bar{F}_{kl}$ and $\bar{G}_{l1}$ be the collections of $(k - 1)$-edges and $(l - 1)$-edges of $\{s_1, ..., s_{n-1}\}$ whose union with $\{s_n\}$ lie in $\bar{F}_k$ and $\bar{G}_l$, respectively. Let $\bar{F}_{k0}$ and $\bar{G}_{l0}$ be the collections of k-edges and l-edges of $\{s_1, ..., s_{n-1}\}$ which lie in $\bar{F}_k$ and $\bar{G}_l$. By our hypotheses, each member of $\bar{F}_{k1}$, and each member of $\bar{F}_{k0}$, intersect each member of $\bar{G}_{l1}$, and each member of $\bar{G}_{l0}$. We may therefore apply our induction hypothesis to each of the four pairs $(\bar{F}_{ki}, \bar{G}_{lj})$ (since $k + l < n$, we have $k + l \leqslant n - 1$) and with the number of members of $\bar{F}_{k0}, \bar{F}_{k1}, \bar{G}_{k0}$, and $\bar{G}_{k1}$ denoted, respectively, by $f_{k0}, f_{k1}, g_{k0}, g_{k1}$, we find that either both $f_{k0} \leqslant \binom{n-2}{k-1}$ and $f_{k1} \leqslant \binom{n-2}{k-2}$ or both $g_{k0} < \binom{n-2}{k-1}$ and $g_{k1} < \binom{n-2}{l-2}$ (for $l \geqslant 2$).

We conclude that either

$$f_k = f_{k0} + f_{k1} \leqslant \binom{n-1}{k-1}$$

or

$$g_l = g_{l0} + g_{l1} < \binom{n-1}{l-1},$$

which proves our result.

It can be seen that the argument here yields a somewhat stronger result, namely, if $l \neq 1$, we could make the second alternative above $g_l < \binom{n-1}{l-1} - 1$. By pursuing the reasoning involved here we can strengthen our result to the following one:

THEOREM. *If F_k, G_l are collections of k-edges and l-edges of S such that each member of F_k intersects each member of G_l, then either $f_k \leqslant \binom{n-1}{k-1}$ or $g_l \leqslant \binom{n-1}{l-1} - \binom{n-1-k}{l-1}$, where f_k, g_l represent the number of members of F_k and G_l, respectively.*

REFERENCES

1. P. ERDÖS, CHAO KO, R. RADO, Intersection Theorems for Systems of Finite Sets, *Quart. J. Math. Oxford Ser.* **12** (1961), 48.