Perron–Frobenius property of copositive matrices, and a block copositivity criterion

Immanuel M. Bomze

Department of Statistics and Decision Support Systems, University of Vienna, Austria

Received 21 November 2007; accepted 5 February 2008
Available online 25 March 2008
Submitted by R.A. Brualdi

Abstract

Haynsworth and Hoffman proved in 1969 that the spectral radius of a symmetric copositive matrix is an eigenvalue of this matrix. This note investigates conditions which guarantee that an eigenvector corresponding to this dominant eigenvalue has no negative coordinates, i.e., whether the Perron–Frobenius property holds. Also a block copositivity criterion using the Schur complement is specified which may be helpful to reduce dimension in copositivity checks and which generalizes results proposed by Andersson et al. in 1995, and Johnson and Reams in 2005. Apparently, the latter five researchers were unaware of the more general results by the author precedingly published in 1987 and 1996, respectively.

© 2008 Published by Elsevier Inc.

Keywords: Dominant eigenvalue; Positive eigenvector; Schur complement; Spectral radius

The Perron–Frobenius property has many well-studied consequences, also for matrices with some negative entries, see [6–9] and references therein. In [4], Haynsworth and Hoffman prove, among other results and in a more general setting, that a symmetric copositive matrix has a dominant (positive) eigenvalue. This way, part of the Perron–Frobenius theorem is extended to a class of matrices larger than the nonnegative ones, as copositive matrices also may have some negative entries, even if they are positive-semidefinite. For some of the latter class, the eigenvectors to the largest eigenvalue may have negative coordinates. So there is no hope to generalize the full Perron–Frobenius theorem to all copositive matrices, and we need additional conditions. Taking a closer look at the proof in [4], this can be accomplished quite easily. For convenience of the readers, we present the full argument.

E-mail address: immanuel.bomze@univie.ac.at

0024-3795/$ see front matter © 2008 Published by Elsevier Inc.
doi:10.1016/j.laa.2008.02.003
Recall that a real symmetric $n \times n$ matrix $Q$ is said to be copositive if it generates a quadratic form taking no negative value over the positive orthant $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i\}$, i.e., $x^T Q x \geq 0$ for all $x \in \mathbb{R}_+^n$. Denote by $\lambda_i(Q)$ the eigenvalues of $Q$ and by $\rho = \max_i |\lambda_i(Q)|$ its spectral radius. By $\|x\|$ we denote the Euclidean norm of a vector $x$, by $x^+ = \left\{\max\{x_i, 0\}\right\}_i \in \mathbb{R}_+^n$, and by $x^- = \left\{\max\{-x_i, 0\}\right\}_i \in \mathbb{R}_+^n$, so that $x = x^+ - x^-$ is a representation of an arbitrary vector as a difference of two orthogonal vectors in $\mathbb{R}_+^n$.

First we deal with the case that $-\rho$ is an eigenvalue, which is the case, for example, for the $2 \times 2$ permutation matrix differing from the identity. Then there is no strict dominance of $\rho$, as shown in [4]:

**Proposition 1.** If $Q$ is copositive and $-\rho < 0$ is an eigenvalue of $Q$, then the Perron–Frobenius property holds: there is a $y \in \mathbb{R}_+^n \setminus \{0\}$ such that $Qy = \rho y$.

**Proof.** Let $Qx = -\rho x$ with $\|x\| = 1$ and define $y = x^+ + x^- \in \mathbb{R}_+^n$. Since $x^+ \perp x^-$ by construction, also $\|y\| = 1$. By copositivity of $Q$, we get

$$y^T Q y + x^T Q x = 2(x^+)^T Q (x^+) + 2(x^-)^T Q (x^-) \geq 0,$$

so that $y^T Q y \geq -x^T Q x = \rho$, which ensures that $y$ is an eigenvector to the dominant eigenvalue $\rho$. The latter is a slight sharpening of the argument in [4], but suffices to establish the claim. □

Note that in the above case there are two eigenvalues of maximal absolute value. Further, since also $x \perp y$, we get $\|x^+\| = \|x^-\| = 1/\sqrt{2}$, so that up to positive scaling there is only one copositive $2 \times 2$ matrix which satisfies the assumptions of the above proposition. However, for larger dimensions this leaves room for more such matrices.

Next, we deal with the case where $-\rho$ is no eigenvalue of $Q$. By virtue of symmetry of $Q$, we then know that $\rho$ must be a dominant eigenvalue, but without any further conditions the corresponding eigenvector may have negative coordinates.

**Proposition 2.** For a real symmetric matrix $Q$, let $x$ be an eigenvector to the dominant positive eigenvalue $\rho > 0$, satisfying $(x^-)^T Q x^+ \geq 0$. Then the Perron–Frobenius property holds: there is a $y \in \mathbb{R}_+^n \setminus \{0\}$ such that $Qy = \rho y$.

**Proof.** We assume without loss of generality $\|x\| = 1$ and construct $y = x^+ + x^-$ as in the above proposition. Consider now $y^T Q y - x^T Q x = 4(x^-)^T Q x^+ \geq 0$, so that $y^T Q y \geq x^T Q x = \rho$, and the result follows as above. □

Note that the argument above does not necessarily mean that $\rho$ is a multiple eigenvalue, since $x^+$ or $x^-$ may be zero. However, if $\rho$ has multiplicity one, then the condition that

$$(x^-)^T Q x^+ \geq 0 \quad \text{holds for any eigenvector } x \in \mathbb{R}^n \text{ to the eigenvalue } \rho \quad (1)$$

is necessary and sufficient for the Perron–Frobenius property. The following example shows that (1) does not hold in general if $n \geq 3$, even if $Q$ is copositive and has the Perron–Frobenius property:
Example 1. The singular matrix

\[
Q = \begin{bmatrix}
481 & 108 & -240 \\
108 & 544 & 180 \\
-240 & 180 & 225
\end{bmatrix}
\]

is positive-semidefinite with a double eigenvalue of 625, corresponding to the (unscaled) eigenvectors \( x = [-16, 12, 15]^T \) and \( y = [3, 4, 0]^T \in \mathbb{R}_+^n \), so, despite of the relation \( (x^-)^T Q x^+ = -36864 < 0 \), the Perron–Frobenius property holds. Note that the sufficient condition specified in [9], which may be violated by nonnegative matrices like \( \begin{bmatrix} 51 \\ 10 \end{bmatrix} \), is also violated by this example: if \( e = [1, 1]^T \) and \( \|Q\|_F \) denotes the Frobenius norm, we get \( e^T Q e = 1346 < 1976 < \sqrt{(n - 1)^2 + 1\|Q\|_F} \). Finally notice that \( Q = \begin{bmatrix} 76 & 6 \\ 6 & -2 \end{bmatrix} \) satisfies the (strong) Perron–Frobenius property as \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) is an eigenvector to the dominant eigenvalue 10, and also satisfies the condition \( e^T Q e = 17 > \sqrt{250} = \sqrt{(n - 1)^2 + 1\|Q\|_F} \) from [9], but \( Q \) clearly is not copositive.

Combining Propositions 1 and 2 we arrive the following full generalization of the Perron–Frobenius theorem, since the conditions on \( Q \) below are satisfied if \( Q \) has no negative entries.

**Theorem 1.** Let \( Q \) be a symmetric copositive matrix with spectral radius \( \rho \) and suppose that one (or both) of the following conditions holds:

(a) \( -\rho \) is an eigenvalue of \( Q \);
(b) there is an eigenvector \( x \in \mathbb{R}^n \) to the eigenvalue \( +\rho \) with \( (x^-)^T Q x^+ \geq 0 \).

Then \( Q \) has the Perron–Frobenius property.

In [5, Theorem 11], the authors prove a similar result under the additional assumption that \( Q \) has no positive eigenvalue except the dominating one, \( \rho \). This assumption of course implies condition (1) above. On the other hand, for instance all positive-definite matrices with no negative entries violate the condition in [5, Theorem 11], but of course satisfy (1) even if \( \rho \) may be a multiple eigenvalue.

In this paper, also a block copositivity criterion is treated [5, Theorem 4]. However, a more general result can be found in an earlier publication [3]. We present this result in a more compact form below. The example on [5, p. 277] clearly illustrates the gap between conditions (a) and (b) below.

**Theorem 2.** Let \( Q \) be a real symmetric \( n \times n \) matrix with block structure

\[
Q = \begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix},
\]

where \( A \) is a symmetric positive-definite \( k \times k \) matrix. Define \( Q_{\emptyset} = C - B^T A^{-1} B \) (the Schur complement of \( A \) in \( Q \)) and \( \Gamma_{\emptyset} = \{ z \in \mathbb{R}^{n-k} : A^{-1} B z \leq 0 \} \). Then

(a) if \( Q \) is copositive, then \( Q_{\emptyset} \) is \( \Gamma_{\emptyset} \)-copositive, i.e. \( z^T Q_{\emptyset} z \geq 0 \) for all \( z \in \Gamma_{\emptyset} \);
(b) if \( Q_{\emptyset} \) is copositive, then \( Q \) is copositive.

Further, if \( A^{-1} B \leq 0 \), then \( \mathbb{R}^{n-k}_+ \subseteq \Gamma_{\emptyset} \), so that (a) and (b) together imply the criterion of [5, Theorem 4]. This criterion could even be extended to the following equivalence (which may be less appealing as it involves generalized copositivity):
If $A^{-1}B \leq 0$, then $Q$ is copositive if and only if $Q_{\emptyset}$ is $\Gamma_{\emptyset}$-copositive.

**Proof.** (a) follows immediately from [3, Theorem 5 and (22)]. To derive (b), observe that the cones $\Gamma_I$ defined in [3, (32)] are contained in $\mathbb{R}_{++}^{n-k}$. Further, the matrices $Q_I$ defined in [3, (21)] are such that $Q_I - Q_{\emptyset}$ are positive-semidefinite. Hence copositivity of $Q_{\emptyset}$ implies $\Gamma_I$-copositivity of $Q_I$ for all $I \subset \{1, \ldots, k\}$. A similar argument holds for the case $I = \{1, \ldots, k\}$, see [3, p. 175]. Assertion (b) follows more directly from the proof of [5, Theorem 4], valid without assuming $A^{-1}B \leq 0$. On the other hand, this assumption ($A^{-1}B^\top \leq 0$ in the notation of [5, Theorem 4]) implies $\mathbb{R}_{++}^{n-k} \subseteq \Gamma_{\emptyset}$, which establishes the last assertion. □

Whether one may view Theorem 1 as a generalization of [5, Theorem 11] or as a complementary result, may be a matter of taste. By contrast, [5, Theorem 2] – which generalizes [1, Theorem 2.1] – can be found in more general form already in [2], which seems to have gone unnoticed by the authors of [1,5]. Finally, it should be noted that [5, Theorem 4] can easily be derived from the more general results in [3, Theorems 5 and 8], as detailed in Theorem 2.

**Acknowledgement**

Thanks are due to an anonymous referee whose care considerably improved the paper. She or he also corrected some previously incorrect numerical values in the example. All remaining errors are mine, of course.

**References**