The topology of moduli spaces of group representations: The case of compact surface

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Received 9 February 2011
Available online 19 February 2011

Abstract

Let $G$ be a connected complex semisimple affine algebraic group, and let $K$ be a maximal compact subgroup of $G$. Let $X$ be a noncompact oriented surface. The main theorem of Florentino and Lawton (2009) [3] says that the moduli space of flat $K$-connections on $X$ is a strong deformation retraction of the moduli space of flat $G$-connections on $X$. We prove that this statement fails whenever $X$ is compact of genus at least two.

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MSC: 14D22

Keywords: Flat connection; Higgs bundle; Representation

1. Introduction

In [3], the following is proved: Let $F$ be a free group of finitely many generators, let $G$ be a connected complex reductive affine algebraic group, and let $K$ be a maximal compact subgroup of $G$. Then $\text{Hom}(F, K)/K$ is a strong deformation retraction of $\text{Hom}(F, G)//G$. (See [3, Theorem 1.1].) Since the fundamental group of a noncompact oriented surface is a free group, this result has the following reformulation.

Let $X$ be a noncompact oriented surface. Then the moduli space of flat $K$-connections on $X$ is a strong deformation retraction of the moduli space of flat $G$-connections on $X$. 
It is natural to ask whether the above result remains valid for a compact oriented surface.

Let $X$ be a compact connected oriented surface of genus $g$, with $g \geq 2$. We assume that $G$ is nontrivial and semisimple. Fix a complex structure on $X$. The representation space $\text{Hom}(\pi_1(X), K)/K$ is homeomorphic to the moduli space $M_G(X)$ of topologically trivial semistable principal $G$-bundles on $X$. The representation space $\text{Hom}(\pi_1(X), G)//G$ is homeomorphic to the moduli space $H_G(X)$ of semistable Higgs $G$-bundles $(E_G, \theta)$ on $X$ such that $E_G$ is topologically trivial. From Corollary 2.3 it follows immediately that $M_G(X)$ is not a deformation retraction of $H_G(X)$.

2. Moduli of Higgs bundles and the nilpotent cone

Let $G$ be a connected semisimple affine algebraic group defined over $\mathbb{C}$. We assume that $G \neq e$. Fix a maximal compact subgroup $K \subset G$. Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$.

Let $M_G(X)$ be the moduli space of topologically trivial semistable principal $G$-bundles over $X$. See [6] for the definition of semistable principal $G$-bundles; a construction of the moduli space $M_G(X)$ can be found in [7]. We know that $M_G(X)$ is homeomorphic to the equivalence classes of homomorphisms from $\pi_1(X)$ to $K$; see [6].

The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. The holomorphic cotangent bundle of $X$ will be denoted by $K_X$.

Let $E_G \rightarrow X$ be a holomorphic principal $G$-bundle. Let $\text{ad}(E_G) := E_G \times^G \mathfrak{g}$ be the adjoint vector bundle for $E_G$. A Higgs field on $E_G$, a Higgs field, is a holomorphic section of $\text{ad}(E_G) \otimes K_X$. A Higgs $G$-bundle on $X$ is a pair of the form $(E_G, \theta)$, where $E_G$ is a principal $G$-bundle on $X$, and $\theta$ is a Higgs field on $E_G$.

A Higgs $G$-bundle $(E_G, \theta)$ is called semistable if for every pair of the form $(Q, E_Q)$, where $Q$ is a (proper) maximal parabolic subgroup, and $E_Q \subset E_G$ is a holomorphic reduction of structure group to $Q$ such that

$$\theta \in H^0(X, \text{ad}(E_Q) \otimes K_X),$$

the inequality

$$\text{degree}(\text{ad}(E_G)/\text{ad}(E_Q)) \geq 0$$

holds, where $\text{ad}(E_Q)$ is the adjoint bundle for $E_Q$.

Let $H_G(X)$ denote the moduli space of semistable Higgs $G$-bundles $(E_G, \theta)$ such that $E_G$ is topologically trivial; see [9,2] for the construction of $H_G(X)$. The moduli space $H_G(X)$ is homeomorphic to $\text{Hom}(\pi_1(X), G)//G$, the space of $S$-equivalence classes representations of $\pi_1(X)$ in $G$ [8].

Fix generators

$$\beta_{n_1}, \ldots, \beta_{n_\ell} \in \bigoplus_{i \geq 1} \text{Sym}^i(\mathfrak{g}^*)^G$$

(2.1)

of the $\mathbb{C}$-algebra of $G$-invariant polynomial functions on $\mathfrak{g}$; the degree of $\beta_{n_j}$, $1 \leq j \leq \ell$, is $n_j$. Using $\beta_{n_j}$, we get a morphism

$$H_G(X) \longrightarrow H^0(X, K_X^{\otimes n_j}), \quad (E_G, \theta) \longmapsto \beta_{n_j}(\theta).$$

These morphisms combine together to define a morphism
which is known as the Hitchin map; see [4,5,2].

The inverse image

\[ N := \mathcal{H}^{-1}(0) \subset \mathcal{H}_G(X) \]  

(2.3)

is known as the nilpotent cone.

**Theorem 2.1.** The moduli space \( \mathcal{H}_G(X) \) admits a deformation retraction to the nilpotent cone \( N \).

**Proof.** Fix a Hermitian structure \( h \) on \( X \). We note that \( h \) is Kähler because \( \dim_\mathbb{C} X = 1 \). The Hermitian structure \( h \) induces a Hermitian structure on each line bundle \( K_X \otimes^n i \). Therefore, we obtain an inner product on the vector space \( H^0(X, K_X \otimes^n i) \).

The group \( \mathbb{C}^* \) has a natural action on \( \mathcal{H}_G(X) \). The action of any \( \lambda \in \mathbb{C}^* \) sends any \((E_G, \theta)\) to \((E_G, \lambda \cdot \theta)\). This action is clearly algebraic. Restrict this action of \( \mathbb{C}^* \) to the subgroup \( \mathbb{R}^+ \subset \mathbb{C}^* \).

Consider the map

\[ \Phi : \bigoplus_{j=1}^\ell H^0(X, K_X \otimes^n j) \longrightarrow \mathbb{R}_{\geq 0}, \]

defined by

\[ \Phi \left( \sum_{j=1}^\ell \omega_j \right) := \sum_{j=1}^\ell \|\omega_j\|_{nj}. \]

Clearly \( \Phi \) is continuous, proper, and \( \Phi^{-1}(0) = 0 \). We have

\[ \Phi \left( t \cdot \sum_{j=1}^\ell \omega_j \right) = t \cdot \Phi \left( \sum_{j=1}^\ell \omega_j \right) \]

for all \( t \in \mathbb{R}^+ \). Hence for all \( \epsilon > 0 \), the inverse image

\[ V_\epsilon := \Phi^{-1}(\{0, \epsilon\}) \]

is a compact neighborhood of the origin. Since the map \( \mathcal{H} \) in (2.2) is proper (see [4]),

\[ U_\epsilon := \mathcal{H}^{-1}(V_\epsilon) \subset \mathcal{H}_G(X) \]

is a compact neighborhood of the nilpotent cone.

Any open neighborhood of \( 0 \in \bigoplus_{j=1}^\ell H^0(X, K_X \otimes^n j) \) contains \( V_\epsilon \) whenever \( \epsilon \) is sufficiently small. Since the map \( \mathcal{H} \) is proper, this implies that any open neighborhood of \( \mathcal{H}^{-1}(0) \) contains \( U_\epsilon \) provided \( \epsilon \) is sufficiently small.

We have a retraction of \( \mathcal{H}_G(X) \) onto \( U_\epsilon \) defined as follows:

\[ R : \mathcal{H}_G(X) \times [0, 1] \longrightarrow \mathcal{H}_G(X), \]

\[ ((E_G, \theta), t) \longmapsto \begin{cases} (E_G, t \cdot \theta), & t \in [0, 1], \ t \geq \frac{\epsilon}{\Phi(\mathcal{H}(E_G, \theta))}, \\ (E_G, t_0 \cdot \theta), & t \in [0, 1], \ t \leq t_0 = \frac{\epsilon}{\Phi(\mathcal{H}(E_G, \theta))} \leq 1, \\ (E_G, \theta), & t \in [0, 1], \ \Phi(\mathcal{H}(E_G, \theta)) \leq \epsilon. \end{cases} \]
Note that in the first two cases, either \( t \neq 0 \) or \( t_0 \neq 0 \); this ensures that the map is well defined. For any \((E_G, \theta) \in U_\epsilon\), we have \( R((E_G, \theta), t) = (E_G, \theta)\). Also, \( R((E_G, \theta), 1) = (E_G, \theta)\) for each \((E_G, \theta) \in H_G(X)\).

We claim that \( R((E_G, \theta), 0) \in U_\epsilon\) for each \((E_G, \theta) \in H_G(X)\). To prove this, first note it is evident for all \((E_G, \theta)\) with \( \Phi(H(E_G, \theta)) \leq \epsilon \). Now, if \( \Phi(H(E_G, \theta)) \geq \epsilon \), then it also holds because \( \Phi(R((E_G, \theta), 0)) = \Phi(H(E_G, t_0 \cdot \theta)) = t_0 \cdot \Phi(H(E_G, \theta)) = \epsilon \).

This proves the claim.

The nilpotent cone \( \mathcal{N} \) in (2.3) is a closed subvariety of \( H_G(X) \). Therefore there exists an analytic open neighborhood \( U \) of \( \mathcal{N} \) in the Euclidean topology such that \( U \) retracts to \( \mathcal{N} \). Fix a retraction \( R' \) of \( U \) to \( \mathcal{N} \). Take \( \epsilon > 0 \) small enough so that \( U_\epsilon \subset U \). The above retraction \( R \) followed by the retraction \( R' \) (as composition of two homotopies) gives a retraction of \( H_G(X) \) onto the nilpotent cone.

We have \( \dim H_G(X) = 2 \dim G \cdot (g - 1) \), and \( \dim \mathcal{N} = \dim G \cdot (g - 1) \).

The following lemma is a consequence of Theorem 2.1.

**Lemma 2.2.** For any \( i > \dim G \cdot (g - 1) \),

\[ H^i(H_G(X), \mathbb{Z}) = 0. \]

Also,

\[ H^{\dim G \cdot (g - 1)}(H_G(X), \mathbb{Z}) = \mathbb{Z}^N, \]

where \( N \) is the number of conjugacy classes of nilpotent elements in \( g \).

**Proof.** We have \( H^i(\mathcal{N}, \mathbb{Z}) = 0 \) for \( i > \dim G \cdot (g - 1) \), because \( \dim \mathcal{N} = \dim G \cdot (g - 1) \). Hence the first statement follows immediately from Theorem 2.1.

The irreducible components of \( \mathcal{N} \) are parametrized by the conjugacy classes of nilpotent elements in \( g \) [5]. Also, each irreducible component of \( \mathcal{N} \) is Lagrangian [5] (see also [1]); in particular, the dimension of each irreducible component of \( \mathcal{N} \) is \( \dim G \cdot (g - 1) \). Hence the second statement follows.

Lemma 2.2 has the following corollary:

**Corollary 2.3.** \( \text{rank } H^{\dim G \cdot (g - 1)}(H_G(X), \mathbb{Z}) > \text{rank } H^{\dim G \cdot (g - 1)}(M_G(X), \mathbb{Z}) \).

**Proof.** Since \( M_G(X) \) is an irreducible projective variety of dimension \( \dim G \cdot (g - 1) \),

\[ H^{\dim G \cdot (g - 1)}(M_G(X), \mathbb{Z}) = \mathbb{Z}. \]

On the other hand, \( N \) in Lemma 2.2 is at least two.

**Acknowledgements**

The first author wishes to thank Instituto Superior Técnico, where the work was carried out, for its hospitality. The visit to IST was funded by the FCT project PTDC/MAT/099275/2008.
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