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Separable Rings

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TO NATHAN JACOBSON ON HIS 70TH BIRTHDAY

As a natural generalization of the classical theory of simple algebras, Auslander and Goldman established the theory of separable algebras in [1]. They proved in particular that if Λ is a separable algebra then it has the following two properties (a) By any ring epimorphism of Λ onto another ring the center of Λ is mapped onto the center of the image ring of Λ. (b) Every two-sided ideal of Λ is generated by an ideal of its center. In this paper we are concerned with rings having these two properties, and indeed we call a ring Λ a separable ring if it satisfies both the conditions (a) and (b). Every separable algebra is thus a separable ring, but the converse is not true. Our purpose is, however, to show that separable rings behave quite similarly to simple rings as well as separable algebras; also ideal algebras, introduced by Rao, are considered in connection with separable rings. For the theory of separable algebras, as is well-known, the Morita theory for projective modules gave a nice background, while in developing our study the recent Fuller theory which extends the Morita theory to the case of quasi-projective modules provides a substantial background.

1. FULLER'S THEOREM AND ITS COROLLARIES

Let R be a ring, and let U be a left R-module. We denote by $\mathrm{Gen}_R(U)$ the class of all left R-modules which are expressed as sums of homomorphic images of ${}_RU$. Let S be another ring and let U now be an R-S-bimodule. Then for every left R-module X and for every left S-module Y we define the canonical homomorphisms

$$\rho(X)\colon {}_RU \otimes_S \mathrm{Hom}_R(U, X) \to {}_RX,$$
$$\sigma(Y)\colon {}_SY \to {}_S\mathrm{Hom}_R(U, U \otimes_S Y)$$

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by $\rho(X)(u \otimes f) = f(u)$ for $u \in U$, $f \in \mathrm{Hom}_R(U, X)$ and $(\sigma(Y)y)u = u \otimes y$ for $y \in Y$, $u \in U$, respectively. Concerning these homomorphisms an important theorem was obtained by Fuller in [5, Theorem 2.6] and recently some improvement and refinement have been added to it by Sato [8] and Azumaya [2]. Thus the improved Fuller's theorem is

THEOREM 1.1. *Let U be a left R-module and let $S = \mathrm{End}_R(U)$; we view U as an R-S-bimodule. Then the following conditions are equivalent:*

(1) *${}_RU$ is finitely generated quasi-projective and every submodule of ${}_RU$ is in* $\mathrm{Gen}_R(U)$.

(2) *$\rho(X)$ is an isomorphism for all X in* $\mathrm{Gen}_R(U)$ *and $\sigma(Y)$ is an isomorphism for all left S-modules Y.*

(3) $\mathrm{Gen}_R(U)$ *is closed under submodules and U_S is a weak generator* (*i.e., $U \otimes_S Y = 0$ for a left S-module Y implies $Y = 0$*).

(4) *$\rho(X)$ is an isomorphism for all X in* $\mathrm{Gen}_R(U)$ *and U_S is a weak generator.*

(5) *$\rho(X)$ is an isomorphism for all X in* $\mathrm{Gen}_R(U)$ *and U_S is faithfully flat* (*i.e., flat and a weak generator*).

Remark. For convenience, we assumed throughout that $S = \mathrm{End}_R(U)$ in Theorem 1.1, but this assumption is superfluous for conditions (2), (4), and (5) as a matter of fact. In this connection, Zimmermann-Huisgen's theorem [10, Lemma 1.4] that $\mathrm{Gen}_R(U)$ is closed under submodules if and only if $\rho(X)$ is an isomorphism for all X in $\mathrm{Gen}_R(U)$ and U_S is flat should also be taken into account.

We can now derive the following corollaries from Theorem 1.1:

COROLLARY 1.2. *Let ${}_RU$ be finitely generated quasi-projective, $S = \mathrm{End}_R(U)$, and every submodule of ${}_RU$ be in $\mathrm{Gen}_R(U)$. Let X be in $\mathrm{Gen}_R(U)$ and let $Y = \mathrm{Hom}_R(U, X)$, Then between submodules X_0 of ${}_RX$ and submodules Y_0 of ${}_SY$ there is a one-to-one correspondence by the following relations:*

$$X_0 = \sum_{f \in Y_0} f(U), \qquad Y_0 = \{f \in Y \mid f(U) \subset X_0\}.$$

Proof. Let X_0 be a submodule of ${}_RX$ and put $Y_0 = \mathrm{Hom}_R(U, X_0)$. Then clearly $Y_0 = \{f \in Y \mid f(U) \subset X_0\}$. Since $\mathrm{Gen}_R(U)$ is closed under submodules by Theorem 1.1, X_0 is also in $\mathrm{Gen}_R(U)$, which means that $X_0 = \sum_{f \in Y_0} f(U)$. Let, conversely, Y_0 be any submodule of ${}_SY$ and put $X_0 = \sum_{f \in Y_0} f(U)$. Then X_0 is in $\mathrm{Gen}_R(U)$ and therefore $\rho(X_0)$: ${}_RU \otimes {}_S\mathrm{Hom}_R(U, X_0) \to {}_RX_0$ is an isomorphism by Theorem 1.1. On the other hand, since Y_0 is a submodule of ${}_S\mathrm{Hom}_R(U, X_0)$ and since U_S is flat by Theorem 1.1, $U \otimes_S Y_0$ is regarded as a submodule of $U \otimes_S \mathrm{Hom}_R (U, X_0)$ in the natural manner. Since, however, the image of $u \otimes f$ ($u \in U$, $f \in Y_0$) by $\rho(X_0)$ is $f(u)$, it is clear that $U \otimes_S Y_0$ is mapped onto X_0 by $\rho(X_0)$ and therefore $U \otimes_S Y_0 = U \otimes_S \mathrm{Hom}_R(U, X_0)$. Let h: $U \to X_0$ be an R-homomorphism. Then since $\rho(X_0)$: $U \otimes_S Y_0 \to X_0$ is an R-isomorphism, there exists a (unique) R-homomorphism g: $U \to U \otimes_S Y_0$ such that $h = \rho(X_0) \circ g$. Since, furthermore, $\sigma(Y_0)$: $Y_0 \to \mathrm{Hom}_R(U, U \otimes_S Y_0)$ is an isomorphism by Theorem 1.1, there corresponds to g an element f of Y_0 such that $g(u) = u \otimes f$ for all $u \in U$. Thus we have $h(u) = \rho(X_0)(u \otimes f) = f(u)$ for all $u \in U$, i.e., $h = f$, and this implies that $Y_0 = \mathrm{Hom}_R(U, X_0)$.

Corollary 1.3. *Let ${}_RU_S$ be as in Corollary 1.2. Let Y be a left S-module and let $X = U \otimes_S Y$. Then between submodules X_0 of ${}_RX$ and submodules Y_0 of ${}_SY$ there is a one-to-one correspondence by the following relations*:

$$X_0 = U \otimes Y_0, \qquad Y_0 = \{y \in Y \mid U \otimes y \subset X_0\}.$$

Proof. Since X is in $\mathrm{Gen}_R(U)$ and since $\sigma(Y)$: $Y \to \mathrm{Hom}_R(U, X)$ is an isomorphism by Theorem 1.1, it is clear that our corollary is an immediate consequence of Corollary 1.2 by identifying each $y \in Y$ with the corresponding map $\sigma(Y)y = [u \mapsto u \otimes y] \in \mathrm{Hom}_R(U, X)$.

Corollary 1.4. *Let ${}_RU_S$ be as in Corollary 1.2. Then between submodules U_0 of ${}_RU$ and left ideals L of S there is a one-to-one correspondence by the following relations*:

$$U_0 = UL, \qquad L = \{s \in S \mid Us \subset U_0\}.$$

Proof. This is the particular case of Corollary 1.2 where $X = U$ and $Y = S$.

2. Separable Rings and Ideal Algebras

Let Λ be a ring and Z the center of Λ. Then Λ is considered an algebra over Z (central Z-algebra). Let Λ^0 be the opposite ring of Λ. Then Λ^0 is also a central Z-algebra and we can define the enveloping algebra $\Lambda^e = \Lambda \otimes_Z \Lambda^0$ of Λ (over Z). Let X be a Λ-bimodule. We call X a *central Λ-bimodule* if X is element-wise commutative with Z. It is well-known that every central Λ-bimodule is regarded as a left Λ^e-module and conversely every left Λ^e-module is converted into a central Λ-bimodule in the usual way. In particular, Λ is a central Λ-bimodule and so becomes a left Λ^e-module. Furthermore, for a central Λ-bimodule X, there is a natural isomorphism between Z-modules $\mathrm{Hom}_{\Lambda^e}(\Lambda, X)$ and $X^\Lambda = \{x \in X \mid ax = xa \text{ for all } a \in \Lambda\}$ given by the mapping $f \mapsto f(1)$, $f \in \mathrm{Hom}_{\Lambda^e}(\Lambda, X)$, and the inverse of this isomorphism is obtained by associating each $x \in X^\Lambda$ with $f \in \mathrm{Hom}_{\Lambda^e}(\Lambda, X)$ defined by $f(a) = ax (= xa)$, $a \in \Lambda$.

PROPOSITION 2.1. *Let X be a central Λ-bimodule. Then X is in* $\mathrm{Gen}_{\Lambda^e}(\Lambda)$ *if and only if* $X = \Lambda X^\Lambda$.

Proof. Suppose that X is in $\mathrm{Gen}_{\Lambda^e}(\Lambda)$. Then X is a sum of submodules $f(\Lambda)$ with $f \in \mathrm{Hom}_{\Lambda^e}(\Lambda, X)$. But then $f(\Lambda) = \Lambda f(1)$ and $f(1) \in X^\Lambda$. Therefore we have $X = \Lambda X^\Lambda$. Assume conversely that $X = \Lambda X^\Lambda$. Then X is a sum of submodules Λx with $x \in X^\Lambda$. But, for each $x \in X^\Lambda$, there is an $f \in \mathrm{Hom}_{\Lambda^e}(\Lambda, X)$ such that $f(1) = x$, so we have $\Lambda x = f(\Lambda)$ and thus $X \in \mathrm{Gen}_{\Lambda^e}(\Lambda)$.

COROLLARY 2.2. *Let T be a two-sided ideal of Λ. Then T is in* $\mathrm{Gen}_{\Lambda^e}(\Lambda)$ *if and only if* $T = \Lambda(T \cap Z)$.

PROPOSITION 2.3. *The left Λ^e-module Λ is quasi-projective if and only if for every ring epimorphism $f: \Lambda \to \Lambda'$ the image $f(Z)$ of Z coincides with the center of Λ'.*

Proof. Let $f: \Lambda \to \Lambda'$ be a ring epimorphism, and let Z' be the center of the ring Λ'. Then clearly $f(Z) \subset Z'$ and so, by means of f, Λ' can be made into a central Λ-bimodule, or a left Λ^e-module, and in this case f becomes a Λ^e-epimorphism. Let z' be any element of $Z' = (\Lambda')^\Lambda$. Then there corresponds a Λ^e-homomorphism $g: \Lambda \to \Lambda'$ such that $g(1) = z'$. Assume that the Λ^e-module Λ is quasi-projective. Then there must exist a Λ^e-endomorphism $h: \Lambda \to \Lambda$ such that $f \circ h = g$. It follows therefore that $z' = g(1)$ is the image of $h(1) \in \Lambda^\Lambda = Z$ by f.

Next let T be any Λ^e-submodule of Λ, and let f be the natural Λ^e-epimorphism $\Lambda \to \Lambda/T$. Since T is nothing but a two-sided ideal of Λ, Λ/T can also be considered a factor ring and then f becomes a ring epimorphism. Let $g: \Lambda \to \Lambda/T$ be any Λ^e-homomorphism. Then $g(1)$ is in the center $(\Lambda/T)^\Lambda$ of Λ/T. Assume now that there exists a $z \in Z = \Lambda^\Lambda$ such that $f(z) = g(1)$. Let $h: \Lambda \to \Lambda$ be a Λ^e-endomorphism such that $h(1) = z$. Then we have $f(h(1)) = f(z) = g(1)$, which is, however, equivalent to $f \circ h = g$ because both $f \circ h$ and g are in $\mathrm{Hom}_{\Lambda^e}(\Lambda, \Lambda/T)$ This completes the proof of our proposition.

We now call Λ a *separable ring* if it satisfies the following two conditions:

(a) For any ring epimorphism $f: \Lambda \to \Lambda'$ the image $f(Z)$ of the center Z of Λ coincides with the center of Λ'.

(b) Every two-sided ideal T of Λ is generated by the ideal $T \cap Z$ of the center Z: $T = \Lambda(T \cap Z)$.

PROPOSITION 2.4. *Let Λ satisfy condition* (a) *or* (*b*). *Then every homomorphic image of Λ also satisfies* (a) *or* (b), *respectively. In particular, if Λ is a separable ring then every homomorphic image of Λ is a separable ring too.*

Proof. Let $f: \Lambda \to \Lambda'$ be a ring epimorphism and Z' the center of the ring Λ'. Suppose that Λ satisfies (a). Then $f(Z) = Z'$. Let $g: \Lambda' \to \Lambda''$ be a ring epimorphism. Then $g \circ f: \Lambda \to \Lambda''$ is also a ring epimorphism and therefore $g(f(Z)) = g(Z')$ is the center of Λ''. This implies that Λ' satisfies the condition (a).

Suppose next that Λ satisfies (b). Let T' be a two-sided ideal of Λ'. Then the inverse image $T = f^{-1}(T')$ is a two-sided ideal of Λ and so $T = \Lambda I$ with $I = Z \cap T$. From this it follows that $T' = f(T) = f(\Lambda)f(I) = \Lambda' f(I)$. But since $f(Z) \subset Z'$ we have $\Lambda' f(I) \subset \Lambda'(Z' \cap T') \subset T'$ and hence $\Lambda'(Z' \cap T') = T'$, which shows that Λ' satisfies (b).

Now, according to Proposition 2.3, Λ satisfies the condition (a) if and only if the left Λ^e-module Λ is quasi-projective, while that Λ satisfies the condition (b) is, according to Corollary 2.2, equivalent to the condition that every submodule of the left Λ^e-module Λ is in $\mathrm{Gen}_{\Lambda^e}(\Lambda)$. Therefore, by applying Theorem 1.1 to the left Λ^e-module Λ and $Z = \mathrm{End}_{\Lambda^e}(\Lambda)$ and taking the canonical identifications $X^\Lambda = \mathrm{Hom}_{\Lambda^e}(\Lambda, X)$ and $\mathrm{Hom}_{\Lambda^e}(\Lambda, \Lambda \otimes_Z Y) = (\Lambda \otimes_Z Y)^\Lambda$ into account, we have

THEOREM 2.5. *The following conditions are equivalent*:

(1) *Λ is a separable ring*:

(2) *For every Λ-bimodule X such that $X = \Lambda X^\Lambda$ the Λ-bimodule homomorphism $\rho(X): \Lambda \otimes_Z X^\Lambda \to X$, defined by $\rho(X)(a \otimes x) = ax$ for $a \in \Lambda$, $x \in X$, is an isomorphism, and for every Z-module Y the Z-homomorphism $\sigma(Y): Y \to (\Lambda \otimes_Z Y)^\Lambda$, defined by $\sigma(Y)y = 1 \otimes y$ for $y \in Y$, is an isomorphism*:

Similarly, by specializing Corollaries 1.2, 1.3, and 1.4 to the case where $R = \Lambda^e$, $U = \Lambda$, and $S = Z$, we have the following three propositions.

PROPOSITION 2.6. *Let Λ be a separable ring. Let X be a Λ-bimodule such that $X = \Lambda X^\Lambda$ and let $Y = X^\Lambda$. Then between Λ-bisubmodules X_0 of X and Z-submodules Y_0 of Y there is a one-to-one correspondence by the following relations*:

$$X_0 = \Lambda Y_0, \qquad Y_0 = X_0^\Lambda.$$

PROPOSITION 2.7. *Let Λ be a separable ring. Let Y be a Z-module and let $X = \Lambda \otimes_Z Y$. Then between Λ-bisubmodules X_0 of X and Z-submodules Y_0 of Y there is a one-to-one correspondence by the following relations*:

$$X_0 = \Lambda \otimes Y_0, \qquad Y_0 = \{y \in Y \mid 1 \otimes y \in X_0\}.$$

PROPOSITION 2.8. *Let Λ be a separable ring. Then between two-sided ideals T of Λ and ideals I of the center Z there is a one-to-one correspondence by the following relations*:

$$T = \Lambda I, \qquad I = Z \cap T;$$

moreover, Λ is faithfully flat as a Z-module.

The last assertion of Proposition 2.8 follows from [that (1) implies (5) in] Theorem 1.1.

We now consider another algebra Γ over the center Z of the separable ring Λ. Then $\Lambda \otimes_Z \Gamma$ becomes a Z-algebra. By Theorem 2.5 the mapping $b \mapsto 1 \otimes b$ $(= b(1 \otimes 1) = (1 \otimes 1)b)$ for $b \in \Gamma$ gives a Z-isomorphism $\sigma(\Gamma)\colon \Gamma \to (\Lambda \otimes_Z \Gamma)^{\Lambda}$, but this is clearly an algebra isomorphism in our case. Therefore, by identifying b with $1 \otimes b$ we can and shall regard Γ as a subalgebra of $\Lambda \otimes_Z \Gamma$.

PROPOSITION 2.9. *Let Λ be a separable ring with center Z and let Γ be a Z-algebra with center C. Then C is also the center of $\Lambda \otimes_Z \Gamma$, and between two-sided ideals P of $\Lambda \otimes_Z \Gamma$ and two-sided ideals Q of Γ there is a one-to-one correspondence by the following relations*:

$$P = \Lambda \otimes Q, \qquad Q = \Gamma \cap P.$$

Proof. Let c be an element of $\Lambda \otimes_Z \Gamma$. Then that c is in the center of $\Lambda \otimes_Z \Gamma$ is equivalent to that c is element-wise commutative with Λ and Γ. But since $(\Lambda \otimes_Z \Gamma)^{\Lambda} = \Gamma$, this condition means that c is in $\Gamma^{\Gamma} = C$. Thus C is the center of $\Lambda \otimes_Z \Gamma$. According to Proposition 2.7, between Λ-bisubmodules P of $\Lambda \otimes_Z \Gamma$ and Z-submodules Q of Γ there is a one-to-one correspondence by the relations given in our proposition; observe that each $b \in \Gamma$ is identified with $1 \otimes b$. It is then clear that P is Γ-bisubmodule, or equivalently a two-sided ideal of $\Lambda \otimes_Z \Gamma$ if and only if the corresponing Q is a two-sided ideal of Γ.

Remark. In Proposition 2.9, the mapping $a \mapsto a \otimes 1$ for $a \in \Lambda$ gives an algebra homomorphism $\Lambda \to \Lambda \otimes_Z \Gamma$. Let T be the kernel of this homomorphism. Then T is a two-sided ideal of Λ and so we have $T = \Lambda I$ by Proposition 2.8, where $I = Z \cap T$ is the ideal of Z consisting of those element z of Z for which $z \otimes 1 = 0$. But since $z \otimes 1 = 1 \otimes z1$ for every $z \in Z$, that $z \otimes 1 = 0$ implies that $z1 = 0$, 1 being the unit element of Γ. Thus I is nothing but the annihilator ideal of the Z-module Γ; in particular, the map $a \mapsto a \otimes 1$ is a monomorphism if and only if the Z-module Γ is faithful.

PROPOSITION 2.10. *Let Λ be a separable ring with center Z and let Γ be a Z-algebra which is a separable ring with center C. Then $\Lambda \otimes_Z \Gamma$ is also a separable ring with center C.*

Proof. Let P be a two-sided ideal of $\Lambda \otimes_Z \Gamma$. Then by Proposition 2.9 $P = \Lambda \otimes Q$ with the two-sided ideal $Q = \Gamma \cap P$ of Γ. Since, however, Γ is a separable ring with center C, we have $Q = \Gamma(C \cap Q)$ and therefore $P = \Lambda \otimes \Gamma(C \cap Q) = (\Lambda \otimes_Z \Gamma)(C \cap Q)$; if we observe that $C \cap Q = C \cap \Gamma \cap P = C \cap P$ we have then $P = (\Lambda \otimes_Z \Gamma)(C \cap P)$, which shows that $\Lambda \otimes_Z \Gamma$ satisfies the condition (b).

Let next $f\colon \Lambda \otimes_Z \Gamma \to (\Lambda \otimes_Z \Gamma)/P$ be the natural (ring) epimorphism. Let $\Gamma' = \Gamma/Q$ and let $g\colon \Gamma \to \Gamma'$ be the natural epimorphism. Consider then the following exact sequence of Z-modules:

$$0 \to Q \xrightarrow{i} \Gamma \xrightarrow{g} \Gamma' \to 0,$$

where i is the inclusion map. Since Λ is a separable ring, Λ is a flat Z-module by Proposition 2.8. Therefore we have the following exact sequence:

$$0 \longrightarrow \Lambda \otimes_Z Q \xrightarrow{1 \otimes i} \Lambda \otimes_Z \Gamma \xrightarrow{1 \otimes g} \Lambda \otimes_Z \Gamma' \longrightarrow 0.$$

The isomorphic image of $\Lambda \otimes_Z Q$ by $1 \otimes i$ is then $P = \Lambda \otimes Q$ and thus we have the natural isomorphism $(\Lambda \otimes_Z \Gamma)/P \to \Lambda \otimes_Z \Gamma'$ such that the following diagram is commutative:

$$\begin{array}{ccc} & \Lambda \otimes_Z \Gamma & \\ {}^{f}\swarrow & & \searrow^{1 \otimes g} \\ (\Lambda \otimes_Z \Gamma)/P & \longrightarrow & \Lambda \otimes_Z \Gamma'. \end{array}$$

Let C' be the center of Γ'. Then since Γ is a separable ring g maps C onto C', C and C' are also the centers of $\Lambda \otimes_Z \Gamma$ and $\Lambda \otimes_Z \Gamma'$, respectively, by Proposition 2.9. From this it follows that the center of $\Lambda \otimes_Z \Gamma$ is mapped onto the center of $(\Lambda \otimes_Z \Gamma)/P$ by f, and this means that the condition (a) is satisfied by $\Lambda \otimes_Z \Gamma$. Thus $\Lambda \otimes_Z \Gamma$ is a separable ring.

It is to be mentioned that every simple ring is trivially a separable ring and its center is a field and conversely every separable ring whose center is a field is a simple ring, and therefore Theorems 2.5, Propositions 2.6, 2.7, 2.9, and 2.10 remain true if the term "separable" is replaced by the term "simple."

THEOREM 2.11. *Let Λ be a ring with center Z. Then the following conditions are equivalent*:

(1) *Λ is a separable Z-algebra.*

(2) *Λ is a separable ring and is a finitely generated Z-module.*

Proof. Assume (1). Then [1, Propmosition 1.4] implies that Λ satisfies(a), while [1, Corollary 3.2] implies that Λ satisfies (b); thus Λ is a separable ring. Furthermore, by [1, Theorem 2.1] Λ is a finitely generated Z-module.

Assume conversely (2). Let M be a maximal ideal of Z. Then by Proposition 2.8 ΛM is a maximal two-sided ideal of Λ, so the factor ring $\Lambda/\Lambda M$ becomes a finite-dimensional central simple algebra over the field Z/M. As is well-known, this means that $\Lambda/\Lambda M$ is a central separable algebra over Z/M. Therefore it follows from Endo and Watanabe [4, Proposition 1.1] that Λ is a separable algebra over Z.

Let K be a commutative ring and Λ an algebra over K. Rao [6] defined Λ to be an ideal K-algebra if the mapping $I \mapsto \Lambda I$ gives a one-to-one correspondence between ideals I of K and two-sided ideals of Λ. In this case, Λ is clearly a faithful K-module, or equivalently K is regarded as a subring of the center Z of Λ. In particular, Proposition 2.8 implies that every separable ring is a central ideal algebra.

PROPOSITION 2.12. *A K-algebra Λ is an ideal algebra if and only if it is faithfully flat as a K-module and every two-sided ideal T of Λ is generated by $K \cap T$: $T = \Lambda(K \cap T)$.*

Proof. That every ideal K-algebra is a faithfully flat K-module is proved in [6, Proposition 1.2]. Let Λ be an ideal K-algebra and T a two-sided ideal of Λ. Then $T = \Lambda I$ for some ideal I of K. Then $K \cap T$ is an ideal of K satisfying $I \subset K \cap T \subset T$. Therefore it follows that $\Lambda(K \cap T) = T$.

In order to prove the "if" part, assume that the K-algebra Λ is a faithfully flat K-module. Then Λ is K-faithful, so K is regarded as a subring of Λ. Let I be an ideal of K. Put $J = K \cap \Lambda I$. Then J is an ideal of K such that $I \subset J$ and hence $\Lambda I = \Lambda J$. Consider the following obvious exact sequence of K-modules:

$$0 \to I \to J \to J/I \to 0.$$

Since Λ is K-flat, we have the following exact sequence:

$$0 \to \Lambda \otimes_K I \to \Lambda \otimes_K J \to \Lambda \otimes_K (J/I) \to 0.$$

Furthermore the K-flatness of Λ implies that $\Lambda \otimes_K I$ and $\Lambda \otimes_K J$ are canonically identified with ΛI and ΛJ, respectively. Therefore it follows that $\Lambda \otimes_K (J/I) = 0$. Since Λ is K-faithfully flat, this implies that $J/I = 0$, i.e., $I = J = K \cap \Lambda I$. Thus it is shown that the mapping $I \mapsto \Lambda I$, for ideals of K, is one-to-one. Now to assume further that every two-sided ideal T of Λ satisfies $T = \Lambda(K \cap T)$ clearly means that every two-sided ideal of Λ is an image of this mapping, i.e., Λ is an ideal K-algebra.

PROPOSITION 2.13. *Let Λ be an ideal K-algebra. If, as a K-module, Λ is projective then K is a direct summand of Λ.*

Proof. Since Λ is an ideal K-algebra, $K \cap \Lambda I = I (= KI)$ for every ideal I of K. Since Λ is K-projective and hence K-flat, this implies that the K-module Λ/K is K-flat by Rotman [7, Theorem 3.37, p. 59]. Applying then [7, Theorem 3.39, p. 61] to the exact sequence $0 \to K \to \Lambda \to \Lambda/K \to 0$, we know that there exists a K-homomorphism h: $\Lambda \to K$ such that $h(1) = 1$. This equality implies that the restriction of h to K is the identity map. Therefore it follows that K is a direct summand of the K-module Λ. (In the above Theorem 3.39 (Villamayor's theorem) in [7], F is assumed to be free. But if we use the fact that every projective module is a direct summand of a free module, we can easily derive that the same theorem remains true even if F is assumed to be projective.)

COROLLARY 2.14. *Let Λ be a separable ring with center Z and suppose Λ is projective as a Z-module. Then Z is a direct summand of the Z-module Λ.*

3. SEPARABLE SUBALGEBRAS OF A SEPARABLE RING

We shall first prove the following lemmas, which may be of some interest for themselves:

LEMMA 3.1. *Let A be a separable algebra over a commutative ring K and let M be a left A-module. If M is injective as a K-module then M is an injective A-module.*

Proof. Let X and Y be left A-modules and let $h: Y \to X$ be an A-monomorphism. Assume that M is K-injective. Then considering X, Y as K-modules and h as a K-monomorphism, we have a K-epimorphism $\mathrm{Hom}_K(h, M)$: Hom_K $(X, M) \to \mathrm{Hom}_K(Y, M)$. But, as is well known, both $\mathrm{Hom}_K(X, M)$ and Hom_K (Y, M) are converted into A-bimodules in the natural manner and besides $\mathrm{Hom}_K(h, M)$ becomes an A-bimodule-epimorphism. Since A is separable over K, it follows that $\mathrm{Hom}_K(h, M)$ induces an epimorphism $\mathrm{Hom}_K(X, M)^A \to \mathrm{Hom}_K$ $(Y, M)^A$ [3, Corollary 1.5, p. 43]. If we observe, however, that every $f \in \mathrm{Hom}_K$ (X, M) satisfies $(af)(x) = af(x)$ and $(fa)(x) = f(ax)$ for all $a \in A$, $x \in X$, we know that f is in $\mathrm{Hom}_K(X, M)^A$ if and only if f is in $\mathrm{Hom}_A(X, M)$, that is, we have $\mathrm{Hom}_K(X, M)^A = \mathrm{Hom}_A(X, M)$. Similarly we have $\mathrm{Hom}_K(Y, M)^A = \mathrm{Hom}_A(Y, M)$, and also the restriction of $\mathrm{Hom}_K(h, M)$ to $\mathrm{Hom}_A(X, M)$ is clearly nothing but $\mathrm{Hom}_A(h, M)$. Thus it is shown that M is A-injective.

LEMMA 3.2. *Let A be a separable K-algebra and let M be a left A-module. If M is flat as a K-module then M is a flat A-module.*

Proof. Let Q be the additive group of rationals and Z the additive group of integers. Put $M^* = \mathrm{Hom}_Z(M, Q/Z)$. Then M^* is a right A-module in the natural manner. Assume that M is K-flat. Then M^* is K-injective by [7, Theorem 3.35, p. 58]. Since A is separable K-algebra, we know that M^* is A-injective by applying Lemma 3.1 to M^*. Then again by the above cited theorem we can conclude that M is A-flat.

Remark. Lemmas 3.1 and 3.2 can be regarded apparently as those propositions which are obtained from DeMeyer and Ingraham [3, Proposition 2.31, p. 48] by replacing the projectivity with the injectivity and the flatness, respectively.

LEMMA 3.3. *Let U be a finitely generated quasi-projective left R-module such that every R-submodule of U is in* $\mathrm{Gen}_R(U)$, *and let* $S = \mathrm{End}_R(U)$. *Then between R-S-submodules U_0 of U and two-sided ideals S_0 of S there is a one-to-one correspondence by the following relations*:

$$U_0 = US_0, \qquad S_0 = \{s \in S \mid Us \subset U_0\}.$$

And, if U_0 corresponds to S_0, the factor module U/U_0 is also a finitely generated quasi-projective left R-module such that every R-submodule of U/U_0 is in $\mathrm{Gen}_R(U/U_0)$, *and the factor ring S/S_0 can be identified with* $\mathrm{End}_R(U/U_0)$ *in the natural manner.*

Proof. The first assertion about one-to-one correspondence is an immediate consequence of Corollary 1.4, since it is clear in the corollary that an R-submodule U_0 is an S-submodule if and only if the corresponding left ideal L is a two-sided ideal of S.

Let $s \in S$ be an R-endomorphism of U. Then since $U_0 s \subset U_0$, s induces an R-endomorphism of U/U_0. Since furthermore $Us \subset U_0$ if and only if $s \in S_0$, S/S_0 can be identified with a subring of $\mathrm{End}_R(U/U_0)$. Let conversely h be any R-endomorphism of U/U_0. Let $p\colon U \to U/U_0$ be the natural R-epimorphism. Then the quasi-projectivity of ${}_RU$ implies that there exists an endomorphism $s \in S = \mathrm{End}_R(U)$ such that $p \circ s = h \circ p$, which means nothing but that s induces h. Thus we have that $S/S_0 = \mathrm{End}_R(U/U_0)$. The remaining assertions can be proved in a routine way.

LEMMA 3.4. *Let U be an R-S-bimodule of Morita type (i.e., ${}_RU$ is a finitely generated projective generator and* $S = \mathrm{End}_R(U)$, *or equivalently, U_S is a finitely generated projective generator and* $R = \mathrm{End}_S(U)$). *Then*:

(i) *Between two-sided ideals R_0 of R, R-S-submodules U_0 of U and two-sided ideals S_0 of S, there is a one-to-one correspondence by the following relations*:

$$R_0 U = U_0 = US_0,$$
$$R_0 = \{r \in R \mid rU \subset U_0\}, \qquad S_0 = \{s \in S \mid Us \subset U_0\};$$

and, if R_0, U_0, and S_0 correspond, the factor module U/U_0 is also of Morita type when regarded as an R/R_0-S/S_0-bimodule.

(ii) *For any element z of the center Z of R there exists a unique element z^* of S such that $zu = uz^*$ for all $u \in U$, and the mapping $z \mapsto z^*$ gives a canonical isomorphism of Z onto the center Z^* of S.*

(iii) *R is a separable ring if and only if S is a separable ring.*

Proof. (i) The assertion about one-to-one-correspondence and that $S/S_0 = \mathrm{End}_R(U/U_0)$ follow from Lemma 3.3, while that the left R/R_0-module

U/U_0 is a finitely generated projective generator is more or less well known and can easily be proved. (ii) Let z be in the center Z of R. Then the mapping $u \mapsto zu$, $u \in U$, is an endomorphism of ${}_RU$, that is, there is a unique element z^* of $S = \mathrm{End}_R(U)$ such that $zu = uz^*$ for all $u \in U$. It then follows that for each $s \in S$ we have $usz^* = zus = uz^*s$ for all $u \in U$ and hence $sz^* = z^*s$, which means that z^* is in the center Z^* of S. In the same way, we can associate with each $z^* \in Z^*$ a unique $z \in Z$ such that $zu = uz^*$ for all $u \in U$. Thus the mapping $z \mapsto z^*$ gives a ring isomorphism $Z \to Z^*$. (iii) Let R_0, U_0, and S_0 be a triple of corresponding two-sided ideal of the R, R-S-submodule of U and two-sided ideal of S, respectively. Let z be an element of Z and z^* the corresponding element of Z^*. Then z is in R_0 if and only if $zU \subset U_0$, and also z^* is in S_0 if and only if $Uz^* \subset U_0$; but since $zU = Uz^*$, this implies that z is in R_0 if and only if z^* is in S_0. Thus, if we put $I = Z \cap R_0$ and I^* the corresponding ideal of Z^*, we have $I^* = Z^* \cap S_0$. Suppose now that R is a separable ring. Then we have $R_0 = RI$ and therefore $U_0 = R_0U = IRU = IU = UI^* = USI^*$. But this means that to the two-sided ideal SI^* of S there corresponds the R-S-submodule U_0 ($= US_0$) of U, so that we have $SI^* = S_0$ since the correspondence is one-to-one. On the other hand, the separability of the ring R implies that the subring Z/I of the factor ring R/R_0 is equal to its center. Applying then (ii) to the R/R_0-S/S_0-bimodule U/U_0 of Morita type, we find that Z/I is, by the mapping $z + I \mapsto z^* + I^*$, carried isomorphically onto Z^*/I^* and Z^*/I^* coincides with the center of S/S_0. Since these are the case for every two-sided ideal S_0 of S, we see that S is a separable ring.

Remark. As for the one-to-one correspondence between two-sided ideals of R and two-sided ideals of S as in Lemma 3.4, (i), cf. [3, Corollary 3.5, p. 22].

THEOREM 3.5. *Let Λ be a separable ring with center Z, and let A be a subring of Λ containing Z such that A is a separable Z-algebra and is a pure Z-submodule of Λ. Let Γ be the centralizer of A in Λ, that is, $\Gamma = \Lambda^A$. Then*

(i) *Γ is a separable ring,*

(ii) *A is the centralizer of Γ in Λ, that is, $A = \Lambda^\Gamma$,*

(iii) *the right (or left) Γ-module Λ is finitely generated projective and Γ is a Γ-direct summand of Λ.*

Proof. Let Λ^0 be an opposite ring of Λ, and let $x \mapsto x^0$, $x \in \Lambda$, be a fixed opposite isomorphism $\Lambda \to \Lambda^0$. We may assume that Z is also the center of Λ^0 and $z^0 = z$ for all $z \in Z$. We denote by A^0 the image of the subring A of Λ by the fixed opposite isomorphism. Then A^0 is a subring of Λ^0 containing Z. Consider then the enveloping algebra $\Lambda^e = \Lambda \otimes_Z \Lambda^0$ of Λ. Since Λ is flat as a Z-module by Proposition 2.8, $\Lambda \otimes_Z A^0$ is regarded as a subring of Λ^e in the natural manner. Furthermore since A is a pure Z-submodule of Λ by assumption, $A \otimes_Z A^0$ is also considered as a subring of $\Lambda \otimes_Z \Lambda^0$ in the natural manner.

We now regard Λ as a left Λ^e-module in the usual way. Since Λ^e is a separable ring with center Z by Proposition 2.10, the Λ^e-module Λ is faithful because every two-sided ideal of Λ^e is generated by an ideal of Z. We next consider Λ as a left module over the subring $\Lambda \otimes_Z A^0$ of Λ^e. The module Λ is a cyclic module geneated by 1, or what is the same thing, the mapping $\xi \mapsto \xi 1$ for $\xi \in \Lambda \otimes_Z A^0$ gives a $\Lambda \otimes_Z A^0$-epimorphism $\varphi: \Lambda \otimes_Z A^0 \to \Lambda$. We can also consider A as a left module over the subring $A \otimes_Z A^0$. Then A is clearly a submodule of Λ, and if we denote by μ the restriction of φ to $A \otimes_Z A^0$ μ is also an $A \otimes_Z A^0$-epimorphism $A \otimes_Z A^0 \to A$. Since A is a separable Z-algebra, i.e., the $A \otimes_Z A^0$-module A is projective, the epimorphism μ must split, which means that there exists an $A \otimes_Z A^0$-homomorphism $\nu: A \to A \otimes_Z A^0$ such that $\mu \circ \nu =$ identity map of A. Put $\epsilon = \nu(1)$. Then $\epsilon \in A \otimes_Z A^0$ and satisfies the following three conditions: (1) $\epsilon 1 = 1$, (2) $\epsilon^2 = \epsilon$, (3) $(a \otimes 1 - 1 \otimes a^0)\epsilon = 0$ for all $a \in A$. For $\epsilon 1 = \mu(\epsilon) = \mu(\nu(1)) = 1$, $\epsilon^2 = \epsilon\nu(1) = \nu(\epsilon 1) = \nu(1) = \epsilon$, and $(a \otimes 1 - 1 \otimes a^0)\epsilon = (a \otimes 1 - 1 \otimes a^0)\,\nu(1) = \nu((a \otimes 1 - 1 \otimes a^0)1) = \nu(a - a) = 0$ for all $a \in A$. (Cf. [3, Proposition 1.1].)

Now we define a homomorphism $\psi: \Lambda \to \Lambda \otimes_Z A^0$ by $\psi(y) = (y \otimes 1)\epsilon$ for $y \in \Lambda$. Then ψ is a $\Lambda \otimes_Z A^0$-homomorphism, because $\psi((x \otimes a^0)y) = \psi(xya) = ((xya) \otimes 1)\epsilon = (xy \otimes 1)(a \otimes 1)\epsilon = (xy \otimes 1)(1 \otimes a^0)\epsilon = (xy \otimes a^0)\epsilon = (x \otimes a^0)(y \otimes 1)\epsilon = (x \otimes a^0)\,\psi(y)$ for all $x, y \in \Lambda$, $a \in A$. Moreover we have $\varphi(\psi(y)) = \varphi((y \otimes 1)\epsilon) = (y \otimes 1)\,\varphi(\epsilon) = (y \otimes 1)(\epsilon 1) = (y \otimes 1)1 = y$ for all $y \in \Lambda$, that is, $\varphi \circ \psi =$ identity map of Λ. Thus it is shown that φ splits and therefore the cyclic $\Lambda \otimes_Z A^0$-module Λ is projective.

Let next T be the trace ideal of the $\Lambda \otimes_Z A^0$-module Λ, i.e., the sum of all $\Lambda \otimes_Z A^0$-homomorphic images of Λ in $\Lambda \otimes_Z A^0$. Then T is a two-sided ideal of $\Lambda \otimes_Z A^0$. Therefore, by Proposition 2.9, there is a two-sided ideal Q^0 of A^0 such that $T = \Lambda \otimes Q^0$. Applying then the homomorphism $\varphi: \Lambda \otimes_Z A^0 \to \Lambda$, we have $\varphi(T) = \varphi(\Lambda \otimes Q^0) = (\Lambda \otimes Q^0)1 = \Lambda Q$ where Q is a two-sided ideal of A corresponding to Q^0. On the other hand, since $\psi: \Lambda \to \Lambda \otimes_Z A^0$ is a $\Lambda \otimes_Z A^0$-homomorphism, $\epsilon = \psi(1)$ is in T. Applying again φ, we have $\varphi(\epsilon) = \epsilon 1 = 1 \in \varphi(T) = \Lambda Q$. But since ΛQ is a left ideal of Λ, this implies that $\Lambda Q = \Lambda$. Now A is a pure submodule of the flat Z-module Λ by assumption. Therefore the factor module Λ/A is also a flat Z-module by Stenström [9, Proposition 11.1] Since A is a separable Z-algebra, it follows from Lemma 3.2 that both Λ and Λ/A are flat even as right A-modules. Then applying [7, Theorem 3.37], we have $Q = AQ = A \cap \Lambda Q = A \cap \Lambda = A$ and therefore $T = \Lambda \otimes_Z A^0$. Thus we have shown that the left $\Lambda \otimes_Z A^0$-module Λ is a generator.

Since for every $x \in \Lambda$ and $a \in A$ the left multiplications of $x \otimes 1$ and $1 \otimes a^0$ on Λ are the same as the left multiplication of x and the right multiplication of a, respectively, and since the endomorphism ring of the left Λ-module Λ is identified with Λ, as the right multiplication ring, it is clear that the endomorphism ring of the left $\Lambda \otimes_Z A^0$-module Λ coincides with $\Gamma = \Lambda^A$. Thus Λ is a $\Lambda \otimes_Z A^0$-Γ-bimodule of Morita type. In particular, Λ is finitely generated and projective as a right Γ-module. Furthermore, since $\Lambda \otimes_Z A^0$ is a separable ring by Proposition 2.10, it follows from Lemma 3.4(iii) that Γ is a separable ring too. Let now x and a be any elements of Λ and A, respectively. Then ϵx is in Λ and we have $a(\epsilon x) - (\epsilon x)a = (a \otimes 1 - 1 \otimes a^0)\, \epsilon x = 0$, which implies that $\epsilon\, \Lambda \subset \Lambda^A = \Gamma$. Let conversely y be any element of Γ. Since Λ is a $\Lambda \otimes_Z A^0$-Γ-bimodule, we have $\epsilon y = \epsilon(1y) = (\epsilon 1)y = 1y = y$. Thus we have $\Gamma \subset$ and therefore $= \epsilon\Lambda$. Since ϵ is an idempotent, this implies that Γ is a Γ-direct summand of Λ; indeed, we have the following well-known direct decomposition: $\Lambda = \Gamma \oplus (1-\epsilon)\, \Lambda$. Finally, let b be any element of Λ^Γ. Then clearly $1 \otimes b^0(\epsilon\Lambda \otimes_Z \Lambda^0)$ is an endomorphism of the right Γ-module Λ. Since $\Lambda \otimes_Z A^0 = \mathrm{End}_\Gamma(\Lambda)$, this means that $1 \otimes b^0 \in \Lambda \otimes_Z A^0$. However, since Λ is a separable ring, it follows from Theorem 2.5 that the mapping $y \mapsto 1 \otimes y$ for $y \in \Lambda^0$ defines an isomorphism $\sigma(\Lambda^0)\colon \Lambda^0 \to (\Lambda \otimes_Z \Lambda^0)^\Lambda$. Therefore we have $1 \otimes b^0 \in (\Lambda \otimes_Z \Lambda^0)^\Lambda \cap (\Lambda \otimes_Z A^0) = (\Lambda \otimes_Z A^0)^A$. On the other hand, again by Theorem 2.5 the restriction of $\sigma(\Lambda^0)$ to A^0 defines also an isomorphism $\sigma(A^0)\colon A^0 \to (\Lambda \otimes_Z A^0)$. This implies in particular that $1 \otimes b^0 = 1 \otimes a^0$ for some $a^0 \in A^0$. But since $\sigma(\Lambda^0)$ is a monomorphism, this implies that $b^0 = a^0$ and so $b \in A$. Thus we have that $\Lambda^\Gamma = A$, and this completes the proof of our theorem.

From Theorem 3.5(ii) it follows that the center of Γ coincides with the center of A. If we notice this, we can derive the following corollary:

COROLLARY 3.6. *Let Λ be a simple ring with center Z, and let A be a simple subring of Λ containing Z such that A is finite dimensional over Z and the center of A is a separable finite extension field of Z. Let $\Gamma = \Lambda^A$. Then* (i) *Γ is a simple ring,* (ii) *$A = \Lambda^\Gamma$, and* (iii) *the right* (*or left*) *Γ-module Λ is finitely generated projective and Γ is a Γ-direct summand of Λ.*

In this connection, we prove the following theorem, in which the separability for the simple subalgebra A is not assumed:

THEOREM 3.7. *Let Λ be a simple ring with center Z, and let A be a simple subring of Λ containing Z which is finite dimensional over Z. Let $\Gamma = \Lambda^A$. Then*

(i) $A = \Lambda^\Gamma$,

(ii) *the right* (*or left*) *Γ-module Λ is finitely generated and projective.*

Proof. We regard Λ as a left module over the enveloping algebra $\Lambda^e = \Lambda \otimes_Z \Lambda^0$ of Λ, as in the proof of Theorem 3.5. Since A is a finite-dimensional simple algebra over Z, the enveloping algebra $A^e = A \otimes_Z A^0$ of A is a Frobenius algebra, and therefore A^e, as a left A^e-module, contains an isomorphic image of every simple left A^e-module. In particular, since A is a simple left A^e-module, there exists an A^e-monomorphism $\nu\colon A \to A^e$. Put $\eta = \nu(1)$. Then $0 \neq \eta \in A^e$

and satisfies $(a \otimes 1 - 1 \otimes a^0)\eta = 0$, because $(a \otimes 1 - 1 \otimes a^0)1 = a - a = 0$. By using η instead of ϵ, we can define a $\Lambda \otimes_Z \Lambda^0$-homomorphism ψ: $\Lambda \to \Lambda \otimes_Z \Lambda^0$ as in the proof of Theorem 3.5; indeed, ψ is defined by $\psi(y) = (y \otimes 1)\eta$ for $y \in \Lambda$. It follows then that $\eta = \psi(1)$ is in the trace ideal T of the left $\Lambda \otimes_Z \Lambda^0$-module Λ and in particular $T \neq 0$. But since T is a two-sided ideal of $\Lambda \otimes_Z \Lambda^0$ and $\Lambda \otimes_Z \Lambda^0$ is a simple ring, it follows that $T = \Lambda \otimes_Z \Lambda^0$, i.e., the $\Lambda \otimes_Z \Lambda^0$-module Λ is a generator. It can, however, be seen in the same way as in the proof of Theorem 3.5 that $\Gamma = \Lambda^\Lambda$ is the endomorphism ring of the $\Lambda \otimes_Z \Lambda^0$-module Λ. Therefore, by Morita's theorem, we know that the right Γ-module Λ is finitely generated projective and besides $\Lambda \otimes_Z \Lambda^0$ coincides with the Γ-endomorphism ring of Λ. By using the last fact, it is also possible to prove that $\Lambda^\Gamma = \Lambda$ in exactly the same way as in the proof of Theorem 3.5.

Remark. If Λ is an Artinian simple ring, then Theorem 3.7 is well known. It is known more precisely in this case that Γ is also an Artinian simple ring and the right (or left) Γ-module Λ is finitely generated free.

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