NON-SINGULAR MORSE–SMALE FLOWS ON 3-DIMENSIONAL MANIFOLDS

JOHN W. MORGAN

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In this paper we study which compact, orientable 3-manifolds \( W \) have non-singular Morse–Smale flows which are transverse to \( \partial W \) and pointing inward exactly on \( \partial_- W \) (\( \partial_- W \) is a union of components of \( \partial W \)). There is an obvious necessary condition—namely that \( \chi(W, \partial_- W) = 0 \). If \( W \) has dimension different from 3, then Asimov\([1, 2]\) showed that \( (W, \partial_- W) \) does indeed have a non-singular Morse–Smale flow provided that the Euler characteristic condition is satisfied. His method was to compare Morse–Smale flows with what he called round handle decompositions, and then to prove that manifolds satisfying the Euler characteristic condition admit round handle decompositions. Unfortunately, his argument could not be adapted to work in dimension 3. Here, we shall show that the result which Asimov obtained in all other dimensions is not true in dimension 3.

Recall that a 3-manifold \( P \) is prime if every \( S^2 \subset P^3 \) which separates bounds a 3-ball. Every 3-manifold \( M^3 \) has a prime decomposition, that is \( M = P_1 \# \ldots \# P_k \) with each \( P_i \) being prime\([4]\). The prime factors \( \{P_i\}_{i=1}^k \) are unique up to order\([5]\).

A Seifert fibration is a map \( \pi: M^3 \to X^2 \) between compact manifolds such that:

1. for some neighborhood \( U \) of \( \partial X \), \( \pi^{-1}(U) \to U \) is a fibration by circles; and
2. for each \( x \in \text{int} X^2 \) there is a neighborhood \( D_x \) of \( X \), and homeomorphisms forming a commutative diagram

\[
\begin{array}{ccc}
\pi^{-1}(D_x) & \xrightarrow{\pi} & S^1 \times D^2 \\
\downarrow & & \downarrow \\
(D_x, x) & \xrightarrow{\pi} & (D^2, 0)
\end{array}
\]

(Here \( \xi \) and \( z \) are complex numbers with \( |\xi| = 1 \) and \( |z| \leq 1 \).) The numbers \( p \) and \( q \) modulo \( p \) are invariants of the fibration. (If we reverse the orientation then \( q \) changes by sign.) The multiplicity of the fiber over \( x \) is \( p \). All but a finite number of fibers have multiplicity one. They are the regular fibers. All other fibers are multiple fibers. A Seifert fibration is trivial if there is a component \( \partial_0 M \) of \( \partial M \) such that \( \pi|_{\partial_0 M} \to \pi(M) \) is not injective. This is possible if and only if \( X^2 = D^2 \) and there is at most 1 multiple fiber (in which case \( M^3 \) is \( S^1 \times D^2 \).

Our main results are contained in the following theorems.

**Theorem A.** Let \( P \) be an orientable, prime 3-manifold with the Euler characteristic of every boundary component of \( P \) equal to zero. Let \( \partial_- P \) be an arbitrary union of these components. Suppose \( P \) is not \( S^1 \times D^2 \). The pair \((P, \partial_- P)\) admits a non-singular Morse–Smale flow if and only if \( P^3 \) is a union of non-trivial Seifert spaces attached to one another along components of their boundaries.

**Theorem B.** Let \( W^3 \) be an orientable connected 3-manifold with the Euler characteristic of every component of \( \partial W \) equal to zero. Let \( W = P_1 \# \ldots \# P_T \) be a prime decomposition for \( W \). Given \( \partial_- W \) a union of components of \( \partial W \) define \( \partial_- P \) to be
If no \( P_i \) is \( S^2 \times S^1 \), then \((W, \partial_\ast W)\) admits a non-singular Morse–Smale flow if and only if each \((P_n, \partial_\ast P_n)\) does.

The condition in Theorem B that no \( P_i \) be \( S^2 \times S^1 \) is necessary for the "only if" part of the theorem as the next result shows.

**Theorem C.** Let \( \chi(W^3, \partial_\ast W^3) = 0 \). Then for some \( N \geq 0 \), \((W^3 \neq (S^2 \times S^1), \partial_\ast W)\) carries a non-singular Morse–Smale flow.

Manifolds which are unions of non-trivial Seifert fiber spaces along boundary components were completely classified in [8]. In light of the more recent work in the theory of 3-manifolds we can re-interpret this class of 3-manifolds. First, the only non-sufficiently large 3-manifolds which have non-singular Morse–Smale flows are Seifert fiber spaces. According to [3] any sufficiently large 3-manifold \( M \) can be written as \( C \cup N \), where \( C \) is a disjoint union of Seifert fiber spaces maximal in \( M \). The unpublished results of Thurston say that every component of \( N \) is either \( T^2 \times I \) or has an interior which has a complete hyperbolic structure (constant negative sectional curvatures-1) of finite volume. In these terms a sufficiently large 3-manifold has a non-singular Morse–Smale flow if and only if it has no hyperbolic piece, i.e. if and only if \( N = \emptyset \cup T^2 \times I \).

This paper is organized along the following lines. In §1 we discuss Morse–Smale flows and round handle decompositions. The main result of that section is that there is a correspondence between these two concepts. §2 concentrates on the elementary properties of round handle decompositions. In §3 we prove the main technical lemmas about round handle decompositions for manifolds without nonseparating two spheres. We show that if the round handle decomposition is indecomposable and minimal then all the boundary components of each step in the filtration are tori and that the attaching maps for the round 1-handles are non-trivial. This implies that associated to each round handle in the decomposition is a Seifert fiber space. Unfortunately, those associated with the round 0-handles and the round 2-handles are just solid tori. We can however amalgamate these with the Seifert fibrations for the round 1-handles to decompose the 3-manifold into a union of non-trivial Seifert fiber spaces. In §4 we discuss the well-known theory of these Seifert fibrations and prove Theorem B. §5 consists of a proof of Theorem C. We finish by classifying the possible knots \( K \subset S^3 \) which can be attracting closed orbits for a NMS flow.

§1

A non-singular Morse–Smale flow (or NMS flow for short) on \((W, \partial_\ast W)\) is a flow without fixed points in \( W \) which is transverse to \( \partial W \), pointing inward on \( \partial_\ast W \) and outward on \( \partial_\ast W = \partial W - \partial_\ast W \). In addition, it satisfies the following properties (compare [7], p. 798):

1. The non-wandering set consists entirely of closed orbits.
2. The Poincaré map for each closed orbit is hyperbolic (i.e. has no eigenvalue on the unit circle).
3. If \( c \) is a closed orbit, then denote by \( S(c) \) and \( U(c) \) respectively the stable and unstable manifolds of \( c \). Then \( U(c) \) and \( S(c') \) are transversally intersecting submanifolds for \( W \) for all \( c \) and \( c' \).

In a manifold with a flow the result of beginning with a set \( S \) and flowing for time \( t \) is denoted \( \psi_t(S) \). Of course for \( t \) sufficiently large \( \psi_t(x) \) may be undefined for some or all of the elements \( x \in S \). In this case we use \( \psi_t(S) \) to denote the set obtained by flowing for time \( t \) for all those points of \( S \) for which this is possible. We employ a similar notation and convention for \( \psi_t(S) \) for \( t \) negative.

The connection, here as in [1] and [2], between NMS flows and the topology of the underlying manifold is achieved by using round handle decompositions. We expand Asimov's definition slightly to allow for non-orientability in the stable and unstable manifolds.
Definitions. (a) Let $Y^n$ be a manifold. $X^n$ is obtained from $Y^n$ by attaching a round $k$-handle if

1. there are disk bundles over $S^1$, $E^k_r$ and $E^{n-k-1}_t$, and
2. an embedding $\varphi: (\partial E^k_r \times E^{n-k-1}_t) \to \partial Y^{n-1}$ such that $X^n \cong Y^{n-1} \cup (E^k_r \oplus E^{n-k-1}_t)$.

(b) A round handle decomposition for $(X, \partial X)$ is a filtration

$$\partial X \times I = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_T = X,$$

where each $X_i$ is obtained from $X_{i-1}$ by attaching a round handle.

**Proposition (Asimov).** If $(X, \partial X)$ has a round handle decomposition, then $(X, \partial X)$ has a NMS flow whose closed orbits are exactly the core $S^1$'s of the round handles.

A proof is given in [2] for the case when the bundles $E^k_r$ and $E^{n-k-1}_t$ are trivial. As is easily seen this condition is unnecessary.

**Proposition.** Let $(X, \partial X)$ have a NMS flow. Then $(X, \partial X)$ has a round handle decomposition whose core circles are the closed orbits of the flow.

**Proof I.** Let $E_r$ and $E_u$ be flat disk bundles over $S^1$ with vector fields $\rho_r$ and $\rho_u$ which are fiberwise. Suppose that when restricted to any fiber they are linear and have contracting fixed points at the origin. Denote by $\partial/\partial \theta$ the lift of the constant vector field on $S^1$ to $E_r \oplus E_u$. The vector field $\partial/\partial \theta + \rho_r - \rho_u$ on $E_r \oplus E_u$ is without singularities and has one closed orbit—the zero section. This is a hyperbolic closed orbit. Given any hyperbolic closed orbit $c$ in a flow there is a neighborhood of $c, N(c)$, a flow of the above type, and a diffeomorphism between neighborhoods of the closed orbits in the two flows which carries flow lines to flow lines. Of course the stable and unstable manifolds of $c$ are carried to $E_r$ and $E_u$ respectively.

II. Let $c_1, \ldots, c_T$ be the closed orbits of a NMS flow on $X$. Introduce a relationship $c_i \leq c_j$ if $U(c_i) \cap S(c_j) \neq \emptyset$. This produces a partial order on the $\{c_i\}_{i=1}^T$.

**Proof of II.** (This is the “no cycle condition”.) It says that there are not closed orbits $c_1 \leq c_2 \leq \cdots \leq c_n \leq c_1$ with $c_i \neq c_j$. A proof of the no cycle condition can be found in [7], p. 780. Briefly, one shows that if $U$ is an open set containing a point $x \in S(c)$ then $\bigcup_{t>0} \psi_t(U)$ contains $U(c)$. This is proved using the local hyperbolic structure around $c$. Consequently, if $S(c')$ crosses $U(c)$ transversally then $\bigcup_{t>0} \psi_t(U)$ must meet $S(c')$ in an open set containing $U(c) \cap S(c')$ in its closure. Iterating this we find that if there is a circuit $c_1 < c_2 < \cdots < c_n < c_1$ and $U$ is any open set about $x \in S(c_n) \cap U(c_1)$ then $\bigcup_{t>0} \psi_t(U)$ contains $x$. This proves that $x$ is non-wandering, and hence contradicts property (1) for a NMS flow.

Choose a total ordering of the closed orbits which is compatible with the above partial order, $\{c_1, c_2, \ldots, c_T\}$. Suppose inductively that we have a $C^\infty$-submanifold $X_l$, $l = 1, \ldots, r$ with $\partial X \times I \subset X_l \subset X$ such that

- $c_1, \ldots, c_l \subset X_l$,
- $c_i \cap X_l = \emptyset$, $i > l$, and
- the flow on $X$ is transverse to $\partial X_l$ and pointing outward on $\partial X_l - \partial X$.

Let $F^k_r \oplus E^{n-k-1}_t$ be a small neighborhood of $c_{r+1}$ where the flow has the form $(\partial/\partial \theta)^{\partial X_l} + \rho_r - \rho_u$. Let $W^u = E^k_r \oplus E^{n-k-1}_t \cup (\bigcup_{t>0} \psi_t(\partial E^k_r \times E_u))$. Since any point in $\partial E^k_r \times E_u$ must flow in negative time to one of the $c_j$ for $j < r + 1$ or must flow off $\partial X$ we see that $W^u \cap \partial X_l = \emptyset$. Since $W$ is a union of partial flow lines $W^u$ crosses $\partial X$ transversally. Let $W_0 = W - W \cap \text{int} X_r$. Since $W_0 - E^k_r \oplus E_u$ has a NMS flow without closed orbits we see that $W_0 = E^k_r \oplus E_u \cup (\partial E^k_r \times E_u) \times I$ where the flow on the second factor is $\partial_x t$ (Here $t$ is the variable in the interval.) Consequently $X_l \cup W_0$ is obtained from $X$ by attaching a round $k$-handle.
The manifold $X_r \cup W_0$ satisfies conditions (a) and (b). However, it is not a $C^\infty$-submanifold since it has corners along $\partial(W_0 \cap \partial X_r)$ and $\partial E_r \cup \partial E_\alpha$. Also, condition (c) is not satisfied because $(\partial E_r \times \partial E_\alpha) \subset \partial(X_r \cup W_0)$ is invariant under the flow.

To correct these defects we simply round the corners as in the following picture.

§2. GENERALITIES ON ROUND HANDLE DECOMPOSITIONS

**Lemma 2.1.** Given round handle decompositions for $(X^r, \partial X)$ and $(Y^r, \partial Y)$ and an embedding $\varphi: A \hookrightarrow \partial X$ where $A \subset \partial Y$ is a union of components, then there is a round handle decomposition for $(X \cup Y, \partial (X \cup A))$.

**Proof.** One begins with $(\partial X \times I) \cup (\partial Y \times A) \times I$ and adds to $\partial X \times I$ the round handles forming $X \cup (\partial Y \times A) \times I$. Now instead of adding the round handles for $(Y, \partial Y)$ to $\partial Y \times I$ add them to $X \cup (\partial Y \times A) \times I$ by viewing $A \subset \partial X$ via $\varphi$.

**Lemma 2.2.** If $(X, \partial X)$ has a round handle decomposition, then so does $(X, \partial X)$.

**Proof.** Invert the round handle decomposition.

**Proposition 2.3.** Suppose every component of $X^3$ has Euler characteristic zero. $(X^3, \partial X)$ has a round handle decomposition if and only if $(X, \emptyset)$ does.

**Proof.** Let $M^2$ be either $S^1 \times S^1$ or the Klein bottle. Since $M^2$ is fibered by circles, $(M^2 \times I, \emptyset)$ has a round handle decomposition. This implies that $(\partial X \times I, \emptyset)$ has a round handle decomposition. Applying 2.2 we see that $(X, \emptyset)$ has a round handle decomposition provided that $(X, \partial X)$ does. Conversely, if $(X, \emptyset)$ has a round handle decomposition then by 2.1 so does $(X, \partial X)$. Since, as above, $(\partial X \times I, \emptyset)$ has a round handle decomposition, $(\partial X \times I \cup X, \partial X)$ has one also. The latter pair is homeomorphic to $(X, \partial X)$.

**Proposition 2.4.** If $\pi: M^3 \to X^3$ is a Seifert fibration then $(M^3, \emptyset)$ has a round handle decomposition.

**Proof.** Take an ordinary handle decomposition for $(X^3, \emptyset)$ so that each point of multiplicity $> 1$ is the center of a 0-handle. Let $h_0, \ldots, h_i$ be the handles in this
decomposition. Then \( \emptyset \subset \pi^{-1}(h_0) \subset \pi^{-1}(h_0 \cup h_1) \subset \cdots \subset \pi^{-1}(h_0 \cup \cdots \cup h_l) = M \) is a round handle decomposition for \( M^3 \).

**Corollary 2.5.** If \( M = \bigcup_{i=1}^{l} F_i \) where each \( F_i \) is a Seifert fibration and \( F_i \cap F_j \) is a union of boundary components of both \( F_i \) and \( F_j \) then \( M \) has a round handle decomposition.

**Proof.** This is an immediate consequence of 2.4, 2.3 and 2.1.

**Proposition 2.6.** If \( (X^3, \partial X) \) and \( (Y^3, \partial Y) \) admit round handle decompositions then so does \( (X^3 \# Y^3, \partial X \# \partial Y) \).

**Proof.** Let us assume for simplicity that \( X \) and \( Y \) are oriented. First note that if \( h \subset X^3 \) and \( h' \subset Y^3 \) are round 0-handles then \( ((X^3 - \text{int } h) \bigcup (Y^3 - \text{int } h')) \) admits a round handle decomposition. We claim in addition that \( (h \# h', \emptyset) \) also admits one. If that is so then 2.6 follows from 2.1. The manifold \( S^1 \times D^2 \# S^1 \times D^2 \) is a round 1-handle attached to a round 0-handle along the following curves:

\[ \begin{array}{c}
\text{In this section we concentrate on compact, orientable 3-manifolds whose boundary is a disjoint union of tori. In addition, we assume that all embeddings } S^2 \hookrightarrow M^3 \text{ separate. Since } M^3 \text{ is oriented, its round 0- and 2-handles are solid tori. There are two possibilities for round one handles—}\ E_i \oplus E_s \text{ where } E_i \text{ and } E_s \text{ are both trivial or are both non-trivial. We call the first kind orientable and the second kind non-orientable. For technical reasons we wish to “fatten up” the round 1-handles. We change the definition of adjoining a round 1-handle to read: } X^3 \text{ is obtained from } Y^3 \text{ by attaching a round 1-handle } h \text{ if } X^3 = Y^3 \bigcup \bigcup_{s \in \{0\}} A \times I \bigcup \bigcup_{s \in \{0\}} (E_i \oplus E_s), \text{ where } A \times \{0\} \text{ is a union of boundary components of } Y, \text{ and } \varphi: \partial E_i \times E_s \to A \times \{1\} \text{ meets every component of } A \times \{1\}. \text{ In such a decomposition } C(h) \text{ denotes } A \times I \bigcup E_i \oplus E_s. \text{ When } h \text{ is a round 0- or 2-handle, we let } C(h) \text{ be the round handle } h \text{ itself.}
\end{array} \]

**Lemma 3.1.** Let \( \partial M \times I \subset M_1 \subset M_2 \subset \cdots \subset M_T = M \) be a round handle decomposition. For every \( i, \partial M \) is a disjoint union of tori.

**Proof.** If \( \partial M \) has a non-torus component then (since \( M_i \) is orientable) it has components of non-zero Euler characteristic. But \( \chi(\partial M_i) = 0 \), and thus it must have components of positive Euler characteristic, i.e. spheres. By assumption any such \( S^2 \subset \partial M_i \) separates \( M \), say \( M = A \cup B \). The component of \( M_i \) which contains the \( S^2 \) lies in one of the two sides. We arrange, by renaming if necessary, that it be contained in \( A \). Restricting the round handle decomposition of \( (M, \partial M) \) to \( A \) gives a round handle decomposition of \( (A, \partial A \cap A) \). Thus \( \chi(A, \partial A \cap A) = 0 \). Since every component of \( \partial M \) is a torus it follows that \( \chi(A) = 0 \). On the other hand \( \partial A \) consists of tori in \( \partial M \) together with one \( S^2 \). Thus \( \chi(\partial A) = 2 \) and hence \( \chi(A) = 1 \). This contradiction establishes the lemma.

**Corollary 3.2.** Let \( (M^3, \partial M) \) have a round handle decomposition with round handles \( h_1, \ldots, h_T \). Then \( M^3 = (\partial M \times I) \bigcup_{i=1}^{T} C(h_i) \) where the intersections of the
various submanifolds in the union are boundary tori. Each term, except possibly some of the $C(h^i)$, are $S^1$-fibrations.

Unfortunately, in the above decomposition the $C(h^0)$ and $C(h^1)$ are trivial Seifert fibrations. Thus our task is two-fold: first to show that the $C(h^1)$ are all $S^1$-fibrations and second to amalgamate the $C(h^0)$ and the $C(h^1)$ with the $C(h^1)$'s so that $M^3$ is a union of non-trivial Seifert fibrations. For these two results to be true we need some assumptions on our round handle decomposition, namely, indecomposability and minimality. A round handle decomposition for $(M^3, \partial_M)$ is minimal if no other round handle decomposition for $(M^3, \partial_M)$ has fewer round handles. A round handle decomposition for $(M, \partial_M)$ is indecomposable if any time $(M, \partial_M) = (A, \partial_A) \# (B, \partial_B)$ with both $(A, \partial_A)$ and $(B, \partial_B)$ having round handle decompositions; then either $A$ or $B$ is $S^3$. (Here, as always in this paper, $\#$ denotes interior connected sum.)

To prove that the $C(h^1)$'s are $S^1$-fibrations and to amalgamate the $C(h^1)$ into non-trivial Seifert fibrations we must study the attaching maps for round $1$-handles. The key idea is that of incompressibility of surfaces in a $3$-manifold. Let $X^2 \subset M^3$ be a closed surface of genus $\geq 1$. We say that $X^2$ is incompressible in $M^3$ if $\pi_1(X^2) \to \pi_1(M^3)$ is an injection. Otherwise $X$ is compressible. If $X^2 \subset M^3$ is two-sided and if $X^2$ is compressible, then by Dehn's lemma there is an embedding $(D^2, \partial D^2) \hookrightarrow (M, X)$ with $D^2 \cap X = \partial D^2$ a non-trivial loop in $X$.

If $X^2 \subset M^3$ separates $M^3$ into $2$ components $A^3$ and $B^3$, then $X$ is incompressible if and only if $\pi_1(X) \to \pi_1(A)$ and $\pi_1(X) \to \pi_1(B)$ are injections. We say that $X$ is compressible on the side containing $A$ if $\pi_1(X) \to \pi_1(A)$ is not an injection.

**Lemma 3.3.** Let $\partial_- M \times I \subset M_1 \subset M_2 \subset \cdots \subset M_r = M$ be a minimal, indecomposable round handle decomposition for $M$. Suppose a torus $T$ in $\partial M_1$ is compressible in $M^3$. Then $T$ separates $M^3$, and one side of $T$ in $M^3$ must be a round $0$-handle or a round $2$-handle.

**Proof.** If $T \subset M^3$ is compressible, then there is $(D^2, S^1) \hookrightarrow (M^3, T)$ with $S^1 \subset T$ non-trivial. A neighborhood of $T \cup D^2$ is a punctured solid torus with boundary in $T \cup S^2$. Since every $S^2 \subset M^3$ separates, $T$ must separate. Let $M = P \cup R$. Suppose that the component of $M$, containing $T$ is contained in $P$. Then the round handle decomposition for $M$ induces one for $(P, P \cap \partial_M)$ and $(R, R \cap \partial_M \cup T)$. Applying Lemma 2.1 we see that $(R, R \cap \partial_M)$ and $(P, P \cap \partial_M \cup T)$ also have round handle decompositions. $T$ is compressible at least one of the two sides. Renaming if necessary, we can assume that $T \subset P$ is compressible. Thus, as we have seen, $P = A_0 \cup (S^1 \times D^2)_0$. (Here $W_0$ means $(W \text{-int } D^3)$ where $D^3$ is a closed $3$-ball contained in the interior of $W$.) Let $B_0 = (S^1 \times D^2)_0 \cup R$. We have $M = A_0 \cup B_0$ or $M = A \# B$. Since $B = S^1 \times D^2 \cup R$, we see that $(B, \partial_- M \cap B)$ has a round handle decomposition. Likewise, since $A \approx A_0 \cup (S^1 \times D^2)_0 \cup (D^2 \times S^1)$ it follows that $(A, \partial_- (M \setminus T))$ is a round handle decomposition. By the indecomposability one of $A$ or $B$ must be $S^3$. If it is $A$, then $P = S^1 \times D^2$ and by minimality $P$ must be a single round handle. If it is $B$ which is $S^3$, then $A \sim M$. The number of round handles in $A$ is the number in $P$ plus $1$. By minimality this implies that $R$ is a single round handle.

**Lemma 3.4.** Suppose $\partial_- M \times I \subset M_1 \subset \cdots \subset M_r = M$ is a minimal, indecomposable round handle decomposition and that $T \subset \partial M_i$ is a torus which is compressible on both sides. If $\pi_1(T^3) \to \pi_1(M^3)$ is non-trivial, then $M = (S^1 \times D^2) \cup (S^1 \times D^2)$.

**Proof.** As before let $M = P \cup R$ and then rewrite this as $A \# B$ with $T \subset B$. Since $\pi_1(T) \to \pi_1(M)$ is non-trivial $B$ cannot be $S^1$. Hence, $M = S^3$ and $P = S^1 \times D^2$. Since $T \subset R$ is also compressible, we see that $M = (S^1 \times D^2) \cup (S^1 \times D^2) \# B'$. Arguing as in
3.3 we see that $B'$ has a round handle decomposition. This time however $(S' \times D^2) \cup (S^1 \times D^3)$ cannot be $S^3$ since $\pi_1(T) \to \pi_1(M)$ is non-zero. Thus $B'$ must be $S^3$ and $M = (S^1 \times D^3) \cup (S^1 \times D^3)$.

**Proposition 3.5.** Let $\partial M \times I \subset M_1 \subset M_2 \subset \ldots \subset M_R = M$ be an indecomposable, minimal round handle decomposition for $M$.

1. No round 1-handle is attached to a torus boundary component along a loop trivial in that torus.
2. No round 1-handle is attached to a round 0-handle along a loop trivial in that round handle.

**Proof.** (1) There are several cases to consider.

(1a) The round 1-handle $h$ is orientable and is attached to two different boundary components $T$ and $T'$ with each attaching map trivial.

There are two ways to attach the round one handle $h$ to $T$ and $T'$: The first case is ruled out by 3.1 since the resulting boundary components are $S^2$ and $T \neq T'$. In the other case $C(h) = (T \times I) \cup h \cup (T' \times I)$ is homeomorphic to $(T \times I) \# (T' \times I)$. We parameterize $(T \times I) \# (T' \times I)$ so that $T \times \{0\}$ and $T' \times \{0\}$ are the intersection of $C(h)$ with all the previous round handles. The boundary components at this stage are $T \times \{1\}$ and $T' \times \{1\}$. Since the $S^2$ of the connected sum must separate $M^3$, we see that $M^3 = (A \cup B) \# _T \cup (T \times I) \#_T (T' \times I)$ where $A$ and $A'$ are the union of the previous thickened round bundles.

The round handle decomposition for $(M^3, \partial M)$ induces one for $(A, A \cap \partial M)$ and for $(B, (B \cap \partial M) \cup (T \times \{1\}))$. If we identify $T \times \{0\}$ and $T \times \{1\}$ these two round handle decompositions merge to form one for $(A \cup B, (A \cup B) \cap \partial M)$. The number of round handles in this decomposition is the number in $A$ and $B$ originally and hence is fewer than the total number in $M$. Likewise $(A' \cup B', (A' \cup B') \cap \partial M)$ has a round handle decomposition with fewer round handles than the given decomposition for $M$. Since $M = ((A \cup B) \# A' \cup B')$ this contradicts either the minimality or the indecomposability of the given decomposition for $M$.

(1b) The round 1-handle is orientable and attached to different boundary components, $T$ and $T'$, with the attaching map for $T$ trivial and the one for $T'$ non-trivial.
These attaching maps make $T'$ compressible on the side containing $h$. Since this side is not a single round 0- or 2-handle the other side of $T'$ must be. Hence $T'$ is the boundary of a round 0-handle $\tau$. Since $T'$ is compressible on both sides and the round handle decomposition has at least 3 round handles (or has non-empty boundary) it follows from 3.4 that $\pi_1(T') \to \pi_1(M)$ must be zero. Hence the attaching map for $h$ to $T'$ must generate $\pi_1(\tau)$. In this case $T \times I \cup h \cup \tau$ is diffeomorphic to $T \times I$. Thus we could remove $h$ and $\tau$ from the decomposition without altering $M$. This contradicts minimality.

(1c) $h$ is orientable and attached twice to the same boundary component by trivial circles.

There are two possibilities for the relative positions of the attaching circles—either they bound disjoint disks or they are nested. Consider $C(h) = (T \times I) \cup h$. The first possibility yields either a solid handlebody of genus 2 with a solid torus and a 3-ball removed or $(T \times I) \# (S^2 \times S^1)$ for $C(h)$ (depending on the choice of orientations of the attaching circles). Both these are ruled out—one by 3.1 and the other by the hypothesis that every $S^2 \subset M$ separates. If the circles are nested the $C(h)$ is either $(T \times I) \# (S^2 \times S^1)$ or $(T \times I) \# (S^1 \times D^2)$. Again the fact that every $S^2 \subset M$ separates rules out the first possibility. To exclude the second we argue similarly to the way we did in case 1a. Namely, we have $M = A \cup (T \times I) \# (S^1 \times D^2) \cup B$ where both $A \cup (T \times I)$ and $(S^1 \times D^2) \cup B$ have round handle decompositions with fewer round handles than in the decomposition for $M$. This contradicts either minimality or indecomposability.

(1d) $h$ is orientable and is attached twice to the same boundary component once by a non-trivial circle and once by a trivial one.

Let $T$ be the boundary component to which $h$ is attached. Clearly, $h$ makes $T$ compressible in $M$ on the side containing $h$. Since this side is not a single round 0- or 2-handle the other side of $T$ must be a round 0-handle $\tau$. Thus $T$ is compressible on both sides. Since the round handle decomposition has more than 2-handles (or has non-empty boundary), $\pi_1(T) \to \pi_1(M)$ must be trivial (3.4). Thus the nontrivial attaching curve for $h$ must generate $\pi_1(\tau)$. The union $\tau \cup h$ is then itself a solid torus. Consequently, in the round handle decomposition for $(M^3, \partial M^3)$ we can replace $\tau \cup h$ by a single round 0-handle. This contradicts minimality.

(1e) $h$ is non-orientable.

This time $h$ is attached to a single torus $T$ along a single curve. Actually attaching a non-orientable handle to $T$ along $\gamma$ is the same as adding a round 0-handle and then a round 1-handle connecting $\gamma$ and the curve $(2,1)$ on the round 0-handle. Thus $T \times I \cup h = T \times I \# \mathbb{R}P^3$, and $(M^3, \partial M^3) = (A, \partial A) \# \mathbb{R}P^3$. As in 3.3 $(A, \partial A)$ has a round handle decomposition. Thus either minimality or indecomposability is contradicted.

Proof of 2. Here also there are several cases analogous to those in 1 to eliminate. (2a) $h$ is orientable; one end is attached to $\tau$ along a curve trivial in $\tau$, and the other end is attached to $\tau'$ along a curve trivial in $\tau'$. (Here $\tau$ and $\tau'$ are round 0-handles.)

By part 1 both these attaching maps must be non-trivial in the boundaries, $\partial \tau$ and $\partial \tau'$. In $\tau \cup h \cup \tau'$ we then have a non-separating $S^2$. This is impossible.

(2b) $h$ is orientable; one end is attached to $\tau$ (a round zero handle) along a curve trivial in $\tau$, and the other end is attached to a different boundary component $T$.

Part 1 implies that $h$ is attached non-trivially to $T$. Thus $T$ is compressible on the side containing $h$. This side is not a single round handle. The other side must be a round 0-handle $\tau'$. By 3.4 $\pi_1(T) \to \pi_1(M)$ is zero, and thus $h$ must be attached to $\tau'$ by a curve generating $\pi_1(\tau')$. In this case $\tau \cup h \cup \tau' = S^1 \times D^2$. This contradicts minimality.

(2c) $h$ is orientable and attached twice to curves trivial in $\tau$ (a round 0-handle). These curves must be non-trivial in $\partial \tau$ and parallel. The result is either $S^1 \times$
D^2 \# S^1 \times S^2 or S^1 \times D^2 \# S^1 \times D^2. The first is not allowed because of the non-separating S^2. If the second occurs then the S^2 along which the connected sum is taken must separate. As in 3.3 either indecomposability or minimality would be contradicted.

(2d) h is non-orientable and is attached to a round zero handle \( \tau \) along a curve trivial in \( \tau \).

In this case \( \tau \cup h \equiv \text{RP}^2 \# S^1 \times D^2 \) and \( M^3 = \text{RP}^3 \# S^1 \times D^2 \cup \mathcal{A} \). Clearly, \( (A, \partial A \cup S^1 \times S^2) \) has a round handle decomposition, and hence so does \( A \cup S^1 \times D^2, \partial A \). Indecomposability requires then that \( A \cup S^1 \times D^2 \) be S^3, and \( M^3 \) be \( \text{RP}^3 \). But \( \text{RP}^3 \) has a round handle decomposition with only 2 round handles. The given decomposition for \( M \) has at least 3. This contradicts minimality.

This completes the proof of 3.5.

If \( h \) is an orientable round 1-handle then \( h \) has a natural fibration by circles with base the 2-disk. If \( h \) is a non-orientable round 1-handle then \( h \) has a natural Seifert fibration structure. Here \( h = ([S^1 \times D^2]/{(\theta, \bar{x}) \sim (\theta + \pi, -\bar{x})}) \). The fibration of \( S^1 \times D^2 \) by \( S^1 \times \{x\} \) gives a Seifert fibration of the quotient. It has one multiple fiber \([S^1 \times \{0\}]\) of multiplicity 2. In both these cases \( \partial E_{1g} \times E_{au} \) and \( E_{1g} \times \partial E_{au} \) are sub-fiber bundles of the Seifert fibration.

**Corollary 3.6.** Let \( h \) be a round 1-handle in a minimal, indecomposable round handle decomposition. The natural Seifert fibration of \( h \) extends to one for \( C(h) \).

**Proof.** Case 1: \( h \) is orientable and attached to different boundary components \( T \) and \( T' \). Here \( C(h) = (T \times I) \cup h \cup (T' \times I) \). Since the attaching circles for \( h \) in \( T \) and \( T' \) are non-trivial, there are fibrations of \( T \times I \) and \( T' \times I \) so that the attaching circles for \( h \) are fibers. These fibrations match with the given one on \( h \) to define one on all of \( C(h) \).

Case 2: \( h \) is orientable and attached twice to the same boundary component \( T \). Here \( C(h) = (T \times I) \cup h \). Since the attaching maps for \( h \) are disjoint and both non-trivial they must be parallel. Thus we can fiber \( T \times I \) so that both attaching maps become fibers. This extends the natural fibration of \( h \) to all of \( C(h) \).

Case 3: \( h \) is non-orientable. Let \( T \) be the boundary component to which \( h \) is attached. Then \( C(h) = (T \times I) \cup h \). Here again we can fiber \( T \times I \) so that the attaching map for \( h \) is a fiber. This extends the Seifert fiber structure for \( h \) over \( C(h) \).

Now we turn to the problem of amalgamating the \( C(h^0) \)'s and the \( C(h^1) \)'s into non-trivial Seifert fibrations. Associate to \( h \) all the round zero handles \( \tau_i \) with the property that \( h \) is the first round 1-handle attached to them. Likewise associate to \( h \) all the round two handles \( \tau_i \) with the property that in the inverted round handle decomposition the round handle dual to \( h \) is the first round 1-handle attached to them. Thus we are associating to \( h \) 0, 1 or 2 round 0-handles and 0, 1 or 2 round 2-handles. Define \( C(h) \) to be \( C(h) \) union all the round 0- and 2-handles associated to \( h \).

**Proposition 3.7.** Let \( h \) be a round 1-handle in a minimal, indecomposable round handle decomposition. The natural Seifert fiber structure on \( C(h) \) extends over \( C(h) \).

**Proof.** Suppose that \( \tau \) is a round zero handle associated to \( h \). The Seifert fibering on \( C(h) \) induces a fibering of \( \partial \tau \) by circles. These circles are non-trivial in \( H_i(\tau) \) by 3.5 part 2. Thus this fibration extends to a Seifert fibration of \( \tau \). There will be a multiple fiber at the core of \( \tau \) if and only if the fibers in \( \partial \tau \) do not generate \( H_i(\tau) \).

Inverting the round handle decomposition, the above argument applies to the round 2-handles associated to \( h \).

**Theorem 3.8.** Let \( M^3 \) be a connected, orientable 3-manifold with all boundary components tori and every \( S^2 \subset M^3 \) separating. Suppose \( \partial M \times I \subset M_1 \subset M_2 \subset \cdots \subset M_n = M \) is a minimal, indecomposable round handle decomposition for \( M \).
(1) If $M$ has no round 1-handles then either $M^3 = S^1 \times D^2$ or $M^3$ is a circle bundle over $S^2$ or $S^1 \times I$.

(2) If $M$ has at least 1-round 1-handle then
\[ M^3 = \partial M \times I \cup \nabla(h_i). \]

Each submanifold in this union is a non-trivial Seifert fiber space and the intersection of any two of these submanifolds is union of boundary tori in each.

**Proof.** Since $M$ is connected $M \cup \text{(round 2-handles)}$ is connected. If there are no round 1-handles, then $\partial M \times I \cup \text{(round 0-handles)}$ must be connected. Thus either $\partial M \times I \cup \text{(round 0-handles)}$ is $T^2 \times I$ or $S^1 \times D^2 \cup S^1 \times S^1 \times D^2$. Consequently $M^3 = T^2 \times I$, $S^1 \times D^2$ or $S^1 \times D^2 \cup S^1 \times S^1 \times D^2$. The last is always an $S^1$-bundle over $S^2$.

If $M$ has a round 1-handle and is connected then every round 0-handle must have a round 1-handle attached to it. The same is true for the round 2-handles when the decomposition is inverted. Thus we have
\[ M^3 = \partial M \times I \cup \nabla(h_i). \]

Furthermore, since each round 0-handle and round 2-handle is associated to exactly 1 round 1-handle, $\nabla(h_i) \cap \nabla(h_i)$ is either empty or a union of boundary tori in $\nabla(h_i)$ and $\nabla(h_i)$.

Lastly, we must show that each $\nabla(h_i)$ is a non-trivial Seifert fiber space. But if $\nabla(h_i)$ is a trivial Seifert fiber space then it must be $S^1 \times D^2$. There are two cases: $\nabla(h_i) = C(h_i)$ and $\nabla(h_i) \neq C(h_i)$. In the second case $\nabla(h_i)$ is a union of two or more round handles. Hence it cannot be $S^1 \times D^2$ by minimality. If $\nabla(h_i) = C(h_i)$, then $\nabla(h_i)$ has at least two boundary components and hence is not $S^1 \times D^2$.

§4. SEIFERT FIBERINGS AND 3-MANIFOLDS

**Theorem 4.1** (see [8]): (a) A union of two or more non-trivial Seifert fiber spaces attached to each other along boundary components is a prime 3-manifold.

(b) The only 3-manifold which is a Seifert fibration and is not prime is $\mathbb{RP}^3 \# \mathbb{RP}^3$.

**Proof.** If $X$ is a Seifert fiber space over a 2-manifold with non-empty boundary, then the universal cover of $X$, $\tilde{X}$, is fibered by $\mathbb{R}^n$'s over a contractible 2-manifold. As such $\tilde{X}$ is homeomorphic to a contractible subspace of $\mathbb{R}^3$ and hence every $S^2 \subseteq \tilde{X}^3$ bounds a 3-ball in $\tilde{X}$. This implies that any $S^2 \subseteq \mathbb{X}^3$ bounds a 3-ball in $\mathbb{X}^3$.

To prove part (a) one shows that if $A$ and $B$ are prime and if $T$ is an incompressible component of $\partial A$ and $\partial B$, then $A \cup B$ is prime. Suppose to the contrary that some $S^2 \subseteq A \cup B$ does not bound a 3-ball. We can assume that $S^2$ crosses $T$ transversally so that every component of intersection is a circle. Since $T$ is incompressible in $A \cup B$ all these circles must be trivial in $T$. Take an innermost circle in $T$ which separates $S^2$ into $H_+$ and $H_-$ and which bounds $B^2$ in $T$. If both $H_+ \cup B$ and $H_- \cup B$ bound 3-balls in $A \cup B$ then so would the original $S^2$. Both $H_+ \cup B$ and $H_- \cup B$ can be shifted by a slight isotopy to have fewer components of intersection with $T$. Arguing inductively we find $S^2 \subseteq A \cup B$ which misses $T$ and does not bound a 3-ball in $A \cup B$. This contradicts the fact that $A$ and $B$ are prime. Thus the result of gluing prime 3-manifolds together along incompressible boundary components is always prime. This provides an inductive proof of part (a) of 4.1.

(b) We break the study of Seifert fibrations up into several cases.

1. If the base is $D^2$ at least 2 multiple fibers; if the base is $S^2$ at least 4 multiple fibers; if the base is $\mathbb{RP}^2$, then at least 2 multiple fibers; if $\chi(\text{base}) \leq 0$ then no condition.
In all these cases some finite cover of the Seifert fibration is a fibration without multiple fibers over a base of non-positive Euler characteristic. Every embedded $S^2$ in such a cover bounds a 3-ball. The same must be true for the original manifold.

II. Base is $S^2$ with 3 multiple fibers.

This time there is a finite covering which is either an $S^1$-fibration over an oriented surface of non-positive Euler characteristic or is $S^3$. Arguing as in I we see that all such manifolds are prime.

III. Base $S^2$ with at most 2 multiple fibers.

In this case the 3-manifold is the union of two solid tori. Thus it is either $S^1$, $RP^3$, a lens space, or $S^2 \times S^1$. These are all prime.

IV. Base $RP^2$ and at most 1 multiple fiber.

These manifolds have double covers which are manifolds of type 3. Thus one of these manifolds is either prime or has a 2-sheeted cover $S^2 \times S^1$. In the latter case, it must be $RP^3 \# RP^3$.

V. Base $D^2$ with at most 1 multiple fiber.

In this case the manifold is $S^1 \times D^2$.

**Theorem 4.2.** If $(M^3, \partial M^3)$ has no non-separating, embedded 2-spheres; if $\partial M^3$ is a union of tori; and if $(M^3, \partial M^3) = (P_1, \partial P_1) \# \cdots \# (P_n, \partial P_n)$ with the $P_i$ prime then $(M^3, \partial M^3)$ admits a round handle decomposition if and only if only if each $(P_i, \partial P_i)$ does.

**Proof.** Suppose $(M^3, \partial M^3)$ has a round handle decomposition which is indecomposable. Then by 3.8 it is either $S^1 \times D^2$, or a union of non-trivial Seifert fibrations. All of the latter are prime except for $RP^3 \# RP^3$. But if $M^3$ were $RP^3 \# RP^3$ then its round handle decomposition would be decomposable. This proves that if the round handle decomposition for $M^3$ is indecomposable then $M$ is prime. To prove the “only if” direction of 4.2 one applies this result together with induction on $k$.

The “if” direction follows immediately from 2.5.

§5

In this section we prove the following theorem.

**Theorem 5.1.** Suppose that $(M^3, \partial M^3)$ is a connected, orientable 3-manifold with $\chi(M, \partial M) = 0$. For some $k > 0$ $(M \# (S^2 \times S^1)_{i=1}^k, \partial M)$ has a non-singular Morse-Smale flow.

**Proof.** We show that $(M \# (S^2 \times S^1)_{i=1}^k, \partial M)$ has a round handle decomposition. From this it follows that $(M, \partial M)$ has a NMS flow. Assume first of all that $\partial M$ and $\partial_+ M = \partial M - \partial_- M$ are both non-empty. This allows us to find a self-indexing Morse function without local maxima or local minima, [6], $f$: $(M, \partial_+ M, \partial_- M) \rightarrow ([0, 3], 0, 3)$. Let $X = f^{-1}([1, 3]), A = f^{-1}([0, 1]),$ and $B = f^{-1}([1, 3])$.

$A$ is obtained from $\partial_- M$ by attaching 1-handles:

$$A = (\partial_- M \times I) \bigcup_{i=1}^l h_i^1.$$

$B$ is obtained from $X$ by attaching 2-handles:

$$B = (X \times I) \bigcup_{i=1}^l h_i^2.$$

The condition that $\chi(M, \partial M) = 0$ is just the condition that $k = l$. In each one handle $h_i^1 = I \times D^2$, remove a neighborhood of the core $I \times D^2 \subset I \times D^2$. Removing these disks from $A$ punctures it $l$ times. Let $A_0 = A - \bigcup_{i=1}^l (I \times D^2)$, each $h_i^1 - I \times D^2$ is a round 1-handle attached to $\partial_- M \times I$. Thus $(A_0, \partial M)$ has a round handle decomposition. Likewise $(B_0, \partial_+ M)$ has a round handle decomposition. Inverting this gives one for
Let $W$ be the manifold obtained from $A \cup \ldots \cup A_k$ by identifying the boundaries of the $i$th puncture in $B_i$ with the boundary $i$th puncture in $A_i$ for $i = 1, \ldots , k$. By 2.1 we see that $W$ has a round handle decomposition. $W$ is nothing but $M \# (S^2 \times S')_i$.

If either $\partial_- M$ or $\partial_+ M$ is empty we simply remove two solid tori, $T_0$ and $T_1$, from $M$, and add $\partial T_0$ to $\partial_- M$ and $\partial T_1$ to $\partial_+ M$. The above result then shows that $([M - (\tau_0 \cup \tau_1)] \neq (S^1 \times S^2)_i \cup \partial_- M \cup \partial_+ M)$ admits a round handle decomposition. Insert $\tau_0$ making it a round 0-handle, and insert $\tau_1$ making it a round 2-handle. The result is a round handle decomposition for $M \# (S^2 \times S')_i$, $\partial_+ M$.

§6. KNOTS IN $S^3$

In this section we shall show that if $K \subset S^3$ is an attracting closed orbit of a NMS flow on $S^3$ then $K$ is an iterated torus knot. This application was suggested by the referee. Recall that a knot $K \subset S^3$ is an iterated torus knot if there is a sequence of solid tori $T_0, T_1, \ldots, T_n$ such that the core of $T_0$ is unknotted, the core of $T_i$ lies on a torus in $T_{i-1}$ which is parallel to a $(p, q)$-torus knot, and the core of $T_n$ is $K$. We define an iterated torus knot in a torus $T$ to be one for which there is a sequence of solid tori, $T = T_0 \cup T_1 \cup \cdots \cup T_n$, with analogous properties. Clearly if $T \subset S^3$ has a core which is an iterated torus knot and if $T' \subset T \subset S^3$, then $T'$ is also an iterated torus knot. Similarly, if $T \subset T_0$ has such a core, then so does $T' \subset T \subset T_0$.

The first step in proving the result about knots in $S^3$ is to understand which knot complements can be Seifert fibrations. As we shall see the only ones are the complements of torus knots. Define a map $S^3 \to S^2$ by considering $S^3$ as the unit sphere in $C^2$, and sending $(z_1, z_2) \to [p_\infty, z_1, z_2]$ in $\mathbb{P}^1(C) = S^2$, $(p$ and $q$ must be relatively prime). If $\alpha$ is a point on the torus $|z_1| = \epsilon, \epsilon < 1$, then the fiber containing $\alpha$ is a $(p, q)$-torus knot. Removing an invariant neighborhood of this fiber produces a map

$$\pi(p, q): C(p, q) \to D^2,$$

where $C(p, q)$ is the complement of the torus knot of type $(p, q)$. This map has two multiple fibers of multiplicity $p$ and $q$. The local invariants of the two multiple fibers are $(p, q \mod p)$ and $(q, p \mod q)$ respectively. There is a unique way to attach a solid torus $\tau_0 \supset \tau_1 \supset \cdots \supset \tau_n$ to $C(p, q)$ so that the result is $S^3$. Hence, when we attach a solid torus to $C(p, q)$ to make $S^3$ that solid torus has a core which is the $(p, q)$-torus knot.

If $\pi: F \to X$ is any non-trivial Seifert fibration with the property that $F$ is a knot complement, then $\pi: F \to X$ is isomorphic (as a Seifert fibration) to $\pi(p, q)$. Briefly, one shows that since $H_1(F) \cong \mathbb{Z}$ the base, $X$, must be $D^2$. Since $\pi_1(F)$ is normally generated by one element there can be at most 2 multiple fibers. If there are less than two such fibers then $F$ is a solid torus and hence a trivial Seifert fibration. Further study of the fundamental group shows that the multiplicities of the two multiple fibers must be relatively prime and that the local invariants must agree with those of $\pi(p, q)$ for some $p$ and $q$. It is then an easy matter to show that Seifert fibrations over $D^2$ with 2 multiple fibers are classified by their local invariants at the multiple fibers.

In the case of knots in a solid torus the situation is similar. Let $K \subset T$ be a $(p, q)$-torus knot with complement $C^*(p, q)$. Parameterize the solid torus as $(\xi, z)$ where $|\xi| = 1$ and $|z| \leq 1$. Define $\pi: \tau \to D^2$ by $\pi(\xi, z) = \xi^p z^q$. If $\alpha$ is a point on some torus $|z| = \epsilon, \epsilon < 1$, then the fiber through $\alpha$ is a $(p, q)$-torus knot lying on this torus. If we remove a neighborhood, $\nu(K)$, of this knot then $\pi$ induces $\pi': C^*(p, q) \to S^1 \times I$. It has one multiple fiber whose local invariants are $(p, q \mod p)$. As before, any Seifert
A fiber space which is homeomorphic to the complement of one solid torus in another is in fact equivalent to one of the above Seifert fibrations, or is the product circle bundle over $S^1 \times I$.

This discussion shows that any iterated torus knot occurs as an attracting closed orbit for some NMS flow on $S^3$. The next theorem proves the converse.

**Theorem 6.1.** (a) If $K \subset S^3$ is an attracting closed orbit for a NMS flow on $S^3$, then $K$ is an iterated torus knot. (b) If $K \subset \tau$ is an attracting closed orbit for a NMS flow on $\tau$, then $K$ is an iterated torus knot in $\tau$.

**Proof.** In either case the NMS flow gives one on the complement of a neighborhood of the knot. On this complement then we have a round handle decomposition which we can assume is minimal. This produces a decomposition of $S^3 - \nu(K)$, or $\tau - \nu(K)$ as

$$
\bigcup_{i=1}^{r} F_i
$$

where each $F_i$ is a union of thickened round handles and is a non-trivial Seifert fibration. (We ignore the trivial case where the complement is a solid torus.) We shall prove parts $a$ and $b$ together by induction on the number of round handles. In case $r$ equals 1 there is no need for induction. This case was dealt with by the preliminary discussion. If $r > 1$, then we can find a boundary component, $T$, which is not $\partial \nu(K)$ and not $\partial \tau$ (in the case of part $b$). If we are considering the case of $K \subset S^3$, then $T$ bounds a solid torus in $S^3$. This solid torus contains $K$ in its interior. Thus we have

$$
S^3 = (\bigcup F_i) \cup (\bigcup F_i \cup \nu(K)).
$$

By induction on the number of round handles we can assume that the core of $\tau'$ is an iterated torus knot in $S^3$. Likewise, since $\tau' - \nu(K) = \bigcup F_i$, induction allows us to assume that $K \subset \tau'$ is an iterated torus knot. It follows that $K \subset S^3$ is also an iterated torus knot.

If we are considering the case $K \subset \tau$, then there are two possibilities for the decomposition given by $T \subset \tau$. Either it bounds a solid torus containing $K$, or it divides $\tau$ into

$$(S^3 - \text{solid torus}) \cup \tau_1 \neq \tau_2,$$

with $\nu(K)$ contained in $\tau_1 \neq \tau_2$, and with the $(S^3 - \text{solid torus})$ being a non-trivial knot complement. The first case is handled by induction just as in the spherical case. The second cannot arise because the round handle decomposition for $\tau - \nu(K)$ was assumed minimal. If we replace $(S^3 - \text{solid torus})$ in the above decomposition by a solid torus, we get a round handle decomposition for $\tau - \nu(K)$ with fewer round handles.

**References**


Columbia University
New York