Bounds on the Spectrum of Nonnegative Matrices and Certain Z-Matrices

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ABSTRACT

Spectral bounds are obtained for a type of Z-matrix using Perron-Frobenius theory on the inverse. These lead to spectral bounds for nonnegative matrices and M-matrices. These results are then extended using the result of Rothblum and Tan concerning the coefficient of ergodicity of a nonnegative matrix.

I. INTRODUCTION AND PRELIMINARIES

Throughout we deal with \( n \times n \) real nonnegative matrices and \( n \times n \) real Z-matrices, i.e. matrices whose off-diagonal elements are nonpositive.

By the Perron-Frobenius theorem, if \( B \) is nonnegative (i.e. \( B \geq 0 \)), then \( \rho(B) \), the spectral radius of \( B \), is an eigenvalue of \( B \) of maximum modulus. Thus, in the literature that concerns the localization of eigenvalues of square nonnegative matrices, much attention has been devoted to determining upper bounds on the moduli of the remaining eigenvalues, i.e. the identification of a disk centered at the origin which includes all the eigenvalues of the corresponding matrix except the eigenvalue of largest modulus. There have also been studies directed toward the identification of sets which contain these same eigenvalues but which have a different geometry than that of a disk centered at the origin; the purpose of this paper is to make a further contribution in this latter direction.

Note that if \( B \geq 0 \) is irreducible and \( Bx = \rho(B)x \), where \( x = (x_1, \ldots, x_n)^T \geq 0 \), then \( B\bar{e} = \rho(B)e \), where \( \bar{B} = D^{-1}BD \), \( D = \text{diag}(x_1, \ldots, x_n) \), and \( e = (1, \ldots, 1)^T \). Hence, we may consider the nonnegative matrix \( B \) to be general-
ized stochastic in trying to localize its eigenvalues, and most of the prior results on localization of the eigenvalues of a nonnegative matrix are stated for stochastic matrices.

Fréchet [8, 9] showed that all the eigenvalues of a stochastic matrix $B$ lie in the interior or on the boundary of the circle

$$|z - b_{kk}| \leq 1 - b_{kk}, \quad (1.1)$$

where $b_{kk}$ is the smallest element of the main diagonal. Dmitriev and Dynkin [5] proved that no eigenvalue of a stochastic matrix of order less than or equal to $n$ can lie in the interior of the segments bounded by the unit circle and the chords joining the point $z = 1$ with the points $z = \exp(2\pi i/n)$ and $z = \exp(-2\pi i/n)$, and that any other point of the sector $-2\pi/n \leq \arg z \leq 2\pi/n$, including the chords, can be an eigenvalue. Brauer [1] showed that all the eigenvalues of a stochastic matrix $B$ lie in the interior or on the boundary of the oval of Cassini:

$$|z - b_{ii}| |z - b_{jj}| \leq (1 - b_{ii})(1 - b_{jj}), \quad (1.2)$$

where $b_{ii}$ and $b_{jj}$ are the two smallest elements of the main diagonal. If $b_{ii} \neq b_{jj}$, then the oval lies in the interior of Frechet’s circle. Lastly, Deutsch and Zenger [4] showed that if $B$ satisfies $Be = \rho(B)e$, then all eigenvalues (other than $\rho(B)$) lie in

$$G(B) = \bigcup_{i,k} G_{ik}(B) \quad (1.3)$$

with $G_{ik}(B) = \{ m_{ik} + ur_{ik} + zh_{ik} | z \in \mathbb{Z}, |u| \leq 1 \}$, where

$$m_{ik} := \frac{1}{2} (a_{ii} + a_{kk} - a_{ik} - a_{ki}),$$

$$h_{ik} := |a_{ii} + a_{ik} - a_{ki} - a_{kk}|,$$

$$d_{ik} := \frac{1}{2} \sum_{j \neq i, j \neq k} |a_{ij} - a_{kj}|,$$

$$r_{ik} := d_{ik} - \frac{1}{2} h_{ik},$$

$$z := \{ z : |z - \frac{1}{2}| \leq 1, |z + \frac{1}{2}| \leq 1 \}.$$
Fiedler and Pták [7, Theorem 4.8] showed that if \( A = (a_{ij}) = tI - B \), where \( B \geq 0 \) and \( t > \rho(B) \) (i.e., \( A \) is an \( M \)-matrix) and \( a = \max\{ a_{ii} \} \), then \( \text{sp} A \), the spectrum of \( A \), is contained in the circle with the center \( a \) and radius \( \rho(aI - A) \). Following Fiedler and Pták, we let \( K \) denote the class of \( M \)-matrices and \( K_0 \) denote the closure of \( K \), i.e. \( K_0 = \{ A \in Z | A = tI - B \text{ where } B \geq 0 \text{ and } t \geq \rho(B) \} \).

For \( B \geq 0 \), we shall let \( \rho_k(B) \), \( 1 \leq k \leq n \), denote the maximum spectral radius of all \( k \times k \) principal submatrices of \( B \). \( N_\text{cr} \)-matrices were defined by Johnson [11] to be matrices \( A \) of the form \( A = tI - B \) where \( B \geq 0 \) and \( \rho_{n-1}(B) < t < \rho(B) \). It is well known that if \( B \geq 0 \), then

\[
\rho_1(B) \leq \rho_2(B) \leq \cdots \leq \rho_{n-1}(B) \leq \rho_n(B) = \rho(B). \tag{1.4}
\]

Moreover, if \( B > 0 \), then the inequalities are strict. Also, it is well known that if \( B \geq 0 \) is irreducible, then \( \rho_{n-1}(B) < \rho(B) \); however, the other inequalities in (1.4) are not necessarily strict. To illustrate this, consider

\[
B = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
0 & 1 & 4
\end{pmatrix}.
\]

Here, \( \rho_1(B) = \rho_2(B) = 4 \).

Fan [6] examined those matrices in \( Z \) whose determinant is negative, whose principal minors of order \( n - 1 \) are negative, and whose principal minors of order \( n - 2 \) or less are positive. Following this vein for \( n > 3 \), Johnson [11] examined those matrices \( A \) such that

(i) \( A \in Z \),
(ii) all principal submatrices of order \( n - 2 \) or less are in \( K_0 \), and
(iii) at least one principal submatrix of order \( n - 1 \) is in \( N_0 \).

In Section II we note that for \( n \geq 3 \) this is equivalent to \( A \) having the form \( A = tI - B \) where \( B \geq 0 \) and \( \rho_{n-2}(B) \leq t < \rho_{n-1}(B) \), and we call such a matrix an \( F_0 \)-matrix in honor of Fan. Johnson [11, Theorem 3.1] then proved the following result:

**Theorem 1.1.** Let \( M \) be a matrix satisfying (i), (ii), (iii), and \( n \geq 3 \). If \( M \) is nonsingular, then \( \det M < 0 \) and \( M^{-1} \in Z \) with at least one positive diagonal entry.
Concerning the spectrum of $N_0$-matrices, the author [12, Theorem 2.2] proved the following result:

**Theorem 1.2.** Let $A$ be an $N_0$-matrix with $n(A)$ the negative root of minimum modulus, let $a = \max \{a_{ii}\}$, and let

$$R(A) = \{ z \in \mathbb{C} : |z - a| < a - n(A), |z| > -n(A) \}.$$  

Then $R(A)$ contains the spectrum of $A$.

In Section II bounds on the spectrum of an $F_0$-matrix $F = sI - B$ where $B \geq 0$ and $\rho_{n-2}(B) \leq s < \rho_{n-1}(B)$ are obtained in terms of $\rho_{n-2}(B)$, $\rho_{n-1}(B)$, and $\rho(B)$ by applying Perron-Frobenius theory to $F^{-1}$. These bounds lead directly to bounds on the real eigenvalues of a nonnegative matrix. In Section III, using Theorem 1.2 and the results on $F_0$-matrices obtained in Section II, bounds are obtained for the spectrum of both a nonnegative matrix $B$ and an $M$-matrix $A = tI - B$, where $B \geq 0$ and $t > \rho(B)$, in terms of $\rho_{n-2}(B)$, $\rho_{n-1}(B)$, and $\rho(B)$. These results are then extended using the results of Rothblum and Tan [10] concerning the coefficient of ergodicity of a nonnegative matrix. Lastly, an example is given in Section III which illustrates these bounds.

II. $F_0$-MATRICES

We first prove the following characterization of $F_0$ matrices.

**Theorem 2.1.** Suppose $n \geq 3$. Then $M \in F_0$ if and only if $M = tI - B$ where $\rho_{n-2}(B) \leq t < \rho_{n-1}(B)$.

**Proof.** First suppose that $M \in F_0$; then

(i) $M \in Z$,
(ii) all principal submatrices of order $n - 2$ or less are in $K_0$, and 
(iii) at least one principal submatrix of order $n - 1$ is in $N_0$.

Now by (i), $M = tI - B$ for some $B \geq 0$. If $t \geq \rho_{n-1}(B)$, then $M \in N_0$ or $M \in K_0$, which contradicts (iii). If $t < \rho_{n-2}(B)$, let $B_1$ be an $(n - 2) \times (n - 2)$
submatrix of $B$ such that $\rho(B_1) = \rho_{n-2}(B)$. Then the $(n-2)\times(n-2)$ principal submatrix $M_1 = tl - B_1$ of $M$ is not in $K_0$, which contradicts (ii). Thus, $\rho_{n-2}(B) \leq t < \rho_{n-1}(B)$. Obviously the converse holds.

We note that this disproves a counterexample given by Johnson [11, p. 213].

It was shown by the author in [12, Theorem 2.5] that each $F_0$-matrix $C$ has exactly one negative eigenvalue $n(C)$. Using this, we now prove:

**Lemma 2.2.** Suppose $n \geq 3$. Let $C$ be a singular $F_0$-matrix, say $C = (c_{ij}) = tl - B$, where $B \geq 0$ and $\rho_{n-2}(B) \leq t < \rho_{n-1}(B)$. Then $t = \rho_{n-2}(B)$.

**Proof.** Since $C$ is singular, $t \in \text{sp } B$. Suppose $t > \rho_{n-2}(B)$; then the $F_0$-matrix $G = \rho_{n-2}(B)l - B$ has two negative eigenvalues, namely $\rho_{n-2}(B) - \rho(B)$ and $\rho_{n-2}(B) - t$, which is a contradiction.

Immediately we have the following corollary:

**Corollary 2.3.** Suppose $B \geq 0$ is irreducible. If $\lambda$ is a real eigenvalue of $B$ and $\lambda \neq \rho(B)$, then $\lambda$ lies in the interval $(-\rho(B), \rho_{n-2}(B)]$.

**Proof.** Since $B \geq 0$ is irreducible, each real eigenvalue $\lambda$ such that $\lambda \neq \rho(B)$ lies in the interval $(-\rho(B), \rho(B))$. By the lemma, $\lambda \notin (\rho_{n-2}(B), \rho_{n-1}(B))$. Further, $\lambda \notin [\rho_{n-1}(B), \rho(B))$, since $tl - B$ is a (nonsingular) $N_0$-matrix for each $t \in [\rho_{n-1}(B), \rho(B))$.

**Example 2.4.** Consider the symmetric matrix

$$B = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$  

The eigenvalues of $B$ are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \rho(B) = 10$. Note that $1 \in (-\rho(B), \rho(B)] = (-10, 10]$, as guaranteed by Corollary 2.3.

We now obtain bounds on the spectrum of an $F_0$-matrix. Here, $\langle n \rangle = \{1, 2, 3, \ldots, n \}$, and $C(i, i)$ denotes the $(n-1)\times(n-1)$ minor of $C$ obtained by deleting the $i$th row and the $i$th column.
Theorem 2.5. Let \( C \in F_0 \), where \( n \geq 3 \); say \( C = (c_{ij}) = tI - B \), where \( B \geq 0 \) and \( \rho_{n-2}(B) \leq t < \rho_{n-1}(B) \). Then \( n(C) = t - \rho(B) \), and for all \( \lambda \in \text{sp}\ C \),

(i) \( |\lambda - c| \leq c - n(C) \), where \( c = \max\{ c_{ii} \} \), and either

(ii) \( |\lambda - \lambda_1| \geq r_1 \), where \( \gamma = \max\{ C(i, i)/\det C \} \)

\[
\lambda_1 = -\frac{\gamma [n(C)]^2}{1 - 2\gamma n(C)}
\]

and

\[
r_1 = \frac{n(C)[\gamma n(C) - 1]}{1 - 2\gamma n(C)},
\]

if \( C \) is nonsingular, or

(iii) \( |\lambda - \lambda_2| \geq r_2 \), where \( \lambda_2 = n(C)/2 \) and \( r_2 = -n(C)/2 \), if \( C \) is singular.

Proof. Suppose that \( C = tI - B \) where \( B \geq 0 \) and \( \rho_{n-2}(B) \leq t < \rho_{n-1}(B) \). Then \( n(C) = t - \rho(B) \). Let \( c = \max\{ c_{ii} \} \). Since \( cI - C \geq 0 \), all eigenvalues of \( cI - C \) have modulus less than or equal to the Perron root of \( cI - C \), i.e., \( |c - \lambda| \leq \rho(cI - C) = c - n(C) \) for all \( \lambda \in \text{sp}\ C \), which establishes (i).

If \( C \) is nonsingular, then \( C^{-1} = (c_{ij}') \in Z \) with at least one positive diagonal entry, and \( \det C < 0 \) by Theorem 1.1. Let \( \gamma = \max\{ c_{ii}' \} = \max\{ C(i, i)/\det C \} > 0 \). Now \( \gamma I - C^{-1} \geq 0 \), and thus if \( \mu \in \text{sp}\{ \gamma I - C^{-1} \} \), then \( |\mu| \leq \rho(\gamma I - C^{-1}) = \gamma - 1/n(C) \). It can be shown that the circle \( (x - r)^2 + y^2 = s^2 \) is transformed into the circle

\[
\left( \frac{u + r}{s^2 - r^2} \right)^2 + v^2 = \left( \frac{s}{s^2 - r^2} \right)^2
\]

under the transformation \( u + iv = 1/(x + iy) \). In our case, with \( \mu = x + iy \in \text{sp}\ C^{-1} \) and \( \lambda = 1/\mu = 1/(x + iy) = u + iv \in \text{sp}\ C \), we have \( r = \gamma \) and \( s = \gamma - 1/n(C) \), and it then follows by direct computation that if \( \lambda \in \text{sp}\ C \), then \( |\lambda - \lambda_1| \geq r_1 \), where \( \lambda_1 = -\gamma [n(C)]^2/[1 - 2\gamma n(C)] \) and \( r_1 = n(C)[\gamma n(C) - 1]/[1 - 2\gamma n(C)] \), which establishes (ii).
Now, to establish (iii), suppose that $C$ is singular. By Lemma 2.2, 
$C = \rho_{n-2}(B)I - B$. Let $C(i_0, i_0) = \min\{C(i, i)\} < 0$, and let $\alpha = \{i : C(i, i) = C(i_0, i_0)\}$. There exists a positive number $\epsilon$ such that for all $s \in (0, \epsilon)$,

(a) $C + sl$ is a nonsingular $F_\alpha$-matrix,
(b) $\rho_{n-2}(B) < t + s < \rho_{n-1}(B)$, and
(c) $(C + sl)(i, i) < (C + sl)(j, j)$ for all $i \in \alpha$ and $j \in \{n\} \setminus \alpha$.

By Theorems 1.1 and 2.1, $(C + sl)^{-1} \in Z$ with at least one positive diagonal element and $\det(C + sl) < 0$. Let

$$D^{(s)} = (C + sl)^{-1} = \left(d^{(s)}_{ij}\right)$$

and

$$d_s = \max\{d^{(s)}_{ii}\} = \max\left\{\frac{(C + sl)(i, i)}{\det(C + sl)}\right\} > 0.$$

Note that this maximum occurs for some $i \in \alpha$. Then, $d_s I - (C + sl)^{-1} \geq 0$ and hence $\rho(d_s I - (C + sl)^{-1}) = d_s - 1/[n(C + sl)]$, which in turn implies that if $\mu \in \text{sp}\{d_s I - (C + sl)^{-1}\}$, then $|\mu| \leq d_s - 1/[n(C + sl)]$. It then follows (again by direct computation) that if $\lambda \in \text{sp}\{C + sl\}$, then $|\lambda - \mu| \geq r_s$, where

$$n(C + sl) = \rho_{n-2}(B) + s - \rho(B), \quad \mu_s = -\frac{d_s [n(C + sl)]^2}{1 - 2d_s n(C + sl)},$$

and

$$r_s = \frac{n(C + sl)[d_s n(C + sl) - 1]}{1 - 2d_s n(C + sl)}.$$ 

Now as $s$ approaches $0^+$, $\det(C + sl)$ approaches $0^-$, $(C + sl)(i, i)$ approaches $C(i_0, i_0)$ for all $i \in \alpha$, and $n(C + sl)$ approaches $n(C)^+$. Thus, $d_s$ approaches $+\infty$, which implies $\mu_s$ approaches $n(C)/2$ and $r_s$ approaches $-n(C)/2$, and (iii) follows.

We note that $n(C)$ is the leftmost point of the circular region in all three cases of Theorem 2.1, and (by the Perron-Frobenius theorem) if $C$ is irreducible, then equality holds in (i) or (ii) if and only if $\lambda = n(C)$.
Following Rothblum and Tan [10], let $P$ be an $n \times n$ nonnegative, irreducible matrix having spectral radius $\rho$, and let $w \in \mathbb{R}^n$ be a positive right eigenvector of $P$ corresponding to the eigenvalue $\rho$. For a norm $\| \cdot \|$ defined on $\mathbb{R}^n$, define the coefficient $\tau_{\| \cdot \|} (P)$ by

$$
\tau_{\| \cdot \|} (P) = \max_{\|x\| \leq 1} \|x^T P\|, \quad x^T w = 0, \quad x \in \mathbb{R}^n
$$

where $\|y^T\| = \|y\|$. Let $\xi (P)$, the coefficient of ergodicity of $P$, be defined by

$$
\xi (P) = \max_{\lambda \in \text{sp } P} |\lambda|, \quad \lambda \neq \rho (P)
$$

Rothblum and Tan [10, Theorem 3.1] showed that $\xi (P) \leq \tau_{\| \cdot \|} (P)$ for any norm $\| \cdot \|$ defined on $\mathbb{R}^n$. In particular, if $\tau_i (P)$ corresponds to the $l_1$ norm on $\mathbb{R}^n$ and $w = (w_1, \ldots, w_n)^T$ is a positive right eigenvector of $P$ corresponding to the eigenvalue $\rho$, they showed [10, Corollary 4.2] that

$$
\tau_1 (P) = \max_{i, j = 1, \ldots, n} \left( w_i + w_j \right)^{-1} \left( \sum_k |w_j P_{ik} - w_i P_{jk}| \right).
$$

Thus, if we use $\tau_{\| \cdot \|} (P)$ in the proof of Theorem 2.5(i) and (ii) rather than the Perron-Frobenius bound $\rho (P)$, we obtain the sharper bounds on $\{ \lambda \in \text{sp } P, \lambda \neq \rho (R) \}$ as follows:

(i') $|c - \lambda| \leq \tau_{\| \cdot \|} (c I - C) = T_1$;

(ii') $|\lambda - \lambda_1| \geq r_1$, where

$$
\lambda_1 = -\frac{\gamma}{T_2^2 - \gamma^2},
$$

$$
r_1 = \frac{T_2}{T_2^2 - \gamma^2},
$$

$$
T_2 = \tau_{\| \cdot \|} (\gamma I - C^{-1}),
$$

with

$$
\gamma = \max \left\{ \frac{C(i, i)}{\det C} \right\}.
$$
III. M-MATRICES AND NONNEGATIVE MATRICES.

We next examine the spectrum of irreducible $M$-matrices. As mentioned before, (i) was done by Fiedler and Pták.

**Theorem 3.1.** Let $A$ be an irreducible $n \times n$ $M$-matrix where $n \geq 3$, (say $A = tI - B$ where $B \succeq 0$ and $t > \rho(B)$), and let $a = \max\{a_{ii}\}$. Then, each $\lambda \in \text{sp}A$ satisfies

(i) $|\lambda - a| \leq \rho(aI - A) = a + \rho(B) - t$ and  
(ii) $|\lambda - (t - \rho_{n-1}(B))| \geq \rho(B) - \rho_{n-1}(B)$.

Further, if $\rho_{n-2}(B) < \rho_{n-1}(B)$ and $F = \rho_{n-2}(B)I - B$, then either

(iii) $|\lambda - (t - \rho_{n-2}(B) + \lambda_1)| \geq r_1$ if $F$ is nonsingular, where

$$
\gamma = \max \left\{ \frac{F(i, i)}{\det F} \right\}, \quad n(F) = \rho_{n-2}(B) - \rho(B),
$$

$$
\lambda_1 = -\frac{\gamma n(F)^2}{1 - 2\gamma n(F)}, \quad \text{and} \quad r_1 = \frac{n(F)[\gamma n(F) - 1]}{1 - 2\gamma n(F)},
$$

or

(iv) $|\lambda - \{t - [\rho_{n-2}(B) + \rho(B)]/2\}| \geq [\rho(B) - \rho_{n-2}(B)]/2$ if $F$ is singular.

**Proof.** Let $A = tI - B$ be irreducible, where $B \succeq 0$ and $t > \rho(B)$. Since $B$ is irreducible, $\rho(B) - \rho_{n-1}(B) > 0$. Let $C = \rho_{n-1}(B)I - B = A + (\rho_{n-1}(B) - t)I$. Then $C$ is an $N_0$-matrix with $n(C) = \rho_{n-1}(B) - \rho(B)$. By Theorem 1.2, all $\mu \in \text{sp}C$ satisfy

(i') $|\mu - C| \leq c - n(C)$ and  
(ii') $|\mu| \geq -n(C)$,

where $c = \max\{c_{ii}\} = \max\{a_{ii}\} + \rho_{n-1}(B) - t = a + \rho_{n-1}(B) - t$. Thus, each $\lambda \in \text{sp}A$ satisfies

(i) $|\lambda - a| \leq a + \rho(B) - t - \rho(aI - A)$ and  
(ii) $|\lambda - \{t - \rho_{n-1}(B)\}] > \rho(B) - \rho_{n-1}(B)$.
Now assume $\rho_{n-2}(B) < \rho_{n-1}(B)$, and let $F = \rho_{n-2}(B)I - B = A + (\rho_{n-2}(B) - t)I$. Then $F$ is an $F_0$-matrix with $n(F) = \rho_{n-2}(B) - \rho(B)$. Thus, by Theorem 2.5, each $\lambda \in \text{sp} A$ satisfies either

(iii) $|\lambda - (t - \rho_{n-2}(B) + \lambda_1)| \geq r_1$ if $F$ is nonsingular, where

$$\gamma = \max \left\{ \frac{F(i, i)}{\text{det} F} \right\}, \quad \lambda_1 = -\frac{\gamma [n(F)]^2}{1 - 2\gamma n(F)}, \quad \text{and} \quad r_1 = \frac{n(F)[\gamma n(F) - 1]}{1 - 2\gamma n(F)},$$

or

(iv) $|\lambda - \{t - [\rho_{n-2}(B) + \rho(B)]/2\}| \geq [\rho(B) - \rho_{n-2}(B)]/2$ if $F$ is singular.

We note that equality holds in (i), (ii), or (iii) if and only if $\lambda = t - \rho(B)$, and that $t - \rho(B)$ is the leftmost point of the circular region in all four cases of Theorem 3.1.

For an irreducible $M$-matrix $A$, denote by $\gamma(A)$ the region defined in Theorem 3.1 by (i), (ii), and either (iii) or (iv). If $A$ is an arbitrary $M$-matrix, there is a permutation matrix $P$ such that

$$PAP^T = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1m} \\
0 & A_{22} & \cdots & A_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & A_{mm}
\end{pmatrix},$$

where each $A_{ii}$ is an irreducible $n_i \times n_i$ $M$-matrix, $1 \leq i \leq m$. Then $\text{sp} A \subseteq \bigcup_{i=1}^{m} \gamma(A_{ii})$.

By an analogous argument we have the following theorem for irreducible nonnegative matrices.

**Theorem 3.2.** Let $B = (b_{ij})$ be an irreducible $n \times n$ nonnegative matrix where $n \geq 3$, and let $b = \min\{b_{ii}\}$. Then, each $\lambda \in \text{sp} B$ satisfies

(i) $|\lambda - b| \leq \rho(B) - b,$
(ii) $|\lambda - \rho_{n-1}(B)| \geq \rho(B) - \rho_{n-1}(B).$
and if $\rho_{n-2}(B) < \rho_{n-1}(B)$ and $F = \rho_{n-2}(B)I - B$, either

(iii) $|\lambda - [\rho_{n-2}(B) - \lambda_1]| \geq r_1$ if $F$ is nonsingular, where

$$
\gamma = \max \left\{ \frac{F(i, i)}{\det F} \right\}, \quad n(F) = \rho_{n-2}(B) - \rho(B),
$$

$$
\lambda_1 = -\frac{\gamma [n(F)]^2}{1 - 2\gamma n(F)}, \quad \text{and} \quad r_1 = \frac{n(F)[\gamma n(F) - 1]}{1 - 2\gamma n(F)},
$$
or

(iv) $|\lambda - [\rho_{n-2}(B) + \rho(B)]/2| \geq [\rho(B) - \rho_{n-2}(B)]/2$ if $F$ is singular.

Equality holds in (i), (ii), and (iii) if and only if $\lambda = \rho(B)$.

Note that $\rho(B) > \rho_{n-1}(B) \geq \rho_{n-2}(B) \geq \rho_1(B) \geq b$ and thus

$$
\rho(B) - b \geq \rho(B) - \rho_{n-1}(B) \quad \text{and} \quad \rho(B) - b > r_1 > \frac{\rho(B) - \rho_{n-2}(B)}{2}.
$$

Therefore, since $\rho(B)$ is the rightmost point of all the circles, the circular region in (i) contains the interior and boundary of the circular regions in (ii) and (iii) or (iv). Also, the spectrum of an arbitrary nonnegative matrix can be bounded in a manner identical to the one used for arbitrary $M$-matrices.

We remark that by utilizing the bound $\tau_{\|}(P)$ for an irreducible, nonnegative matrix $P$ we can improve Theorems 3.1 and 3.2 in a manner similar to that given for $F_0$-matrices subsequent to Theorem 2.5. For example, for Theorem 3.2 with $F = \rho_{n-2}(B)I - B$ nonsingular we obtain the following: for each $\lambda \in \text{sp} B$ with $\lambda \neq \rho(B)$,

(i') $|\lambda - b| \leq \tau_{\|}(B - bI)$,

(ii') $|\lambda - \rho_{n-1}(B)| \geq \{ \tau_{\|}([B - \rho_{n-1}(B)I]^{-1}) \}^{-1}$,

(iii') $|\lambda - [\rho_{n-2}(B) - \lambda_1]| \geq r_1$, where

$$
\lambda_1 = \frac{-\gamma}{T^2 - \gamma^2},
$$

$$
r_1 = \frac{T}{T^2 - \gamma^2},
$$

$$
T = \tau_{\|}(\gamma I - F^{-1}),
$$
The following example illustrates Theorem 3.2.

**Example 3.3.** Consider the matrix

$$B = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix}.$$

Both the Fréchet and Brauer bounds yield the circular region \( \{ z : |z - 1| < 4 \} \), and the Dmitriev-Dynkin bounds yield the region inside the circle \( \{ z : |z| < 5 \} \) but not in the interior of the region bounded by the circle and the chords joining \( z = 5 \) with the points \( z = 5 \exp(2\pi i/3) \) and \( z = 5 \exp(-2\pi i/3) \).

Obviously \( \rho_{n-2}(B) = \rho_4(B) = 1 \), and direct computation yields \( \rho_{n-1}(B) = \rho_6(B) = 1 + \sqrt{6} \). Since \( B \) is generalized stochastic, \( \mu_1 = \rho(B) = 5 \) and the other eigenvalues of \( B \) are calculated to be \( \mu_2 = -1 + i \) and \( \mu_3 = -1 - i \). Also,

$$F = \rho_{n-2}(B)I - B = \begin{pmatrix} 0 & -2 & -2 \\ -3 & 0 & -1 \\ -1 & -3 & 0 \end{pmatrix},$$

and direct computation yields \( n(F) = -4 \), \( \det F = -20 \), \( \gamma = \frac{3}{10} \), \( \lambda_1 = -\frac{24}{17} \), and \( \tau_1 = \frac{11}{17} \). By (i), (ii), and (iii) of Theorem 3.2, each \( \lambda \in \text{sp} B \) satisfies

(a) \(|\lambda - 1| < 4 \),
(b) \(|\lambda - (1 + \sqrt{6})| > 4 - \sqrt{6} \), and
c(c) \(|\lambda + \frac{11}{17}| > \frac{11}{17} \).

The upper-half-plane portion of this region is depicted as the shaded portion of Figure 1.

To improve these bounds consider

$$C = \rho_{n-1}(B)I - B = \begin{pmatrix} \sqrt{6} & -2 & -2 \\ -3 & \sqrt{6} & -1 \\ -1 & -3 & \sqrt{6} \end{pmatrix}.$$
Then

\[
-C^{-1} = \frac{1}{5(4+\sqrt{6})} \begin{bmatrix}
3 & 6+2\sqrt{6} & 2+2\sqrt{6} \\
1+3\sqrt{6} & 4 & 6+\sqrt{6} \\
9+\sqrt{6} & 2+3\sqrt{6} & 0
\end{bmatrix},
\]

and direct computation yields \( \tau_1(-C^{-1}) = (9-\sqrt{6})/25 \approx 0.2620 \), which in Theorem 3.2(ii') yields

\[
|\lambda - (1+\sqrt{6})| \geq \frac{25}{9-\sqrt{6}} = 3.8165 \quad \text{for} \quad \lambda \in \text{sp } B \setminus \{ \rho(B) \}.
\]

Moreover,

\[
F^{-1} = \frac{1}{20} \begin{bmatrix}
3 & -6 & -2 \\
-1 & 2 & -6 \\
-9 & -2 & 6
\end{bmatrix},
\]

so that

\[
\gamma I - F^{-1} = \frac{1}{20} \begin{bmatrix}
3 & 6 & 2 \\
1 & 4 & 6 \\
9 & 2 & 0
\end{bmatrix}.
\]
Direct computation yields $\tau_1(\gamma I - F^{-1}) = \frac{2}{3}$, and substituting this into Theorem 3.2(iii'), we get

$$|\lambda - \frac{37}{7}| \geq \frac{40}{7} \quad \text{for} \quad \lambda \in \text{sp} B \setminus \{\rho(B)\}.$$  

Lastly, since $\tau_1(B) = 2$, we have $|\lambda| \leq 2$ for $\lambda \in \text{sp} B \setminus \rho\{B\}$.

The upper-half-plane portion of this region is depicted as the shaded portion of Figure 2.

To compute the Deutsch-Zenger bounds for $B$, we note that

- $m_{12} = -\frac{3}{2}, \quad r_{12} = \frac{3}{2}, \quad h_{12} = 1, \quad d_{12} = 2,$
- $m_{13} = -\frac{1}{2}, \quad r_{13} = \frac{3}{2}, \quad h_{13} = 1, \quad d_{13} = 1,$
- $m_{23} = -1, \quad r_{23} = 1, \quad h_{23} = 2, \quad d_{23} = 2,$

so that for $Z = \{z: |z - \frac{1}{2}| \leq 1, \ |z + \frac{1}{2}| \leq 1\}$,

- $G_{13} = \left\{ -\frac{3}{2} + \frac{3}{2}u + z: z \in Z, \ |u| \leq 1 \right\},$
- $G_{13} = \left\{ -\frac{1}{2} + \frac{3}{2}u + z: z \in Z, \ |u| \leq 1 \right\},$
- $G_{23} = \left\{ -1 + u + 2z: z \in Z, \ |u| \leq 1 \right\}.$
The upper-half-plane portion of the region $G(B) = G_{12} \cup G_{13} \cup G_{23}$ intersected with $|z| < \gamma(B) = 2$ is depicted as the shaded portion of Figure 3, and we note that the region given in Figure 2 is significantly smaller.

Immediately we have a refinement of Corollary 2.3.

**Corollary 3.4.** Suppose that $B \geq 0$ is irreducible, $b = \min\{ b_{ii} \}$, and $\lambda \neq \rho(B)$ is a real eigenvalue of $B$. Then,

(i) $-\rho(B) + 2b \leq \lambda < \rho(B)$,

(ii) $\lambda \leq 2\rho_{n-1}(B) - \rho(B)$,

and if $\rho_{n-2}(B) < \rho_{n-1}(B)$ and $F = \rho_{n-2}(B)I - B$, either

(iii) $\lambda \leq \rho_{n-2}(B) + [\rho_{n-2}(B) - \rho(B)]/\{1 + 2\gamma[\rho(B) - \rho_{n-2}(B)]\} < \rho_{n-2}(B)$, where $\gamma = \max\{ F(i,i) / \det F \}$, if $F$ is nonsingular, or

(iv) $\lambda \leq \rho_{n-2}(B)$ if $F$ is singular.

On the other hand, if $\rho_{n-2}(B) = \rho_{n-1}(B)$, then

(v) $\lambda < \rho_{n-2}(B)$.

**Proof.** (i), (ii), (iii), and (iv) follow directly from Theorem 3.2. To prove (v) assume $\rho_{n-2}(B) = \rho_{n-1}(B)$. If $\rho_{n-1}(B) = \rho(B)$, then $\lambda < \rho_{n-2}(B) = \rho(B)$ by assumption. If $\rho_{n-1}(B) < \rho(B)$, assume that $\rho_{n-1}(B) < \lambda < \rho(B)$. Then $\lambda I - B \in \mathbb{N}_0$, which implies that $\lambda I - B$ is nonsingular and hence that $\lambda \notin \text{sp} B$. Thus, $\lambda < \rho_{n-2}(B)$. \[\square\]
We note that these could be refined further using Theorem 3.2 (i'), (ii'), and (iii').

Considering Example 2.4 again, direct computation yields $b = \min b_{ii} = 2$, $\rho_1(B) = 5$, $\rho_2(B) = 9$, $\rho(B) = 10$, and $\gamma = 0.9$. Thus, if $\lambda \in \text{sp} B$ and $\lambda \neq \rho(B)$, then by Corollary 3.4 we have $-6 \leq \lambda \leq 4.5$.

The author greatly appreciates the diligence of the referee in reviewing this manuscript.

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Received 15 January 1988; final manuscript accepted 4 January 1989