Journal of Algebra 324 (2010) 1334–1346

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Binomial generation of the radical of a lattice ideal

Anargyros Katsabekis [a], Marcel Morales [b,c], Apostolos Thoma [d,*]

[a] Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece
[b] Université de Grenoble I, Institut Fourier, UMR 5582, B.P. 74, 38402 Saint-Martin D'Hères Cedex, France
[c] IUFM de Lyon, 5 rue Anselme, 69317 Lyon Cedex, France
[d] Department of Mathematics, University of Ioannina, Ioannina 45110, Greece

ARTICLE INFO

Article history:
Received 19 November 2008
Available online 16 July 2010
Communicated by Steven Dale Cutkosky

Keywords:
Lattice ideal
Toric varieties
Arithmetical rank

ABSTRACT

Let $I_{L,\rho}$ be a lattice ideal. We provide a necessary and sufficient criterion under which a set of binomials in $I_{L,\rho}$ generates the radical of $I_{L,\rho}$ up to radical. We apply our results to the problem of determining the minimal number of generators of $I_{L,\rho}$ or of the $rad(I_{L,\rho})$ up to radical.

1. Introduction

Lattice ideals are an important class of binomial ideals with a lot of applications in several areas like algebraic statistics, dynamical systems, graph theory, hypergeometric differential equations, integer programming and toric geometry [7]. Lattice ideals are generalizations of toric ideals, see [17,22] for details about these ideals. The generation of the radical of a lattice ideal up to radical by binomials was the subject of several recent papers, see [1–11,14,15,19] and sections of books [23,24]. In [8] S. Eliahou and R. Villarreal (see Theorem 2.5) provide two necessary and sufficient conditions for a set of binomials to generate a toric ideal up to radical. This result was later generalized by K. Eto [9,10] for lattice ideals. The last years appeared results in the literature [1,14] that required a huge number of binomials to generate the radical of lattice ideals up to radical. In this article we approach the problem in a different manner to understand these results.

Let (L,ρ) be a partial character on $\mathbb{Z}^m$, then we associate to any lattice ideal $I_{L,\rho}$ a rational polyhedral cone $\sigma_L = pos_{\mathbb{Q}}(A)$, where $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$; see Section 2 for details. Let S be a subset of the cone σ_L, then $\mathbb{E}_S := \{i \in \{1, \ldots, m\} \mid \mathbf{a}_i \in S\}$. Given a vector $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{Z}^m$, its *support* is the set $\operatorname{supp}(\mathbf{u}) = \{i \in \{1, \ldots, m\} \mid u_i \neq 0\}$. We can write every vector $\mathbf{u}$ in $\mathbb{Z}^m$ uniquely as

* Corresponding author.
E-mail addresses: katsabek@aegean.gr (A. Katsabekis), morales@ujf-grenoble.fr (M. Morales), athoma@cc.uoi.gr (A. Thoma).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.07.005

$\mathbf{u} = \mathbf{u}_+ - \mathbf{u}_-$, where $\mathbf{u}_+$ and $\mathbf{u}_-$ are non-negative and have disjoint support. We shall denote by $\mathbb{Z}^E$ the set $\{\mathbf{u} \in \mathbb{Z}^m \mid \mathrm{supp}(\mathbf{u}) \subset E\}$.

Definition 1.1. We say that the set of binomials $\{B(\mathbf{u}_1), B(\mathbf{u}_2), \ldots, B(\mathbf{u}_q)\} \subset I_{L,\rho}$ is a cover of A if and only if for every $E \subset \{1, \ldots, m\}$, that is not in the form $\mathbb{E}_{\mathcal{F}}$ for a face $\mathcal{F}$ of σ_L, there exists an $i \in \{1, \ldots, q\}$ such that $(\mathbf{u}_i)_+ \in \mathbb{Z}^E$ and $(\mathbf{u}_i)_- \notin \mathbb{Z}^E$ or $(\mathbf{u}_i)_- \in \mathbb{Z}^E$ and $(\mathbf{u}_i)_+ \notin \mathbb{Z}^E$.

The next theorem improves and generalizes results of S. Eliahou and R. Villarreal [8] and K. Eto [9, 10]. It shows how the binomial generation of the radical of a lattice ideal is related with the geometry of the cone σ_L and the algebra of the lattice L.

Theorem 3.5. *Let (L, ρ) be a partial character on $\mathbb{Z}^m$, $\mathbf{u}_1, \ldots, \mathbf{u}_q$ be elements of the lattice L and $\sigma_L = pos_{\mathbb{Q}}(A)$ the rational polyhedral cone associated to $I_{L,\rho}$. Then*

$$rad(I_{L,\rho}) = rad\big(B(\mathbf{u}_1), B(\mathbf{u}_2), \ldots, B(\mathbf{u}_q)\big)$$

if and only if

(i) *$\{B(\mathbf{u}_1), B(\mathbf{u}_2), \ldots, B(\mathbf{u}_q)\}$ is a cover of A,*
(ii) *for every face $\mathcal{F}$ of σ_L we have*

$$L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}} = \sum_{\mathbf{u}_i \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}} \mathbb{Z}\mathbf{u}_i,$$

in characteristic zero. While in characteristic $p > 0$,

$$\big(L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}\big) : p^\infty = \left(\sum_{\mathbf{u}_i \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}} \mathbb{Z}\mathbf{u}_i\right) : p^\infty.$$

The two conditions depend also on the geometry of the cone σ_L. In particular our first condition depends on the non-faces of σ_L and the second involves sublattices associated to faces of the cone. The new conditions explain why there are lattices that require a huge number of binomials to generate the radical of a lattice ideal up to radical; see the remark after Proposition 3.6. In Section 4 we provide a necessary and sufficient condition for a lattice ideal to be complete intersection. Finally we prove that lattice ideals with associated full vector configuration are either set-theoretic complete intersection or almost set-theoretic complete intersection in characteristic zero, while they are set-theoretic complete intersection in positive characteristic.

2. Basics on lattice ideals

Let K be a field of any characteristic. A *lattice* is a finitely generated free abelian group. A *partial character* (L, ρ) on $\mathbb{Z}^m$ is a homomorphism ρ from a sublattice L of $\mathbb{Z}^m$ to the multiplicative group $K^* = K - \{0\}$. Given a partial character (L, ρ) on $\mathbb{Z}^m$, we define the ideal

$$I_{L,\rho} := \big(\big\{B(\alpha) := \mathbf{x}^{\alpha_+} - \rho(\alpha)\mathbf{x}^{\alpha_-} \;\big|\; \alpha = \alpha_+ - \alpha_- \in L\big\}\big) \subset K[x_1, \ldots, x_m]$$

called *lattice ideal*. There $\mathbf{x}^\beta = x_1^{\beta_1} \cdots x_m^{\beta_m}$ for $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$. The height of $I_{L,\rho}$ is equal to the rank of L. Given a finite subset C of L, define the ideal

$$J_{C,\rho} := \big(\big\{B(\alpha) \;\big|\; \alpha = \alpha_+ - \alpha_- \in C\big\}\big) \subset I_{L,\rho}.$$

Lemma 2.1. *(See [22].) A subset C spans the lattice L if and only if*

$$J_{C,\rho} : (x_1 \cdots x_m)^\infty = I_{L,\rho}.$$

If L is a sublattice of $\mathbb{Z}^m$, then the *saturation* of L is the lattice

$$Sat(L) := \{\alpha \in \mathbb{Z}^m \mid d\alpha \in L \text{ for some } d \in \mathbb{Z},\ d \neq 0\}.$$

We say that the lattice L is *saturated* if $L = Sat(L)$. The lattice ideal $I_{L,\rho}$ is prime if and only if L is saturated.

For a prime number p the p-saturation of L is the lattice

$$(L : p^\infty) := \{\alpha \in \mathbb{Z}^m \mid p^k\alpha \in L \text{ for some } k \in \mathbb{N}\}.$$

Throughout this paper we assume that L is a non-zero *positive sublattice* of $\mathbb{Z}^m$, that is $L \cap \mathbb{N}^m = \{\mathbf{0}\}$. This means that the lattice ideal $I_{L,\rho}$ is homogeneous with respect to some positive grading.

The group $\mathbb{Z}^m/Sat(L)$ is free abelian, therefore is isomorphic to $\mathbb{Z}^n$, where $n = m - rank(L)$. Let ψ be the above isomorphism, $\mathbf{e}_1, \dots, \mathbf{e}_m$ the unit vectors of $\mathbb{Z}^m$ and $\mathbf{a}_i := \psi(\mathbf{e}_i + Sat(L))$ for $1 \leqslant i \leqslant m$. We call $A = \{\mathbf{a}_i \mid 1 \leqslant i \leqslant m\} \subset \mathbb{Z}^n$ *the configuration of vectors associated to the lattice* L.

We grade $K[x_1, \dots, x_m]$ by setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \dots, m$. We define the *A-degree* of the monomial $\mathbf{x}^{\mathbf{u}}$ to be

$$\deg_A(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A,$$

where $\mathbb{N}A$ is the semigroup generated by A. A polynomial $F \in K[x_1, \dots, x_m]$ is called *A-homogeneous* if the monomials in each non-zero term of F have the same A-degree. An ideal I is A-homogeneous if it is generated by A-homogeneous polynomials. The lattice ideal $I_{L,\rho}$, as well as every binomial in it, is A-homogeneous.

The *binomial arithmetical rank* of a binomial ideal I (written $\mathrm{bar}(I)$) is the smallest integer s for which there exist binomials $f_1, \dots, f_s$ in I such that $rad(I) = rad(f_1, \dots, f_s)$. Hence the binomial arithmetical rank is an upper bound for the *arithmetical rank* of a binomial ideal (written $\mathrm{ara}(I)$), which is the smallest integer s for which there exist $f_1, \dots, f_s$ in I such that $rad(I) = rad(f_1, \dots, f_s)$. From the definitions, the generalized Krull's principal ideal theorem and the graded version of Nakayama's lemma we deduce the following inequality for a lattice ideal $I_{L,\rho}$:

$$h(I_{L,\rho}) \leqslant \mathrm{ara}(I_{L,\rho}) \leqslant \mathrm{bar}(I_{L,\rho}) \leqslant \mu(I_{L,\rho}).$$

Here $h(I)$ denotes the height and $\mu(I)$ denotes the minimal number of generators of an ideal I. When $h(I) = \mathrm{ara}(I)$ the ideal I is called a *set-theoretic complete intersection*; when $h(I) = \mu(I)$ it is called a *complete intersection*. The ideal I is called an *almost set-theoretic complete intersection* if $\mathrm{ara}(I) \leqslant h(I) + 1$.

We associate to the lattice ideal $I_{L,\rho}$ the rational polyhedral cone

$$\sigma_L = pos_{\mathbb{Q}}(A) := \{\lambda_1\mathbf{a}_1 + \cdots + \lambda_m\mathbf{a}_m \mid \lambda_i \in \mathbb{Q} \text{ and } \lambda_i \geqslant 0\}.$$

The *dimension* of σ_L is equal to the dimension of the $\mathbb{Q}$-vector space

$$\mathrm{span}_{\mathbb{Q}}(\sigma_L) = \{\lambda_1\mathbf{a}_1 + \cdots + \lambda_m\mathbf{a}_m \mid \lambda_i \in \mathbb{Q}\}.$$

A *face* of σ_L is any set of the form

$$\mathcal{F} = \sigma_L \cap \{\mathbf{x} \in \mathbb{Q}^n \colon \mathbf{cx} = 0\},$$

where $\mathbf{c} \in \mathbb{Q}^n$ and $\mathbf{cx} \geqslant 0$ for all $x \in \sigma_L$. Faces of dimension one are called *extreme rays*. If the number of the extreme rays of a cone coincides with the dimension (i.e. the extreme rays are linearly independent), the cone is called *simplex cone*. A cone σ is *strongly convex* if $\sigma \cap -\sigma = \{\mathbf{0}\}$. The condition that the lattice L is positive, is equivalent with the condition that the cone σ_L is strongly convex.

We decompose the affine space K^m into 2^m *coordinate cells*,

$$(K^*)^E := \{(q_1, \ldots, q_m) \in K^m \mid q_i \neq 0 \text{ for } i \in E,\ q_i = 0 \text{ for } i \notin E\},$$

where E runs over all subsets of $\{1, \ldots, m\}$. Let $P = (x_1, \ldots, x_m)$ be a point of K^m, then

$$P_E := (\delta_1^E x_1, \delta_2^E x_2, \ldots, \delta_m^E x_m) \in K^m,$$

where $\delta_i^E = 1$ if $i \in E$ and $\delta_i^E = 0$ if $i \notin E$. Note that if $P \in (K^*)^{\{1,\ldots,m\}}$ then $P_E \in (K^*)^E$.

The n-dimensional algebraic torus $(K^*)^n$ acts on the affine m-space K^m via

$$(x_1, \ldots, x_m) \to (x_1\mathbf{t}^{\mathbf{a}_1}, \ldots, x_m\mathbf{t}^{\mathbf{a}_m}).$$

Let $\mathcal{K}$ denote the algebraic closure of K. For the lattice algebraic set $V(I_{L,\rho}) \subset \mathcal{K}^m$ we have that $V(I_{L,\rho}) = \bigcup_{j=1}^{g} \mathbf{X}_{A,j}$, see [7] and [14], where the affine toric variety $\mathbf{X}_{A,j}$ is the Zariski-closure of the $(\mathcal{K}^*)^n$-orbit of a point $P_j = (c_{j1}, c_{j2}, \ldots, c_{jm}) \in \mathcal{K}^m$ for appropriate c_{ji} all different from zero. Actually the toric variety $\mathbf{X}_{A,j}$ is the disjoint union of the orbits of the points $(P_j)_{\mathbb{E}_\mathcal{F}}$, for every face $\mathcal{F}$ of σ_L. There are no points of the toric varieties $\mathbf{X}_{A,j}$ which are in the cells $(\mathcal{K}^*)^E$, where E is not in the form $\mathbb{E}_\mathcal{F}$ for a face $\mathcal{F}$ of σ_L. Thus the lattice algebraic set $V(I_{L,\rho})$ has points only on the cells in the form $(\mathcal{K}^*)^{\mathbb{E}_\mathcal{F}}$ for some face $\mathcal{F}$ of the cone σ_L.

3. Radical generation by binomials

Set $K[E] = K[\{x_i \mid i \in E\}]$, where E is a subset of $\{1, \ldots, m\}$.

Lemma 3.1. *(Cf. [14, Lemma 2.3].) Let $\mathcal{F}$ be a face of the rational polyhedral cone $\sigma_L = pos_\mathbb{Q}(A)$. The monomial $\mathbf{x}^\mathbf{u} \in K[\mathbb{E}_\mathcal{F}]$ if and only if $\deg_A(\mathbf{x}^\mathbf{u}) \in \mathcal{F}$.*

To every face $\mathcal{F}$ of σ_L we can associate the ideal $I_{L_\mathcal{F},\rho} := I_{L,\rho} \cap K[\mathbb{E}_\mathcal{F}]$. The next proposition shows how the geometry of the cone affects the radical generation by A-homogeneous polynomials. Note that binomials in a lattice ideal are always A-homogeneous.

Proposition 3.2. *Let $B = \{G_1, G_2, \ldots, G_q\} \subset I_{L,\rho}$ be a set of A-homogeneous polynomials and $\sigma_L = pos_\mathbb{Q}(A)$ the rational polyhedral cone associated to $I_{L,\rho}$. Then B generates $rad(I_{L,\rho})$ up to radical if and only if for every face $\mathcal{F}$ of σ_L the set $B \cap K[\mathbb{E}_\mathcal{F}]$ generates $rad(I_{L_\mathcal{F},\rho})$ up to radical.*

Proof. ($\Leftarrow$) For $\mathcal{F} = \sigma_L$ we have that $K[\mathbb{E}_{\sigma_L}] = K[x_1, \ldots, x_m]$, so $I_{L_{\sigma_L},\rho} = I_{L,\rho}$ and also $B \cap K[\mathbb{E}_{\sigma_L}] = B$.

($\Rightarrow$) Obviously $B \cap K[\mathbb{E}_\mathcal{F}] \subset I_{L_\mathcal{F},\rho}$ and therefore $rad(B \cap K[\mathbb{E}_\mathcal{F}]) \subset rad(I_{L_\mathcal{F},\rho})$. It is enough to show that $I_{L_\mathcal{F},\rho} \subset rad(B \cap K[\mathbb{E}_\mathcal{F}])$. Let $f \in I_{L_\mathcal{F},\rho} \subset I_{L,\rho}$, so there exist a positive integer k and polynomials $A_1, \ldots, A_q \in K[x_1, \ldots, x_m]$ such that

$$f^k = A_1G_1 + \cdots + A_qG_q.$$

For every polynomial $h \in K[x_1, \dots, x_m]$, we let $h_{\mathcal{F}}$ be the polynomial in $K[\mathbb{E}_{\mathcal{F}}]$ taken from h by setting all the variables x_i, where $i \notin \mathbb{E}_{\mathcal{F}}$, equal to zero. There are two cases for the polynomial G_i:

(1) G_i belongs to $K[\mathbb{E}_{\mathcal{F}}]$. In this case $(G_i)_{\mathcal{F}} = G_i$.
(2) The polynomial G_i does not belong to $K[\mathbb{E}_{\mathcal{F}}]$. Then, from Lemma 3.1, every monomial of G_i does not belong to $K[\mathbb{E}_{\mathcal{F}}]$ since the polynomial G_i is A-homogeneous. In this case $(G_i)_{\mathcal{F}} = 0$.

It follows that $B \cap K[\mathbb{E}_{\mathcal{F}}]$ consists of the non-zero polynomials among $(G_1)_{\mathcal{F}}, \dots, (G_q)_{\mathcal{F}}$. We have that

$$f^k = (f_{\mathcal{F}})^k = (A_1)_{\mathcal{F}}(G_1)_{\mathcal{F}} + \dots + (A_q)_{\mathcal{F}}(G_q)_{\mathcal{F}},$$

since $f \in I_{L_{\mathcal{F}},\rho} = I_{L,\rho} \cap K[\mathbb{E}_{\mathcal{F}}]$. Thus f^k belongs to the ideal generated by the set $B \cap K[\mathbb{E}_{\mathcal{F}}]$. □

Lemma 3.3. *Let (L, ρ) be a partial character on $\mathbb{Z}^m$ and $\mathcal{F}$ a face of $\sigma_L = pos_{\mathbb{Q}}(A)$. Then a vector $\mathbf{u} = \mathbf{u}_+ - \mathbf{u}_- \in L$ belongs to $\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$ if and only if $\mathbf{u}_+ \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$ if and only if $\mathbf{u}_- \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$.*

Proof. It suffices to prove $\mathbf{u}_+ \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$ implies that $\mathbf{u}_- \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$. The other parts are similar or obvious.

From Lemma 3.1 we have that $\deg_A(\mathbf{x}^{\mathbf{u}_+})$ belongs to $\mathcal{F}$ and therefore $\deg_A(\mathbf{x}^{\mathbf{u}_-})$ belongs to $\mathcal{F}$, since the binomial $B(\mathbf{u})$ is A-homogeneous.

Let $\mathbf{u}_- = (u_{-,1}, \dots, u_{-,m})$. For any vector $\mathbf{c}_{\mathcal{F}}$ which defines the face $\mathcal{F}$ we have that $\mathbf{c}_{\mathcal{F}} \cdot (\sum_{i=1}^m u_{-,i}\mathbf{a}_i) = 0$. Thus $\sum_{i=1}^m u_{-,i}(\mathbf{c}_{\mathcal{F}} \cdot \mathbf{a}_i) = 0$. But $u_{-,i} \geqslant 0$ for all i, and $\mathbf{c}_{\mathcal{F}} \cdot \mathbf{a}_i > 0$ for every $i \notin \mathbb{E}_{\mathcal{F}}$, while $\mathbf{c}_{\mathcal{F}} \cdot \mathbf{a}_i = 0$ for every $i \in \mathbb{E}_{\mathcal{F}}$. So we conclude that $u_{-,i} = 0$ for every $i \notin \mathbb{E}_{\mathcal{F}}$. Consequently $\mathbf{u}_- \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$. □

Proposition 3.4. *Let (L, ρ) be a partial character on $\mathbb{Z}^m$. For a face $\mathcal{F}$ of $\sigma_L = pos_{\mathbb{Q}}(A)$, the ideals $I_{L_{\mathcal{F}},\rho}$ and $I_{L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}},\rho}$ coincide.*

Proof. From Corollary 1.3 in [7] the ideal $I_{L_{\mathcal{F}},\rho}$ has a generating set

$$\{B(\mathbf{u}_1), \dots, B(\mathbf{u}_q)\} \subset I_{L,\rho} \cap K[\mathbb{E}_{\mathcal{F}}].$$

So $\mathbf{u}_i$ belongs to L, for every $i = 1, \dots, q$. If $\mathbf{u}_i \notin \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$, then, from Lemma 3.3, both $(\mathbf{u}_i)_+ \notin \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$, $(\mathbf{u}_i)_- \notin \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$ and therefore $B(\mathbf{u}_i)$ is not in $I_{L_{\mathcal{F}},\rho}$, a contradiction, so $\mathbf{u}_i \in L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$. Conversely if $B(\mathbf{v})$ belongs to $I_{L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}},\rho}$, then $\mathbf{v}$ belongs to L and therefore $B(\mathbf{v}) \in I_L$. Moreover $\mathbf{v} \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$, so, again from Lemma 3.3, both $\mathbf{v}_+$ and $\mathbf{v}_-$ belong to $\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$. Thus $B(\mathbf{v}) \in K[\mathbb{E}_{\mathcal{F}}]$ and therefore belongs to $I_{L_{\mathcal{F}},\rho}$. □

The next theorem generalizes results of S. Eliahou and R. Villarreal [8] and K. Eto [9,10].

Theorem 3.5. *Let (L, ρ) be a partial character on $\mathbb{Z}^m$, $\mathbf{u}_1, \dots, \mathbf{u}_q$ be elements of the lattice L and $\sigma_L = pos_{\mathbb{Q}}(A)$ the rational polyhedral cone associated to $I_{L,\rho}$. Then*

$$rad(I_{L,\rho}) = rad\big(B(\mathbf{u}_1), B(\mathbf{u}_2), \dots, B(\mathbf{u}_q)\big)$$

if and only if

(i) $\{B(\mathbf{u}_1), B(\mathbf{u}_2), \dots, B(\mathbf{u}_q)\}$ *is a cover of A,*
(ii) *for every face $\mathcal{F}$ of σ_L we have*

$$L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}} = \sum_{\mathbf{u}_i \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}} \mathbb{Z}\mathbf{u}_i,$$

in characteristic zero. While in characteristic $p > 0$,

$$\left(L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}\right) : p^{\infty} = \left(\sum_{\mathbf{u}_i \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}} \mathbb{Z}\mathbf{u}_i\right) : p^{\infty}.$$

Proof. ($\Rightarrow$) (i) Let $E \subset \{1, \ldots, m\}$ that is not in the form $\mathbb{E}_{\mathcal{F}}$ for any face $\mathcal{F}$ of σ_L. Then for all j the point $(P_j)_E$ is not a point of $V(I_{L,\rho})$, since $(P_j)_E$ belongs to the cell $(\mathcal{K}^*)^E$ and E is of the above form. Thus $(P_j)_E$ is not a point of $V(B(\mathbf{u}_1), B(\mathbf{u}_2), \ldots, B(\mathbf{u}_q))$ and therefore there is an $i \in \{1, \ldots, q\}$ such that $B(\mathbf{u}_i)((P_j)_E) \neq 0$. On the other hand

$$B(\mathbf{u}_i)(P_j) = 0, \quad \text{since } P_j \in \mathbf{X}_{A,j}.$$

If $\operatorname{supp}(\mathbf{u}_i) \subset E$ then we have

$$B(\mathbf{u}_i)\big((P_j)_E\big) = B(\mathbf{u}_i)(P_j) = 0,$$

a contradiction. So $\operatorname{supp}(\mathbf{u}_i) \not\subset E$ which implies that

$$\mathbf{x}^{(\mathbf{u}_i)_-}\big((P_j)_E\big) = 0 \quad \text{or} \quad \mathbf{x}^{(\mathbf{u}_i)_+}\big((P_j)_E\big) = 0.$$

Since $B(\mathbf{u}_i)((P_j)_E) \neq 0$ we have

$$\mathbf{x}^{(\mathbf{u}_i)_+}\big((P_j)_E\big) \neq 0 \quad \text{and} \quad \mathbf{x}^{(\mathbf{u}_i)_-}\big((P_j)_E\big) = 0 \quad \text{or}$$
$$\mathbf{x}^{(\mathbf{u}_i)_+}\big((P_j)_E\big) = 0 \quad \text{and} \quad \mathbf{x}^{(\mathbf{u}_i)_-}\big((P_j)_E\big) \neq 0.$$

Thus $(\mathbf{u}_i)_+ \in \mathbb{Z}^E$ and $(\mathbf{u}_i)_- \notin \mathbb{Z}^E$ or $(\mathbf{u}_i)_- \in \mathbb{Z}^E$ and $(\mathbf{u}_i)_+ \notin \mathbb{Z}^E$. Consequently $\{B(\mathbf{u}_1), B(\mathbf{u}_2), \ldots, B(\mathbf{u}_q)\}$ is a cover of A.

(ii) Let $\mathcal{F}$ be a face of σ_L. Set $G = \sum_{\mathbf{u}_i \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}} \mathbb{Z}\mathbf{u}_i$, $B = \{B(\mathbf{u}_1), \ldots, B(\mathbf{u}_q)\}$ and $B \cap K[\mathbb{E}_{\mathcal{F}}] = \{B(\mathbf{u}_{i_1}), \ldots, B(\mathbf{u}_{i_t})\}$. In any characteristic we have $I_{G,\rho} \subset I_{L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}},\rho}$ and also

$$\big(B(\mathbf{u}_{i_1}), \ldots, B(\mathbf{u}_{i_t})\big) \subset I_{G,\rho} \subset I_{L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}},\rho}.$$

Thus

$$rad\big(B(\mathbf{u}_{i_1}), \ldots, B(\mathbf{u}_{i_t})\big) \subset rad(I_{G,\rho}) \subset rad(I_{L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}},\rho}).$$

But $rad(I_{L,\rho}) = rad(B(\mathbf{u}_1), B(\mathbf{u}_2), \ldots, B(\mathbf{u}_q))$, so combining Propositions 3.2 and 3.4, since every binomial in $I_{L,\rho}$ is A-homogeneous, we deduce that

$$rad(I_{L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}},\rho}) = rad\big(B(\mathbf{u}_{i_1}), \ldots, B(\mathbf{u}_{i_t})\big)$$

and therefore

$$rad(I_{G,\rho}) = rad\big(B(\mathbf{u}_{i_1}), \ldots, B(\mathbf{u}_{i_t})\big).$$

Thus $rad(I_{L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}},\rho}) = rad(I_{G,\rho})$. In characteristic zero we have that $I_{G,\rho} = I_{L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}},\rho}$, so $L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}} = G$, while in characteristic $p > 0$ it holds $I_{G:p^{\infty},\rho} = I_{L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}:p^{\infty},\rho}$, so $G : p^{\infty} = L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}} : p^{\infty}$, see [7].

($\Leftarrow$) It is enough to prove that $V(B(\mathbf{u}_1), B(\mathbf{u}_2), \ldots, B(\mathbf{u}_q)) \subset V(I_{L,\rho})$. Let $\mathbf{y} \in V(B(\mathbf{u}_1), B(\mathbf{u}_2), \ldots, B(\mathbf{u}_q)) \subset \mathcal{K}^m$ and assume that $\mathbf{y} \in (\mathcal{K}^*)^E$, where E is not of the form $\mathbb{E}_{\mathcal{F}}$ for a face $\mathcal{F}$ of σ_L. Then there is an $i \in \{1, \ldots, q\}$ such that $(\mathbf{u}_i)_+ \in \mathbb{Z}^E$ and $(\mathbf{u}_i)_- \notin \mathbb{Z}^E$ since $\{B(\mathbf{u}_1), B(\mathbf{u}_2), \ldots, B(\mathbf{u}_q)\}$ is a cover of A. This means that $B(\mathbf{u}_i)(\mathbf{y}) \neq 0$, a contradiction to the fact that $\mathbf{y} \in V(B(\mathbf{u}_1), B(\mathbf{u}_2), \ldots, B(\mathbf{u}_q))$. Thus $\mathbf{y} \in (\mathcal{K}^*)^{\mathbb{E}_{\mathcal{F}}}$, for a face $\mathcal{F}$ of σ_L. Let $\mathbf{v} \in L$. We will prove that $B(\mathbf{v})(\mathbf{y}) = 0$. If $\mathbf{v} \notin \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$, then by Lemma 3.3 both $\mathbf{v}_+$, $\mathbf{v}_-$ do not belong to $\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$ and therefore $B(\mathbf{v})(\mathbf{y}) = 0$. Suppose now that $\mathbf{v} \in \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$, which implies that $B(\mathbf{v}) \in I_{L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}},\rho}$. Set $C = \{\mathbf{u}_i \mid \mathrm{supp}(\mathbf{u}_i) \subset \mathbb{E}_{\mathcal{F}}\} \subset \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$. Then

(1) by hypothesis in characteristic zero we have that C spans the lattice $L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}$. Thus, from Lemma 2.1, the binomial $B(\mathbf{v})$ belongs to the ideal $J_{C,\rho} : \langle \prod_{j\in\mathbb{E}_{\mathcal{F}}} x_j \rangle^\infty$. But $\mathbf{y}$ belongs to $V(B(\mathbf{u}_1), B(\mathbf{u}_2), \ldots, B(\mathbf{u}_q))$, which in particular implies that $\mathbf{y} \in V(B(\mathbf{u}_i) \mid \mathbf{u}_i \in C)$. Consequently $B(\mathbf{v})(\mathbf{y}) = 0$;
(2) by hypothesis in characteristic $p > 0$ we have that

$$\left(L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}\right) : p^\infty = \left(\sum_{\mathbf{u}_i \in C} \mathbb{Z}\mathbf{u}_i\right) : p^\infty,$$

so $I_{(L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}}):p^\infty,\rho} = I_{G:p^\infty,\rho}$ and therefore $rad(I_{L\cap\mathbb{Z}^{\mathbb{E}_{\mathcal{F}}},\rho}) = rad(I_{G,\rho})$ where $G = \sum_{\mathbf{u}_i\in C} \mathbb{Z}\mathbf{u}_i$. Thus $(B(\mathbf{v}))^k \in I_{G,\rho}$, so $B(\mathbf{v})(y) = 0$ since $I_{G,\rho} = J_{C,\rho} : p^\infty$. $\square$

It is well known, see Proposition 8.7 in [7] and Proposition 3.4 in [9], that every lattice ideal is the set-theoretic intersection of its circuits. Therefore a cover of A always exists. In the case that the σ_L is a simplex cone it is easy to find a cover. For an $\mathbf{a}_i \in A$, $1 \leqslant i \leqslant m$, we define $\mathcal{F}_{\mathbf{a}_i}$ to be the *minimal face* of σ_L that contains $\mathbf{a}_i$, i.e.

$$\mathcal{F}_{\mathbf{a}_i} = \bigcap_{\mathbf{a}_i \in \mathcal{F}} \mathcal{F},$$

since any intersection of faces of σ_L is a face of σ_L.

Proposition 3.6. *Let (L, ρ) be a partial character on $\mathbb{Z}^m$ such that the rational polyhedral cone $\sigma_L = pos_{\mathbb{Q}}(A) \subset \mathbb{Q}^n$ associated to $I_{L,\rho}$ is a simplex cone of dimension n, then there exists a cover of A consisting of $m - n$ binomials.*

Proof. The strongly convex cone σ_L is n-dimensional and also it is a simplex cone, so we choose one vector for each extreme ray of σ_L and obtain a linearly independent set $B \subset A$ consisting of n vectors such that $\sigma_L = pos_{\mathbb{Q}}(B)$. Remark that the choice of the above vectors is not unique. Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$. We rearrange the vectors in A such that the first n vectors, i.e. $\mathbf{a}_1, \ldots, \mathbf{a}_n$, are the elements of B. Every $\mathcal{F}_{\mathbf{a}_i}$ is a rational polyhedral cone, so $\mathcal{F}_{\mathbf{a}_i}$ is of the form $pos_{\mathbb{Q}}(\mathbf{a}_j \mid j \in \mathbb{E}_{\mathcal{F}_{\mathbf{a}_i}} \cap \{1, \ldots, n\})$. For every $i \in \{n+1, \ldots, m\}$ the vector $\mathbf{a}_i$ belongs to $\mathcal{F}_{\mathbf{a}_i}$, so there exists a binomial $B(\mathbf{u}_i) = x_i^{g_i} - N_i \in I_{L,\rho}$ where $x_i^{g_i} = \mathbf{x}^{(\mathbf{u}_i)_+}$, $N_i = \mathbf{x}^{(\mathbf{u}_i)_-}$ and the support of $(\mathbf{u}_i)_-$ equals $\mathbb{E}_{\mathcal{F}_{\mathbf{a}_i}} \cap \{1, \ldots, n\}$. We claim that the set $\{B(\mathbf{u}_{n+1}), \ldots, B(\mathbf{u}_m)\}$ is a cover of A.

Let $E \subset \{1, \ldots, n, n+1, \ldots, m\}$ which is not of the form $\mathbb{E}_{\mathcal{F}}$ for a face $\mathcal{F}$ of the cone σ_L. Let $\mathcal{H}_E = pos_{\mathbb{Q}}(\mathbf{a}_j \mid j \in E \cap \{1, \ldots, n\})$, then $\mathcal{H}_E$ is a face of the simplex cone σ_L. We have that $E \cap \{1, \ldots, n\} \subsetneqq E$ and therefore E has at least one element belonging to the set $\{n+1, \ldots, m\}$. There are two cases:

(1) The set E has an element $i \in \{n+1, \ldots, m\}$ such that $\mathbf{a}_i \notin \mathcal{H}_E$. Then $\mathbb{E}_{\mathcal{F}_{\mathbf{a}_i}} \cap \{1, \ldots, n\}$ is not a subset of E, since if it was the vector $\mathbf{a}_i$ should belong to $\mathcal{H}_E$. For the binomial $B(\mathbf{u}_i)$ we have $(\mathbf{u}_i)_+ \in \mathbb{Z}^E$ and $(\mathbf{u}_i)_- \notin \mathbb{Z}^E$.

(2) For every element $i \in \{n+1, \ldots, m\}$ of E the vector $\mathbf{a}_i \in \mathcal{H}_E$. In this case there exists a $j \in \{n+1, \ldots, m\}$, which does not belong to E, such that $\mathbf{a}_j \in \mathcal{H}_E$. If not, we have that $\mathcal{H}_E = pos_{\mathbb{Q}}(\mathbf{a}_j \mid j \in E)$ and therefore $E = \mathbb{E}_{\mathcal{H}_E}$ contradiction. For the binomial $B(\mathbf{u}_j) \in I_{L,\rho}$ we have $(\mathbf{u}_j)_- \in \mathbb{Z}^E$, since $\mathcal{F}_{\mathbf{a}_j} \subset \mathcal{H}_E$, and $(\mathbf{u}_j)_+ \notin \mathbb{Z}^E$.

The preceding discussion yields that the set $\{B(\mathbf{u}_{n+1}), \ldots, B(\mathbf{u}_m)\}$ is a cover of A. $\square$

Let us explain why there are lattices which require a huge number of binomials to generate the radical of a lattice ideal up to radical. The first condition of Theorem 3.5 states that you need a number of binomials to cover A. In the case that σ_L is a simplex cone of dimension n Proposition 3.6 provides a cover consisting of $m-n$ binomials. But if the geometry of the cone is complicated, then the number of binomials increases. In [14] we studied an example of a family of toric ideals of dimension n and proved, by explicit computation of $5\binom{n}{3} + 6\binom{n}{4}$ binomials generating the toric ideal up to radical, that the binomial arithmetical rank is equal to $5\binom{n}{3} + 6\binom{n}{4}$. The above binomials constitute a cover and actually one can easily prove that there is no cover with less than $5\binom{n}{3} + 6\binom{n}{4}$ binomials. The second condition depends also on the characteristic of the field and it states that you do not only need to generate the lattice or the p-saturation of it in the characteristic p case, but also sublattices or their p-saturations that are associated to the faces of the cone σ_L. The examples given by M. Barile in [1] show this effect very clearly and the dependence also on the characteristic. In [1] M. Barile provides a family of toric ideals of dimension n and height n that their cones σ_L are simplex cones, so to cover A you need only n binomials; see Proposition 3.6. While to generate the lattice and the sublattices associated to the faces of the cone σ_L you need $n + \binom{n-1}{2}$ binomials when the characteristic of the field is not equal to p. In characteristic p exactly n elements of the lattice provide a cover and generate also the p-saturations of all the sublattices associated to the faces of the cone σ_L. In this case the toric ideals are binomial set theoretic complete intersections and the lattice is completely p-glued [19].

To compute the exact value of the binomial arithmetical rank is not usually an easy process, since one can use the same or modify some binomials to satisfy more than one conditions of Theorem 3.5, as the following example shows. Note that the procedure may depend also on the characteristic of the field.

Example 3.7. In this example we apply our methods to compute the exact value of the binomial arithmetical rank of a lattice ideal. For a different approach to the same example see also [10]. Given an $n \times m$ matrix M with columns the vectors of the set $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{Z}^n$ and a saturated lattice $L = ker_{\mathbb{Z}}(M) \subset \mathbb{Z}^m$, the toric ideal $I_{L,1}$ will be denoted by I_A. For a face $\mathcal{F}$ of $pos_{\mathbb{Q}}(A)$ we shall denote by $A_{\mathcal{F}}$ the set $\{\mathbf{a}_i \mid i \in \mathbb{E}_{\mathcal{F}}\}$. We consider the set of vectors $A = \{(a_1, a_2, a_3) \in \mathbb{N}^3 \mid a_1 + a_2 + a_3 = 3\}$. The vectors of A are the transpose of the columns of the matrix

$$M_{3,3} = \begin{pmatrix} 3 & 0 & 0 & 2 & 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 3 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 1 & 2 & 1 & 2 & 1 \end{pmatrix}.$$

Consider the toric ideal $I_A \subset K[x_1, \ldots, x_{10}]$. The toric variety $V(I_A) \subset K^{10}$ is the so-called (3, 3)-Veronese toric variety.

The cone $pos_{\mathbb{Q}}(A)$ is a three-dimensional simplicial cone with three facets $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$. For $\mathbb{E}_{\mathcal{F}_1} = \{1, 2, 4, 5\}$ we can see, using [18], that the set $C_1 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans the lattice $L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}_1}}$ where $\mathbf{u}_1 = (2, 1, 0, -3, 0, 0, 0, 0, 0, 0)$, $\mathbf{u}_2 = (1, 0, 0, -2, 1, 0, 0, 0, 0, 0)$ and $\mathbf{u}_3 = (1, 2, 0, 0, -3, 0, 0, 0, 0, 0)$. Moreover

$$I_{A_{\mathcal{F}_1}} = \big(B(\mathbf{u}_1), B(\mathbf{u}_2), B(\mathbf{u}_3)\big).$$

For $\mathbb{E}_{\mathcal{F}_2} = \{2, 3, 6, 7\}$ we can see, using [18], that the set $C_2 = \{\mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\}$ spans the lattice $L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}_2}}$ where $\mathbf{u}_4 = (0, 2, 1, 0, 0, -3, 0, 0, 0, 0)$, $\mathbf{u}_5 = (0, 1, 0, 0, 0, -2, 1, 0, 0, 0)$ and $\mathbf{u}_6 = (0, 1, 2, 0, 0, 0, -3, 0, 0, 0)$. In fact

$$I_{A_{\mathcal{F}_2}} = \big(B(\mathbf{u}_4), B(\mathbf{u}_5), B(\mathbf{u}_6)\big).$$

For $\mathbb{E}_{\mathcal{F}_3} = \{1, 3, 8, 9\}$ we use [18] to deduce that the set $C_3 = \{\mathbf{u}_7, \mathbf{u}_8, \mathbf{u}_9\}$ spans the lattice $L \cap \mathbb{Z}^{\mathbb{E}_{\mathcal{F}_3}}$ where $\mathbf{u}_7 = (2, 0, 1, 0, 0, 0, 0, -3, 0, 0)$, $\mathbf{u}_8 = (1, 0, 0, 0, 0, 0, 0, -2, 1, 0)$ and $\mathbf{u}_9 = (1, 0, 2, 0, 0, 0, 0, 0, -3, 0)$. In fact

$$I_{A_{\mathcal{F}_3}} = \big(B(\mathbf{u}_7), B(\mathbf{u}_8), B(\mathbf{u}_9)\big).$$

For $\mathbb{E}_\sigma = \{1, \ldots, 10\}$ we can see that

$$L = L \cap \mathbb{Z}^{\mathbb{E}_\sigma} = \mathbb{Z}\mathbf{u}_1 + \mathbb{Z}\mathbf{u}_2 + \mathbb{Z}\mathbf{u}_4 + \mathbb{Z}\mathbf{u}_5 + \mathbb{Z}\mathbf{u}_8 + \mathbb{Z}\mathbf{u}_{10} + \mathbb{Z}\mathbf{u}_{11},$$

where $\mathbf{u}_{10} = (1, 0, 0, -1, 0, 0, 0, -1, 0, 1)$ and $\mathbf{u}_{11} = (0, 0, 0, 1, 0, 0, 1, 0, 0, -2)$. For the singleton $E = \{10\}$ there is a binomial, namely $B(\mathbf{u}_{11})$, in the toric ideal I_A such that $(\mathbf{u}_{11})_- \in \mathbb{Z}^E$ and $(\mathbf{u}_{11})_+ \notin \mathbb{Z}^E$. Therefore we can check that the set

$$\big\{B(\mathbf{u}_1), B(\mathbf{u}_3), B(\mathbf{u}_4), B(\mathbf{u}_6), B(\mathbf{u}_7), B(\mathbf{u}_9), B(\mathbf{u}_{11})\big\}$$

is a cover of A. Thus $rad(I_A) = rad(B(\mathbf{u}_1), \ldots, B(\mathbf{u}_{11}))$ and so $\mathrm{bar}(I_A) \leqslant 11$.

Suppose that K is a field of characteristic zero. We will prove that $\mathrm{bar}(I_A) = 11$. Let $\mathcal{B}$ be a set of binomials which generate I_A up to radical. Using the fact that every $I_{A_{\mathcal{F}_i}}$ is generated up to radical by 3 binomials and it is not a set-theoretic complete intersection on binomials (see [4]), we take, from Proposition 3.2, that $\mathcal{B}$ has at least 9 binomials. Let $B(\mathbf{v}_i^1), B(\mathbf{v}_i^2), B(\mathbf{v}_i^3) \in I_{\mathcal{F}_i}$, $1 \leqslant i \leqslant 3$, be those 9 binomials. Note that in all these binomials the variable x_{10} does not appear. Since the set $\mathcal{B}$ must cover the set $\{10\}$ it must contain also a monic binomial $B(\mathbf{w}) \in I_A$ in x_{10}. Note that for such $\mathbf{w} = (w_1, \ldots, w_{10}) \in \mathbb{Z}^{10}$ we have $w_{10} > 1$. Consider the vector $\mathbf{z} = (1, 0, 0, -1, 0, 0, 0, -1, 0, 1) \in L$. It does not belong to the lattice generated by the 9 vectors $\mathbf{v}_i^j$ plus the vector $\mathbf{w}$, since the last coordinate is 1. Consequently in $\mathcal{B}$ there are more than 10 binomials. Therefore $\mathrm{bar}(I_A) \geqslant 11$, we conclude that $\mathrm{bar}(I_A) = 11$.

In the case that K is a field of characteristic $p > 0$ we have that $\mathrm{ara}(I_{A_{\mathcal{F}_i}}) = \mathrm{bar}(I_{A_{\mathcal{F}_i}}) = 2$, see [3]. In fact when K is a field of characteristic 3 we have that $\mathrm{bar}(I_A) = 7$ since

$$rad(I_A) = rad\big(B(\mathbf{u}_1), B(\mathbf{u}_3), B(\mathbf{u}_4), B(\mathbf{u}_6), B(\mathbf{u}_7), B(\mathbf{u}_9), B(\mathbf{v})\big),$$

where $\mathbf{v} = (1, 1, 1, 0, 0, 0, 0, 0, 0, -3)$, and the height of I_A equals 7. This means also that the semigroup generated by A is completely 3-glued, see [4]. When K is a field of characteristic $p \neq 3$ the toric ideal I_A is generated up to radical by 8 binomials.

4. Applications

Complete intersection lattice ideals have been characterized in [13,19], either in terms of semigroup gluing or in terms of mixed dominating matrices. Both characterizations show that the problem of determining the complete intersection property for lattice ideals is in the NP-class [13,21]. Therefore it is interesting to find better criteria for establishing that a lattice ideal is or is not a complete intersection. In this direction such criteria were given in [16] which can be read off from the geometry of the cone σ_L. Our next theorem provides a criterion depending on the lattices associated to the faces of the cone σ_L. It generalizes in a more geometric setting a result by K. Eto (see Lemma 1.6

in [11]) for complete intersection lattice ideals and the proof, contrary to the proof in [11], uses mixed dominating matrices. Note that lattice divisors correspond to facets of the cone σ_L.

Theorem 4.1. *Let L be a non-zero positive sublattice of $\mathbb{Z}^m$ and (L,ρ) be a partial character on $\mathbb{Z}^m$. Then $I_{L,\rho}$ is complete intersection if and only if for every face $\mathcal{F}$ of σ_L the lattice ideal $I_{L\cap\mathbb{Z}^{\mathbb{E}_\mathcal{F}},\rho}$ is complete intersection.*

Proof. ($\Leftarrow$) For $\mathcal{F}=\sigma_L$ we have that $I_{L\cap\mathbb{Z}^{\mathbb{E}_{\sigma_L}},\rho}=I_{L,\rho}$ is complete intersection.

($\Rightarrow$) Suppose that $I_{L,\rho}=(B(\mathbf{u}_1),\dots,B(\mathbf{u}_r))$ is complete intersection, where $r=rank(L)$. Given a set $\{\mathbf{v}_1,\dots,\mathbf{v}_q\}\subset\mathbb{Z}^m$, we shall denote by $M(\mathbf{v}_1,\dots,\mathbf{v}_q)$ the $q\times m$ matrix whose rows are the vectors $\mathbf{v}_1,\dots,\mathbf{v}_q$. From Theorem 3.9 in [19] we have that the matrix $M(\mathbf{u}_1,\dots,\mathbf{u}_r)$ is mixed dominating while from Theorem 3.5 we deduce that $L\cap\mathbb{Z}^{\mathbb{E}_\mathcal{F}}=\sum_{\mathbf{u}_i\in\mathbb{Z}^{\mathbb{E}_\mathcal{F}}}\mathbb{Z}\mathbf{u}_i$. Set $U=\{\mathbf{u}_1,\dots,\mathbf{u}_r\}$ and $U\cap\mathbb{Z}^{\mathbb{E}_\mathcal{F}}=\{\mathbf{u}_{i_1},\dots,\mathbf{u}_{i_k}\}$. Recall that a matrix M with coefficients in $\mathbb{Z}$ is called mixed dominating if it is mixed, i.e. every row has a positive and negative entry, and also it does not contain any square mixed submatrix. By Corollary 2.8 in [12] the vectors $\mathbf{u}_1,\dots,\mathbf{u}_r$ are linearly independent, so in particular the set $U\cap\mathbb{Z}^{\mathbb{E}_\mathcal{F}}$ is linearly independent. Thus it is a $\mathbb{Z}$-base for the lattice $L\cap\mathbb{Z}^{\mathbb{E}_\mathcal{F}}$. Also $M(\mathbf{u}_1,\dots,\mathbf{u}_r)$ is mixed dominating, so $M(\mathbf{u}_{i_1},\dots,\mathbf{u}_{i_k})$ is mixed dominating. Thus again from Theorem 3.9 in [19] we have that $I_{L\cap\mathbb{Z}^{\mathbb{E}_\mathcal{F}},\rho}$ is complete intersection. $\square$

Next we will use the results of Section 3 to generalize Theorems 1 and 2 of [3].

Proposition 4.2. *Let $E_1\subset E_2\subset\{1,\dots,m\}$, $L\subset\mathbb{Z}^m$ be a lattice and $L_1=L\cap\mathbb{Z}^{E_1}$, $L_2=L\cap\mathbb{Z}^{E_2}$. Every $\mathbb{Z}$-basis (resp. spanning set) of L_1 can be extended to a $\mathbb{Z}$-basis (resp. spanning set) of L_2.*

Proof. It is enough to consider the case that $E_2=E_1\cup\{i\}$. Let $\{\mathbf{u}_1,\dots,\mathbf{u}_r\}$ be a $\mathbb{Z}$-basis of L_1. There are two cases:

(1) Every element $\mathbf{v}\in L_2$ has the i-coordinate equal to zero. Then $\operatorname{supp}(\mathbf{v})\subset E_1$, so $\mathbf{v}\in L\cap\mathbb{Z}^{E_1}=L_1$. Thus $L_2\subset L_1$ and therefore $L_2=L_1$.

(2) There are elements in L_2 with i-coordinate different from zero. Choose $\mathbf{u}_{r+1}\in L_2$ such that its i-th coordinate $(\mathbf{u}_{r+1})_i$ is positive and this is the least possible i-coordinate among all elements of L_2. Remark that $\mathbf{u}_{r+1}\notin L_1$. If $\mathbf{u}_{r+1}\in L_1$, then there are $\lambda_1,\dots,\lambda_r\in\mathbb{Z}$ such that $\mathbf{u}_{r+1}=\lambda_1\mathbf{u}_1+\cdots+\lambda_r\mathbf{u}_r$. But $(\mathbf{u}_{r+1})_i>0$ while $(\mathbf{u}_1)_i=\cdots=(\mathbf{u}_r)_i=0$.
Let $\mathbf{v}\in L_2$. If $(\mathbf{v})_i=0$, then $\mathbf{v}\in L_1$. Suppose that $(\mathbf{v})_i>0$, the case that $(\mathbf{v})_i<0$ is similar. Divide $(\mathbf{v})_i$ with $(\mathbf{u}_{r+1})_i$, so there exist $\lambda,\mu\in\mathbb{Z}$ such that $(\mathbf{v})_i=\lambda(\mathbf{u}_{r+1})_i+\mu$ and $0\leqslant\mu<(\mathbf{u}_{r+1})_i$. Set $\mathbf{w}=\mathbf{v}-\lambda\mathbf{u}_{r+1}\in L_2$. Remark that $(\mathbf{w})_i=\mu\geqslant0$. Since $\mathbf{w}\in L_2$ and $(\mathbf{w})_i\geqslant0$, we have, from the choice of $\mathbf{u}_{r+1}$, that $\mu=0$. Thus $\mathbf{w}\in L\cap\mathbb{Z}^{E_1}=L_1$ and therefore $\mathbf{v}\in\sum_{i=1}^{r+1}\mathbb{Z}\mathbf{u}_i$. It remains to prove that the vectors $\mathbf{u}_1,\dots,\mathbf{u}_r,\mathbf{u}_{r+1}$ are $\mathbb{Z}$-linearly independent. Consider an equality

$$\lambda_1\mathbf{u}_1+\cdots+\lambda_r\mathbf{u}_r+\lambda_{r+1}\mathbf{u}_{r+1}=\mathbf{0}.$$

Then $\lambda_1(\mathbf{u}_1)_i+\cdots+\lambda_r(\mathbf{u}_r)_i+\lambda_{r+1}(\mathbf{u}_{r+1})_i=0$ and therefore $\lambda_{r+1}=0$. But $\mathbf{u}_1,\dots,\mathbf{u}_r$ are $\mathbb{Z}$-linearly independent, so $\lambda_1=\cdots=\lambda_r=0$. $\square$

Definition 4.3. We say that a configuration of vectors $A=\{\mathbf{a}_1,\dots,\mathbf{a}_m\}$ is full if

(1) the cone $\sigma=pos_{\mathbb{Q}}(A)$ is an n-dimensional simplex cone generated by $\mathbf{a}_1,\dots,\mathbf{a}_n$, i.e. $\sigma=pos_{\mathbb{Q}}(\mathbf{a}_1,\dots,\mathbf{a}_n)$, and
(2) for each $n<i\leqslant m$ we have $\mathcal{F}_{\mathbf{a}_i}=\sigma$. This means that the vectors $\mathbf{a}_i$, $n<i\leqslant m$, are in the relative interior $relint_{\mathbb{Q}}(\mathbf{a}_1,\dots,\mathbf{a}_n)$ of σ, which is the set of all strictly positive rational linear combinations of $\mathbf{a}_1,\dots,\mathbf{a}_n$.

Theorem 4.4. *Let (L,ρ) be a partial character on $\mathbb{Z}^m$ such that the configuration of vectors associated to the lattice L is full, then*

(1) *in characteristic zero $I_{L,\rho}$ is either a set theoretic complete intersection or an almost set theoretic complete intersection,*
(2) *in positive characteristic $I_{L,\rho}$ is a set theoretic complete intersection.*

Proof. Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\} \subset \mathbb{Z}^n$ be the configuration of vectors associated to L, then every face of σ_L is of the form $pos_{\mathbb{Q}}(\mathbf{a}_i \mid i \in E)$, for a subset E of $\{1, \dots, n\}$. Moreover for every face $\mathcal{F} \neq \sigma_L$ we have that $L \cap \mathbb{Z}^{\mathbb{E}_\mathcal{F}} = \{\mathbf{0}\}$, since A is full. Applying Proposition 4.2 to the lattices $L \cap \mathbb{Z}^{E_i}$, where $E_i = \{1, \dots, i\}$ and $n < i \leqslant m$, we obtain a $\mathbb{Z}$-basis $\{\mathbf{u}_{n+1}, \dots, \mathbf{u}_m\}$ of the lattice L. Remark that $\mathrm{supp}(\mathbf{u}_i) \subset E_i$, the i-coordinate of $\mathbf{u}_i$ is positive and also this is the least possible i-coordinate among all elements of $L \cap \mathbb{Z}^{E_i}$. In addition, using the fact that for every $i \in \{n+1, \dots, m\}$ it holds $\mathcal{F}_{\mathbf{a}_i} = \sigma_L$, we take binomials $B(\mathbf{v}_i) \in I_{L,\rho}$, $n < i \leqslant m$, with $\mathrm{supp}((\mathbf{v}_i)_+) = \{1, \dots, n\}$ and $\mathrm{supp}((\mathbf{v}_i)_-) = \{i\}$. Note that $\mathbf{u}_{n+1}$ is in that form, so we will consider $\mathbf{v}_{n+1} := \mathbf{u}_{n+1}$. Now we will distinguish two cases for the characteristic of the field K.

(1) The characteristic of K is equal to zero. We will construct $m - n + 1$ binomials that generate the radical of $I_{L,\rho}$ up to radical. Given an index $i \in \{n+3, \dots, m\}$, there are appropriate large positive integers $r_{n+1}, \dots, r_{i-2}$ such that for every $j \in \{n+1, \dots, i-2\}$ the j-coordinate of $\mathbf{w}_i = \mathbf{u}_i - \sum_{j=n+1}^{i-2} r_j \mathbf{v}_j$ is positive. Furthermore, there exists a big enough positive integer r such that for every $j \in \{1, \dots, n\}$ the j-coordinate of $\mathbf{z}_i = \mathbf{w}_i + r\mathbf{v}_{i-1}$ is positive, while also the j-coordinate, $n+1 \leqslant j \leqslant i-2$, of $\mathbf{z}_i$ is positive, since are the same as $\mathbf{w}_i$. Notice that the vectors $\mathbf{z}_i \in L$ and $\mathbf{u}_i$ have the same i-coordinate and also $\mathrm{supp}(\mathbf{z}_i) = E_i$. But the lattice L is positive, so $L \cap \mathbb{N}^m = \{\mathbf{0}\}$ and therefore the $(i-1)$-coordinate of $\mathbf{z}_i$ is negative. Set $\mathbf{z}_{n+1} = \mathbf{u}_{n+1}$ and $\mathbf{z}_{n+2} = \mathbf{u}_{n+2}$. From the proof of Proposition 4.2 we have that $\{\mathbf{z}_{n+1}, \dots, \mathbf{z}_m\}$ is a $\mathbb{Z}$-basis of L. Let $\mathbf{z}_{m+1} = \mathbf{v}_m$, then for every $i \in \{n+1, n+3, \dots, m+1\}$ the binomial $B(\mathbf{z}_i)$ is of the form $B(\mathbf{z}_i) = x_{i-1}^{g_{i-1}} - N_{i-1}$ where N_{i-1} is a monomial not containing the variable x_{i-1}. Thus the set $\{B(\mathbf{z}_{n+1}), B(\mathbf{z}_{n+3}), \dots, B(\mathbf{z}_{m+1})\}$ is a cover of A. Using Theorem 3.5 we take that the $m - n + 1$ binomials $B(\mathbf{z}_i)$, where $n < i \leqslant m+1$, generate the radical of the lattice ideal up to radical.
(2) The characteristic of K is equal to $p > 0$. We are going to construct $m - n$ binomials that generate the radical of $I_{L,\rho}$ up to radical. Given an index $i \in \{n+2, \dots, m\}$, there are appropriate large positive integers $r_{n+1}, \dots, r_{i-1}$ such that for every $j \in \{n+1, \dots, i-1\}$ the j-coordinate of $\mathbf{w}_i = \mathbf{u}_i + \sum_{j=n+1}^{i-1} r_j \mathbf{v}_j$ is negative. Note that
  (a) the i-coordinate of $\mathbf{w}_i$ coincides with the i-coordinate of $\mathbf{u}_i$ and both of them are positive,
  (b) the i-coordinate of $\mathbf{v}_i$ is equal to $-t(\mathbf{u}_i)_i$ for a positive integer t, where $(\mathbf{u}_i)_i$ is the i-coordinate of $\mathbf{u}_i$, since $(\mathbf{u}_i)_i$ is the least possible i-coordinate among all elements of $L \cap \mathbb{Z}^{E_i}$.

  Given a power p^k of p, there exist integers $r, s \in \mathbb{Z}$ such that $p^k = rt + s$ and $0 \leqslant s < t$. We choose a big enough k such that for every $j \in \{1, \dots, n\}$ the j-coordinate of $\mathbf{y}_i := s\mathbf{w}_i - r\mathbf{v}_i \in L$, $n+1 < i \leqslant m$, is negative. Therefore the binomial $B(\mathbf{y}_i)$ is of the form $B(\mathbf{y}_i) = x_i^{g_i} - N_i$, where N_i is a monomial not containing the variable x_i, $\mathrm{supp}(\mathbf{y}_i) = E_i$ and the i-coordinate of $\mathbf{y}_i$ is $p^k(\mathbf{u}_i)_i$. Let $\mathbf{y}_{n+1} = \mathbf{u}_{n+1}$, then the set $\{B(\mathbf{y}_{n+1}), \dots, B(\mathbf{y}_m)\}$ is a cover of A. It remains to prove that

$$\left(L \cap \mathbb{Z}^{E_i}\right) : p^\infty = \left(\sum_{j=n+1}^{i} \mathbb{Z}\mathbf{y}_j\right) : p^\infty, \quad \text{for every } i = n+1, \dots, m.$$

  The proof is obvious for $i = n+1$, since

$$L \cap \mathbb{Z}^{E_{n+1}} = \mathbb{Z}\mathbf{u}_{n+1} = \mathbb{Z}\mathbf{y}_{n+1}.$$

  Assume that $(L \cap \mathbb{Z}^{E_{i-1}}) : p^\infty = (\sum_{j=n+1}^{i-1} \mathbb{Z}\mathbf{y}_j) : p^\infty$. We have

$$\left(\sum_{j=n+1}^{i} \mathbb{Z}\mathbf{y}_j\right) : p^\infty \subset \left(L \cap \mathbb{Z}^{E_i}\right) : p^\infty,$$

since $(\sum_{j=n+1}^{i} \mathbb{Z}\mathbf{y}_j) \subset L \cap \mathbb{Z}^{E_i}$. Now we will prove that

$$\left(L \cap \mathbb{Z}^{E_i}\right) : p^\infty \subset \left(\sum_{j=n+1}^{i} \mathbb{Z}\mathbf{y}_j \right) : p^\infty.$$

Let $g \in (L \cap \mathbb{Z}^{E_i}) : p^\infty$, then $p^b g \in L \cap \mathbb{Z}^{E_i}$, for a $b \in \mathbb{N}$, and therefore $p^b g = \lambda_{n+1}\mathbf{u}_{n+1} + \cdots + \lambda_i \mathbf{u}_i$ for some integers $\lambda_{n+1}, \ldots, \lambda_i$. Remark that $\lambda_{n+1}\mathbf{u}_{n+1} + \cdots + \lambda_{i-1}\mathbf{u}_{i-1}$ belongs to $L \cap \mathbb{Z}^{E_{i-1}}$. This means that there is a $c \in \mathbb{N}$ such that $p^c(\lambda_{n+1}\mathbf{u}_{n+1} + \cdots + \lambda_{i-1}\mathbf{u}_{i-1})$ belongs to $(\sum_{j=n+1}^{i-1} \mathbb{Z}\mathbf{y}_j) : p^\infty$. Moreover $\mathbf{y}_i - p^k\mathbf{u}_i$ belongs to $L \cap \mathbb{Z}^{E_{i-1}}$, so there is a natural number d such that $p^d(\mathbf{y}_i - p^k\mathbf{u}_i) \in \sum_{j=n+1}^{i-1} \mathbb{Z}\mathbf{y}_j$ and therefore $p^{d+k}\mathbf{u}_i \in \sum_{j=n+1}^{i} \mathbb{Z}\mathbf{y}_j$. Thus $p^{c+d+k} g$ belongs to $\sum_{j=n+1}^{i} \mathbb{Z}\mathbf{y}_j$. Using Theorem 3.5 we take that the $m-n$ binomials $B(\mathbf{y}_i)$, where $n < i \leqslant m$, generate the radical of $I_{L,\rho}$ up to radical. □

Example 4.5. Let m be a fixed integer number greater than or equal to 8 and let $L(m)$ be the sublattice of $\mathbb{Z}^4$ generated by $\mathbf{e}_1 = (m+2q-3, -m+2q+5, -1, -1)$, $\mathbf{e}_2 = (-m-2q+5, m-2q-3, -1, -1)$, $\mathbf{e}_3 = (-m-2q+5, -1, m-3, -1)$, where $q = 0$ when m is even and $q = 1$ otherwise. It is easy to check that $L(m)$ has rank 3; thus, $I_{L(m),1}$ is a lattice ideal in $K[x_1, \ldots, x_4]$ of codimension 3 where K is a field of any characteristic. Moreover $L(m)$ is not saturated because $(2, 2, -2, -2) = \mathbf{e}_1 + \mathbf{e}_2 \in L(m)$ and it is easy to check that $(1, 1, -1, -1) \notin L(m)$; therefore, the lattice ideal $I_{L(m),1}$ is never toric. In [20] Ojeda proved that the lattice ideal $I_{L(m),1}$ is generic and minimally generated by m elements, i.e.

$$\begin{gathered} f_1 = x_1^{m+2q-3} - x_2^{m-2q-5}x_3x_4, \qquad f_2 = x_2^{m-2q-3} - x_1^{m+2q-5}x_3x_4, \\ f_3 = x_3^{m-3} - x_1^{m+2q-5}x_2x_4, \\ f_{12} = x_1^2x_2^2 - x_3^2x_4^2, \\ f_{23}^i = x_2^{2i-1}x_3^{m-2i-3} - x_1^{m+2q-2i-5}x_4^{2i+1}, \quad i = 1, \ldots, \frac{m}{2} + \frac{q}{2} - 3, \\ f_{13}^j = x_1^{2j}x_3^{m-2j-2} - x_2^{m-2q-2j-2}x_4^{2j}, \quad j = 1, \ldots, \frac{m}{2} - \frac{q}{2} - 2 \end{gathered}$$

and

$$f_{123} = \begin{cases} x_4^{m-3} - x_1x_2^{m-5}x_3 & \text{if } m \text{ is even}, \\ x_4^{m-3} - x_1^{m-3}x_2x_3 & \text{if } m \text{ is odd}. \end{cases}$$

On the other hand for every integer $m \geqslant 8$ consider the homomorphism $\phi_m : \mathbb{Z}^4 \to \mathbb{Z}$ defined by $\phi_m(a, b, c, d) = a + b + c + d$ if m is even, and $\phi_m(a, b, c, d) = a(m-6) + b(m-2) + c(m-4) + d(m-4)$ if m is odd. By direct computation we can check that $L(m) \subset \ker\phi_m$, so that $Sat(L(m)) = \ker\phi_m$. We can associate to $L(m)$ the rational polyhedral cone $\sigma_{L(m)} = pos_{\mathbb{Q}}(A)$ where

(1) $A = \{1, 1, 1, 1\}$ when m is even, and
(2) $A = \{m-6, m-2, m-4, m-4\}$ when m is odd.

In both cases the cone $\sigma_{L(m)} = pos_{\mathbb{Q}}(A)$ has only one non-zero face $\mathcal{F} = pos_{\mathbb{Q}}(A)$. So $L(m) \cap (\mathbb{Z}^4)^{\mathbb{E}_{\sigma_{L(m)}}} = L(m) = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle$. To cover A we need also the binomial f_{123}. So, from Theorem 3.5, we have that $rad(I_{L(m),1}) = rad(f_1, f_2, f_3, f_{123})$. Therefore $I_{L(m),1}$ is an almost set theoretic complete intersection in the characteristic zero case. In positive characteristic we have, from Theorem 4.4, that it is set theoretic complete intersection and the 3 binomials which generate $rad(I_{L(m),1})$ up to radical depend on the characteristic.

References

[1] M. Barile, On a special class of simplicial toric varieties, J. Algebra 308 (2007) 368–382.
[2] M. Barile, G. Lyubeznik, Set-theoretic complete intersections in characteristic p, Proc. Amer. Math. Soc. 133 (2005) 3199–3209.
[3] M. Barile, M. Morales, A. Thoma, On simplicial toric varieties which are set-theoretic complete intersections, J. Algebra 226 (2000) 880–892.
[4] M. Barile, M. Morales, A. Thoma, Set-theoretic complete intersections on binomials, Proc. Amer. Math. Soc. 130 (2002) 1893–1903.
[5] T. Bogart, A. Jensen, R. Thomas, The circuit ideal of a vector configuration, J. Algebra 309 (2007) 518–542.
[6] W. Bruns, J. Gubeladje, N.V. Trung, Problems and algorithms for affine semigroups, Semigroup Forum 64 (2002) 180–212.
[7] D. Eisenbud, B. Sturmfels, Binomial ideals, Duke Math. J. 84 (1996) 1–45.
[8] S. Eliahou, R. Villarreal, On systems of binomials in the ideal of a toric variety, Proc. Amer. Math. Soc. 130 (2001) 345–351.
[9] K. Eto, Binomial arithmetical rank of lattice ideals, Manuscripta Math. 109 (4) (2002) 455–463.
[10] K. Eto, When is a binomial ideal equal to a lattice ideal up to radical?, in: Commutative Algebra, Grenoble/Lyon, 2001, in: Contemp. Math., vol. 331, 2003, pp. 111–118.
[11] K. Eto, Set-theoretic complete intersection lattice ideals in monoid rings, J. Algebra 299 (2006) 689–706.
[12] K. Fischer, J. Shapiro, Mixed matrices and binomial ideals, J. Pure Appl. Algebra 113 (1996) 39–54.
[13] K. Fischer, W. Morris, J. Shapiro, Affine semigroups that are complete intersections, Proc. Amer. Math. Soc. 125 (1997) 3137–3145.
[14] A. Katsabekis, M. Morales, A. Thoma, Stanley–Reisner rings and the radical of lattice ideals, J. Pure Appl. Algebra 204 (2006) 584–601.
[15] A. Katsabekis, A. Thoma, Matchings in simplicial complexes, circuits and toric varieties, J. Combin. Theory Ser. A 114 (2007) 300–310.
[16] N. Michelacakis, A. Thoma, On the geometry of complete intersection toric varieties, Arch. Math. 87 (2006) 113–123.
[17] E. Miller, B. Sturmfels, Combinatorial Commutative Algebra, Grad. Texts in Math., vol. 227, Springer-Verlag, New York, 2005.
[18] M. Morales, Equations des varietes monomiales en codimension deux, J. Algebra 175 (1995) 1082–1095.
[19] M. Morales, A. Thoma, Complete intersection lattice ideals, J. Algebra 284 (2005) 755–770.
[20] I. Ojeda, Examples of generic lattice ideals of codimension 3, Comm. Algebra 36 (2008) 279–287.
[21] G. Scheja, O. Scheja, U. Storch, On regular sequences of binomials, Manuscripta Math. 98 (1999) 115–132.
[22] B. Sturmfels, Gröbner Bases and Convex Polytopes, Univ. Lecture Ser., vol. 8, American Mathematical Society, Providence, RI, 1995.
[23] R. Villarreal, Momomial Algebras, Pure Appl. Math., Marcel Dekker, New York, 2001.
[24] R. Villarreal, Combinatorial optimization methods in commutative algebra, in preparation.