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On the top local cohomology modules ☆

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ABSTRACT

Let $(R,\mathfrak{m})$ be a Noetherian local ring and I an ideal of R. Let M be a finitely generated R-module with $\dim M = d$. It is clear by Matlis duality that if R is complete then $H^d_I(M)$ satisfies the following property:

$$\operatorname{Ann}_R(0:_{H^d_I(M)} \mathfrak{p}) = \mathfrak{p} \quad \text{for all prime ideals } \mathfrak{p} \supseteq \operatorname{Ann}_R H^d_I(M). \qquad (*)$$

However, $H^d_I(M)$ does not satisfy the property $(*)$ in general. In this paper we characterize the property $(*)$ of $H^d_I(M)$ in order to study the catenarity of the ring $R/\operatorname{Ann}_R H^d_I(M)$, the set of attached primes $\operatorname{Att}_R H^d_I(M)$, the co-support $\operatorname{Cos}_R(H^d_I(M))$, and the multiplicity of $H^d_I(M)$. We also show that if $H^d_I(M)$ satisfies the property $(*)$ then $H^d_I(M) \cong H^d_{\mathfrak{m}}(M/N)$ for some submodule N of M.

1. Introduction

Throughout this paper, $(R,\mathfrak{m})$ is a Noetherian local ring, I is an ideal of R and M is a finitely generated R-module with $\dim M = d$. Let $\operatorname{Var}(I)$ denote the set of all prime ideals of R containing I. Denote by $\widehat{R}$ and $\widehat{M}$ the $\mathfrak{m}$-adic completions of R and M respectively.

It is clear that $\operatorname{Ann}_R(M/\mathfrak{p}M) = \mathfrak{p}$ for all prime ideals $\mathfrak{p} \in \operatorname{Var}(\operatorname{Ann}_R M)$. So, it follows by Matlis duality that if R is complete then

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$$\mathrm{Ann}_R(0 :_A \mathfrak{p}) = \mathfrak{p} \quad \text{for all } \mathfrak{p} \in \mathrm{Var}(\mathrm{Ann}_R A) \tag{$*$}$$

for all Artinian R-modules A. Recently, H. Zöschinger [Zos] proved that the property $(*)$ is satisfied for all Artinian R-modules if and only if the natural map $R \to \widehat{R}$ satisfies the going up theorem. However, the property $(*)$ is not satisfied in general, cf. [CN, Example 4.4].

We know that the local cohomology module $H^i_{\mathfrak{m}}(M)$ is Artinian for all i. It is shown that the top local cohomology module $H^d_{\mathfrak{m}}(M)$ satisfies the property $(*)$ if and only if the ring $R/\mathrm{Ann}_R(H^d_{\mathfrak{m}}(M))$ is catenary, cf. [CDN, Main Theorem]. Also, $H^i_{\mathfrak{m}}(M)$ satisfies the property $(*)$ for all finitely generated R-modules M and all integers $i \geqslant 0$ if and only if R is universally catenary and all its formal fibers are Cohen–Macaulay (see [NA1, Corollary 3.2], [NC, Proposition 3.6]).

Note that the top local cohomology module $H^d_I(M)$ is Artinian, but it may not satisfy the property $(*)$ even when R is a quotient of a regular local ring (see Example 3.9). In this paper, we characterize the property $(*)$ of $H^d_I(M)$ in order to study the catenarity of the ring $R/\mathrm{Ann}_R H^d_I(M)$, the set of attached primes $\mathrm{Att}_R(H^d_I(M))$, the co-support $\mathrm{Cos}_R(H^d_I(M))$ and the multiplicity of $H^d_I(M)$.

Following I.G. Macdonald [Mac], the set of attached primes of an Artinian R-module A is denoted by $\mathrm{Att}_R A$. From now on, we keep the following notations.

Notations 1.1. Let $0 = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M} N(\mathfrak{p})$ be a reduced primary decomposition of the submodule 0 of M. Set

$$\mathrm{Ass}_R(I, M) = \left\{\mathfrak{p} \in \mathrm{Ass}_R M \;\middle|\; \dim(R/\mathfrak{p}) = d,\ \sqrt{\mathfrak{p} + I} = \mathfrak{m}\right\}.$$

Set $N = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R(I,M)} N(\mathfrak{p})$. Note that N does not depend on the choice of the reduced primary decomposition of 0 because $\mathrm{Ass}_R(I, M) \subseteq \min \mathrm{Ass}_R M$. The co-support of $H^d_I(M)$, denoted by $\mathrm{Cos}(H^d_I(M))$, is defined by

$$\mathrm{Cos}_R\big(H^d_I(M)\big) = \left\{\mathfrak{p} \in \mathrm{Spec}(R) \;\middle|\; H^{d-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M/N)_{\mathfrak{p}} \neq 0\right\}.$$

The following theorem is the main result of this paper.

Theorem 1.2. *The following statements are equivalent*:

(i) $H^d_I(M)$ *satisfies the property* $(*)$.
(ii) *The ring* $R/\mathrm{Ann}_R H^d_I(M)$ *is catenary and* $\sqrt{\mathfrak{p} + I} = \mathfrak{m}$ *for all* $\mathfrak{p} \in \mathrm{Att}_R H^d_I(M)$.
(iii) *The ring* $R/\mathrm{Ann}_R H^d_I(M)$ *is catenary and* $H^d_I(M) \cong H^d_{\mathfrak{m}}(M/N)$.
(iv) $\mathrm{Cos}_R(H^d_I(M)) = \mathrm{Var}(\mathrm{Ann}_R H^d_I(M))$.

This theorem will be proved in Sections 2 and 3 (Theorems 2.7 and 3.7). As consequences, we show that if $H^d_I(M)$ satisfies the property $(*)$ then the results on the attached primes, the co-support and the multiplicity of $H^d_I(M)$ are as good as when R is complete.

2. The property $(*)$ of $H^d_I(M)$

Let us recall the following notion introduced in [CN, Definition 4.2].

Definition 2.1. Let A be an Artinian R-module. A is said to satisfy *the property* $(*)$ if

$$\mathrm{Ann}_R(0 :_A \mathfrak{p}) = \mathfrak{p} \quad \text{for all } \mathfrak{p} \in \mathrm{Var}(\mathrm{Ann}_R A). \tag{$*$}$$

In this section, we study the property $(*)$ for $H^d_I(M)$. Before presenting the results, we need some preliminaries.

The theory of secondary representation was introduced by I.G. Macdonald [Mac]. This theory is in some sense dual to the more known theory of primary decomposition for Noetherian modules. Note that every Artinian R-module A has a minimal secondary representation $A = A_1 + \cdots + A_n$, where A_i is $\mathfrak{p}_i$-secondary. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the choice of the minimal secondary representation of A. This set is called the set of *attached primes* of A, and denoted by $\operatorname{Att}_R A$.

Lemma 2.2. *(See [Mac].) Let A be an Artinian R-module. Then*:

(i) $A \neq 0$ *if and only if* $\operatorname{Att}_R A \neq \emptyset$.
(ii) $\min \operatorname{Att}_R A = \min \operatorname{Var}(\operatorname{Ann}_R A)$.
(iii) *If* $0 \to A' \to A \to A'' \to 0$ *is an exact sequence of Artinian R-modules then*

$$\operatorname{Att}_R A'' \subseteq \operatorname{Att}_R A \subseteq \operatorname{Att}_R A' \cup \operatorname{Att}_R A''.$$

If A is an Artinian R-module then A has a natural structure as an $\widehat{R}$-module. With this structure, a subset of A is an R-submodule if and only if it is an $\widehat{R}$-submodule of A. Therefore A is an Artinian $\widehat{R}$-module.

Lemma 2.3. *(See [BS, 8.2.4 and 8.2.5].)* $\operatorname{Att}_R A = \{P \cap R \mid P \in \operatorname{Att}_{\widehat{R}} A\}$.

Denote by $\widehat{\mathfrak{m}}$ the unique maximal ideal of $\widehat{R}$. The attached primes of the $\widehat{R}$-module $H^d_I(M)$ can be described as follows.

Lemma 2.4. *(See [DSc, Corollary 3.3].)*

$$\operatorname{Att}_{\widehat{R}} H^d_I(M) = \left\{P \in \operatorname{Ass}_{\widehat{R}} \widehat{M} \;\middle|\; \dim(\widehat{R}/P) = d,\ \sqrt{P + I\widehat{R}} = \widehat{\mathfrak{m}}\right\}.$$

The following result shows that the property $(*)$ of the top local cohomology module $H^d_{\mathfrak{m}}(M)$ can be characterized by the catenarity of the base ring.

Lemma 2.5. *(See [CDN, Main Theorem].) The following statements are equivalent*:

(i) $H^d_{\mathfrak{m}}(M)$ *satisfies the property* $(*)$.
(ii) *The ring* $R/\operatorname{Ann}_R H^d_{\mathfrak{m}}(M)$ *is catenary.*

Following T.N. An and the first author [NA2], an Artinian R-module A is *quasi unmixed* if $\dim(\widehat{R}/P) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A)$ for all $P \in \min \operatorname{Att}_{\widehat{R}} A$. If $\dim(\widehat{R}/P) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A)$ for all $P \in \operatorname{Att}_{\widehat{R}} A$ then A is called *unmixed*.

Lemma 2.6. *(See [NA2, Theorem 1.1].) Assume that A is quasi unmixed. If A satisfies the property $(*)$ then the ring $R/\operatorname{Ann}_R A$ is catenary and* $\dim(R/\operatorname{Ann}_R A) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A)$.

The following theorem is the first main result of this paper.

Theorem 2.7. *Let N be defined as in Notations* 1.1, *the following statements are equivalent*:

(i) $H^d_I(M)$ *satisfies the property* $(*)$.
(ii) *The ring* $R/\operatorname{Ann}_R H^d_I(M)$ *is catenary and* $\sqrt{\mathfrak{p} + I} = \mathfrak{m}$ *for all* $\mathfrak{p} \in \operatorname{Att}_R H^d_I(M)$.
(iii) *The ring* $R/\operatorname{Ann}_R H^d_I(M)$ *is catenary and* $H^d_I(M) \cong H^d_{\mathfrak{m}}(M/N)$.

Proof. If $H_I^d(M) = 0$ then the result is clear. So, assume that $H_I^d(M) \neq 0$.

(i) ⇒ (ii). Note that $\dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}} H_I^d(M)) = d$ by Lemmas 2.2 and 2.4. So, it follows by Lemma 2.4 that $H_I^d(M)$ is unmixed. Since $H_I^d(M)$ satisfies the property $(*)$, we get by Lemma 2.6 that $R/\operatorname{Ann}_R H_I^d(M)$ is catenary.

It is clear that

$$\operatorname{Rad}\big(\operatorname{Ann}_R(0 :_{H_I^d(M)} I)\big) \supseteq \operatorname{Rad}\big(I + \operatorname{Ann}_R H_I^d(M)\big).$$

Let $\mathfrak{q} \in \operatorname{Spec}(R)$ such that $\mathfrak{q} \supseteq I + \operatorname{Ann}_R H_I^d(M)$. Since $H_I^d(M)$ satisfies the property $(*)$, we have $\operatorname{Ann}_R(0 :_{H_I^d(M)} I) \subseteq \operatorname{Ann}_R(0 :_{H_I^d(M)} \mathfrak{q}) = \mathfrak{q}$. It follows that

$$\operatorname{Rad}\big(\operatorname{Ann}_R(0 :_{H_I^d(M)} I)\big) \subseteq \bigcap_{\substack{\mathfrak{q} \in \operatorname{Spec}(R) \\ \mathfrak{q} \supseteq I + \operatorname{Ann}_R H_I^d(M)}} \mathfrak{q} = \operatorname{Rad}\big(I + \operatorname{Ann}_R H_I^d(M)\big).$$

Therefore $\operatorname{Rad}(\operatorname{Ann}_R(0 :_{H_I^d(M)} I)) = \operatorname{Rad}(I + \operatorname{Ann}_R H_I^d(M))$. Since $H_I^d(M)$ is Artinian, so is $0 :_{H_I^d(M)} I$. Because $H_I^d(M)$ is I-cofinite by [DM, Theorem 3], it follows that $0 :_{H_I^d(M)} I$ is a finitely generated R-module. Hence $0 :_{H_I^d(M)} I$ is of finite length, and hence $\operatorname{Ann}_R(0 :_{H_I^d(M)} I)$ is an $\mathfrak{m}$-primary ideal of R. Therefore $I + \operatorname{Ann}_R H_I^d(M)$ is $\mathfrak{m}$-primary. Let $\mathfrak{p} \in \operatorname{Att}_R H_I^d(M)$. Then $\mathfrak{p} \supseteq \operatorname{Ann}_R H_I^d(M)$ by Lemma 2.2. Therefore $I + \mathfrak{p}$ is $\mathfrak{m}$-primary.

(ii) ⇒ (iii). As in Notations 1.1, let $0 = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M} N(\mathfrak{p})$ be a reduced primary decomposition of the submodule 0 of M and set $N = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R(I,M)} N(\mathfrak{p})$, where

$$\operatorname{Ass}_R(I, M) = \big\{\mathfrak{p} \in \operatorname{Ass}_R M \;\big|\; \dim(R/\mathfrak{p}) = d,\ \sqrt{I + \mathfrak{p}} = \mathfrak{m}\big\}.$$

It is easy to check that $\operatorname{Ass}_R(M/N) = \operatorname{Ass}_R(I, M)$ and $\operatorname{Ass}_R N = \operatorname{Ass}_R M \setminus \operatorname{Ass}_R(I, M)$. From the exact sequence $0 \to N \to M \to M/N \to 0$ we get the exact sequence

$$H_I^d(N) \to H_I^d(M) \to H_I^d(M/N) \to 0.$$

We claim that $H_I^d(N) = 0$. Suppose that $H_I^d(N) \neq 0$ and we look for a contradiction. Then there exists by Lemma 2.2 an attached prime $P \in \operatorname{Att}_{\widehat{R}} H_I^d(N)$. By Lemma 2.4, we have $P \in \operatorname{Ass}_{\widehat{R}} \widehat{N}$, $\dim(\widehat{R}/P) = d$ and $\sqrt{I\widehat{R} + P} = \widehat{\mathfrak{m}}$. Since $\operatorname{Ass}_{\widehat{R}} \widehat{N} \subseteq \operatorname{Ass}_{\widehat{R}} \widehat{M}$, it follows that $P \in \operatorname{Ass}_{\widehat{R}} \widehat{M}$. Therefore $P \in \operatorname{Att}_{\widehat{R}} H_I^d(M)$ by Lemma 2.4. Set $\mathfrak{p} = P \cap R$. Then $\mathfrak{p} \in \operatorname{Att}_R H_I^d(M)$ by Lemma 2.3. By the assumption (ii), we have $\sqrt{\mathfrak{p} + I} = \mathfrak{m}$. Since $P \in \operatorname{Ass}_{\widehat{R}} \widehat{M}$ and $\dim(\widehat{R}/P) = d$, we get $\mathfrak{p} \in \operatorname{Ass}_R M$ and $\dim(R/\mathfrak{p}) = d$. Therefore $\mathfrak{p} \in \operatorname{Ass}_R(I, M)$. On the other hand, because $P \in \operatorname{Att}_{\widehat{R}} H_I^d(N)$, we have by Lemma 2.4 that $P \in \operatorname{Ass}_{\widehat{R}} \widehat{N}$. Hence $\mathfrak{p} \in \operatorname{Ass}_R N$ and hence $\mathfrak{p} \in \operatorname{Ass}_R M \setminus \operatorname{Ass}_R(I, M)$. This gives a contradiction. Therefore $H_I^d(N) = 0$, the claim is proved.

It follows by the above exact sequence and by the claim that $H_I^d(M) \cong H_I^d(M/N)$. Since $\operatorname{Ass}_R(I, M)$ is a finite set and $\sqrt{I + \mathfrak{p}} = \mathfrak{m}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(I, M)$, we can check that the ideal $I + \bigcap_{\mathfrak{p} \in \operatorname{Ass}(I,M)} \mathfrak{p}$ is $\mathfrak{m}$-primary. Because $\operatorname{Ass}_R(M/N) = \operatorname{Ass}_R(I, M)$, we have $\operatorname{Rad}(\operatorname{Ann}_R(M/N)) = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R(I,M)} \mathfrak{p}$. Therefore, $I + \operatorname{Ann}_R(M/N)$ is $\mathfrak{m}$-primary. So, we have by the Independence Theorem [BS, Theorem 4.2.1] that

$$H_I^d(M/N) \cong H^d_{I + \operatorname{Rad}(\operatorname{Ann}_R M/N)}(M/N) \cong H^d_{\mathfrak{m}}(M/N).$$

Thus, $H_I^d(M) \cong H_{\mathfrak{m}}^d(M/N)$.

(iii) ⇒ (i). Since $H^d_I(M) \cong H^d_{\mathfrak{m}}(M/N)$, we have $\operatorname{Ann}_R(H^d_I(M)) = \operatorname{Ann}_R(H^d_{\mathfrak{m}}(M/N))$. Since $R/\operatorname{Ann}_R H^d_I(M)$ is catenary, the ring $R/\operatorname{Ann}_R H^d_{\mathfrak{m}}(M/N)$ is catenary. By Lemma 2.5, $H^d_{\mathfrak{m}}(M/N)$ satisfies the property $(*)$ and hence so does $H^d_I(M)$. □

Corollary 2.8. *Let* $\operatorname{Ass}_R(I, M)$ *be defined as in Notations* 1.1. *Then we have*:

(i) $\operatorname{Ass}_R(I, M) \subseteq \operatorname{Att}_R H^d_I(M)$. *In particular, if* $\operatorname{Ass}_R(I, M) \neq \emptyset$ *then* $H^d_I(M) \neq 0$.
(ii) *Suppose that* $\operatorname{Ass}_R(I, M) = \emptyset$. *Then* $H^d_I(M)$ *satisfies* $(*)$ *if and only if* $H^d_I(M) = 0$.

Proof. (i) Let $\mathfrak{p} \in \operatorname{Ass}_R(I, M)$. Then $\mathfrak{p} \in \operatorname{Ass}_R M$, $\dim(R/\mathfrak{p}) = d$ and $\sqrt{I+\mathfrak{p}} = \mathfrak{m}$. Let $P \in \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ such that $\dim(\widehat{R}/P) = d$. Then $P \cap R = \mathfrak{p}$. Since

$$\operatorname{Ass}_{\widehat{R}} \widehat{M} = \bigcup_{\mathfrak{q} \in \operatorname{Ass}_R M} \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{q}\widehat{R})$$

by [Mat, Theorem 23.2(ii)], we have $P \in \operatorname{Ass}_{\widehat{R}} \widehat{M}$. Because $\sqrt{I+\mathfrak{p}} = \mathfrak{m}$, it follows that $\sqrt{I\widehat{R}+P} = \widehat{\mathfrak{m}}$. Therefore $P \in \operatorname{Att}_{\widehat{R}} H^d_I(M)$ by Lemma 2.4. Hence $\mathfrak{p} \in \operatorname{Att}_R H^d_I(M)$ by Lemma 2.3. Therefore $\operatorname{Ass}_R(I, M) \subseteq \operatorname{Att}_R H^d_I(M)$. The rest assertion follows by Lemma 2.2.

(ii) Suppose that $\operatorname{Ass}_R(I, M) = \emptyset$. It is clear that if $H^d_I(M) = 0$ then $H^d_I(M)$ satisfies the property $(*)$. Suppose that $H^d_I(M) \neq 0$. Then there exists $\mathfrak{p} \in \operatorname{Att}_R H^d_I(M)$ by Lemma 2.2. It follows by Lemmas 2.3 and 2.4 that $\mathfrak{p} \in \operatorname{Ass}_R M$ and $\dim(R/\mathfrak{p}) = d$. If $H^d_I(M)$ satisfies the property $(*)$ then $\sqrt{I+\mathfrak{p}} = \mathfrak{m}$ by Theorem 2.7, and hence $\mathfrak{p} \in \operatorname{Ass}_R(I, M)$. This is impossible. □

Remark 2.9. If $\dim R \leqslant 1$ then $H^1_I(M)$ satisfies the property $(*)$. This follows by Lemma 2.5 and the fact that R is catenary and $H^1_I(M) \cong H^1_I(M_1) \cong H^1_{\mathfrak{m}}(M_1)$, where $M_1 = M/\bigcup_{n \geqslant 1}(0 :_M I^n)$. Note that R is not necessarily complete when $\dim R = 1$. Moreover, there exists by D. Ferrand and M. Raynaud [FR] a Noetherian local ring $(R, \mathfrak{m})$ of dimension 1 which cannot be expressed as a quotient of a Gorenstein local ring.

3. Attached primes, co-support and multiplicity

Using Theorem 2.7, we have the following description of the attached primes of $H^d_I(M)$.

Corollary 3.1. *If* $H^d_I(M)$ *satisfies the property* $(*)$ *then*

$$\operatorname{Att}_R H^d_I(M) = \left\{\mathfrak{p} \in \operatorname{Ass}_R M \;\middle|\; \dim(R/\mathfrak{p}) = d,\ \sqrt{I+\mathfrak{p}} = \mathfrak{m}\right\}.$$

Proof. Let $\operatorname{Ass}_R(I, M)$ be defined as in Notations 1.1. Then $\operatorname{Ass}(I, M) \subseteq \operatorname{Att}_R H^d_I(M)$ by Corollary 2.8(i). Let $\mathfrak{p} \in \operatorname{Att}_R H^d_I(M)$. Then $\mathfrak{p} \in \operatorname{Ass}_R M$ and $\dim(R/\mathfrak{p}) = d$. Since $H^d_I(M)$ satisfies the property $(*)$, $\sqrt{I+\mathfrak{p}} = \mathfrak{m}$ by Theorem 2.7. Therefore $\mathfrak{p} \in \operatorname{Ass}(I, M)$. Thus,

$$\operatorname{Att}_R H^d_I(M) = \operatorname{Ass}_R(I, M) = \left\{\mathfrak{p} \in \operatorname{Ass}_R M \;\middle|\; \dim(R/\mathfrak{p}) = d,\ \sqrt{I+\mathfrak{p}} = \mathfrak{m}\right\}. \qquad \square$$

Note that $\operatorname{Ass}_{\widehat{R}}(\widehat{M}) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R M} \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ for all finitely generated R-modules M, cf. [Mat, Theorem 23.2(ii)]. However, the dual formula $\operatorname{Att}_{\widehat{R}} A = \bigcup_{\mathfrak{p} \in \operatorname{Att}_R A} \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ for an Artinian R-module A is not true in general. Below, we give a characterization for $H^d_I(M)$ to satisfy this formula. Following M. Nagata [Na], R is called *unmixed* if $\dim(\widehat{R}/P) = \dim(\widehat{R})$ for all $P \in \operatorname{Ass}(\widehat{R})$.

Proposition 3.2. *Let* $\operatorname{Ass}_R(I, M)$ *be defined as in Notations* 1.1. *Then the following statements are equivalent*:

(i) $\operatorname{Att}_{\widehat{R}} H^d_I(M) = \bigcup_{\mathfrak{p}\in \operatorname{Att}_R H^d_I(M)} \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$.

(ii) $H^d_I(M)$ *satisfies the property* $(*)$ *and* $R/\mathfrak{p}$ *is unmixed for all* $\mathfrak{p} \in \operatorname{Ass}_R(I, M)$.

Proof. (i) $\Rightarrow$ (ii). Let $\mathfrak{q} \in \operatorname{Var}(\operatorname{Ann}_R H^d_I(M))$. By Lemma 2.2, there exists $\mathfrak{p} \in \operatorname{Att}_R H^d_I(M)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Let $Q \in \operatorname{Ass}(\widehat{R}/\mathfrak{q}\widehat{R})$. Then $Q \cap R = \mathfrak{q}$. Since the natural map $R \to \widehat{R}$ is flat, it satisfies the going down theorem [Mat, Theorem 9.5]. Therefore there exists $P \in \operatorname{Spec}(\widehat{R})$ such that $P \subseteq Q$ and $P \cap R = \mathfrak{p}$. Because $P \supseteq \mathfrak{p}\widehat{R}$, there exists $P' \in \min \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ such that $P' \subseteq P$. Since $\mathfrak{p} \in \operatorname{Att}_R H^d_I(M)$, we get by the hypothesis (i) that $P' \in \operatorname{Att}_{\widehat{R}} H^d_I(M)$. Hence $P' \supseteq \operatorname{Ann}_{\widehat{R}} H^d_I(M)$ by Lemma 2.2 and hence $Q \supseteq \operatorname{Ann}_{\widehat{R}} H^d_I(M)$. Since $\widehat{R}$-module $H^d_I(M)$ satisfies the property $(*)$, we have $\operatorname{Ann}_{\widehat{R}}(0 :_{H^d_I(M)} Q) = Q$. Therefore

$$\mathfrak{q} \subseteq \operatorname{Ann}_R(0 :_{H^d_I(M)} \mathfrak{q}) \subseteq \operatorname{Ann}_{\widehat{R}}(0 :_{H^d_I(M)} Q) \cap R = Q \cap R = \mathfrak{q}.$$

Hence $\operatorname{Ann}_R(0 :_{H^d_I(M)} \mathfrak{q}) = \mathfrak{q}$. Thus $H^d_I(M)$ satisfies the property $(*)$.

Let $\mathfrak{p} \in \operatorname{Ass}_R(I, M)$. Then $\mathfrak{p} \in \operatorname{Att}_R H^d_I(M)$ by Corollary 2.8(i). Let $P \in \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$. Then $P \in \operatorname{Att}_{\widehat{R}} H^d_I(M)$ by the assumption (i), and hence $\dim(\widehat{R}/P) = d$ by Lemma 2.4. Therefore $R/\mathfrak{p}$ is unmixed.

(ii) $\Rightarrow$ (i). Let $P \in \operatorname{Att}_{\widehat{R}} H^d_I(M)$. Set $\mathfrak{p} = P \cap R$. Then $\mathfrak{p} \in \operatorname{Att}_R H^d_I(M)$ by Lemma 2.3 and $\dim(\widehat{R}/P) = d$ by Lemma 2.4. It follows that $\dim(R/\mathfrak{p}) = d$. Hence $P \in \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$.

Conversely, let $\mathfrak{p} \in \operatorname{Att}_R H^d_I(M)$ and $P \in \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$. Since $H^d_I(M)$ satisfies the property $(*)$ by (ii), we have $\mathfrak{p} \in \operatorname{Ass}_R(I, M)$ by Corollary 3.1. Therefore $\mathfrak{p} \in \operatorname{Ass}_R M$, $\dim(R/\mathfrak{p}) = d$ and $\sqrt{I + \mathfrak{p}} = \mathfrak{m}$. Hence $P \in \operatorname{Ass}_{\widehat{R}} \widehat{M}$ by [Mat, Theorem 23.2(ii)]. Since $R/\mathfrak{p}$ is unmixed by (ii), $\dim(\widehat{R}/P) = \dim(R/\mathfrak{p}) = d$. Since $\sqrt{I + \mathfrak{p}} = \mathfrak{m}$, it follows that $\sqrt{P + I\widehat{R}} = \widehat{\mathfrak{m}}$. Hence $P \in \operatorname{Att}_{\widehat{R}} H^d_I(M)$ by Lemma 2.4. $\square$

Let $\mathfrak{p} \in \operatorname{Spec} R$. In [Sm], K.E. Smith studied a functor called "dual to localization"

$$F_{\mathfrak{p}}(-) = \operatorname{Hom}_R\big(\operatorname{Hom}_R\big(-, E(R/\mathfrak{m})\big), E(R/\mathfrak{p})\big)$$

from the category of R-modules to the category of $R_{\mathfrak{p}}$-modules, where $E(-)$ is the injective hull. Note that this functor $F_{\mathfrak{p}}$ is linear exact, $F_{\mathfrak{p}}(A) \neq 0$ if and only if $\mathfrak{p} \supseteq \operatorname{Ann}_R A$, and when R is complete then $F_{\mathfrak{p}}(A)$ is Artinian for any Artinian R-module A.

Proposition 3.3. *Let* $\mathfrak{p} \in \operatorname{Spec}(R)$ *and let* $F_{\mathfrak{p}}(-)$ *be the above dual to localization. Let* N *be defined as in Notations* 1.1. *Suppose that* R *is complete. Then*

$$F_{\mathfrak{p}}\big(H^d_I(M)\big) \cong H^{d\text{-}\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M/N)_{\mathfrak{p}}.$$

Proof. Since R is complete, $H^d_I(M)$ satisfies the property $(*)$. Hence $H^d_I(M) \cong H^d_{\mathfrak{m}}(M/N)$ by Theorem 2.7. Since R is complete, it follows from the local duality [BS, 11.2.6] that

$$F_{\mathfrak{p}}\big(H^d_I(M)\big) \cong F_{\mathfrak{p}}\big(H^d_{\mathfrak{m}}(M/N)\big) \cong H^{d\text{-}\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M/N)_{\mathfrak{p}}. \qquad \square$$

Proposition 3.3 suggests the following notion of co-support of $H^d_I(M)$.

Definition 3.4. Let N be defined as in Notations 1.1. The *co-support* of $H^d_I(M)$, denoted by $\operatorname{Cos}_R(H^d_I(M))$, is defined as follows

$$\operatorname{Cos}_R\big(H^d_I(M)\big) = \big\{\mathfrak{p} \in \operatorname{Spec}(R) \;\big|\; H^{d\text{-}\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M/N)_{\mathfrak{p}} \neq 0\big\}.$$

In general we have the following inclusion.

Lemma 3.5. $\operatorname{Cos}_R(H_I^d(M)) \subseteq \operatorname{Var}(\operatorname{Ann}_R H_I^d(M))$.

Proof. Let $\operatorname{Ass}_R(I, M)$ and N be defined as in Notations 1.1. Let $\mathfrak{p} \in \operatorname{Cos}_R(H_I^d(M))$. Then we have $H_{\mathfrak{p}R_{\mathfrak{p}}}^{d\text{-}\dim(R/\mathfrak{p})}(M/N)_{\mathfrak{p}} \neq 0$. Therefore there exists $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Att}_{R_{\mathfrak{p}}}(H_{\mathfrak{p}R_{\mathfrak{p}}}^{d\text{-}\dim(R/\mathfrak{p})}(M/N)_{\mathfrak{p}})$ by Lemma 2.2. It follows by the weak general shifted localization principle [BS, 11.3.8] that $\mathfrak{q} \in \operatorname{Att}_R H_{\mathfrak{m}}^d(M/N)$. Hence $\mathfrak{q} \in \operatorname{Ass}_R(M/N)$ by Lemmas 2.3 and 2.4. Since $\operatorname{Ass}_R(M/N) = \operatorname{Ass}_R(I, M)$, we have $\mathfrak{q} \in \operatorname{Ass}_R(I, M)$. Hence $\mathfrak{q} \in \operatorname{Att}_R H_I^d(M)$ by Corollary 2.8(i) and hence $\mathfrak{q} \supseteq \operatorname{Ann}_R H_I^d(M)$ by Lemma 2.2. Therefore $\mathfrak{p} \in \operatorname{Var}(\operatorname{Ann}_R H_I^d(M))$. □

Let $i \geqslant 0$ be an integer. Recall that the i-th *pseudo support* of M, denoted by $\operatorname{Psupp}^i(M)$, is defined by

$$\operatorname{Psupp}_R^i(M) = \left\{\mathfrak{p} \in \operatorname{Spec}(R) \mid H_{\mathfrak{p}R_{\mathfrak{p}}}^{i\text{-}\dim(R/\mathfrak{p})}(M)_{\mathfrak{p}} \neq 0\right\},$$

cf. [BS1]. The following lemma follows easily by Definition 3.4.

Lemma 3.6. *Let N be defined as in Notations* 1.1. *Let $U_M(0)$ be the largest submodule of M of dimension less than d. Then we have*:

(i) $\operatorname{Cos}_R(H_I^d(M)) = \operatorname{Psupp}_R^d(M/N)$.
(ii) $\operatorname{Cos}_R(H_{\mathfrak{m}}^d(M)) = \operatorname{Psupp}_R^d(M/U_M(0)) = \operatorname{Psupp}_R^d(M)$.

The following theorem, which is the second main result of this paper, characterizes the property $(*)$ of $H_I^d(M)$ in term of the co-support.

Theorem 3.7. *The following statements are equivalent*:

(i) $H_I^d(M)$ *satisfies the property* $(*)$.
(ii) $\operatorname{Cos}_R(H_I^d(M)) = \operatorname{Var}(\operatorname{Ann}_R H_I^d(M))$.

Proof. (i) ⇒ (ii). Let $\operatorname{Ass}_R(I, M)$ and N be defined as in Notations 1.1. Since $H_I^d(M)$ satisfies the property $(*)$, we have $H_I^d(M) \cong H_{\mathfrak{m}}^d(M/N)$ by Theorem 2.7. It follows that $H_{\mathfrak{m}}^d(M/N)$ satisfies the property $(*)$. So, we have by [NA1, Theorem 3.1] and Lemma 3.6 that

$$\operatorname{Var}\bigl(\operatorname{Ann}_R H_{\mathfrak{m}}^d(M/N)\bigr) = \operatorname{Psupp}_R^d(M/N) = \operatorname{Cos}_R\bigl(H_I^d(M)\bigr).$$

Hence $\operatorname{Var}(\operatorname{Ann}_R H_I^d(M)) = \operatorname{Var}(\operatorname{Ann}_R H_{\mathfrak{m}}^d(M/N)) = \operatorname{Cos}_R(H_I^d(M))$.

(ii) ⇒ (i). Let $\mathfrak{q} \supseteq \operatorname{Ann}_R(H_I^d(M))$. Then $\mathfrak{q} \in \operatorname{Cos}_R(H_I^d(M))$ by the assumption (ii), and hence $H_{\mathfrak{q}R_{\mathfrak{q}}}^{d\text{-}\dim(R/\mathfrak{q})}(M/N)_{\mathfrak{q}} \neq 0$. Let $Q \in \operatorname{Ass}(\widehat{R}/\mathfrak{q}\widehat{R})$ such that $\dim(\widehat{R}/Q) = \dim(R/\mathfrak{q})$. Then $Q \cap R = \mathfrak{q}$ and Q is a minimal prime ideal of $\mathfrak{q}\widehat{R}$. Since the induced map $R_{\mathfrak{q}} \to \widehat{R}_Q$ is faithfully flat, we have by the Flat Base Change Theorem [BS, 4.3.2] that

$$H_{Q\widehat{R}_Q}^{d\text{-}\dim(\widehat{R}/Q)}\bigl(\widehat{M/N}\bigr)_Q \cong H_{\mathfrak{q}R_{\mathfrak{q}}}^{d\text{-}\dim(R/\mathfrak{q})}(M/N)_{\mathfrak{q}} \otimes \widehat{R}_Q \neq 0. \tag{1}$$

Let $0 = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M} N(\mathfrak{p})$ be a reduced primary decomposition of 0. Then $N = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R(I,M)} N(\mathfrak{p})$. For each $\mathfrak{p} \in \operatorname{Ass}_R M$, we get by [Mat, Theorem 23.2(ii)] that $\operatorname{Ass}_{\widehat{R}}(\widehat{M}/\widehat{N(\mathfrak{p})}) = \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$. Therefore $\widehat{N(\mathfrak{p})}$

has a reduced primary decomposition $\widehat{N(\mathfrak{p})} = \bigcap_{P \in \mathrm{Ass}(\widehat{R}/\mathfrak{p}\widehat{R})} K(\mathfrak{p}, P)$, where $K(\mathfrak{p}, P)$ is P-primary. Since $R \to \widehat{R}$ is faithfully flat, it follows that $\widehat{N} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R(I,M)} \widehat{N(\mathfrak{p})}$ and $0 = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M} \widehat{N(\mathfrak{p})}$. Therefore we can check that

$$\widehat{N} = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Ass}_R(I,M) \\ P \in \mathrm{Ass}(\widehat{R}/\mathfrak{p}\widehat{R})}} K(\mathfrak{p}, P) \quad \text{and} \quad 0 = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Ass}_R M \\ P \in \mathrm{Ass}(\widehat{R}/\mathfrak{p}\widehat{R})}} K(\mathfrak{p}, P)$$

are reduced primary decompositions respectively of $\widehat{N}$ and 0. Let K_1 be the intersection of all primary components $K(\mathfrak{p}, P)$ where $\mathfrak{p} \in \mathrm{Ass}_R(I, M)$ and $P \in \mathrm{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ such that $\dim(\widehat{R}/P) = d$. It is clear that $K_1 \supseteq \widehat{N}$ and $\dim_{\widehat{R}} K_1 < d$. Therefore

$$\dim(K_1/\widehat{N})_Q < d\text{-}\dim(\widehat{R}/Q).$$

So, from the exact sequence $0 \to K_1/\widehat{N} \to \widehat{M}/\widehat{N} \to \widehat{M}/K_1 \to 0$, we have an isomorphism

$$H^{d\text{-}\dim(\widehat{R}/Q)}_{Q\widehat{R}_Q}(\widehat{M/N})_Q \cong H^{d\text{-}\dim(\widehat{R}/Q)}_{Q\widehat{R}_Q}(\widehat{M}/K_1)_Q.$$

Therefore we get by (1) that

$$H^{d\text{-}\dim(\widehat{R}/Q)}_{Q\widehat{R}_Q}(\widehat{M}/K_1)_Q \neq 0. \tag{2}$$

As in Notations 1.1, set

$$\mathrm{Ass}_{\widehat{R}}(I\widehat{R}, \widehat{M}) = \big\{P \in \mathrm{Ass}_{\widehat{R}} \widehat{M} \bigm| \dim(\widehat{R}/P) = d,\ \sqrt{I\widehat{R} + P} = \widehat{\mathfrak{m}}\big\}.$$

Then we have by [Mat, Theorem 23.2(ii)] that

$$\mathrm{Ass}_{\widehat{R}}(I\widehat{R}, \widehat{M}) = \big\{P \in \mathrm{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R}) \bigm| \mathfrak{p} \in \mathrm{Ass}_R M,\ \dim(\widehat{R}/P) = d,\ \sqrt{P + I\widehat{R}} = \widehat{\mathfrak{m}}\big\}.$$

Set $K_2 = \bigcap_{P \in \mathrm{Ass}_{\widehat{R}}(I\widehat{R},\widehat{M})} K(\mathfrak{p}, P)$. Since $\sqrt{I\widehat{R} + P} = \widehat{\mathfrak{m}}$ for all $P \in \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(I,M)} \mathrm{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$, it follows that

$$\mathrm{Ass}_{\widehat{R}}(I\widehat{R}, \widehat{M}) \supseteq \big\{P \in \mathrm{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R}) \bigm| \mathfrak{p} \in \mathrm{Ass}_R(I, M),\ \dim(\widehat{R}/P) = d\big\}.$$

Therefore $K_2 \subseteq K_1$. Since $\dim K_1 < d$, we have $\dim(K_1/K_2)_Q < d\text{-}\dim(\widehat{R}/Q)$. So, from the exact sequence

$$0 \to K_1/K_2 \to \widehat{M}/K_2 \to \widehat{M}/K_1 \to 0$$

we get an isomorphism

$$H^{d\text{-}\dim(\widehat{R}/Q)}_{Q\widehat{R}_Q}(\widehat{M}/K_2)_Q \cong H^{d\text{-}\dim(\widehat{R}/Q)}_{Q\widehat{R}_Q}(\widehat{M}/K_1)_Q.$$

Therefore we have by (2) that $H^{d\text{-}\dim(\widehat{R}/Q)}_{Q\widehat{R}_Q}(\widehat{M}/K_2)_Q \neq 0$. It means that $Q \in \mathrm{Cos}_{\widehat{R}}(H^d_{I\widehat{R}}(\widehat{M}))$. As $H^d_{I\widehat{R}}(\widehat{M})$ satisfies the property $(*)$, we have $\mathrm{Cos}_{\widehat{R}}(H^d_{I\widehat{R}}(\widehat{M})) = \mathrm{Var}(\mathrm{Ann}_{\widehat{R}}(H^d_{I\widehat{R}}(\widehat{M})))$ by the proof of (i) $\Rightarrow$ (ii).

Hence $Q \supseteq \mathrm{Ann}_{\widehat{R}}(H^d_{I\widehat{R}}(\widehat{M}))$. Since $H^d_{I\widehat{R}}(\widehat{M})$ satisfies the property $(*)$ and $H^d_I(M) \cong H^d_{I\widehat{R}}(\widehat{M})$ as $\widehat{R}$-modules, we have

$$\mathrm{Ann}_{\widehat{R}}(0 :_{H^d_I(M)} Q) = \mathrm{Ann}_{\widehat{R}}(0 :_{H^d_{I\widehat{R}}(\widehat{M})} Q) = Q.$$

So, we have

$$\mathfrak{q} \subseteq \mathrm{Ann}_R(0 :_{H^d_I(M)} \mathfrak{q}) \subseteq \mathrm{Ann}_{\widehat{R}}(0 :_{H^d_I(M)} Q) \cap R = Q \cap R = \mathfrak{q}.$$

It follows that $\mathrm{Ann}_R(0 :_{H^d_I(M)} \mathfrak{q}) = \mathfrak{q}$. Thus $H^d_I(M)$ satisfies the property $(*)$. $\square$

Remark 3.8. Theorem 3.7 asserts that if $H^d_I(M)$ satisfies the property $(*)$ then its co-support is a closed subset of $\mathrm{Spec}(R)$ in the Zariski topology. Note that $\mathrm{Cos}_R H^d_I(M)$ may not closed even when $I = \mathfrak{m}$, cf. [BS1, Example 3.2]. In general, if $R/\mathrm{Ann}_R H^d_{\mathfrak{m}}(M)$ is not catenary then $\mathrm{Cos}_R H^d_{\mathfrak{m}}(M)$ is not closed, cf. [NA1, Corollary 3.4].

Even R is a quotient of a regular local ring and $\mathrm{Cos}_R(H^d_I(M))$ is closed, $H^d_I(M)$ may not satisfy the property $(*)$. Here is an example.

Example 3.9. Let K be a field of characteristic 0. Let $S = K[X_1, X_2, X_3]$ denote the ring of polynomials over K. Set $\mathfrak{n} = (X_1, X_2, X_3)$, $\mathfrak{a} = (X_2^2 - X_1^2 - X_1^3)$, $\mathfrak{b} = (X_2)$ and $\mathfrak{c} = \mathfrak{a} \cap \mathfrak{b}$. Let x_i denote the image of X_i in $S/\mathfrak{c}$. Let $R = (S/\mathfrak{c})_{\mathfrak{n}/\mathfrak{c}}$, $\mathfrak{m} = (x_1, x_2, x_3)R$ and

$$I = (x_1 + x_2 - x_2x_3)R + \big((x_3 - 1)^2(x_1 + 1) - 1\big)R.$$

Then $(R, \mathfrak{m})$ is a Noetherian local ring with $\dim R = 2$ and

(i) $\mathrm{Att}_R H^2_I(R) = \{\mathfrak{a}R, \mathfrak{b}R\}$;
(ii) $\mathrm{Var}(\mathrm{Ann}_R H^2_I(R)) = \mathrm{Spec}(R)$ and $\mathrm{Cos}_R(H^2_I(R)) = \mathrm{Var}(\mathfrak{b}R)$;
(iii) $H^2_I(R)$ does not satisfy the property $(*)$.

Proof. Note that $R/\mathfrak{a}R$ is a domain, cf. [BS1, 8.2.9]. It is clear that $R/\mathfrak{b}R$ is a domain. Therefore $\mathrm{Ass}(R) = \{\mathfrak{a}R, \mathfrak{b}R\}$. So, $\dim R = 2$.

(i) From the exact sequence $0 \to R \to R/\mathfrak{a}R \oplus R/\mathfrak{b}R \to R/(\mathfrak{a}R + \mathfrak{b}R) \to 0$ with notice that $\dim R/(\mathfrak{a}R + \mathfrak{b}R) = 1$, we have an exact sequence

$$H^1_I\big(R/(\mathfrak{a}R + \mathfrak{b}R)\big) \to H^2_I(R) \to H^2_I(R/\mathfrak{a}R) \oplus H^2_I(R/\mathfrak{b}R) \to 0.$$

Therefore it follows by Lemma 2.2(iii) that

$$\mathrm{Att}_R H^2_I(R) = \mathrm{Att}_R H^2_I(R/\mathfrak{a}R) \cup \mathrm{Att}_R H^2_I(R/\mathfrak{b}R).$$

Since $H^2_I(R/\mathfrak{a}R) \neq 0$ by [BS1, 8.2.9], we have

$$\emptyset \neq \mathrm{Att}_R H^2_I(R/\mathfrak{a}R) \subseteq \mathrm{Ass}_R(R/\mathfrak{a}R) = \{\mathfrak{a}R\}.$$

So, $\mathrm{Att}_R H^2_I(R/\mathfrak{a}R) = \{\mathfrak{a}R\}$. Because $I + \mathfrak{b}R$ is $\mathfrak{m}$-primary, $H^2_I(R/\mathfrak{b}R) \cong H^2_{\mathfrak{m}}(R/\mathfrak{b}R)$. Hence $\mathrm{Att}_R H^2_I(R/\mathfrak{b}R) = \{\mathfrak{b}R\}$. Therefore $\mathrm{Att}_R H^2_I(R) = \{\mathfrak{a}R, \mathfrak{b}R\}$.

(ii) Since $\mathrm{Att}_R H_I^2(R) = \{\mathfrak{a}R, \mathfrak{b}R\} = \mathrm{Ass}(R)$, we have $\mathrm{Var}(\mathrm{Ann}_R H_I^2(R)) = \mathrm{Spec}(R)$ by Lemma 2.2. Note that $0 = \mathfrak{a}R \cap \mathfrak{b}R$ is a reduced primary decomposition of the ideal 0 of R, $\dim(R/(I + \mathfrak{b}R)) = 0$ and $\dim(R/(I + \mathfrak{a}R)) = 1$ by [BS1, 8.2.9]. Therefore,

$$\mathrm{Cos}_R\big(H_I^2(R)\big) = \big\{\mathfrak{p} \in \mathrm{Spec}(R) \bigm| H_{\mathfrak{p}R_\mathfrak{p}}^{2\text{-}\dim(R/\mathfrak{p})}(R/\mathfrak{b}R) \neq 0\big\} = \mathrm{Psupp}_R^2(R/\mathfrak{b}R).$$

As R is catenary, $H_\mathfrak{m}^2(R/\mathfrak{b}R)$ satisfies the property $(*)$ by Lemma 2.5. So,

$$\mathrm{Psupp}_R^2(R/\mathfrak{b}R) = \mathrm{Var}\big(\mathrm{Ann}_R H_\mathfrak{m}^2(R/\mathfrak{b}R)\big) = \mathrm{Var}(\mathfrak{b}R)$$

by [NA1, Theorem 3.1]. Thus, $\mathrm{Cos}_R(H_I^2(R)) = \mathrm{Var}(\mathfrak{b}R)$.

(iii) Since $\mathrm{Cos}_R(H_I^2(R)) \neq \mathrm{Var}(\mathrm{Ann}_R H_I^2(R))$, it follows by Theorem 3.7 that $H_I^2(R)$ does not satisfy the property $(*)$. $\square$

R.N. Roberts [R] introduced the concept of Krull dimension for Artinian modules. D. Kirby [K] changed the terminology of Roberts and referred to Noetherian dimension to avoid confusion with Krull dimension defined for finitely generated modules. The Noetherian dimension of an Artinian R-module A is denoted by $\text{N-dim}_R(A)$. Note that if $\mathfrak{q}$ is an ideal of R such that $\ell(0 :_A \mathfrak{q}) < \infty$ then $\ell(0 :_A \mathfrak{q}^n)$ is a polynomial with rational coefficients for $n \gg 0$, cf. [K, Proposition 2] and

$$\text{N-dim}_R(A) = \deg\big(\ell\big(0 :_A \mathfrak{q}^n\big)\big) = \inf\big\{t \bigm| \exists x_1, \ldots, x_t \in \mathfrak{m}\colon\ \ell\big(0 :_A (x_1, \ldots, x_t)R\big) < \infty\big\},$$

cf. [R, Theorem 6]. Assume that $\text{N-dim}_R(A) = t$. Let a_t be the leading coefficient of the polynomial $\ell(0 :_A \mathfrak{q}^n)$ for $n \gg 0$. Following Brodmann and Sharp [BS1], the *multiplicity* of A with respect to $\mathfrak{q}$, denoted by $e'(\mathfrak{q}, A)$, is defined by the formula $e'(\mathfrak{q}, A) := a_t t!$.

As a consequence, we have the following associativity formula for the multiplicity of $H_I^d(M)$ when $H_I^d(M)$ satisfies the property $(*)$.

Corollary 3.10. *Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. Let $\mathrm{Ass}_R(I, M)$ and N be defined as in Notations* 1.1. *If $H_I^d(M)$ satisfies the property $(*)$ then*

$$e'\big(\mathfrak{q}, H_I^d(M)\big) = \sum_{\substack{\mathfrak{p} \in \mathrm{Cos}\, H_I^d(M) \\ \dim(R/\mathfrak{p}) = d}} \ell_{R_\mathfrak{p}}\big(H_{\mathfrak{p}R_\mathfrak{p}}^0(M/N)_\mathfrak{p}\big) e(\mathfrak{q}, R/\mathfrak{p}).$$

In this case, $e'(\mathfrak{q}, H_I^d(M)) = e(\mathfrak{q}, M/N) = \sum_{\mathfrak{p} \in \mathrm{Ass}_R(I,M)} \ell_{R_\mathfrak{p}}(M_\mathfrak{p}) e(\mathfrak{q}, R/\mathfrak{p})$.

Proof. Since $H_I^d(M)$ satisfies the property $(*)$, $H_I^d(M) \cong H_\mathfrak{m}^d(M/N)$ by Theorem 2.7. So, $H_\mathfrak{m}^d(M/N)$ satisfies the property $(*)$. Note that $\mathrm{Cos}_R(H_I^d(M)) = \mathrm{Psupp}_R^d(M/N)$ by Lemma 3.6. Therefore we get by [NA1, Corollary 3.4] that

$$e'\big(\mathfrak{q}, H_I^d(M)\big) = e'\big(\mathfrak{q}, H_\mathfrak{m}^d(M/N)\big) = \sum_{\substack{\mathfrak{p} \in \mathrm{Cos}\, H_I^d(M) \\ \dim(R/\mathfrak{p}) = d}} \ell_{R_\mathfrak{p}}\big(H_{\mathfrak{p}R_\mathfrak{p}}^0(M/N)_\mathfrak{p}\big) e(\mathfrak{q}, R/\mathfrak{p}).$$

Let $\mathfrak{p} \in \mathrm{Cos}(H_I^d(M))$ such that $\dim(R/\mathfrak{p}) = d$. Then $\mathfrak{p} \in \min \mathrm{Var}(\mathrm{Ann}_R H_I^d(M))$ by Lemma 3.5. Hence $\mathfrak{p} \in \mathrm{Att}_R H_I^d(M)$ by Lemma 2.2, and hence $\mathfrak{p} \in \mathrm{Ass}_R(I, M)$ by Corollary 3.1. Therefore $\mathfrak{p} \notin \mathrm{Ass}_R(N)$. Since $\dim(R/\mathfrak{p}) = d$, we have $\mathfrak{p} \notin \mathrm{Supp}_R N$. So, $M_\mathfrak{p} \cong (M/N)_\mathfrak{p}$. Note that $\ell(M_\mathfrak{p}) < \infty$. Hence $H_{\mathfrak{p}R_\mathfrak{p}}^0(M_\mathfrak{p}) = M_\mathfrak{p}$. Combining these facts with notice that

$$\mathrm{Ass}_R(M/N) = \mathrm{Ass}_R(I, M) = \{\mathfrak{p} \in \mathrm{Cos}(H_I^d(M)) \mid \dim(R/\mathfrak{p}) = d\},$$

the rest assertion follows by the associativity formula for the multiplicity $e(\mathfrak{q}, M/N)$ of M/N with respect to $\mathfrak{q}$. □

References

[BS] M. Brodmann, R.Y. Sharp, Local Cohomology: an Algebraic Introduction with Geometric Applications, Cambridge University Press, 1998.
[BS1] M. Brodmann, R.Y. Sharp, On the dimension and multiplicity of local cohomology modules, Nagoya Math. J. 167 (2002) 217–233.
[CDN] N.T. Cuong, N.T. Dung, L.T. Nhan, Top local cohomology and the catenaricity of the unmixed support of a finitely generated module, Comm. Algebra 35 (2007) 1691–1701.
[CN] N.T. Cuong, L.T. Nhan, On Noetherian dimension of Artinian modules, Vietnam J. Math. 30 (2002) 121–130.
[DM] D. Delfino, T. Marley, Cofinite modules and local cohomology, J. Pure Appl. Algebra 121 (1997) 45–52.
[DSc] K. Divaani-Aazar, P. Schenzel, Ideal topology, local cohomology and connectedness, Math. Proc. Cambridge Philos. Soc. 131 (2001) 211–226.
[FR] D. Ferrand, M. Raynaud, Fibres formelles d'un anneau local Noetherian, Ann. Sci. Ecole Norm. Sup. 3 (1970) 295–311.
[K] D. Kirby, Dimension and length of Artinian modules, Quart. J. Math. Oxford. 41 (1990) 419–429.
[Mac] I.G. Macdonald, Secondary representation of modules over a commutative ring, Symp. Math. 11 (1973) 23–43.
[Mat] H. Matsumura, Commutative Ring Theory, Cambridge University Press, 1986.
[Na] M. Nagata, Local Rings, Interscience, New York, 1962.
[NA1] L.T. Nhan, T.N. An, On the unmixedness and the universal catenaricity of local rings and local cohomology modules, J. Algebra 321 (2009) 303–311.
[NA2] L.T. Nhan, T.N. An, On the catenaricity of Noetherian local rings and quasi unmixed Artinian modules, Comm. Algebra 38 (2010) 3728–3736.
[NC] L.T. Nhan, T.D.M. Chau, On the Noetherian dimension and co-localization of Artinian modules over local rings, preprint.
[R] R.N. Roberts, Krull dimension for Artinian modules over quasi local commutative rings, Quart. J. Math. Oxford 26 (1975) 269–273.
[Sm] K.E. Smith, Test ideals in local rings, Trans. Amer. Math. Soc. 347 (1995) 3453–3472.
[Zos] H. Zöschinger, Über die Bedingung Going up für $R \subset \widehat{R}$, Arch. Math. 95 (2010) 225–231.