More Bounds for Eigenvalues Using Traces*

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ABSTRACT

Let the $n\times n$ complex matrix $\mathbf{A}$ have complex eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$. Upper and lower bounds for $\Sigma(\operatorname{Re}\lambda_i)^2$ and $\Sigma(\operatorname{Im}\lambda_i)^2$ are obtained, extending similar bounds for $\Sigma|\lambda_i|^2$ obtained by Eberlein (1965), Henrici (1962), and Kress, de Vries, and Wegmann (1974). These bounds involve the traces of $\mathbf{A}^*\mathbf{A}$, $\mathbf{B}^2$, $\mathbf{C}^2$, and $\mathbf{D}^2$, where $\mathbf{B}=\frac{1}{2}(\mathbf{A}+\mathbf{A}^*)$, $\mathbf{C}=\frac{1}{2}(\mathbf{A}-\mathbf{A}^*)/i$, and $\mathbf{D}=\mathbf{A}\mathbf{A}^*-\mathbf{A}^*\mathbf{A}$, and strengthen some of the results in our earlier paper "Bounds for eigenvalues using traces" in *Linear Algebra and Appl.* [12].

1. INTRODUCTION

Let $\mathbf{A}=(a_{ij})$ be an $n\times n$ (nonzero) complex matrix with conjugate transpose $\mathbf{A}^*$, and let $\lambda_1,\lambda_2,\dots,\lambda_n$ be the eigenvalues of $\mathbf{A}$. Then

$$\sum_i|\lambda_i|^2\leq\|\mathbf{A}\|^2=\sum_{i,j}|a_{ij}|^2=\operatorname{tr}\mathbf{A}^*\mathbf{A} \tag{1.1}$$

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(cf. Schur [10]), where $\|\mathbf{A}\|$ is the Euclidean norm of $\mathbf{A}$. Let

$$\mathbf{B}=\tfrac{1}{2}(\mathbf{A}+\mathbf{A}^*),$$
$$\mathbf{C}=\frac{1}{2i}(\mathbf{A}-\mathbf{A}^*).$$

We call $\mathbf{B}$ the *Hermitian real part* and $\mathbf{C}$ the *Hermitian imaginary part* of $\mathbf{A}$. Then (cf. [7, p. 309])

$$\sum_i (\operatorname{Re}\lambda_i)^2 \leqslant \|\mathbf{B}\|^2=\sum_{i,j}\left|\tfrac{1}{2}(a_{ij}+\bar{a}_{ji})\right|^2=\operatorname{tr}\mathbf{B}^2, \tag{1.2}$$

$$\sum_i (\operatorname{Im}\lambda_i)^2 \leqslant \|\mathbf{C}\|^2=\sum_{i,j}\left|\tfrac{1}{2}(a_{ij}-\bar{a}_{ji})\right|^2=\operatorname{tr}\mathbf{C}^2. \tag{1.3}$$

Equality in any one of the three inequalities (1.1), (1.2), and (1.3) implies equality in all three and occurs if any only if $\mathbf{A}$ is normal, i.e., $\mathbf{AA}^*=\mathbf{A}^*\mathbf{A}$.

In [12] the above inequalities were used to deduce bounds for the eigenvalues of $\mathbf{A}$ of the following type:

$$\frac{|\operatorname{tr}\mathbf{A}|}{n}-s_A\left(\frac{k-1}{n-k+1}\right)^{1/2}\leqslant|\lambda_k|\leqslant\left(\frac{\operatorname{tr}\mathbf{A}^*\mathbf{A}}{n}\right)^{1/2}+s_A\left(\frac{n}{k}-1\right)^{1/2},$$

$$\frac{\operatorname{Re}\operatorname{tr}\mathbf{A}}{n}-s_B\left(\frac{k-1}{n-k+1}\right)^{1/2}\leqslant\mu_k\leqslant\frac{\operatorname{Re}\operatorname{tr}\mathbf{A}}{n}+s_B\left(\frac{n}{k}-1\right)^{1/2},$$

$$\frac{\operatorname{Im}\operatorname{tr}\mathbf{A}}{n}-s_C\left(\frac{k-1}{n-k+1}\right)^{1/2}\leqslant\nu_k\leqslant\frac{\operatorname{Im}\operatorname{tr}\mathbf{A}}{n}+s_C\left(\frac{n}{k}-1\right)^{1/2},$$

where $|\lambda_k|$, μ_k, and ν_k are the kth ordered moduli, real parts, and imaginary parts of the eigenvalues of $\mathbf{A}$ respectively, while

$$s_T^2=\left\{\frac{\operatorname{tr}\mathbf{T}^*\mathbf{T}}{n}-\frac{|\operatorname{tr}\mathbf{T}|^2}{n^2}\right\},\qquad \mathbf{T}=\mathbf{A},\mathbf{B},\mathbf{C}.$$

In [1], Eberlein showed that

$$\sum_i |\lambda_i|^2\leqslant\|\mathbf{A}\|^2-\frac{1}{6}\frac{\|\mathbf{D}\|^2}{\|\mathbf{A}\|^2}, \tag{1.4}$$

where

$$\mathbf{D}=\mathbf{AA}^*-\mathbf{A}^*\mathbf{A}.$$

This inequality was strengthened by Kress, de Vries, and Wegmann [4]:

$$\sum_i |\lambda_i|^2 \leqslant \left(\|\mathbf{A}\|^4 - \tfrac{1}{2}\|\mathbf{D}\|^2\right)^{1/2}. \tag{1.5}$$

A corresponding lower bound given in [2] is

$$\|\mathbf{A}\|^2 - \left(\frac{n^3-n}{12}\right)^{1/2}\|\mathbf{D}\| \leqslant \sum_i |\lambda_i|^2. \tag{1.6}$$

The purpose of this paper is to first deduce additional inequalities for $\Sigma_i(\operatorname{Re}\lambda_i)^2$ and $\Sigma_i(\operatorname{Im}\lambda_i)^2$, which improve (1.2) and (1.3), and then use these inequalities to improve the bounds given in [12].

Section 2 presents several preliminary inequalities for real eigenvalues as well as the upper and lower bounds for $\Sigma_i|\lambda_i|^2$, $\Sigma_i(\operatorname{Re}\lambda_i)^2$, and $\Sigma_i(\operatorname{Im}\lambda_i)^2$. The eigenvalue bounds are given in Theorems 3.1 and 3.2 in Sec. 3. We conclude with several examples in Sec. 4.

2. PRELIMINARIES

Our bounds will be deduced from the following bounds for real eigenvalues, presented in [12]. (For related results see, e.g., [5], [8], [11].)

LEMMA 2.1. *Let* **A** *be an* $n\times n$ *complex matrix with real ordered eigenvalues*

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n.$$

Let

$$m = \sum_i \frac{\lambda_i}{n} = \frac{\operatorname{tr}\mathbf{A}}{n}$$

and

$$s^2 = \frac{\sum_i \lambda_i^2}{n} - \left[\frac{\sum_i \lambda_i}{n}\right]^2 = \frac{\operatorname{tr}\mathbf{A}^2}{n} - \left(\frac{\operatorname{tr}\mathbf{A}}{n}\right)^2$$

be their mean and variance respectively. Then for $1 \leqslant j \leqslant k \leqslant n$,

$$m - s\left(\frac{j-1}{n-j+1}\right)^{1/2} \leqslant \frac{1}{k-j+1}\sum_{i=j}^{k}\lambda_i \leqslant m + s\left(\frac{n-k}{k}\right)^{1/2}, \tag{2.1}$$

$$m + (n-k)r^{-1}(n-1)^{-1/2}s \leqslant \frac{1}{k}\sum_{i=1}^{k}\lambda_i, \tag{2.2a}$$

where $r = \max(k, n-k)$,

$$\frac{1}{n-k+1}\sum_{i=k}^{n}\lambda_i \leqslant m-(k-1)r^{-1}(n-1)^{-1/2}s, \tag{2.2b}$$

where $r = \max(n-k+1, k-1)$,

$$(\lambda_j - \lambda_k) \leqslant sn^{1/2}\left(\frac{1}{j} + \frac{1}{n-k+1}\right)^{1/2}, \tag{2.3}$$

$$2s \leqslant \lambda_1 - \lambda_n, \tag{2.4}$$

$$\frac{2sn}{(n^2-1)^{1/2}} \leqslant \lambda_1 - \lambda_n \qquad \textit{if } n \textit{ is odd}. \tag{2.5}$$

Now suppose that $\mathbf{A}$ is an $n\times n$ complex matrix with (complex) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define

$$\sum_i |\lambda_i^A|^2 = \sum_i |\lambda_i|^2, \quad \sum_i |\lambda_i^B|^2 = \sum_i (\mathrm{Re}\lambda_i)^2, \quad \sum_i |\lambda_i^C|^2 = \sum_i (\mathrm{Im}\lambda_i)^2,$$

$$K_A^u = \left(\|\mathbf{A}\|^4 - \tfrac{1}{2}\|\mathbf{D}\|^2\right)^{1/2}, \tag{2.6}$$

$$K_A^l = \|\mathbf{A}\|^2 - \left(\frac{n^3-n}{12}\right)^{1/2}\|\mathbf{D}\|,$$

$$K_B^u = \begin{cases} \left(\|\mathbf{B}\|^4 - \frac{1}{8}\|\mathbf{D}\|^2\right)^{1/2} & \text{if} \quad \|\mathbf{B}\| \geqslant \|\mathbf{C}\|, \\ \|\mathbf{B}\|^2 - \dfrac{1}{12}\dfrac{\|\mathbf{D}\|^2}{\|\mathbf{A}\|^2} & \text{otherwise}, \end{cases}$$

$$K_B^l = \|\mathbf{B}\|^2 - \left(\frac{n^3-n}{48}\right)^{1/2}\|\mathbf{D}\|,$$

$$K_C^u = \begin{cases} \left(\|\mathbf{C}\|^4 - \frac{1}{8}\|\mathbf{D}\|^2\right)^{1/2} & \text{if} \quad \|\mathbf{C}\| \geqslant \|\mathbf{B}\| \\ \|\mathbf{C}\|^2 - \dfrac{1}{12}\dfrac{\|\mathbf{D}\|^2}{\|\mathbf{A}\|^2}, & \text{otherwise}, \end{cases}$$

$$K_C^l = \|\mathbf{C}\|^2 - \left(\frac{n^3-n}{48}\right)^{1/2}\|\mathbf{D}\|.$$

LEMMA 2.2. *If* **A** *is an* $n\times n$ *complex matrix, then for* $T=A,B,C$,

$$K_T^l \leqslant \sum_i |\lambda_i^T|^2 \leqslant K_T^u. \tag{2.7}$$

Proof. If $T=A$, then (2.7) is just (1.5) and (1.6). Now, if $\mathbf{R}=\mathbf{\Lambda}+\mathbf{M}$ is a Schur triangular form of **A**, i.e., $\mathbf{A}=\mathbf{URU}^*$, **U** is unitary, $\mathbf{\Lambda}$ is diagonal and **M** is upper triangular, then

$$\begin{aligned}\|\mathbf{A}\|^2-\sum_i|\lambda_i|^2&=\|\mathbf{\Lambda}\|^2+\|\mathbf{M}\|^2-\sum_i|\lambda_i|^2\\&=\|\mathbf{M}\|^2.\end{aligned}$$

The left-hand side of (2.7) with $T=A$, and Eberlein's inequality (1.4), are therefore equivalent to

$$\frac{1}{6}\frac{\|\mathbf{D}\|^2}{\|\mathbf{A}\|^2}\leqslant\|\mathbf{M}\|^2\leqslant\left(\frac{n^3-n}{12}\right)^{1/2}\|\mathbf{D}\|. \tag{2.8}$$

Furthermore

$$\begin{aligned}\|\mathbf{A}\|^4-\Big(\sum_i|\lambda_i|^2\Big)^2&=\|\mathbf{\Lambda}\|^4+\|\mathbf{M}\|^4+2\|\mathbf{\Lambda}\|^2\|\mathbf{M}\|^2-\Big(\sum_i|\lambda_i|^2\Big)^2\\&=\|\mathbf{M}\|^4+2\|\mathbf{\Lambda}\|^2\|\mathbf{M}\|^2.\end{aligned}$$

The right-hand side of (2.7) with $T=A$ is now equivalent to

$$\|\mathbf{M}\|^4+2\|\mathbf{\Lambda}\|^2\|\mathbf{M}\|^2\geqslant\tfrac{1}{2}\|\mathbf{D}\|^2. \tag{2.9}$$

But

$$\|\mathbf{B}\|^2-\sum_i(\operatorname{Re}\lambda_i)^2=\left\|\tfrac{1}{2}(\mathbf{M}+\mathbf{M}^*)\right\|^2=\tfrac{1}{2}\|\mathbf{M}\|^2 \tag{2.10}$$

and

$$\|\mathbf{C}\|^2-\sum_i(\operatorname{Im}\lambda_i)^2=\left\|\frac{1}{2i}(\mathbf{M}-\mathbf{M}^*)\right\|^2=\tfrac{1}{2}\|\mathbf{M}\|^2. \tag{2.11}$$

Therefore, from (2.8), (2.10), and (2.11), we get that

$$\|\mathbf{T}\|^2-\left(\frac{n^3-n}{48}\right)^{1/2}\|\mathbf{D}\| \leqslant \sum_i |\lambda_i^T|^2 \leqslant \|\mathbf{T}\|^2-\frac{1}{12}\frac{\|\mathbf{D}\|^2}{\|\mathbf{A}\|^2}, \qquad \mathbf{T}=\mathbf{B},\mathbf{C}. \tag{2.12}$$

Similarly

$$\|\mathbf{B}\|^4-\left[\sum_i (\operatorname{Re}\lambda_i)^2\right]^2=\tfrac{1}{4}\left[\|\mathbf{M}\|^4+4\|\tfrac{1}{2}(\Lambda+\Lambda^*)\|^2\|\mathbf{M}\|^2\right]; \tag{2.13}$$

$$\|\mathbf{C}\|^4-\left[\sum_i (\operatorname{Im}\lambda_i)^2\right]^2=\frac{1}{4}\left[\|\mathbf{M}\|^4+4\left\|\frac{1}{2i}(\Lambda-\Lambda^*)\right\|^2\|\mathbf{M}\|^2\right]. \tag{2.14}$$

Now, since

$$\|\mathbf{B}\|^2=\|\tfrac{1}{2}(\Lambda+\Lambda^*)\|^2+\tfrac{1}{2}\|\mathbf{M}\|^2$$

$$=\sum_i (\operatorname{Re}\lambda_i)^2+\tfrac{1}{2}\|\mathbf{M}\|^2,$$

$$\|\mathbf{C}\|^2=\left\|\frac{1}{2i}(\Lambda-\Lambda^*)\right\|^2+\tfrac{1}{2}\|\mathbf{M}\|^2$$

$$=\sum_i (\operatorname{Im}\lambda_i)^2+\tfrac{1}{2}\|\mathbf{M}\|^2$$

and

$$\|\Lambda\|^2=\sum_i |\lambda_i|^2=\sum_i (\operatorname{Re}\lambda_i)^2+\sum_i (\operatorname{Im}\lambda_i)^2,$$

we see that when $\|\mathbf{B}\|^2 \geqslant \|\mathbf{C}\|^2$, then

$$2\|\tfrac{1}{2}(\Lambda+\Lambda^*)\|^2 \geqslant \|\Lambda\|^2. \tag{2.15}$$

Therefore, (2.9), (2.13), and (2.15) imply that

$$\sum_i |\lambda_i^B|^2 \leqslant \left(\|\mathbf{B}\|^4-\tfrac{1}{8}\|\mathbf{D}\|^2\right)^{1/2} \qquad \text{when} \quad \|\mathbf{B}\| \geqslant \|\mathbf{C}\|. \tag{2.16}$$

Similarly,

$$\sum_i |\lambda_i^C|^2 \leqslant \left(\|\mathbf{C}\|^4 - \tfrac{1}{8}\|\mathbf{D}\|^2\right)^{1/2} \qquad \text{when} \quad \|\mathbf{C}\| \geqslant \|\mathbf{B}\|. \tag{2.17}$$

The result now follows by combining (2.12), (2.16), and (2.17). ∎

3. BOUNDS FOR EIGENVALUES

We can now deduce the bounds for the eigenvalues of an arbitrary matrix. Let $\mathbf{A}$ be an $n\times n$ complex matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define

$$\lambda_j^A = |\lambda_i|,$$

$$\lambda_k^B = \operatorname{Re}\lambda_i,$$

$$\lambda_l^C = \operatorname{Im}\lambda_i,$$

so that the *ordered* vectors (λ_j^T) satisfy

$$\lambda_1^T \geqslant \lambda_2^T \geqslant \cdots \geqslant \lambda_n^T, \qquad T = A, B, C.$$

Further, define

$$s_T^2 = \frac{\sum_i |\lambda_i^T|^2}{n} - \frac{\left|\sum_i \lambda_i^T\right|^2}{n^2}, \qquad T = A, B, C,$$

$$m_A^u = \left(\frac{K_A^u}{n}\right)^{1/2}, \qquad m_A^l = \frac{|\operatorname{tr}\mathbf{A}|}{n},$$

$$m_B^u = m_B^l = \frac{\operatorname{tr}\mathbf{B}}{n}, \qquad m_C^u = m_C^l = \frac{\operatorname{tr}\mathbf{C}}{n},$$

$$(s_T^u)^2 = \frac{K_T^u - |\operatorname{tr}\mathbf{T}|^2/n}{n}, \qquad T = A, B, C,$$

$$(s_T^l)^2 = \max\left\{0, \frac{K_T^l - |\operatorname{tr}\mathbf{T}|^2/n}{n}\right\}, \qquad T = A, B, C,$$

where K_T^u, K_T^l are as in (2.6).

THEOREM 3.1. *Let* **A** *be an* $n \times n$ *complex matrix, and let* (λ_i^T), m_T^u, m_T^l, s_T^u, *and* s_T^l *be defined as above. Then for* $T = A, B, C$ *and* $1 \leqslant j \leqslant k \leqslant n$,

$$m_T^l - s_T^u\left(\frac{j-1}{n-j+1}\right)^{1/2} \leqslant \frac{1}{k-j+1}\sum_{i=j}^{k}\lambda_i^T \leqslant m_T^u + s_T^u\left(\frac{n-k}{k}\right)^{1/2}, \tag{3.1}$$

$$m_T^l + (n-k)r^{-1}(n-1)^{-1/2}s_T^l \leqslant \frac{1}{k}\sum_{i=1}^{k}\lambda_i^T, \tag{3.2a}$$

where $r = \max(k, n-k)$,

$$\frac{1}{n-k+1}\sum_{i=k}^{n}\lambda_i^T \leqslant m_T^u - (k-1)r^{-1}(n-1)^{-1/2}s_T^l, \tag{3.2b}$$

where $r = \max(n-k+1, k-1)$,

$$|\lambda_j^T - \lambda_k^T| \leqslant s_T^u n^{1/2}\left(\frac{1}{j} + \frac{1}{n-k+1}\right)^{1/2}, \tag{3.3}$$

$$2s_T^l \leqslant \lambda_1^T - \lambda_n^T, \tag{3.4}$$

$$2s_T^l n/(n^2-1)^{1/2} \leqslant \lambda_1^T - \lambda_n^T \qquad \text{if } n \text{ is odd.} \tag{3.5}$$

Proof. Note that

$$m_A^l = \frac{\left|\sum_i \lambda_i\right|}{n} \leqslant \frac{\sum_i \lambda_i^A}{n} = \frac{\sum_i |\lambda_i|}{n} \leqslant \left[\frac{\sum_i |\lambda_i|^2}{n}\right]^{1/2} \leqslant m_A^u,$$

by the triangle and Cauchy-Schwarz inequalities and Lemma 2.2. Furthermore,

$$s_T^l \leqslant s_T \leqslant s_T^u, \qquad T = A, B, C.$$

The inequalities (3.1) to (3.5) now follow upon substituting the vectors (λ_i^T), $T = A, B, C$, for the vector (λ_i) in Lemma 2.1. ■

When $\mathbf{A}$ is real, then we know that the complex eigenvalues of $\mathbf{A}$ occur in conjugate pairs. Moreover, when $\mathbf{A}$ is nonnegative, then the Perron-Frobenius theorem implies that the largest eigenvalue of $\mathbf{A}$, in modulus, is real and nonnegative. This extra information enables us to strengthen several of the bounds for the imaginary parts of the eigenvalues.

THEOREM 3.2. *Suppose that* $\mathbf{A}$ *is real and*

$$p=\begin{cases}\left[\dfrac{n-1}{2}\right] & \text{if } \mathbf{A} \text{ is nonnegative,}\\[2ex] \left[\dfrac{n}{2}\right] & \text{otherwise,}\end{cases}$$

where $[\cdot]$ *denotes integer part. Then for* $1\leqslant j\leqslant k\leqslant p$,

$$\frac{1}{k-j+1}\sum_{i=j}^{k}\lambda_i^c\leqslant\left(\frac{K_c^u}{2k}\right)^{1/2}, \tag{3.6}$$

$$\left\{\max\left(0,\frac{K_c^l}{2p^2}\right)\right\}^{1/2}\leqslant\frac{1}{k}\sum_{i=1}^{k}\lambda_i^c, \tag{3.7}$$

$$|\lambda_j^c-\lambda_k^c|\leqslant\left\{\frac{pK_c^u-K_c^l}{2p}\right\}^{1/2}\left(\frac{1}{j}+\frac{1}{p-k+1}\right)^{1/2}. \tag{3.8}$$

Proof. Since $\mathbf{A}$ is real, the eigenvalues of $\mathbf{A}$ occur in conjugate pairs. Furthermore, as mentioned above, if $\mathbf{A}$ is nonnegative, then the largest eigenvalue of $\mathbf{A}$ in modulus is real (and nonnegative). Therefore, there are at most $2p$ nonreal eigenvalues, and moreover, $\lambda_i^c=\lambda_{n-i+1}^c$ for $i=1,2,\dots,p$. This implies that

$$2\sum_{i=1}^{p}(\lambda_i^c)^2=\sum_{i=1}^{n}(\operatorname{Im}\lambda_i)^2.$$

From Lemma 2.2, we now conclude that

$$\frac{K_c^l}{2}\leqslant\sum_{i=1}^{p}(\lambda_i^c)^2\leqslant\frac{K_c^u}{2}. \tag{3.9}$$

First, let us prove (3.6):

$$\frac{1}{k-j+1}\sum_{i=j}^{k}\lambda_i^c \leqslant \frac{1}{k}\sum_{i=1}^{k}\lambda_i^c$$

$$\leqslant \frac{1}{k}\left(\sum_{i=1}^{p}(\lambda_i^c)^2\right)^{1/2}k^{1/2}, \qquad \text{by Cauchy-Schwarz}$$

$$\leqslant (K_c^u/2k)^{1/2}, \qquad \text{by (3.9).}$$

Next,

$$\left(\frac{K_c^l}{2p^2}\right)^{1/2} \leqslant \left(\frac{1}{p^2}\sum_{i=1}^{p}(\lambda_i^c)^2\right)^{1/2}, \qquad \text{by (3.9)}$$

$$\leqslant \frac{1}{p}\sum_{i=1}^{p}\lambda_i^c, \qquad \text{since } \lambda_i^c \geqslant 0,\ i=1,\dots,p$$

$$\leqslant \frac{1}{k}\sum_{i=1}^{k}\lambda_i^c.$$

This proves (3.7). To prove (3.8), substitute the p-vector (λ_i^c) for the vector (λ_i) in Lemma 2.1. This gives

$$|\lambda_j^c-\lambda_k^c| \leqslant \left\{\sum_{i=1}^{p}\frac{(\lambda_i^c)^2}{p}-\left(\sum_{i=1}^{p}\frac{\lambda_i^c}{p}\right)^2\right\}^{1/2} p^{1/2}\left(\frac{1}{j}+\frac{1}{p-k+1}\right)^{1/2}$$

$$\leqslant \left\{\frac{pK_c^u-K_c^l}{2p}\right\}^{1/2}\left(\frac{1}{j}+\frac{1}{p-k+1}\right)^{1/2}, \qquad \text{by (3.9).} \qquad \blacksquare$$

4. EXAMPLES

EXAMPLE 4.1. Marcus and Minc [6, p. 148] considered the matrix

$$\mathbf{A}=\begin{bmatrix} 7+3i & -4-6i & -4 \\ -1-6i & 7 & -2-6i \\ 2 & 4-6i & 13-3i \end{bmatrix}$$

and found, using results due to Hirsch (cf. [6, p. 140]) that

$$\left.\begin{aligned} |\lambda(A)| &\leqslant 40.03 \\ |\mathrm{Re}\,\lambda(A)| &\leqslant 39 \\ |\mathrm{Im}\,\lambda(A)| &\leqslant 20.12 \end{aligned}\right\},$$

while Gerŝgorin's discs give

$$\left.\begin{aligned} |z-7-3i| &\leqslant 11.21 \\ |z-7| &\leqslant 12.40 \\ |z-13+3i| &\leqslant 9.21 \end{aligned}\right\}.$$

In our earlier paper [12, Sec. 4], we obtained

$$\left.\begin{aligned} 9 \leqslant \lambda_1^A &\leqslant 25.46 \\ 2.64 \leqslant \lambda_2^A &\leqslant 19.09 \\ 0 \leqslant \lambda_3^A &\leqslant 12.73 \end{aligned}\right\},$$

$$\left.\begin{aligned} 9 \leqslant \lambda_1^B &\leqslant 14.20 \\ 6.40 \leqslant \lambda_2^B &\leqslant 11.60 \\ 3.81 \leqslant \lambda_3^B &\leqslant 9 \end{aligned}\right\},$$

and

$$\left.\begin{aligned} 0 \leqslant \lambda_1^C &\leqslant 11.62 \\ -5.81 \leqslant \lambda_2^C &\leqslant 5.81 \\ -11.62 \leqslant \lambda_3^C &\leqslant 0 \end{aligned}\right\}.$$

Let us now apply Theorem 3.1. First, we find that

$$\left.\begin{aligned} K_A^l &= 256.90, & K_A^u &= 472.31 \\ K_B^l &= 168.95, & K_B^u &= 277.65 \\ K_C^l &= 87.95, & K_C^u &= 198 \end{aligned}\right\},$$

$$\left.\begin{aligned} & m_A^l = 9, \qquad m_A^u = 12.55 \\ & m_B^l = m_B^u = 9 \\ & m_C^l = m_C^u = 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} s_A^l &= 2.15, & s_A^u &= 8.74 \\ s_B^l &= 0, & s_B^u &= 3.40 \\ s_C^l &= 5.41, & s_C^u &= 8.12 \end{aligned}\right\}.$$

Then we have

(a) *moduli:*

$$
\begin{aligned}
10.52 &\leqslant \lambda_1^A \leqslant 24.91,\\
2.82 &\leqslant \lambda_2^A \leqslant 18.73,\\
0 &\leqslant \lambda_3^A \leqslant 11.03,\\
9.76 &\leqslant (\lambda_1^A+\lambda_2^A)/2,\\
9 &\leqslant (\lambda_1^A+\lambda_2^A+\lambda_3^A)/3 \leqslant 12.55,\\
&\quad (\lambda_2^A+\lambda_3^A)/2 \leqslant 11.79,\\
&\quad \lambda_1^A-\lambda_2^A \leqslant 18.55,\\
&\quad \lambda_1^A-\lambda_3^A \leqslant 21.42,\\
&\quad \lambda_2^A-\lambda_3^A \leqslant 18.55,\\
4.57 &\leqslant \lambda_1^A-\lambda_3^A;
\end{aligned}
$$

(b) *real parts:*

$$
\begin{aligned}
9 &\leqslant \lambda_1^B \leqslant 13.81,\\
6.60 &\leqslant \lambda_2^B \leqslant 11.40,\\
4.20 &\leqslant \lambda_3^B \leqslant 9,\\
&\lambda_1^B-\lambda_2^B \leqslant 7.21,\\
&\lambda_1^B-\lambda_3^B \leqslant 8.33,\\
&\lambda_2^B-\lambda_3^B \leqslant 7.21;
\end{aligned}
$$

(c) *imaginary parts:*

$$
\begin{aligned}
3.81 &\leqslant \lambda_1^C \leqslant 11.49,\\
-5.74 &\leqslant \lambda_2^C \leqslant 5.74,\\
-11.49 &\leqslant \lambda_3^C \leqslant -3.83,\\
1.91 &\leqslant (\lambda_1^C+\lambda_2^C)/2,\\
&\quad (\lambda_2+\lambda_3)/2 \leqslant -1.91,\\
&\quad \lambda_1^C-\lambda_2^C \leqslant 17.23,\\
&\quad \lambda_1^C-\lambda_3^C \leqslant 19.90,\\
&\quad \lambda_2^C-\lambda_3^C \leqslant 17.23,\\
11.49 &\leqslant \lambda_1^C-\lambda_3^C.
\end{aligned}
$$

The eigenvalues of A are 9, $9+9i$, $9-9i$. [Note that since $s_B^l=0$, we did not obtain useful bounds from (3.2a), (3.2b), and (3.5) when $T=B$.]

EXAMPLE 4.2. Now let

$$
\mathbf{A}=\begin{bmatrix} 6 & 0 & 0\\ 1 & 3 & 1\\ 2 & 4 & 0 \end{bmatrix}.
$$

This matrix was given in Scheffold [9], to illustrate bounds for the subdominant eigenvalues of a matrix with nonnegative elements. He found that

$$|\lambda_2|, |\lambda_3| \leqslant 5.$$

Using the bounds in [12], it was found that

$$\left.\begin{array}{r}3 \leqslant \lambda_1 \leqslant 9.89\\ 0.89 \leqslant |\lambda_2| \leqslant 7.31\\ 0 \leqslant |\lambda_3| \leqslant 4.73\end{array}\right\}.$$

Let us apply Theorem 3.1 again. First, we obtain

$$\left.\begin{array}{ll}K_A^l = 19.76, & K_A^u = 62.70\\ K_B^l = 36.38, & K_B^u = 58.83\\ K_C^l = -16.62, & K_C^u = 5.61\end{array}\right\},$$

$$\left.\begin{array}{ll}m_A^l = 3.0, & m_A^u = 4.57\\ m_B^l = m_B^u = 3.0 & \\ m_C^l = m_C^u = 0.0 & \end{array}\right\},$$

$$\left.\begin{array}{ll}s_A^l = 0.0, & s_A^u = 3.45\\ s_B^l = 1.77, & s_B^u = 3.26\\ s_C^l = 0.0, & s_C^u = 1.37\end{array}\right\}.$$

Then we have

(a) *moduli:*

$$\begin{aligned}4.80 \leqslant \lambda_1^A = \lambda_1^B &\leqslant 7.61,\\ 0.5608 \leqslant \lambda_2^A &\leqslant 7.01,\\ 0.0 \leqslant \lambda_3^A &\leqslant 4.57,\\ \lambda_1^A - \lambda_2^A &\leqslant 7.32,\\ \lambda_1^A - \lambda_3^A &\leqslant 8.45,\\ \lambda_2^A - \lambda_3^A &\leqslant 7.32;\end{aligned}$$

(b) *real parts:*

$$\begin{aligned} 4.80 &\leqslant \lambda_1^B \leqslant 7.61, \\ .70 &\leqslant \lambda_2^B \leqslant 5.30, \\ -1.6 &\leqslant \lambda_3^B \leqslant 1.75, \\ 3.57 &\leqslant (\lambda_1^B + \lambda_2^B)/2, \\ (\lambda_2^B + \lambda_3^B)/2 &\leqslant 2.37, \\ \lambda_1^B - \lambda_2^B &\leqslant 6.91, \\ \lambda_1^B - \lambda_3^B &\leqslant 7.98, \\ \lambda_2^B - \lambda_3^B &\leqslant 6.91, \\ 3.75 &\leqslant \lambda_1^B - \lambda_3^B; \end{aligned}$$

(c) *imaginary parts:*

$$\begin{aligned} 0 &\leqslant \lambda_1^C \leqslant 0.47, \\ -0.97 &\leqslant \lambda_2^C \leqslant 0.97, \\ -1.93 &\leqslant \lambda_3^C \leqslant 0.0\ , \\ \lambda_1^C - \lambda_2^C &\leqslant 2.90, \\ \lambda_1^C - \lambda_3^C &\leqslant 3.36, \\ \lambda_2^C - \lambda_3^C &\leqslant 2.90. \end{aligned}$$

The eigenvalues of **A** are 6, 4, -1. [Note that since $s_A^l = s_C^l = 0$, we did not obtain useful bounds from (3.2a), (3.2b), and (3.5) when $T = B$ or C. In addition, since **A** is real and nonnegative, we have applied Theorem 3.2 and used the fact that the largest eigenvalue of **A** in modulus is real and positive.]

EXAMPLE 4.3. Our last example is the nonnegative matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 5 \end{bmatrix}.$$

This matrix was used in [6] to compare various bounds for the dominant eigenvalue. The best bounds obtained there were

$$5.162 \leqslant \lambda_1 \leqslant 9.359.$$

The bounds in [12] yield

$$\left.\begin{aligned} 2.33 \leqslant |\lambda_1| \leqslant 9.67 \\ 0 \leqslant |\lambda_2| \leqslant 7.04 \\ 0 \leqslant |\lambda_3| \leqslant 4.40 \end{aligned}\right\}.$$

We obtain

$$\left.\begin{aligned} &K_A^l = 48.62, && K_A^u = 57.81 \\ &K_B^l = 52.81, && K_B^u = 57.45 \\ &K_C^l = -4.19, && K_C^u = \ \ 0.44 \end{aligned}\right\},$$

$$\left.\begin{aligned} &m_A^l = 2.33, && m_A^u = 4.39 \\ &m_B^l = m_B^u = 2.33 && \\ &m_C^l = m_C^u = 0 && \end{aligned}\right\},$$

$$\left.\begin{aligned} &s_A^l = 3.28, && s_A^u = 3.72 \\ &s_B^l = 3.49, && s_B^u = 3.70 \\ &s_C^l = 0, && s_C^u = 0.38 \end{aligned}\right\}.$$

Then we have

(a) *moduli:*

$$\begin{aligned} 4.25 \leqslant \ & \lambda_1^A = \lambda_1^B \leqslant 7.57, \\ 0 \leqslant \ & \lambda_2^A \leqslant 7.02, \\ 0 \leqslant \ & \lambda_3^A \leqslant 2.07, \\ 3.00 \leqslant \ & (\lambda_1^A + \lambda_2^A)/2, \\ 3.00 \leqslant \ & (\lambda_1^A + \lambda_2^A + \lambda_3^A)/3 \leqslant 4.39, \\ & (\lambda_2^A + \lambda_3^A)/2 \leqslant 3.23, \\ & \lambda_1^A - \lambda_2^A \leqslant 7.89, \\ & \lambda_2^A - \lambda_3^A \leqslant 7.89, \\ 6.96 \leqslant \ & \lambda_1^A - \lambda_3^A \leqslant 9.11; \end{aligned}$$

(b) *real parts:*

$$\begin{aligned} 4.25 \leqslant \lambda_1^B &\leqslant 7.57, \\ -0.28 \leqslant \lambda_2^B &\leqslant 4.95, \\ \lambda_3^B &\leqslant -0.13, \\ 3.63 \leqslant (\lambda_1^B + \lambda_2^B)/2, & \\ (\lambda_2^B + \lambda_2^B)/2 &\leqslant 1.10, \\ \lambda_1^B - \lambda_2^B &\leqslant 7.85, \\ \lambda_2^B - \lambda_3^B &\leqslant 7.85, \\ 7.40 \leqslant \lambda_1^B - \lambda_3^B &\leqslant 9.07; \end{aligned}$$

(c) *imaginary parts:*

$$\begin{aligned} 0 \leqslant \lambda_1^C &\leqslant 0.47, \\ -0.27 \leqslant \lambda_2^C &\leqslant 0.27, \\ -0.54 \leqslant \lambda_3^C &\leqslant 0, \\ \lambda_1^C - \lambda_2^C &\leqslant 0.81, \\ \lambda_2^C - \lambda_3^C &\leqslant 0.81, \\ \lambda_1^C - \lambda_3^C &\leqslant 0.93. \end{aligned}$$

The eigenvalues of **A** are 7.531, 0, -0.531. [Note again that since $s_C^l = 0$, we did not obtain useful bounds from (3.2a), (3.2b), and (3.5) when $T = B$, and furthermore we have applied Theorem 3.2 again.]

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