MAPPING THEOREMS ON N-SPACES

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We prove two mapping theorems on N-spaces: (1) N-spaces are preserved under closed, Lindelöf mappings; (2) a perfect inverse image of an N-space is an N-space if and only if it has a $G_{\delta}$-diagonal.

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1. Introduction

The concept of N spaces was first introduced by Meara in [7] as a generalization of metric spaces and $\mathcal{N}_0$-spaces (Michael [6]). The main results of this paper are two mapping theorems on N-spaces:

(1) N-spaces are preserved under closed Lindelöf mappings. This affirmatively answers a question posed by Tanaka in [8].

(2) A perfect inverse image of an N-space is an N-space if and only if it has a $G_{\delta}$-diagonal.

Throughout this paper, all spaces are assumed to be at least $T_1$ and regular. All mappings are continuous and surjective. A mapping $f$ from $X$ onto $Y$ is to be denoted by $f: X \to Y$. $N$ denotes the set of positive integers.

Let $X$ be a topological space. A family $\mathcal{F}$ of closed subsets of $X$ is a $k$-network for $X$ if for every compact set $K \subset X$ and neighborhood $U$ of $K$, there is a finite $\mathcal{F}' \subset \mathcal{F}$ so that $K \subset \bigcup \mathcal{F}' \subset U$. $\mathcal{F}$ is a cs-network for $X$ if for every convergent sequence $Z$ in $X$ and neighborhood $U$ of $Z$, there is a $F \in \mathcal{F}$ so that $Z$ is eventually in $F$ and $F \subset U$. A regular space with $\sigma$-locally-finite $k$-network is called an N-space [7].
2. Closed images

Mapping \( f: X \to Y \) is called Lindelöf if for each \( y \in Y \) fiber \( f^{-1}(y) \) is a Lindelöf subspace of \( X \); \( f \) is called compact-covering \([6]\) if every compact subset of \( Y \) is the image of a compact subset of \( X \).

**Lemma 2.1.** If \( f: X \to Y \) is closed Lindelöf, then \( f \) is a compact-covering.

**Proof.** Let \( K \) be a compact subset of \( Y \); then \( f^{-1}(K) \) is a Lindelöf subset of \( X \). But if \( g = f|_{f^{-1}(K)} \), then \( g \) is a closed mapping from the paracompact space \( f^{-1}(K) \) onto \( K \). By Proposition 7.2 in \([6]\), \( g \) is compact-covering. Since \( K \) is compact, there exists a compact subset \( L \) of \( f^{-1}(K) \) such that \( g(L) = K \). Also, \( L \) is a compact subset of \( X \), and \( f(L) = K \). □

**Theorem 2.2.** \( \mathfrak{N} \)-spaces are preserved under closed Lindelöf mappings.

**Proof.** Suppose \( X \) is an \( \mathfrak{N} \)-space, and \( f: X \to Y \) is closed Lindelöf. \( X \) has a \( \sigma \)-locally-finite closed \( k \)-network \( \mathcal{P} \). Put \( \mathcal{F} = \{ f(P) \mid P \in \mathcal{P} \} \). Since \( f \) is closed Lindelöf, \( \mathcal{F} \) is a \( \sigma \)-closure-preserving and locally-countable collection of closed subsets of \( Y \). It is clear that the compact-covering image of a \( k \)-network is a \( k \)-network.

Hence, by Lemma 2.1, \( \mathcal{F} \) is a \( \sigma \)-closure-preserving and \( \sigma \)-locally-countable closed \( k \)-network. Foged \([1, Theorem 4, (a) \to (d)]\) proved that a space with \( \sigma \)-locally-finite closed \( k \)-network has a \( \sigma \)-discrete cs-network. It is not difficult to check that, in his proof, the condition "\( \sigma \)-locally-finite closed \( k \)-network" can be replaced by "\( \sigma \)-locally-countable and \( \sigma \)-closure-preserving closed \( k \)-network". Therefore a space with \( \sigma \)-locally-countable and \( \sigma \)-closure-preserving closed \( k \)-network is an \( \mathfrak{N} \)-space. Therefore \( Y \) is an \( \mathfrak{N} \)-space. □

**Remark 1.** The following question is posed by Tanaka in \([8]\): Are the spaces which are closed Lindelöf images of metric spaces \( \mathfrak{N} \)-spaces? Theorem 2.2 answers the question affirmatively.

**Remark 2.** For each \( \alpha < \omega_1 \), let \( I_\alpha = [0, 1] \) with usual topology, and let \( X \) be quotient space of \( \bigoplus_{\alpha < \omega_1} I_\alpha \) obtained by identifying \( \{0\} \). Then \( X \) is a Lašnev space and is not an \( \mathfrak{N} \)-space (by \([5, Proposition 6.4]\)). Hence \( \mathfrak{N} \)-spaces are not preserved under closed mappings.

**Theorem 2.3.** The following properties of a space are equivalent:

(a) \( X \) is a Fréchet and \( \mathfrak{N} \)-space.

(b) \( X \) is a closed Lindelöf image of a metric space.

**Proof.** (b) \( \to \) (a). It is known that closed mappings preserve the Fréchet property. By Theorem 2.2, \( X \) is an \( \mathfrak{N} \)-space.
(a) \(\rightarrow\) (b). Suppose \(X\) is a Fréchet and \(\mathbb{N}\)-space. Foged [2, Theorem 1] has shown that \(X\) is a Fréchet space with \(\sigma\)-hereditarily closure-preserving \(k\)-network if and only if \(X\) is a Lašnev space (a space which is a closed image of a metric space). Let \(M\) be a metric space, \(f: M \to X\) a closed mapping. Since \(M\) is a paracompact \(\mathbb{N}\)-space, and \(X\) a \(k\)-space with point-countable closed \(k\)-network, according to [5, Proposition 6.4] for each \(y \in Y\), \(\partial f^{-1}(y)\) (boundary of \(f^{-1}(y)\)) is Lindelöf. Thus there exists a closed subset \(M'\) of \(M\) such that \(g = f|_{M'}: M' \to X\) is closed Lindelöf with \(g(M') = X\). Hence \(X\) is a closed Lindelöf image of a metric space. \(\square\)

3. Perfect inverse images

For a topological space \(X\), let \(\mathcal{K}(X) = \{K \subset X \mid K\) is a nonempty compact subset of \(X\}\). If \(\mathcal{U}\) and \(\mathcal{V}\) are collections of subsets of \(X\), let \(\mathcal{U} \wedge \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}\) and \(V \in \mathcal{V}\). For any \(A \subset X\), let \((\mathcal{U})_A = \{U \subset X \mid U \cap A \neq \emptyset\}\) and \(\text{st}(A, \mathcal{U}) = \bigcup (\mathcal{U})_A\).

We consider the following properties of space \(X\).

(A) For any open cover of \(X\) there exists a \(\sigma\)-discrete refinement \(\mathcal{F}\) such that every compact subset of \(X\) is covered by a finite subcollection of \(\mathcal{F}\).

(B) For any open cover of \(X\) there exists a sequence \((\mathcal{G}_n)\) of open refinements which satisfies the condition that for each \(K \in \mathcal{K}(X)\), there exist \(K_i \in \mathcal{K}(X)_{1 \leq i \leq m}\) such that \(K = \bigcup_{i=1}^{m} K_i\) and \(|(\mathcal{G}_n)|_{K_i}| = 1_{i \leq m}\).

(C) There exists a sequence \((\mathcal{G}_n)\) of open covers such that for each \(K \in \mathcal{K}(X)\), \(K = \bigcap_{n \in \mathbb{N}} \text{st}(K, \mathcal{G}_n)\).

Lemma 3.1. If \(Y\) is an \(\mathbb{N}\)-space and \(f: X \to Y\) is a perfect mapping, then \(X\) has property (A).

Proof. Since \(Y\) is an \(\mathbb{N}\)-space, \(Y\) has a \(\sigma\)-discrete \(k\)-network (by Foged [1, Theorem 4]). Suppose \(\mathcal{P} = \bigcup_n \mathcal{P}_n\) is a \(k\)-network for \(Y\), each \(\mathcal{P}_n\) is a discrete collection of subsets of \(Y\).

Suppose \(\mathcal{U}\) is any open cover of \(X\). For each \(y \in Y\) we can find a finite subcollection \(\mathcal{U}(y) \subset \mathcal{U}\) such that \(f^{-1}(y) \subset \bigcup \mathcal{U}(y)\). Let \(G(y) = Y - f(X - \bigcup \mathcal{U}(y))\), then \(\mathcal{G} = \{G(y) \mid y \in Y\}\) is an open cover of \(Y\). By the definition of \(k\)-network and the regularity of \(Y\), without loss of generality, we may assume \(\mathcal{P}\) is a refinement of \(\mathcal{G}\). Consequently for each \(P \in \mathcal{P}\) there exist \(U(i, P) \in \mathcal{U}\) such that \(f^{-1}(P) \subset \bigcup_{i \leq m} U(i, P)\). Let \(\mathcal{F}(n, i) = \{f^{-1}(P) \cap U(i, P) \mid P \in \mathcal{P}_n\}\). Then \(\mathcal{F} = \bigcup_{n,i} \mathcal{F}(n, i)\) satisfies (A). \(\square\)

Lemma 3.2. (A) \(\rightarrow\) (B).

Proof. Let \(\mathcal{U}\) be an open cover of a space \(X\) and take a \(\sigma\)-discrete refinement \(\mathcal{F} = \bigcup_n \mathcal{F}_n\) of \(\mathcal{U}\) with the property (A). Let \(\mathcal{F}_n = \{F(n, \alpha) \mid \alpha \in A_n\}\). By regularity, we may assume each element of \(\mathcal{F}\) is a closed subset of \(X\). For each \(n \in \mathbb{N}\), \(\alpha \in A_n\),
pick $U(n, \alpha) \in \mathcal{U}$ such that $F(n, \alpha) \subseteq U(n, \alpha)$, and put $W(n, \alpha) = U(n, \alpha) \setminus \bigcup \{ F(n, \beta) \mid \beta \in A_n \setminus \{ \alpha \} \}$. We define

$$W_n = \{ W(n, \alpha) \mid \alpha \in A_n \} \cup \{ U - \bigcup F_n \mid U \in \mathcal{U} \}.$$ 

It follows that $(W_n)$ satisfies (B).

It is clear that $(W_n)$ is the sequence of open refinement of $\mathcal{U}$. To see that $(W_n)$ satisfies (B), let $K \in \mathcal{K}(X)$, by the property (A), there exists a finite subcollection $\mathcal{F} = \{ F_i \mid i \leq m \}$ of $(\mathcal{F})_K$ which covers $K$. For each $i \in \{1, 2, \ldots, m\}$, there exists an $n_i \in \mathbb{N}$ such that $F_i \in \mathcal{F}_n$. Then $K \cap F_i \in \mathcal{K}(X)_{(i \leq m)}$, $K = \bigcup_{i \leq m} K \cap F_i$ and $|\{ W_n \} \cap F_i| = 1$. □

**Lemma 3.3.** $(B) + G_\delta$-diagonal $\rightarrow$ (C).

**Proof.** Suppose a space $X$ with property (B) has a $G_\delta$-diagonal. Clearly $X$ is a submetacompact (i.e., $\theta$-refinable) space with a $G_\delta$-diagonal, so $X$ has a $G_\delta$-diagonal [4, Theorem 2.11]. Let $(\mathcal{G}_n)$ be a $G_\delta$-diagonal sequence, i.e., $\{ x \} = \bigcap_{x \in X} \text{st}(x, \mathcal{G}_n)$ for each $x \in X$. We may assume that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$. Now we prove for each $K \in \mathcal{K}(X)$, $K = \bigcap_{x \in X} \text{st}(x, \mathcal{G}_n)$. Suppose $x \in X - K$; then $\{ x \setminus \text{st}(x, \mathcal{G}_n) \mid n \in \mathbb{N} \}$ is an open cover of the compact subset $K$, so there exists an $n \in \mathbb{N}$ such that $K \subseteq X \setminus \text{st}(x, \mathcal{G}_n)$. Therefore $K \cap \text{st}(x, \mathcal{G}_n) = \emptyset$, i.e., $x \notin \text{st}(K, \mathcal{G}_n)$. Hence $K = \bigcap_{x \in X} \text{st}(x, \mathcal{G}_n)$.

Now, we use the regularity of $X$ and property (B) to inductively define, for each $m \in \mathbb{N}$, a sequence $(\mathcal{V}_{m,n})_n$ of open covers for $X$ such that

(a) for each $n \in \mathbb{N}$, $\{ \mathcal{V} \mid \mathcal{V} \in \mathcal{V}_{m,n} \}$ is a refinement of $(\bigwedge_{i,j \leq m} \mathcal{V}_{i,j}) \setminus (\bigwedge_{k \leq m} \mathcal{G}_k)$;

(b) $(\mathcal{V}_{m,n})_n$ is a sequence satisfying the condition of property (B).

We prove for each $K \in \mathcal{K}(X)$, $\bigcap_{x \in X} \text{st}(K, \mathcal{V}_{m,n}) = K$. For each $n \in \mathbb{N}$, take $s > n$. Since the sequence $(\mathcal{V}_{s,k})_k$ satisfies (b), there exists $K_i \in \mathcal{K}(X)_{(i \leq h)}$ such that $K = \bigcup_{i \leq h} K_i$ with $|\{ \mathcal{V}_{s,k} \} \setminus K_i| = 1$. Then $\text{st}(K_i, \mathcal{V}_{s,k}) = \bigcup \{ \mathcal{V} \mid \mathcal{V} \in (\mathcal{V}_{s,k})_K \} \subseteq \text{st}(K_i, \mathcal{V}_{m,n}) \subseteq \text{st}(K_i, \mathcal{G}_n)$.

Pick $r > \max \{ s, k_1, k_2, \ldots, k_n \}$; consequently,

$$\bigcap_{m,k} \text{st}(K, \mathcal{V}_{m,k}) \subseteq \text{st}(K, \mathcal{V}_{r,1})$$

$$= \bigcup_{i \leq h} \text{st}(K_i, \mathcal{V}_{r,1}) \subseteq \bigcup_{i \leq h} \text{st}(K_i, \mathcal{V}_{s,k}) \subseteq \text{st}(K, \mathcal{G}_n).$$

Hence

$$\bigcap_{m,k} \text{st}(K, \mathcal{V}_{m,k}) \subseteq \bigcap_{n} \text{st}(K, \mathcal{G}_n) = K.$$

So $K = \bigcap_{m,k} \text{st}(K, \mathcal{V}_{m,k})$.

**Theorem 3.4.** Suppose there exists a perfect mapping $f$ from a topological space $X$ onto an $\mathfrak{K}$-space $Y$. Then $X$ is an $\mathfrak{K}$-space if and only if it satisfies any of the following:
(a) $X$ has a $G_\delta$-diagonal.
(b) $X$ has a point-countable $k$-network.

**Proof.** Necessity is obvious.

Sufficiency: Since a $\sigma$-space has a $G_\delta$-diagonal, by Corollary 3.8 in [5], it is sufficient to show that if $X$ has a $G_\delta$-diagonal, then $X$ is an $\aleph$-space.

Suppose $X$ has a $G_\delta$-diagonal. By Lemmas 3.1, 3.2, and 3.3, there exists a sequence $(\mathcal{G}_n)$ of open covers for $X$ such that for each $K \in \mathcal{K}(X)$, $K = \bigcap_n \text{st}(K, \mathcal{G}_n)$. We can assume $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$. For each $n \in \mathbb{N}$, by Lemma 3.1, $\mathcal{G}_n$ has a $\sigma$-locally-finite closed refinement $\mathcal{F}(n)$ such that every compact subset of $X$ is covered by a finite subcollection of $\mathcal{F}(n)$. Denote by $\mathcal{F}(n) = \bigcup_m \mathcal{F}(n, m)$ where each $\mathcal{F}(n, m)$ is a locally-finite collection of subsets of $X$. We can assume $\mathcal{F}(n, m) \subset \mathcal{F}(n, m+1)$ for each $m \in \mathbb{N}$.

Since $Y$ is an $\aleph$-space, let $\bigcup k \mathcal{I}(k)$ be a $k$-network for $Y$ where each $\mathcal{I}(k)$ is locally-finite and $\mathcal{I}(k) \subset \mathcal{I}(k+1)$ for each $k \in \mathbb{N}$. Let $\mathcal{D}(k) = \{f^{-1}(Q) | Q \in \mathcal{I}(k)\}$; then $\mathcal{D}(k)$ is a locally-finite collection of closed subsets of $X$. Put

$$\mathcal{P}(n, m, k) = \mathcal{F}(n, m) \cap \mathcal{D}(k).$$

Clearly $\mathcal{P}(n, m, k)$ is locally-finite for each $n, m, k \in \mathbb{N}$.

We complete the proof by showing that $\mathcal{P} = \bigcup_{n,m,k} \mathcal{P}(n, m, k)$ is a $k$-network for $X$. For an open subset $W$ and a compact subset $K \subset W \subset X$, since $K = \bigcap_n \text{st}(K, \mathcal{G}_n)$, $\{W\} \cup \{X - \text{st}(K, \mathcal{G}_n) | n \in \mathbb{N}\}$ is an open cover of compact subset $f^{-1}f(K)$ of $X$. Thus there exists a $n \in \mathbb{N}$ such that $f^{-1}f(K) = W \cup (X - \text{st}(K, \mathcal{G}_n))$, so $\text{st}(K, \mathcal{G}_n) \subset f^{-1}f(K) \subset W$. For each $x \in f^{-1}f(K) - W$, since $x \notin \text{st}(K, \mathcal{G}_n)$, there exists an open set $V(x)$ containing $x$ with $V(x) \cap \text{st}(K, \mathcal{G}_n) = \emptyset$. Let $G = W \cup (\bigcup \{V(x) | x \in f^{-1}f(K) - W\})$, then $f(K) \subset Y - f(X - G)$. So there exists a finite $\mathcal{D}'(k) \subset \mathcal{D}(k)$ such that $f(K) \subset \bigcup \mathcal{D}'(k) \subset Y - f(X - G)$ for some $k \in \mathbb{N}$. Take $\mathcal{D}'(k) = \{f^{-1}(Q) | Q \in \mathcal{D}'(k)\}$, then $f^{-1}f(K) \subset \bigcup \mathcal{D}'(k) \subset G$. On the other hand, by the property of $\mathcal{F}(n)$, there exists a finite $\mathcal{F}'(n, m) \subset f^{-1}(f(K) \subset \bigcup \mathcal{F}(n, m) \subset \text{st}(K, \mathcal{G}_n)$ for some $m \in \mathbb{N}$. Put $\mathcal{P}'(n, m, k) = \mathcal{F}'(n, m) \cap \mathcal{D}'(k)$. It is easy to check that $K \subset \bigcup \mathcal{P}'(n, m, k) \subset W$. □

**Corollary 3.5.** Suppose $Y$ is an $\aleph$-space and $f : X \to Y$ is an open, closed, and finite to one mapping. Then $X$ is an $\aleph$-space.

**Proof.** Since $\aleph$-space is a $\sigma$-space, $X$ is a $\sigma$-space [3]. Then $X$ has a $G_\delta$-diagonal. By Theorem 3.4, $X$ is an $\aleph$-space. □

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