On the Solution of Algebraic Equations by the Decomposition Method

G. ADOMIAN

Center for Applied Mathematics, University of Georgia, Athens, Georgia 30602

AND

R. RACH

Raytheon Company, Microwave & Power Tube Division, Waltham, Massachusetts 02254

The decomposition method (G. Adomian, "Stochastic Systems," Academic Press, New York, 1983) developed to solve nonlinear stochastic differential equations has recently been generalized to nonlinear (and/or) stochastic partial differential equations, systems of equations, and delay equations and applied to diverse applications. As pointed out previously (see reference above) the methodology is an operator method which can be used for nondifferential operators as well. Extension has also been made to algebraic equations involving real or complex coefficients. This paper deals specifically with quadratic, cubic, and general higher-order polynomial equations and negative, or nonintegral powers, and random algebraic equations. Further work on this general subject appears elsewhere (G. Adomian, "Stochastic Systems II," Academic Press, New York, in press).

I. QUADRATIC EQUATIONS

The decomposition method [1, 2] is applied to generic operator equations of the form \( \mathcal{F} u = g \) where \( \mathcal{F} \) may be a nonlinear (and/or) stochastic operator and \( u \) a stochastic process on an appropriate probability space. The basic equation is considered in the form \( \mathcal{L} u + \mathcal{N} u = g \), or \( Lu + Nu = g \) in the deterministic case where \( L \) is a linear (deterministic) operator and \( N \) a nonlinear (deterministic) operator. (In the case where the operators involve stochasticity, script letters \( \mathcal{L}, \mathcal{N} \) are preferred.) (If we write an ordinary quadratic equation \( ax^2 + bx + c = 0 \) in the form \( Lu + Nu = g \), identifying \( Nx = ax^2 \), \( L = b \), and \( g = -c \) we have \( Lu = g - Nu \) or

\[
bx = -c - ax^2.
\]
The operation $L^{-1}$ in the referenced work for differential equations is an integral operator. Here it is simply division by $b$. Hence

$$x = \frac{(-c)}{b} - \frac{(a)}{b} x^2$$

in the standard format of the referenced work [13]. The solution $x$ in this methodology is now decomposed into components $x_0 + x_1 + \cdots$ where $x_0$ is taken as $(-c/b)$ here and $x_1, x_2, \ldots$, are still to be identified. Thus

$$x_0 = -c/b.$$

We now have

$$x = x_0 - \frac{(a)}{b} x^2$$

with $x_0$ known. In the general methodology, the nonlinear term without the coefficient—in this case $x^2$—is replaced by $\sum_{n=0}^{\infty} A_n(x_0, x_1, \ldots, x_n)$ where $A_n(x_0, x_1, \ldots, x_n)$ are functions of the $x_i$ defined by Adomian [1, 2]. Since the $A_n$ have been determined for large classes of nonlinearities by methods previously published, we will only list the necessary $A_n$ for this paper. Adomian's $A_n$ polynomials are found for the particular non-linearity by a generating scheme just as one might develop Hermite, Lagrange, or Laguerre polynomials. Rules are given in the referenced works. For the example $Nx = x^2$ we have

$$A_0 = x_0^2$$

$$A_1 = 2x_0 x_1$$

$$A_2 = x_1^2 + 2x_0 x_2$$

$$A_3 = 2x_1 x_2 + 2x_0 x_3$$

$$A_4 = x_2^2 + 2x_1 x_3 + 2x_0 x_4$$

$$A_5 = 2x_0 x_5 + 2x_1 x_4 + 2x_2 x_3$$

$$A_6 = x_3^2 + 2x_0 x_6 + 2x_1 x_5 + 2x_2 x_4$$

$$A_7 = 2x_0 x_7 + 2x_1 x_6 + 2x_2 x_5 + 2x_3 x_4$$

$$A_8 = x_4^2 + 2x_0 x_8 + 2x_1 x_7 + 2x_2 x_6 + 2x_3 x_5$$

$$\vdots$$

Examining the subscripts we note the sum of subscripts in each term is $n$. Now

$$x = x_0 - \frac{(a)}{b} \sum_{n=0}^{\infty} A_n$$
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requires

\[ x_1 = -(a/b)A_0 = -(a/b)x_0^2 \]
\[ x_2 = -(a/b)A_1 = -(a/b)(2x_0x_1) \]
\[ x_3 = -(a/b)A_2 = -(a/b)(x_1^2 + 2x_0x_2) \]
\[ x_4 = -(a/b)A_3 = -(a/b)(2x_1x_2 + 2x_0x_3) \]
\[ x_5 = -(a/b)A_4 = -(a/b)(x_2^2 + 2x_1x_3 + 2x_0x_4) \]

\vdots

thus the \( x_i \) are determined.

We note in the example \( N = x^2 \) that if we expand \((x_0 + x_1 + \cdots)^2\) into \( x_0^2 + x_1^2 + x_2^2 + \cdots + 2x_0x_1 + 2x_0x_2 + \cdots + 2x_1x_2 + \cdots \), we must choose \( A_0 = x_0^2 \) but \( A_1 \) could be \( x_1^2 + 2x_0x_1 \). The sum of the subscripts for \( x_1 \) or \( x_1x_2 \) is higher than for the \( x_0x_1 \) term. By choosing for any \( x_n \) only terms summing to \( n - 1 \), we get consistency with our more general schemes which we can use with high-ordered polynomials, trigonometric or exponential terms, and negative or irrational powers, or even multidimensional differential equations. [3, 4]

When the \( N \), or in the quadratic case, \( x^2 \), term is written in terms of Adomian’s \( A_n \) polynomials, the decomposition method solves the equation. (Although it is not necessary to discuss it here, if stochastic coefficients are involved, the decomposition method achieves statistical separability in the averaging process for desired statistics [1] and no truncations are required.)

Let’s look at examples:

**EXAMPLE.** Consider \( x^2 + 3x + 2 = 0 \) whose solutions are obviously \((-1, -2)\). Write it in the form

\[ 3x = -2 - x^2 \]
\[ x = -\frac{2}{3} - \frac{1}{3}x^2 = x_0 + x_1 + x_2 + \cdots \]
\[ = x_0 + \frac{1}{3} \sum_{n=0}^{\infty} A_n \]
\[ = x_0 - \frac{1}{3}A_0 - \frac{1}{3}A_1 - \cdots \]

Substituting the \( A_n \) we have

\[ x_0 = -0.667 \quad x_3 = -0.037 \]
\[ x_1 = -0.148 \quad x_4 = -0.023 \]
\[ x_2 = -0.069 \quad x_5 = -0.015 \]
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\[ x_6 = -0.0106 \quad x_{10} = -0.0033 \]
\[ x_7 = -0.00765 \quad x_{11} = 0.00268 \]
\[ x_8 = -0.00567 \quad x_{12} = -0.0020. \]
\[ x_9 = -0.0043 \]

Since an \( n \)-term approximation (symbolized by \( \phi_n \)) is given by \( \sum_{i=0}^{n-1} x_i \), we define the error \( \psi_n = (x - \phi_n)/x \). We now have

\[ \phi_1 = -0.667 \quad \psi_1 = 33.3\% \]
\[ \phi_2 = -0.815 \quad \psi_2 = 18.5\% \]
\[ \phi_3 = -0.884 \quad \psi_3 = 11.6\% \]
\[ \phi_4 = -0.921 \quad \psi_4 = 7.9\% \]
\[ \phi_5 = -0.944 \quad \psi_5 = 5.6\% \]
\[ \phi_6 = -0.959 \quad \psi_6 = 4.1\% \]
\[ \phi_7 = -0.970 \quad \psi_7 = 3.0\% \]
\[ \phi_8 = -0.977 \quad \psi_8 = 2.3\% \]
\[ \phi_9 = -0.983 \quad \psi_9 = 1.7\% \]
\[ \phi_{10} = -0.987 \quad \psi_{10} = 1.2\%. \]

which is approaching the smallest root which is \(-1\). The error \( \psi_n \) becomes less than 0.5\% by \( m = 12 \). If we take the equation \( x^2 - 3x + 2 = 0 \) we get the same numbers above for the \( x_i \) except they will all be positive.

**EXAMPLE.** Consider \( x^2 - 1.25x + 0.25 = 0 \) or \( (x - \frac{1}{4})(x - 1) = 0 \). In our form it becomes

\[ -\frac{2}{5}x = -\frac{1}{4} - x^2 \]

or

\[ x = (1/5) + (4/5)x^2. \]

Thus

\[ x_0 = 0.2 \]
\[ x_1 = (0.8)(0.2)^2 = 0.032 \]
\[ x_2 = (0.8)(2)(0.2)(0.032) = 0.01024 \]
\[ x_3 = (0.8)[(0.032)^2 + 2(0.2)(0.01)] = 0.004. \]

Thus \( \phi_m = \sum_{n=0}^{m-1} x_n \) is:

\[ \phi_1 = 0.2 \]
\[ \phi_2 = 0.232 \]
\[ \phi_3 = 0.242 \]
\[ \phi_4 = 0.246 \]
rapidly converging to 0.25 as expected.

**Example.** Consider \( x^2 - 20x + 36 = 0 \), which has the roots (2,18). Write

\[ -20x = -36 - x^2 \]
\[ x = \frac{36}{20} + \frac{1}{20}x^2. \]

By the same procedure we get

\[ x_0 = 1.8 \]
\[ x_1 = 0.16. \]

Hence the approximation to only two terms is given by

\[ \phi_2 = x_0 + x_1 = 1.96. \]

A 3-term approximation is \( \phi_3 = 1.98 \), which is already close to the smallest root \( x = 2 \).

**Example.** Consider \( (x - 4)(x - 100) = 0 \) and write

\[ \frac{401}{4}x = 25 + x^2 \]
\[ x = \frac{100}{401} + \frac{4}{401}x^2 \]
\[ x_0 = 0.2493 \]
\[ x_1 = (0.0099)(0.2493)^2 = 0.0006 \]
\[ \phi_2 = x_0 + x_1 = 0.2499 \approx 0.25. \]

From these examples we observe that the method yields the smallest root and that the further apart the two roots the faster the convergence to the correct solution (which we will discuss further in a following section). Of course the second root is found by factoring once we have one root.

Let's examine the quadratic equation in the form \( (x - r_1)(x - r_2) = 0 \) where \( r_1, r_2 \) are real roots. We have then \( x^2 - (r_1 + r_2)x + r_1r_2 = 0 \). Then in the standard form [1]

\[ (r_1 + r_2)x = r_1r_2 + x^2 \]
or

\[ x = \frac{r_1 r_2}{r_1 + r_2} + \frac{1}{r_1 + r_2} x^2. \]

Now since \( x = \sum_{n=0}^{\infty} x_n \) and we identify \( x_0 = \frac{r_1 r_2}{(r_1 + r_2)} \), the \( x_{n+1} \) for \( n = 0, 1, \ldots \) are given by

\[ x_{n+1} = \frac{1}{r_1 + r_2} A_n \]

or

\[ x = x_0 + \sum_{n=0}^{\infty} \frac{1}{r_1 + r_2} A_n \]

where the \( A_n \) have already been given for \( N x = x^2 \).

Since \( r_1 r_2 = c/a \) and \( r_1 + r_2 = -b/a \) in the standard \( ax^2 + bx + c \) form, we have

\[ x = -(c/b) - (a/b) x^2 \]

where

\[ x_0 = c/b \]
\[ x_1 = (a/b) x_0^2 \]
\[ x_2 = (a/b)(2x_0 x_1) \]

etc.

Note, e.g., that in solving \((x-x)(x - 4) = 0\) where we have deliberately chosen the 2nd root to be only a little larger than the root \( r_2 \), we have \( x^2 - (\pi + 4) x + 4\pi = 0 \). We have

\[ x = \frac{4\pi}{\pi + 4} + \frac{1}{\pi + 4} x^2 \]

so that \( x_0 = 1.76 \). If we consider \((x - \pi)(x - 10) = 0\) we get \( x_0 = 2.39 \). If we take the second root as 100, \( x_0 = 3.05 \) and for the second root \( x = 1000, x_0 = 3.13 \), an error of 0.3% with only the \( x_0 \) term to obtain the smaller root. Thus the results converge to the desired solution more and more quickly, i.e., for smaller \( n \), as the roots are further apart. In general for \((x - r_1)(x - r_2) = 0\), or \( x^2 - (r_1 + r_2) x + r_1 r_2 = 0 \), we have the first term

\[ x_0 = \frac{r_1 r_2}{r_1 + r_2}. \]
If \( r_2 \gg r_1 \), we have \( x_0 \approx r_1 r_2 / r_2 = r_1 \). Since the following terms involving the \( A_n \) are divided by the factor \( 1/(r_1 + r_2) \) or approximately \( 1/r_2 \), the other terms vanish early.

**Decimal Roots**

Finally, as we have previously stated, the roots are not limited to integers. Consider, for example,

\[
x^2 - 5.15x + 2.37 = 0
\]

\[
5.15x = 2.37 + x^2
\]

\[
x = \frac{2.37}{5.15} + \frac{1}{5.15} \sum_{n=0}^{\infty} A_n.
\]

We get immediately

\[
x_0 = 0.460
\]

\[
x_1 = 0.0411
\]

\[
x_2 = 0.00735.
\]

Thus the 3-term approximation \( \phi_3 = x_0 + x_1 + x_2 = 0.50845 \). Let's call this \( r_2 \). But \( r_1 r_2 = 2.37 \) hence \( r_1 = 2.37/0.50845 = 4.66 \). The sum of the roots now constitutes a check by comparison with the coefficient of the middle term of the quadratic equation. We observe in doing this an error less than 0.3% and considering we only used a 3-term approximation, the result is excellent.

**Complex Roots**

If we have complex roots \( z_1, z_2 \) then \( (x - z_1)(x - z_2) = 0 \) or \( x^2 - (z_1 + z_2)x + z_1 z_2 = 0 \). Thus the sum of the roots is the coefficient of the \( x \) term and the product of the roots is the constant term. Consider an example with complex roots but real coefficients

\[
x^2 - 2x + 2 = 0.
\]

Solving it in the usual manner with decomposition, we have

\[
x = 1 + \frac{1}{2} x^2 = 1 + \frac{1}{2} \sum_{n=0}^{\infty} A_n.
\]

Therefore we take

\[
x_0 = 1
\]
and obtain immediately

\[ x_1 = \frac{1}{2} \]
\[ x_2 = \frac{1}{2} \]
\[ x_3 = \frac{5}{8} \]
\[ x_4 = \frac{7}{8} \]
\[ x_5 = \frac{21}{16} \]

\[ \vdots \]

i.e., a diverging series (for a quadratic equation with real coefficients) may indicate complex roots. In that case, as complex roots occur in conjugate pairs, e.g., \( a + bi \) and \( a - bi \), their sum is \( 2a \) and their product is \( a^2 + b^2 \).

Comparison with the coefficients in the equation shows \( 2a = 2 \) or \( a = 1 \) and \( a^2 + b^2 = 2 \), hence \( b = 1 \). Therefore the roots are \( 1 + i \) and \( 1 - i \).

**EXAMPLE.** Quadratic equation with complex roots \( c_1, c_2 \) given by

\[(x - c_1)(x - c_2) = 0 \]

or \[ x^2 - (c_1 + c_2)x + c_1c_2 = 0 \]

where \( c_1, c_2 \in \mathbb{C} \), the set of complex numbers. In the standard Adomian decomposition form, we get

\[ x = \mu + \nu x^2 \]

where \( \mu = \alpha + i\beta \) and \( \nu = \gamma + i\delta \) can of course be written in terms of real and imaginary components of \( c_1, c_2 \). We write

\[ \sum_{n=0}^{\infty} x_n = \mu + \nu \sum_{n=0}^{\infty} A_n \]

where

\[ A_0 = x_0^2 \]
\[ A_1 = 2x_0x_1 \]
\[ A_2 = x_1^2 + 2x_0x_2 \]
\[ A_3 = 2x_0x_3 + 2x_1x_2 \]

\[ \vdots \]

\(^1\) The associated equation with different signature, \( x^2 - 2x - 2 = 0 \), which does have real roots, also results in a diverging series. This special case has been handled by an ingenious method discussed in Adomian [2], which also solves equations of the form \( Ny = x \) such as \( e^x = x \), for example.
Thus

\[ x_0 = \mu \]
\[ x_1 = \nu A_0 = \nu x_0^2 = \nu \mu^2 \]
\[ x_2 = \nu A_1 = \nu (2x_0 x_1) = 2\nu^2 \mu^3 \]
\[ x_3 = \nu A_2 = \nu (x_1^2 + 2x_0 x_2) = 5\nu^3 \mu^4 \]
\[ x_4 = \nu A_3 = \nu (2x_0 x_3 + 2x_1 x_2) = 14\nu^4 \mu^5 \]
\[ \vdots \]
\[ x_m = k_m \nu^m \mu^{m+1} \quad \text{for } m \geq 0 \]

where the \( k_m \) are constants as previously defined. The solution is

\[ x = \sum_{n=0}^{\infty} k_n \nu^n \mu^{n+1} \]

where the \( k_n \) are real numbers and the \( \mu, \nu \) are complex numbers, i.e., \( \mu = \alpha + i\beta \) and \( \nu = \gamma + i\delta \). An \( m \)-term approximation is \( \phi_m = \sum_{n=0}^{m-1} x_n \). Now let \( c_1 = 1 + i \) and let \( c_2 = 10 + 10i \). In the equation in standard form

\[ x = \mu + \nu x^2 \]

where \( \mu = \alpha + i\beta \) and \( \nu = \gamma + i\delta \) we find \( \alpha = 10/11 \), \( \beta = 10/11 \), \( \gamma = 1/22 \), \( \delta = -1/22 \). Thus \( \mu = (10/11)(1 + i) \) and \( \nu = (1/22)(1 - i) \) and

\[ x = (10/11)(1 + i) + (1/22)(1 - i) x^2. \]

Then

\[ x_0 = (10/11)(1 + i) \]
\[ x_1 = \nu \mu^2 = (10^2/11^3)(1 + i) \]
\[ x_2 = 2\nu^2 \mu^3 = 2(10^3/11^5)(1 + i) \]
\[ x_3 = 5(10^4/11^7)(1 + i) \]
\[ x_4 = 14(10^5/11^9)(1 + i) \]
\[ \vdots \]
\[ x_m = k_m \nu^m \mu^{m+1} \quad (m \geq 0) \]

(\( k_m (1/22)^m (1 - i)^m (10/11)^{m+1} (1 + i)^{m+1} \) (where the coefficients \( k_m \) are easily calculated not only for the quadratic case but also for cubics in the form \( x = \mu + \nu x^3 \), quartics in the form...
\( x = \mu + vx^4 \), etc.; similarly we can find coefficients for \( x = \mu + v_1x^2 + v_2x^3 + v_3x^4 + \cdots + v_{n-1}x^n \) for real or complex cases). Factoring \( x_m \) we have

\[
x_m = k_m(1/2)^m(10^{m+1}/11^{2m+1})[(1 - i)(1 + i)]^m(1 + i)
\]

\[
= k_m(1/2)^m(10^{m+1}/11^{2m+1})2^m(1 + i)
\]

\[
= k_m(10^{m+1}/11^{2m+1})(1 + i).
\]

Computing the components \( x_m \),

\[
x_0 = (10/11)(1 + i) = 0.9090(1 + i)
\]

\[
x_1 = (10^2/11^3)(1 + i) = (100/1331)(1 + i) = 0.0751(1 + i)
\]

\[
x_2 = (2)(10^3/11^5)(1 + i) = (2000/161051)(1 + i) = 0.0124(1 + i)
\]

\[
x_3 = (5)(10^4/11^7)(1 + i) = 0.0025(1 + i)
\]

\[
x_4 = 0.00059(1 + i).
\]

Thus

\[
\phi_1 = 0.9090(1 + i)
\]

\[
\phi_2 = 0.9842(1 + i)
\]

\[
\phi_3 = 0.9966(1 + i)
\]

\[
\phi_4 = 0.9992(1 + i)
\]

\[
\phi_5 = 0.9998(1 + i).
\]

\( \phi_n \rightarrow (1 + i) = c_1 \), the smallest root. We see the convergence is very rapid indeed. Even by \( \phi_2 \) we have an excellent solution and the method applies well to quadratic equations with complex coefficients (and is easily extended to polynomial equations). The real and imaginary components generally converge at different rates. Suppose \( c_1 = 1 + i \) and \( c_2 = m_1 + m_2i \) where for illustration we choose \( m_1 = 1 \) and \( m_2 = 2 \) so that \( c_2 = 1 + 2i \). Now \( \alpha = 7/13 \), \( \beta = 9/13 \), \( \gamma = 2/13 \), \( \delta = -3/13 \). Hence \( \mu = (7/13) + (9/13)i \) and \( v = (2/13) - (3/13)i \) so that

\[
x = \left( \frac{7 + 9i}{13} \right) + \left( \frac{2 - 3i}{13} \right) x^2 = \mu + vx^2.
\]

Then

\[
x_0 = \mu = (7/13) + (9/13)i = 0.5385 + 0.6923i
\]

\[
x_1 = vx = 0.1429 + 0.1584i
\]
The n-term approximate solutions are:

\[ \phi_1 = 0.5385 + 0.6923i \]
\[ \phi_2 = 0.6814 + 0.8507i \]
\[ \phi_3 = 0.7564 + 0.9225i \]
\[ \vdots \]
\[ \phi_n = 1 + i. \]

It is clear that the imaginary component is converging more rapidly than the real component so we have a differing convergence for the real and imaginary components of complex roots.

II. Cubic Equations

Consider now equations of the type \( z^3 + A_2 z^2 + A_1 z + A_0 = 0 \). The \( z^2 \) term is ordinarily eliminated by substituting \( z = x - A_2/3 \) to get an equation in the form \( x^3 - qx - r = 0 \). Thus, the equation

\[ z^3 + 9z^2 + 23z + 14 = 0 \]

becomes (substituting \( z = x - 3 \))

\[ x^3 - 4x - 1 = 0 \]

whose roots are 2.11, -1.86, -0.254. If we solve this by decomposition we write the equation in the form \( [1] -qx = r - x^3 \) or

\[ -4x = 1 - x^3 \]
\[ x = -\frac{1}{4} + \frac{1}{4} x^3 \]
\[ x = x_0 + \frac{1}{4} \sum_{n=0}^{\infty} A_n. \]
For this nonlinearity (Nx in Adomian's notation [1])

\[ A_0 = x_0^1 \]
\[ A_1 = 3x_0^2 x_1 \]
\[ A_2 = 3x_0^2 x_2 + 3x_0 x_1^2 \]
\[ A_3 = 3x_0^2 x_3 + 6x_0 x_1 x_2 + x_1^3 \]
\[ A_4 = 3x_0^2 x_4 + 3x_0 x_1^2 + 6x_0 x_1 x_3 + 3x_1^2 x_2 \]
\[ A_5 = 3x_0^2 x_5 + 6x_0 x_1 x_4 + 6x_0 x_2 x_3 + 3x_1^2 x_3 + 3x_1 x_2^2. \]

(We caution against simply extrapolating the \( A_n \) to higher \( n \). We cannot include the complete generating scheme for any \( n \) in this paper. It depends on the actual nonlinearity and is lengthy to discuss so it will be dealt with elsewhere. The objective of this paper is to show applicability to algebraic equations, not to provide a handbook.) Thus \( x_0 = -0.25, x_1 = \frac{1}{4} A_0 = -0.004 \), etc. Thus the one-term approximation \( \phi_1 = -0.25 \), the two-term approximation \( \phi_2 = -0.254 \), and \( x_2 \approx 0 \) for an answer to three decimal places so the correct solution is obtained already with \( \phi_2 \) (again for the smallest root). \( \phi_3 \) gives \(-0.254 \) with no more change to 3 decimal places. Computing 6 terms gives \(-0.25410168 \), which doesn't change any further to 8 place accuracy.

If we now divide \( x^3 - 4x - 1 \) by \( x - 0.254 \), we obtain \( x^2 + 0.254x - 3.9375 \), which yields the other two roots by either the quadratic formula or the decomposition method.

The equation \( x^3 - 6x^2 + 11x - 6 = 0 \) has roots \((1, 2, 3)\). Written in the form

\[ x = \frac{6}{11} + \frac{6}{11} x^2 - \frac{1}{11} x^3 \]

and solving by the decomposition method, it yields \( x_0 = 0.5455, x_1 = 0.1475, \ldots \), and the solution \( x = 1 \) in eight terms.

**Example.** \( x^3 + 4x^2 + 8x + 8 = 0 \) is satisfied by \( x = -2 \). Calculating this with appropriate \( A_n \) for the \( x^2 \) and \( x^3 \) terms, we get

\[ x_0 = -1.0 \]
\[ x_1 = -0.375 \]
\[ x_2 = -0.234375 \]
\[ x_3 = -0.1640625 \]
\[ x_4 = -0.1179199 \]
\[ x_5 = -0.0835876. \]
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If we sum these terms we get approximately $x = -1.98$, which makes us guess $x = -2.0$ and try it in the equation. (It is interesting to note, however, that we actually have an oscillating convergence. If we sum 10 terms, we get $x = -2.0876342$, which is a peak departure from $x = -2.0$. At 20 terms we have a peak departure in the opposite direction with $x = -1.9656587$. At 100 terms we have $-1.997313$.)

**Example.**

\[ x^3 - 6x^2 + 11x - 6 = 0 \]

\[ x = \frac{6}{11} + \frac{6}{11} x^2 - \frac{1}{11} x^3 \]

\[ \sum_{n=0}^{\infty} x_n = x_0 + \frac{6}{11} \sum_{n=0}^{\infty} A_n - \frac{1}{11} \sum_{n=0}^{\infty} B_n \]

expanding the $x^2$ and $x^3$ terms in our usual polynomials but using $A_n$ and $B_n$ to distinguish the two.

\[ x_0 = \frac{6}{11} = 0.5455 \]

\[ x_1 = \left(\frac{6}{11}\right) A_0 - \left(\frac{1}{11}\right) B_0 = \left(\frac{10}{11}\right) \left(\frac{6^3}{11^2}\right) = 0.147531 \]

\[ x_2 = \left(\frac{6}{11}\right) A_1 - \left(\frac{1}{11}\right) B_1 = (19)(10) \left(\frac{6^4}{11^3}\right) = 0.0758160 \]

\[ x_3 = \left(\frac{6}{11}\right) A_2 - \left(\frac{1}{11}\right) B_2 = \left(\frac{6^7}{11^{10}}\right)(3610) = 0.038962 \]

\[ \phi_1 = 0.5455 \]

\[ \phi_2 = 0.693031 \]

\[ \phi_3 = 0.768847 \]

\[ \phi_4 = 0.80780 \]

\[ \vdots \]

where $\phi_n \to 1.0$ as $n \to \infty$.

We can write $x^3 - (r_1 + r_2 + r_3) x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3) x - r_1 r_2 r_3 = 0$, then

\[ x = \frac{(r_1 r_2 r_3)}{(r_1 r_2 + r_1 r_3 + r_2 r_3)} + \frac{(r_1 + r_2 + r_3)}{(r_1 r_2 + r_1 r_3 + r_2 r_3)} x^2 \]

\[ \frac{1}{(r_1 r_2 + r_1 r_3 + r_2 r_3)} x^3 \]
choose \( r_1 < r_2 < r_3 \) or \( r_2 = ar_1 \) and \( r_3 = \beta r_1 \) where \( \alpha, \beta \) are appropriate real fractions. Then the equation for \( x \) will become

\[
x = (r_1) \left( \frac{\alpha \beta + (1 + \alpha + \beta)(x^2/r_1^2) - (x^3/r_1^3)}{\alpha + \beta + \alpha \beta} \right)
\]

where the bracketed quantity \( \to 1 \) and the first solution found is \( r_1 \). Thus, letting \( \psi = (x/r_1) \)

\[
\psi = \left( \frac{\alpha \beta}{\gamma} \right) + \left( \frac{1 + \alpha + \beta}{\gamma} \right) \psi^2 - \left( \frac{1}{\gamma} \right) \psi^3
\]

where \( \gamma = \alpha + \beta + \alpha \beta \). Then if \( \psi = \sum_{n=0}^{\infty} \psi_n \) and \( \psi^2 = \sum_{n=0}^{\infty} A_n \) and \( \psi^3 = \sum_{n=0}^{\infty} B_n \),

\[
\psi_0 = \left( \frac{\alpha \beta}{\gamma} \right)
\]

\[
\psi_1 = \left( \frac{1 + \alpha + \beta}{\gamma} \right) \left( \frac{\alpha \beta}{\gamma} \right)^2 - \left( \frac{1}{\gamma} \right) \left( \frac{\alpha \beta}{\gamma} \right)^3
\]

\[
\vdots
\]

If, for example, \( r_1 < \frac{1}{10} r_2 \) and \( r_2 < \frac{1}{10} r_3 \),

\[
\psi_0 = \frac{(10)(100)}{(10 + 100 + 1000)} = \frac{1000}{1110} \approx 1
\]

and

\[
\psi_1 = \left( \frac{111}{1110} \right) \left( \frac{1000}{1110^2} \right) - \left( \frac{1}{1110} \right) 1000^3
\]

so terms are indeed rapidly approaching zero and \( r_1 \) will be the root calculated.

III. Polynomial Operators

Higher-degree polynomial equations are similarly solved. In the cubic case it is, as we see above, not necessary, of course, to eliminate the quadratic term. We can solve the original equation by simply substituting the appropriate \( A_n \) summations for both \( x^2 \) and \( x^3 \) terms. Even higher-degree equations (or nonintegral powers or negative powers) can be equally well
handled by substituting appropriate $A_n$ for each nonlinearity (see Ref. [1]). Let's consider an equation in the form

$$\sum_{\mu=0}^{n} \gamma_{\mu} x^{\mu} = 0$$

where $\gamma_{\mu}$ are given constants and seek the roots $r_1, r_2, \ldots, r_n$ (assumed to be real) satisfying $\prod_{i=1}^{n} (x - r_i) = 0$. We found if $N_n$ or $f(x) = x^2$, for example, $A_0 = x_0^2$ or $f(x_0)$.

Now we have $N_n = \sum_{\mu=0}^{n} \gamma_{\mu} x^{\mu}$ so $A_0 = \sum_{\mu=0}^{n} \gamma_{\mu} x^{\mu}$. Since $\sum_{\mu=0}^{n} \gamma_{\mu} x^{\mu} = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \cdots + \gamma_n x^n$, we can write the $A_n$ for each term or for the entire polynomial. Let's consider a specific example of the form $f(x) = \sum_{\mu=0}^{n} \gamma_{\mu} x^{\mu} = \gamma_n x^n + \gamma_{n-1} x^{n-1} + \cdots + \gamma_1 x + \gamma_0$ with $\gamma_n \neq 0$ and $\gamma_i$ constant for $0 \leq i \leq n$.

**EXAMPLE.** It is interesting to consider a 5th-order polynomial operator since no formula exists for $n = 5$ or higher. The equation $x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 = 0$ has the roots 1, 2, 3, 4, 5. To calculate all the roots we rewrite the equation in Adomian's usual form as

$$x = (120/274) + (225/274) x^2 - (85/274) x^3$$
$$+ (15/274) x^4 - (1/274) x^5$$

$$x = 0.43796 + 0.82117 x^2 - 0.31022 x^3$$
$$+ 0.054745 x^4 - 0.0036496 x^5$$

or

$$x = k + \sum_{n=2}^{5} \gamma_n x^n$$

where

$$k = 0.43796$$
$$\gamma_2 = 0.82117$$
$$\gamma_3 = -0.31022$$
$$\gamma_4 = 0.054745$$
$$\gamma_5 = -0.0036496.$$
The notation $A_0(x^2)$ means the $A_0$ for the $x^2$ term, etc. Thus
\[
\begin{align*}
    x_1 &= 0.82117x_0^2 - 0.31022x_0^3 + 0.054745x_0^4 - 0.0036496x_0^5 \\
    x_1 &= 0.15751 - 0.026060 + 0.0020141 - 0.00005881 \\
    x_1 &= 0.13341.
\end{align*}
\]

Hence $\phi_2 = x_0 + x_1 = 0.57137$.

\[
x_2 = \gamma_2 A_1(x^2) + \gamma_3 A_1(x^3) + \gamma_4 A_1(x^4) + \gamma_5 A_1(x^5)
\]

where
\[
\begin{align*}
    A_1(x^2) &= 2x_0 x_1 \\
    A_1(x^3) &= 3x_0^2 x_1 \\
    A_1(x^4) &= 4x_0^3 x_1 \\
    A_1(x^5) &= 5x_0^4 x_1.
\end{align*}
\]

Consequently,
\[
\begin{align*}
    x_2 &= \left\{ (0.82117)(2) x_0 - (0.31022)(3) x_0^2 + (0.05745)(4) x_0^3 \\
    &\quad - (0.0036496)(5) x_0^4 \right\} x_1 \\
    x_2 &= 0.0746299.
\end{align*}
\]

Then $\phi_3 = 0.6459999 = 0.6460$. Continuing
\[
x_3 = \gamma_2 A_2(x^2) + \gamma_3 A_2(x^3) + \gamma_4 A_2(x^4) + \gamma_5 A_2(x^5)
\]

etc., as necessary. The $A_n$ are generated by rather complex rules necessarily dealt with elsewhere since they are applicable to differential and partial differential equations as well and require much discussion. They are to be viewed here as a special set of polynomials proposed by Adomian for the expression of nonlinear terms in his decomposition method which are given or can be obtained. The first few of these are:
\[
\begin{align*}
    A_0 &= h_0(x_0) \\
    A_1 &= h_1(x_0) x_1 \\
    A_2 &= \frac{1}{2} \left\{ h_2(x_0) x_1^2 + 2h_1(x_0) x_2 \right\} \\
    A_3 &= \frac{1}{6} \left\{ h_3(x_0) x_1^3 + 6h_2(x_0) x_1 x_2 + 6h_1(x_0) x_3 \right\}
\end{align*}
\]
where $h_i = \frac{d^i f}{dx_i}$ for the function $f(x)$.

Let's list final results for the 5th-degree equation above to 10 digit accuracy.

\[
A_4 = \frac{1}{24} \{ h_4(x_0) x_1^4 + 12 h_3(x_0) x_2^3 x_2 + h_2(x_0) \left[ 12 x_2^2 + 24 x_1 x_3 \right] \\
+ 24 h_1(x_0) x_4 \} \\
A_5 = \frac{1}{120} \{ h_5(x_0) x_1^5 + 20 h_4(x_0) x_1^3 x_2 + 60 h_3(x_0) \left[ x_1 x_2^2 + x_1^2 x_3 \right] \\
+ 120 h_2(x_0) \left[ x_2 x_3 + x_1 x_4 \right] + 120 h_1(x_0) x_5 \} \\
\]

$x_0 = 0.4379562044 \quad \phi_0 = 0.4379562044$
$x_1 = 0.1334006838 \quad \phi_1 = 0.5713568882$
$x_2 = 0.0745028484 \quad \phi_2 = 0.6458597366$
$x_3 = 0.0500356263 \quad \phi_3 = 0.6958953629$
$x_4 = 0.0449342233 \quad \phi_4 = 0.7408295862$
$x_5 = 0.0446966625 \quad \phi_5 = 0.7855262487$
$x_6 = 0.0331390668 \quad \phi_6 = 0.8186653155$
$x_7 = 0.0272374949 \quad \phi_7 = 0.8459028104$
$x_8 = 0.022258001 \quad \phi_8 = 0.8681608114$
$x_9 = 0.0196274208 \quad \phi_9 = 0.8877882322$
$x_{10} = 0.0166467228 \quad \phi_{10} = 0.904434955$

$\psi_1 = 56.2\%$
$\psi_2 = 42.9\%$
$\psi_3 = 35.4\%$
$\psi_4 = 30.4\%$
$\psi_5 = 25.9\%$
$\psi_6 = 21.4\%$
$\psi_7 = 18.1\%$
$\psi_8 = 15.4\%$
$\psi_9 = 13.9\%$
$\psi_{10} = 11.2\%$
$\psi_{11} = 9.56\%$
The error \( \psi \) decreases gradually to less than 10% by \( \psi_{11} \) but it can easily be
carried further by computer. The convergence for inversion in this case of a
quintic operator is relatively poor because of the greater number of more
closely spaced roots and the case of equal roots will be the poorest case.

For \( f(x) = x^k \) where \( k \) is an integer \( \geq 2 \), let's write \( h_n = \frac{d^n f}{dx^n} \) for \( n > 0 \).
(We will write \( h_n(x_0) \) for \( \left( \frac{d^n f}{dx^n} \right)_{x_0} \) for the computation of the \( A_n \).) Then
for \( x^k \),

\[
\begin{align*}
h_0 &= x^k \\
h_1 &= kx^{k-1} \\
&\vdots \\
h_n &= k(k-1) \cdots (k-n+1)x^{k-n} = \binom{k}{n} x^{k-n}
\end{align*}
\]

where \( \binom{k}{n} = \frac{k!}{n!(k-n)!} \). Consequently, the \( A_n \) for \( f(x) = x^k \) are given by

\[
\begin{align*}
A_0 &= x_0^k \\
A_1 &= \left\{ \binom{k}{1} x_0^{k-1} \right\} x_1 \\
A_2 &= \frac{1}{2} \left\{ \binom{k}{2} x_0^{k-2} \right\} x_1^2 + \left\{ \binom{k}{1} x_0^{k-1} \right\} x_2 \\
A_3 &= \frac{1}{6} \left\{ \binom{k}{3} x_0^{k-3} \right\} x_1^3 + \left\{ \binom{k}{2} x_0^{k-2} \right\} x_1x_2 + \left\{ \binom{k}{1} x_0^{k-1} \right\} x_3 \\
A_4 &= \frac{1}{24} \left\{ \binom{k}{4} x_0^{k-4} \right\} + \frac{1}{2} \left\{ \binom{k}{3} x_0^{k-3} \right\} x_1^2x_2 \\
&\quad + \left\{ \binom{k}{2} x_0^{k-2} \right\} \left[ \frac{1}{2} x_2^2 + x_1x_3 \right] + \left\{ \binom{k}{1} x_0^{k-1} \right\} x_4 \\
A_5 &= \frac{1}{120} \left\{ \binom{k}{5} x_0^{k-5} \right\} x_1^5 + \frac{1}{6} \left\{ \binom{k}{4} x_0^{k-4} \right\} x_1^3x_2 + \frac{1}{2} \left\{ \binom{k}{3} x_0^{k-3} \right\} x_1^4x_3 + \left\{ \binom{k}{1} x_0^{k-1} \right\} x_5
\end{align*}
\]

We observe that the subscripts for \( A_n \) always add to \( n \) and the superscripts of the \( x_i \)'s always add to \( k \).

The above work will yield the lowest root, or 1, reducing the equation to a
4th power then the root 2, etc. We can do the problem more rapidly as follows.
Let's write a general polynomial in $x$ with constant nonzero coefficients.

$$f(x) = \sum_{i=k}^{0} \gamma_i x^i = \gamma_k x^k + \cdots + \gamma_0.$$ 

Now

$$h_0 = \sum_{i=k}^{0} \gamma_i x^i$$

$$h_1 = \sum_{i=k}^{1} i \gamma_i x^{i-1}$$

$$\vdots$$

$$h_n = \sum_{i=k}^{n} \binom{i}{n} \gamma_i x^{i-n} \quad (k > n)$$

$$\vdots$$

$$h_k = \binom{k}{k} \gamma_k x^{k-k} = \gamma_k k! \quad (k = n)$$

$$h_{k+1} = 0 \quad \text{or} \quad h_n = 0 \quad \text{for} \ n > k.$$

The $A_n$ can now be given

$$A_0 = \sum_{i=k}^{0} \gamma_i x_0^i$$

$$A_1 = \left( \sum_{i=k}^{1} \binom{i}{1} \gamma_i x_0^{i-1} \right) x_1$$

$$A_2 = \frac{1}{2} \left( \sum_{i=k}^{2} \binom{i}{2} \gamma_i x_0^{i-2} \right) x_1^2 + \left( \sum_{i=k}^{1} \binom{i}{1} \gamma_i x_0^{i-1} \right) x_2$$

$$A_3 = \frac{1}{6} \left( \sum_{i=k}^{3} \binom{i}{3} \gamma_i x_0^{i-3} \right) x_1^3 + \left( \sum_{i=k}^{2} \binom{i}{2} \gamma_i x_0^{i-2} \right) x_1 x_2$$

$$+ \left( \sum_{i=k}^{1} \binom{i}{1} \gamma_i x_0^{i-1} \right) x_3$$

$$A_4 = \frac{1}{24} \left( \sum_{i=k}^{4} \binom{i}{4} \gamma_i x_0^{i-4} \right) + \frac{1}{2} \left( \sum_{i=k}^{3} \binom{i}{3} \gamma_i x_0^{i-3} \right) x_1^2 x_2$$

$$+ \left( \sum_{i=k}^{2} \binom{i}{2} \gamma_i x_0^{i-2} \right) \left( \frac{1}{2} x_1^2 + x_1 x_3 \right) + \left( \sum_{i=k}^{1} \binom{i}{1} \gamma_i x_0^{i-1} \right) x_4$$
\[ A_5 = \frac{1}{120} \left\{ \sum_{i=k}^{5} \binom{i}{5} \gamma_i x_0^{i-5} \right\} x_1^5 + \frac{1}{6} \left\{ \sum_{i=k}^{4} \binom{i}{4} \gamma_i x_0^{i-4} \right\} x_1^3 x_2 \]
\[ + \frac{1}{2} \left\{ \sum_{i=k}^{3} \binom{i}{3} \gamma_i x_0^{i-3} \right\} (x_1 x_2^2 + x_1^2 x_3) \]
\[ + \left\{ \sum_{i=k}^{2} \binom{i}{2} \gamma_i x_0^{i-2} \right\} (x_2 x_3 + x_1 x_4) + \left\{ \sum_{i=k}^{1} \binom{i}{1} \gamma_i x_0^{i-1} \right\} x_5 \]

etc.

from which polynomial equations can be solved more rapidly than with individual substitutions for the various powers as we did earlier in this paper.

**Negative powers.** Consider an example like \( x = 2 + x^{-2} \) or the more general form

\[ x = k + x^{-m}. \]

We write

\[ \sum_{n=0}^{\infty} x_n = k + \sum_{n=0}^{\infty} A_n \]

with \( x_0 = k \) and \( x_n = A_{n-1} \) for \( n \geq 1 \). Then

\[ x_1 = A_0 = x_0^{-m} = k^{-m} \]
\[ x_2 = A_1 = -mx_0^{-(m+1)} x_1 \]
\[ x_3 = A_2 = \frac{1}{2} m(m+1) x_0^{-(m+2)} x_1^2 - mx_0^{-(m+1)} x_2 \]
\[ x_4 = A_3 = -\frac{1}{6} m(m+1)(m+2) x_0^{-(m+3)} x_1^3 + m(m+1) x_0^{-(m+2)} x_1 x_2 \]
\[ - mx_0^{-(m+1)} x_3 \]
\[ x_5 = A_4 = \frac{1}{24} m(m+1)(m+2)(m+3) x_0^{-(m+4)} x_1^4 - \frac{1}{2} m(m+1) \]
\[ \times (m+2) x_0^{-(m+3)} x_1^2 x_2 + m(m+1) x_0^{-(m+2)} \left[ \frac{1}{2} x_2^2 + x_1 x_3 \right] \]
\[ - mx_0^{-(m+1)} x_4. \]

If \( k = 2 \) and \( m = 2 \), then \( x_0 = 2 \) and

\[ x_1 = 2^{-2} = 0.25 \]
\[ x_2 = -2(2)^3 (0.25) = -0.0625 \]
\[ x_3 = \frac{1}{2}(2)(3)2^{-4}(0.25)^2 - (2)(2)^{-3} (-0.0625) = 0.02734375 \]
\[ x_4 = -0.0146484375 \]
\[ x_5 = -0.0087280273. \]
By $\phi_6$ we get an excellent approximation to the solution (2.205569431) and note again rapid convergence.

$\phi_1 = 2 \quad \psi_1 = 9.32\%$
$\phi_2 = 2.25 \quad \psi_2 = -2.02\%$
$\phi_3 = 2.1875 \quad \psi_3 = 0.82\%$
$\phi_4 = 2.21484375 \quad \psi_4 = -0.42\%$
$\phi_5 = 2.200189063 \quad \psi_5 = 0.24\%$
$\phi_6 = 2.20891709 \quad \psi_6 = -0.15\%$

*Non integer powers.* Let's now consider inversion of algebraic operator equations involving fractional or noninteger powers, e.g., consider

$$x = k + x^{1/2}.$$  

Write

$$\sum_{n=0}^{\infty} x_n = k + \sum_{n=0}^{\infty} A_n$$

where $x_0 = k$ and $x_n - A_{n-1}$ for $n \geqslant 1$. Then

$$x_1 = A_0 = x_0^{1/2} = k^{1/2}$$
$$x_2 = A_1 = \frac{1}{3}x_0^{1/2}x_1 = \frac{1}{3}(k)^{-1/4}(k)^{1/2} = \frac{1}{3}$$
$$x_3 = A_2 = \frac{1}{6}x_0^{-1/2}x_2 - \frac{1}{6}x_0^{-3/2}x_1^2 = \frac{1}{6}k^{-1/2}$$
$$x_4 = A_3 = 0$$
$$x_5 = A_4 = -\frac{1}{128}k^{-3/2}$$

$$\vdots$$

$$\phi_6 = (k) + (k)^{1/2} + \frac{1}{2} + \frac{1}{4}k^{-1/2} + 0 - \frac{1}{128}k^{-3/2}.$$  

If $k = 2$ we expect the solution to converge to $x = 4$. As verification, we get

$$x_0 = 2$$
$$x_1 = 1.414213562$$
$$x_2 = 0.50$$
$$x_3 = 0.0883883476$$
$$x_4 = 0$$
$$x_5 = -0.0027621359$$
and 
\[ \phi_1 = 2 \quad \psi_1 = 50\% \\
\phi_2 = 3.414213562 \quad \psi_2 = 14.65\% \\
\phi_3 = 3.914213562 \quad \psi_3 = 2.15\% \\
\phi_4 = 4.00260191 \quad \psi_4 = -0.065\% \\
\phi_5 = \phi_4 + 0 \quad \psi_5 = \psi_4 \\
\phi_6 = 3.999839774 \quad \psi_6 = 0.0040\%.
\]

Thus \( \phi_6 \) is an excellent approximation (in fact \( \phi_4 \) is!). With \( \phi_6 \) we have 4/1000 of 1\% error. Let's be more general now and consider nonlinear terms \( Nx = x^{1/m} \) where \( m \) belongs to the set of positive integers.

**EXAMPLE.**

\[
x = k + x^{1/m} \\
\sum_{n=0}^{\infty} x_n = k + \sum_{n=0}^{\infty} A_n
\]

where the \( A_n \) are the usual Adomian polynomials generated for the specific nonlinearity under consideration. Then

\[
x_0 = k \\
x_n = A_{n-1} \quad \text{for} \quad n \geq 1.
\]

The \( A_n \) are given by

\[
A_0 = x_0^{1/m} \\
A_1 = (1/m) x_0^{(1/m)-1} x_1 \\
A_2 = 1/2(1/m)((1/m) - 1) x_0^{(1/m)-2} x_1^2 + (1/m) x_0^{(1/m)-1} x_2 \\
A_3 = 1/6(1/m)((1/m) - 1)((1/m) - 2) x_0^{(1/m)-3} x_1^3 \\
\quad + (1/m)((1/m) - 1) x_0^{(1/m)-2} x_1 x_2 \\
\quad + (1/m) x_0^{(1/m)-1} x_3 \\
\vdots
\]

Since \( x_n = A_{n-1} \) in this problem, we now have the \( x_n \). Their general form of \( x_n \) is \( \sigma(1/m) k^{(n/m) - n + 1} \) where \( \sigma(1/m) \) has the form \( \sum_i \prod_j \alpha_i (1/m - \beta_j) \). If \( m = 2 \) and \( k = 2 \) we get precisely the previous results in the preceding example.

We now also see that we can consider operators involving a cube root, a fourth root, etc.
Decimal powers. Now consider solution of algebraic equations involving (rational or irrational) decimal powers, first taking up the case of rational powers.

\[ x = k + x^{a/b} \]

where \( k \) is real and \( a, b \) are positive integers. Write

\[ \sum_{n=0}^{\infty} x_n = k + \sum_{n=0}^{\infty} A_n \]

where \( x_0 = k \) and \( x_n = A_{n-1} \) for \( n \geq 1 \). (Thus we have \( N x = x^{a/b} = \sum_{n=0}^{\infty} A_n \).

Now

\[ A_0 = x_0^{a/b} \]
\[ A_1 = (a/b) x_0^{(a/b) - 1} x_1 \]
\[ A_2 = (a/b) x_0^{(a/b) - 1} x_2 + \frac{1}{2} (a/b)((a/b) - 1) x_0^{(a/b) - 2} x_1^2 \]
\[ A_3 = (a/b) x_0^{(a/b) - 1} x_3 + (a/b)((a/b) - 1) x_0^{(a/b) - 2} x_1 x_2 \]
\[ \quad + \frac{1}{6} (a/b)((a/b) - 1)((a/b) - 2) x_0^{(a/b) - 3} x_1^3 \]

Hence,

\[ x_0 = k \]
\[ x_1 = A_0 = k^{a/b} \]
\[ x_2 = A_1 = (a/b) k^{2(a/b) - 1} \]
\[ x_3 = A_2 = \{(a/b)^2 + \frac{1}{2} (a/b)((a/b) - 1)\} k^{3(a/b) - 2} \]

As a specific case, choose \( a = 3, b = 11, k = 2 \). Then \( a/b = 0.272727... \) and \( x = 2 + x^{3/11} \)

\[
\begin{align*}
x_0 &= 2 & \phi_1 &= 2 \\
x_1 &= 1.208089444 & \phi_2 &= 3.208089444 \\
x_2 &= 0.1990200144 & \phi_3 &= 3.407109458 \\
x_3 &= -0.0109288172 & \phi_4 &= 3.396180641 \\
\end{align*}
\]

Notice \( \phi_3 \) and \( \phi_4 \) differ very little as we try substitution into the original equation of \( \phi_4 \) as an approximate solution. Then

\[ \bar{\phi}_4 = 2 + \phi_4^{3/11} = 3.395775811. \]
We see \( \tilde{\phi}_4 \) is very close to \( \phi_4 \), differing by about 0.01%. (We have defined \( \tilde{\phi}_n = k + \phi_n^{a/b} \) to see if the approximate solution satisfies the original equation.)

Now we consider the case of irrational powers. Write

\[ x = k - x^\gamma \]

letting \( k \) be real and \( \gamma \) an irrational number such as \( e \) or \( \pi \). Now

\[
\sum_{n=0}^{\infty} x_n = k - \sum_{n=0}^{\infty} A_n
\]

where \( x_0 = k \) and \( x_n = -A_{n-1} \) for \( n \geq 1 \). Since \( Nx = x^\gamma = \sum_{n=0}^{\infty} A_n \),

\[
\begin{align*}
A_0 & = x_0^\gamma \\
A_1 & = \gamma x_0^{\gamma-1} x_1 \\
A_2 & = \gamma x_0^{\gamma-1} x_2 + \frac{1}{2} \gamma(\gamma - 1) x_0^{\gamma-3} x_1^2 \\
A_3 & = \gamma x_0^{\gamma-1} x_3 + \gamma(\gamma - 1) x_0^{\gamma-2} x_1 x_2 \\
& \quad + \frac{1}{6} \gamma(\gamma - 1)(\gamma - 2) x_0^{\gamma-3} x_1^3.
\end{align*}
\]

Now the components \( x_n \) of the solution \( x = \sum_{n=0}^{\infty} x_n \) can be computed

\[
\begin{align*}
x_0 & = k \\
x_1 & = -k^\gamma \\
x_2 & = \gamma k^{2\gamma-1} \\
x_3 & = \gamma^2 + \frac{1}{2} \gamma(\gamma - 1) k^{3\gamma-2} \\
x_4 & = \gamma^2(\gamma - 1) + \frac{1}{6} \gamma(\gamma - 1)(\gamma - 2) + \gamma^3 + \frac{1}{2} \gamma^2(\gamma - 1) k^{4\gamma-3}.
\end{align*}
\]

For a specific example we now choose \( k = 1/\pi \) and \( \gamma = \pi \)

\[ x = (1/\pi) - x^\pi \]

(letting \( \pi = 3.1415927 \) and \( 1/\pi = 0.3183099 \) for the computation). We get \( x = 0.296736 \) which is within a hundredth of 1%.

Random algebraic equations. The treatment of algebraic equations by the decomposition method suggests further generalization to random algebraic equations. Such equations, with random coefficients, arise in engineering, physics, and statistics whenever random errors are involved. Random matrices too are found in finite-dimensional approximation models

\(^2\) Application of the decomposition method also solves equations with random matrix operators. See [5–7].
for random Hamiltonian operators and various engineering applications concerned with systems of linear random equations. Thus suppose one has the equation

\[ x^3 + \alpha x^2 + \beta x + \gamma = 0 \]

where \( \alpha \) is stochastic. Then we have

\[ x = -\left(\frac{\gamma}{\beta}\right) - \left(1/\beta\right)x^3 - \left(\alpha/\beta\right)x^2 \]

where \( \alpha \) is a stochastic process and \( \beta, \gamma \) are constants. We now write

\[ x = x_0 - \left(1/\beta\right) \sum_{n=0}^{\infty} A_n - \left(\alpha/\beta\right) \sum_{n=0}^{\infty} B_n \]

where the \( A_n, B_n \) are the appropriate Adomian polynomials computed for the nonlinear terms \( x^3 \) and \( x^2 \). For example,

\[ x_1 = -\left(1/\beta\right) x_0^3 - \left(\alpha/\beta\right) x_0^2 \]

hence \( x_1 \) involves a stochastic coefficient in the second term. Continuing one writes \( \phi_n \) and appropriate statistics such as \( \langle \phi_n \rangle \).

If we consider a quadratic operator and a forcing term in the form

\[ y^2 + by + c = x(t) \]

where \( b \) and \( c \) can be functions of \( t \), we can write immediately

\[ y = (1/b)(t - c) - \left(1/\beta\right) \sum_{n=0}^{\infty} A_n(y^2), \quad \text{or, since} \quad y_0 = (1/b)(t - c) \quad \text{and} \quad y_1 = -\left(1/\beta\right)A_0, \quad \text{etc.,} \]

\[ y_1 = -\frac{1}{b^3} (x - c)^2 \]

\[ y_2 = \frac{2}{b^3} (x - c)^3 \]

\[ y_3 = -\frac{5}{b^7} (x - c)^4 \]

\[ \vdots \]

\[ y = \sum_{n=0}^{\infty} \frac{(-1)^n k_n}{b^{2n+1}} (x - c)^{n-1} \]

with \( k_n \) as appropriately defined constants. Clearly \( x(t) \) can be stochastic (or \( x(t, \omega) \)). The \( b(t) \) can have a fluctuating or random component and be written as \( b_0(t) + \beta(t, \omega) \), in which case
Conclusions. We have seen that algebraic equations can be handled by the decomposition method and it provides a useful method for computation of roots of polynomial equations often yielding a very rapid convergence. As discussed earlier by Adomian [1], we have a computational and highly convergent system to solve problems of the real world more realistically without assumptions changing the essential nonlinear nature. Whether we deal with differential or partial differential equations [1, 2, 4, 5] or algebraic systems as demonstrated in this paper (and in Ref. [5]3), an accurate methodology is available for physical applications and more realistic modeling.

REFERENCES


3 Reference [5] continues the theory of Ref. [1], applying it to algebraic equations of all kinds, differential and partial differential equations, delay equations, coupled systems, matrix equations, etc. Applications to physics and engineering appear in [7].