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On the Solution of Algebraic Equations by the Decomposition Method

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The decomposition method (G. Adomian, "Stochastic Systems," Academic Press, New York, 1983) developed to solve nonlinear stochastic differential equations has recently been generalized to nonlinear (and/or) stochastic partial differential equations, systems of equations, and delay equations and applied to diverse applications. As pointed out previously (see reference above) the methodology is an operator method which can be used for nondifferential operators as well. Extension has also been made to algebraic equations involving real or complex coefficients. This paper deals specifically with quadratic, cubic, and general higher-order polynomial equations and negative, or nonintegral powers, and random algebraic equations. Further work on this general subject appears elsewhere (G. Adomian, "Stochastic Systems II," Academic Press, New York, in press).

I. QUADRATIC EQUATIONS

The decomposition method [1, 2] is applied to generic operator equations of the form $\mathscr{F}u = g$ where $\mathscr{F}$ may be a nonlinear (and/or) stochastic operator and x a stochastic process on a appropriate probability space. The basic equation is considered in the form $\mathscr{L}u + \mathscr{N}u = g$, or $Lu + Nu = g$ in the deterministic case where L is a linear (deterministic) operator and N a nonlinear (deterministic) operator. (In the case where the operators involve stochasticity, script letters $\mathscr{L}$, $\mathscr{N}$ are preferred.) (If we write an ordinary quadratic equation $ax^2 + bx + c = 0$ in the form $Lu + Nu = g$, identifying $Nx = ax^2$, $L = b$, and $g = -c$ we have $Lu = g - Nu$ or

$$bx = -c - ax^2.$$

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The operation L^{-1} in the referenced work for differential equations is an integral operator. Here it is simply division by b. Hence

$$x = (-c/b) - (a/b)\, x^2$$

in the standard format of the referenced work [1]. The solution x in this methodology is now decomposed into components $x_0 + x_1 + \cdots$ where x_0 is taken as $(-c/b)$ here and $x_1, x_2, \ldots,$ are still to be identified. Thus

$$x_0 = -c/b.$$

We now have

$$x = x_0 - (a/b)\, x^2$$

with x_0 known. In the general methodology, the nonlinear term without the coefficient—in this case x^2—is replaced by $\sum_{n=0}^{\infty} A_n$ where $A_n(x_0, x_1, \ldots, x_n)$ are functions of the x_i defined by Adomian [1, 2]. Since the A_n have been determined for large classes of nonlinearities by methods previously published, we will only list the necessary A_n for this paper. Adomian's A_n polynomials are found for the particular non-linearity by a generating scheme just as one might develop Hermite, Lagrange, or Laguerre polynomials. Rules are given in the referenced works. For the example $Nx = x^2$ we have

$$\begin{aligned}
A_0 &= x_0^2\\
A_1 &= 2x_0x_1\\
A_2 &= x_1^2 + 2x_0x_2\\
A_3 &= 2x_1x_2 + 2x_0x_3\\
A_4 &= x_2^2 + 2x_1x_3 + 2x_0x_4\\
A_5 &= 2x_0x_5 + 2x_1x_4 + 2x_2x_3\\
A_6 &= x_3^2 + 2x_0x_6 + 2x_1x_5 + 2x_2x_4\\
A_7 &= 2x_0x_7 + 2x_1x_6 + 2x_2x_5 + 2x_3x_4\\
A_8 &= x_4^2 + 2x_0x_8 + 2x_1x_7 + 2x_2x_6 + 2x_3x_5\\
&\vdots
\end{aligned}$$

Examining the subscripts we note the sum of subscripts in each term is n. Now

$$x = x_0 - (a/b) \sum_{n=0}^{\infty} A_n$$

requires

$$x_1 = -(a/b)A_0 = -(a/b)x_0^2$$
$$x_2 = -(a/b)A_1 = -(a/b)(2x_0x_1)$$
$$x_3 = -(a/b)A_2 = -(a/b)(x_1^2 + 2x_0x_2)$$
$$x_4 = -(a/b)A_3 = -(a/b)(2x_1x_2 + 2x_0x_3)$$
$$x_5 = -(a/b)A_4 = -(a/b)(x_2^2 + 2x_1x_3 + 2x_0x_4)$$
$$\vdots$$

thus the x_i are determined.

We note in the example $Nx = x^2$ that if we expand $(x_0 + x_1 + \cdots)^2$ into $x_0^2 + x_1^2 + x_2^2 + \cdots + 2x_0x_1 + 2x_0x_2 + \cdots + 2x_1x_2 + \cdots$, we must choose $A_0 = x_0^2$ but A_1 could be $x_1^2 + 2x_0x_1$. The sum of the subscripts for x_1^2 or x_1x_1 is higher than for the x_0x_1 term. By choosing for any x_n only terms summing to $n-1$, we get consistency with our more general schemes which we can use with high-ordered polynomials, trigonometric or exponential terms, and negative or irrational powers, or even multidimensional differential equations. [3, 4]

When the Nx, or in the quadratic case, x^2, term is written in terms of Adomian's A_n polynomials, the decomposition method solves the equation. (Although it is not necessary to discuss it here, if stochastic coefficients are involved, the decomposition method achieves statistical separability in the averaging process for desired statistics [1] and no truncations are required.) Let's look at examples:

EXAMPLE. Consider $x^2 + 3x + 2 = 0$ whose solutions are obviously $(-1, -2)$. Write it in the form

$$3x = -2 - x^2$$
$$x = -\tfrac{2}{3} - \tfrac{1}{3}x^2 = x_0 + x_1 + x_2 + \cdots$$
$$= x_0 + \tfrac{1}{3}\sum_{n=0}^{\infty} A_n$$
$$= x_0 - \tfrac{1}{3}A_0 - \tfrac{1}{3}A_1 - \cdots.$$

Substituting the A_n we have

$$x_0 = -0.667 \qquad x_3 = -0.037$$
$$x_1 = -0.148 \qquad x_4 = -0.023$$
$$x_2 = -0.069 \qquad x_5 = -0.015$$

$$x_6 = -0.0106 \qquad x_{10} = -0.0033$$
$$x_7 = -0.00765 \qquad x_{11} = -0.00268$$
$$x_8 = -0.00567 \qquad x_{12} = -0.0020.$$
$$x_9 = -0.0043$$

Since an n-term approximation (symbolized by ϕ_n) is given by $\sum_{i=0}^{n-1} x_i$, we define the error $\psi_n = (x - \phi_n)/x$. We now have

$$\phi_1 = -0.667 \qquad \psi_1 = 33.3\%$$
$$\phi_2 = -0.815 \qquad \psi_2 = 18.5\%$$
$$\phi_3 = -0.884 \qquad \psi_3 = 11.6\%$$
$$\phi_4 = -0.921 \qquad \psi_4 = 7.9\%$$
$$\phi_5 = -0.944 \qquad \psi_5 = 5.6\%$$
$$\phi_6 = -0.959 \qquad \psi_6 = 4.1\%$$
$$\phi_7 = -0.970 \qquad \psi_7 = 3.0\%$$
$$\phi_8 = -0.977 \qquad \psi_8 = 2.3\%$$
$$\phi_9 = -0.983 \qquad \psi_9 = 1.7\%$$
$$\phi_{10} = -0.987 \qquad \psi_{10} = 1.2\%.$$

which is approaching the smallest root which is -1. The error ψ_n becomes less than 0.5% by $m = 12$. If we take the equation $x^2 - 3x + 2 = 0$ we get the same numbers above for the x_i except they will all be positive.

EXAMPLE. Consider $x^2 - 1.25x + 0.25 = 0$ or $(x - \frac{1}{4})(x - 1) = 0$. In our form it becomes

$$-\tfrac{5}{4}x = -\tfrac{1}{4} - x^2$$

or

$$x = (1/5) + (4/5)\,x^2.$$

Thus

$$x_0 = 0.2$$
$$x_1 = (0.8)(0.2)^2 = 0.032$$
$$x_2 = (0.8)(2)(0.2)(0.032) = 0.01024$$
$$x_3 = (0.8)[(0.032)^2 + 2(0.2)(0.01)] = 0.004.$$

Thus $\phi_m = \sum_{n=0}^{m-1} x_n$ is:

$$\phi_1 = 0.2$$
$$\phi_2 = 0.232$$

$$\phi_3 = 0.242$$
$$\phi_4 = 0.246$$

rapidly converging to 0.25 as expected.

EXAMPLE. Consider $x^2 - 20x + 36 = 0$, which has the roots (2,18). Write

$$-20x = -36 - x^2$$
$$x = \tfrac{36}{20} + \tfrac{1}{20}x^2.$$

By the same procedure we get

$$x_0 = 1.8$$
$$x_1 = 0.16.$$

Hence the approximation to only two terms is given by

$$\phi_2 = x_0 + x_1 = 1.96.$$

A 3-term approximation is $\phi_3 = 1.98$, which is already close to the smallest root $x = 2$.

EXAMPLE. Consider $(x - \frac{1}{4})(x - 100) = 0$ and write

$$\frac{401}{4}x = 25 + x^2$$

$$x = \frac{100}{401} + \frac{4}{401}x^2$$

$$x_0 = 0.2493$$

$$x_1 = (0.0099)(0.2493)^2 = 0.0006$$

$$\phi_2 = x_0 + x_1 = 0.2499 \approx 0.25.$$

From these examples we observe that the method yields the smallest root and that the further apart the two roots the faster the convergence to the correct solution (which we will discuss further in a following section). Of course the second root is found by factoring once we have one root.

Let's examine the quadratic equation in the form $(x - r_1)(x - r_2) = 0$ where r_1, r_2 are real roots. We have then $x^2 - (r_1 + r_2)x + r_1 r_2 = 0$. Then in the standard form [1]

$$(r_1 + r_2)x = r_1 r_2 + x^2$$

or

$$x = \frac{r_1 r_2}{r_1 + r_2} + \frac{1}{r_1 + r_2} x^2.$$

Now since $x = \sum_{n=0}^{\infty} x_n$ and we identify $x_0 = r_1 r_2/(r_1 + r_2)$, the x_{n+1} for $n = 0, 1, \ldots$, are given by

$$x_{n+1} = \frac{1}{r_1 + r_2} A_n$$

or

$$x = x_0 + \sum_{n=0}^{\infty} \frac{1}{r_1 + r_2} A_n$$

where the A_n have already been given for $Nx = x^2$.

Since $r_1 r_2 = c/a$ and $r_1 + r_2 = -b/a$ in the standard $ax^2 + bx + c$ form, we have

$$x = -(c/b) - (a/b)\, x^2$$

where

$$\begin{aligned} x_0 &= c/b \\ x_1 &= (a/b)\, x_0^2 \\ x_2 &= (a/b)(2x_0 x_1) \\ &\text{etc.} \end{aligned}$$

Note, e.g., that in solving $(x - \pi)(x - 4) = 0$ where we have deliberately chosen the 2nd root to be only a little larger than the root π, we have $x^2 - (\pi + 4)\, x + 4\pi = 0$. We have

$$x = \frac{4\pi}{\pi + 4} + \frac{1}{\pi + 4} x^2$$

so that $x_0 = 1.76$. If we consider $(x - \pi)(x - 10) = 0$ we get $x_0 = 2.39$. If we take the second root as 100, $x_0 = 3.05$ and for the second root $x = 1000$, $x_0 = 3.13$, an error of 0.3% with only the x_0 term to obtain the smaller root. Thus the results converge to the desired solution more and more quickly, i.e., for smaller n, as the roots are further apart. In general for $(x - r_1)(x - r_2) = 0$, or $x^2 - (r_1 + r_2)\, x + r_1 r_2 = 0$, we have the first term

$$x_0 = \frac{r_1 r_2}{r_1 + r_2}.$$

If $r_2 \gg r_1$, we have $x_0 \simeq r_1 r_2/r_2 = r_1$. Since the following terms involving the A_n are divided by the factor $1/(r_1 + r_2)$ or approximately $1/r_2$, the other terms vanish early.

Decimal Roots

Finally, as we have previously stated, the roots are not limited to integers. Consider, for example,

$$x^2 - 5.15x + 2.37 = 0$$

$$5.15x = 2.37 + x^2$$

$$x = \frac{2.37}{5.15} + \frac{1}{5.15}\sum_0^\infty A_n.$$

We get immediately

$$x_0 = 0.460$$

$$x_1 = 0.0411$$

$$x_2 = 0.00735.$$

Thus the 3-term approximation $\phi_3 = x_0 + x_1 + x_2 = 0.50845$. Let's call this r_2. But $r_1 r_2 = 2.37$ hence $r_1 = 2.37/0.50845 = 4.66$. The sum of the roots now constitutes a check by comparison with the coefficient of the middle term of the quadratic equation. We observe in doing this an error less than 0.3% and considering we only used a 3-term approximation, the result is excellent.

Complex Roots

If we have complex roots z_1, z_2 then $(x - z_1)(x - z_2) = 0$ or $x^2 - (z_1 + z_2)x + z_1 z_2 = 0$. Thus the sum of the roots is the coefficient of the x term and the product of the roots is the constant term. Consider an example with complex roots but real coefficients

$$x^2 - 2x + 2 = 0.$$

Solving it in the usual manner with decomposition, we have

$$x = 1 + \tfrac{1}{2}x^2 = 1 + \tfrac{1}{2}\sum_0^\infty A_n.$$

Therefore we take

$$x_0 = 1$$

and obtain immediately

$$x_1 = \tfrac{1}{2}$$
$$x_2 = \tfrac{1}{2}$$
$$x_3 = \tfrac{5}{8}$$
$$x_4 = \tfrac{7}{8}$$
$$x_5 = \tfrac{21}{16}$$
$$\vdots$$

i.e., a diverging series (for a quadratic equation with real coefficients) may[1] indicate complex roots. In that case, as complex roots occur in conjugate pairs, e.g., $a+bi$ and $a-bi$, their sum is $2a$ and their product is a^2+b^2.

Comparison with the coefficients in the equation shows $2a=2$ or $a=1$ and $a^2+b^2=2$, hence $b=1$. Therefore the roots are $1+i$ and $1-i$.

EXAMPLE. Quadratic equation with complex roots c_1, c_2 given by $(x-c_1)(x-c_2)=0$ or $x^2-(c_1+c_2)x+c_1c_2=0$ where $c_1, c_2 \in C$, the set of complex numbers. In the standard Adomian decomposition form, we get

$$x = \mu + \nu x^2$$

where $\mu = \alpha + i\beta$ and $\nu = \gamma + i\delta$ can of course be written in terms of real and imaginary components of c_1, c_2. We write

$$\sum_{n=0}^{\infty} x_n = \mu + \nu \sum_{n=0}^{\infty} A_n$$

where

$$A_0 = x_0^2$$
$$A_1 = 2x_0x_1$$
$$A_2 = x_1^2 + 2x_0x_2$$
$$A_3 = 2x_0x_3 + 2x_1x_2$$
$$\vdots$$

[1] The associated equation with different signature, $x^2-2x-2=0$, which does have real roots, also results in a diverging series. This special case has been handled by an ingenious method discussed in Adomian [2], which also solves equations of the form $Ny=x$ such as $e^y=x$, for example.

Thus

$$
\begin{aligned}
x_0 &= \mu \\
x_1 &= \nu A_0 = \nu x_0^2 = \nu\mu^2 \\
x_2 &= \nu A_1 = \nu(2x_0x_1) = 2\nu^2\mu^3 \\
x_3 &= \nu A_2 = \nu(x_1^2 + 2x_0x_2) = 5\nu^3\mu^4 \\
x_4 &= \nu A_3 = \nu(2x_0x_3 + 2x_1x_2) = 14\nu^4\mu^5 \\
&\vdots \\
x_m &= k_m \nu^m \mu^{m+1} \qquad \text{for } m \geqslant 0
\end{aligned}
$$

where the k_m are constants as previously defined. The solution is

$$x = \sum_{n=0}^{\infty} k_n \nu^n \mu^{n+1}$$

where the k_n are real numbers and the μ, ν are complex numbers, i.e., $\mu = \alpha + i\beta$ and $\nu = \gamma + i\delta$. An m-term approximation is $\phi_m = \sum_{n=0}^{m-1} x_n$. Now let $c_1 = 1 + i$ and let $c_2 = 10 + 10i$. In the equation in standard form

$$x = \mu + \nu x^2$$

where $\mu = \alpha + i\beta$ and $\nu = \gamma + i\delta$ we find $\alpha = 10/11$, $\beta = 10/11$, $\gamma = 1/22$, $\delta = -1/22$. Thus $\mu = (10/11)(1 + i)$ and $\nu = (1/22)(1 - i)$ and

$$x = (10/11)(1 + i) + (1/22)(1 - i)\,x^2.$$

Then

$$
\begin{aligned}
x_0 &= (10/11)(1 + i) \\
x_1 &= \nu\mu^2 = (10^2/11^3)(1 + i) \\
x_2 &= 2\nu^2\mu^3 = 2(10^3/11^5)(1 + i) \\
x_3 &= 5(10^4/11^7)(1 + i) \\
x_4 &= 14(10^5/11^9)(1 + i) \\
&\vdots \\
x_m &= k_m \nu^m \mu^{m+1} \qquad (m \geqslant 0) \\
&= k_m(1/22)^m\,(1 - i)^m\,(10/11)^{m+1}\,(1 + i)^{m+1}
\end{aligned}
$$

(where the coefficients k_m are easily calculated not only for the quadratic case but also for cubics in the form $x = \mu + \nu x^3$, quartics in the form

$x = \mu + \nu x^4$, etc.; similarly we can find coefficients for $x = \mu + \nu_1 x^2 + \nu_2 x^3 + \nu_3 x^4 + \cdots + \nu_{n-1} x^n$ for real or complex cases). Factoring x_m we have

$$\begin{aligned} x_m &= k_m (1/2)^m \,(10^{m+1}/11^{2m+1})[(1-i)(1+i)]^m \,(1+i) \\ &= k_m (1/2)^m \,(10^{m+1}/11^{2m+1})(2^m)(1+i) \\ &= k_m (10^{m+1}/11^{2m+1})(1+i). \end{aligned}$$

Computing the components x_m,

$$\begin{aligned} x_0 &= (10/11)(1+i) = 0.9090(1+i) \\ x_1 &= (10^2/11^3)(1+i) = (100/1331)(1+i) = 0.0751(1+i) \\ x_2 &= (2)(10^3/11^5)(1+i) = (2000/161051)(1+i) = 0.0124(1+i) \\ x_3 &= (5)(10^4/11^7)(1+i) = 0.0025(1+i) \\ x_4 &= 0.00059(1+i). \end{aligned}$$

Thus

$$\begin{aligned} \phi_1 &= 0.9090(1+i) \\ \phi_2 &= 0.9842(1+i) \\ \phi_3 &= 0.9966(1+i) \\ \phi_4 &= 0.9992(1+i) \\ \phi_5 &= 0.9998(1+i). \end{aligned}$$

$\phi_n \to (1+i) = c_1$, the smallest root. We see the convergence is very rapid indeed. Even by ϕ_2 we have an excellent solution and the method applies well to quadratic equations with complex coefficients (and is easily extended to polynomial equations). The real and imaginary components generally converge at different rates. Suppose $c_1 = 1 + i$ and $c_2 = m_1 + m_2 i$ where for illustration we choose $m_1 = 1$ and $m_2 = 2$ so that $c_2 = 1 + 2i$. Now $\alpha = 7/13$, $\beta = 9/13$, $\gamma = 2/13$, $\delta = -3/13$. Hence $\mu = (7/13) + (9/13)\,i$ and $\nu = (2/13) - (3/13)\,i$ so that

$$x = \left(\frac{7+9i}{13}\right) + \left(\frac{2-3i}{13}\right) x^2 = \mu + \nu x^2.$$

Then

$$\begin{aligned} x_0 &= \mu = (7/13) + (9/13)\,i = 0.5385 + 0.6923i \\ x_1 &= \nu\mu^2 = 0.1429 + 0.1584i \end{aligned}$$

$$x_2 = 2v^2\mu^3 = 0.0749 + 0.0718i$$
$$x_3 = 5v^3\mu^4$$
$$x_4 = 14v^4\mu^5$$
$$\vdots$$

The n-term approximate solutions are:

$$\phi_1 = 0.5385 + 0.6923i$$
$$\phi_2 = 0.6814 + 0.8507i$$
$$\phi_3 = 0.7564 + 0.9225i$$
$$\vdots$$
$$\phi_\infty = 1 + i.$$

It is clear that the imaginary component is converging more rapidly than the real component so we have a differing convergence for the real and imaginary components of complex roots.

II. Cubic Equations

Consider now equations of the type $z^3 + A_2 z^2 + A_1 z + A_0 = 0$. The z^2 term is ordinarily eliminated by substituting $z = x - A_2/3$ to get an equation in the form $x^3 - qx - r = 0$. Thus, the equation

$$z^3 + 9z^2 + 23z + 14 = 0$$

becomes (substituting $z = x - 3$)

$$x^3 - 4x - 1 = 0$$

whose roots are 2.11, −1.86, −0.254. If we solve this by decomposition we write the equation in the form [1] $-qx = r - x^3$ or

$$-4x = 1 - x^3$$
$$x = -\tfrac{1}{4} + \tfrac{1}{4}x^3$$
$$x = x_0 + \tfrac{1}{4} \sum_{n=0}^{\infty} A_n.$$

For this nonlinearity (Nx in Adomian's notation [1])

$$A_0 = x_0^3$$
$$A_1 = 3x_0^2 x_1$$
$$A_2 = 3x_0^2 x_2 + 3x_0 x_1^2$$
$$A_3 = 3x_0^2 x_3 + 6x_0 x_1 x_2 + x_1^3$$
$$A_4 = 3x_0^2 x_4 + 3x_0 x_2^2 + 6x_0 x_1 x_3 + 3x_1^2 x_2$$
$$A_5 = 3x_0^2 x_5 + 6x_0 x_1 x_4 + 6x_0 x_2 x_3 + 3x_1^2 x_3 + 3x_1 x_2^2.$$

(We caution against simply extrapolating the A_n to higher n. We cannot include the complete generating scheme for any n in this paper. It depends on the actual nonlinearity and is lengthy to discuss so it will be dealt with elsewhere. The objective of this paper is to show applicability to algebraic equations, not to provide a handbook.) Thus $x_0 = -0.25$, $x_1 = \frac{1}{4}A_0 = -\frac{1}{4}^4 = -0.004$, etc. Thus the one-term approximation $\phi_1 = -0.25$, the two-term approximation $\phi_2 = -0.254$, and $x_2 \simeq 0$ for an answer to three decimal places so the correct solution is obtained already with ϕ_2 (again for the smallest root). ϕ_3 gives -0.254 with no more change to 3 decimal places. Computing 6 terms gives -0.25410168, which doesn't change any further to 8 place accuracy.

If we now divide $x^3 - 4x - 1$ by $x - 0.254$, we obtain $x^2 + 0.254x - 3.9375$, which yields the other two roots by either the quadratic formula or the decomposition method.

The equation $x^3 - 6x^2 + 11x - 6 = 0$ has roots $(1, 2, 3)$. Written in the form

$$x = \tfrac{6}{11} + \tfrac{6}{11}x^2 - \tfrac{1}{11}x^3$$

and solving by the decomposition method, it yields $x_0 = 0.5455$, $x_1 = 0.1475, \ldots$, and the solution $x = 1$ in eight terms.

EXAMPLE. $x^3 + 4x^2 + 8x + 8 = 0$ is satisfied by $x = -2$. Calculating this with appropriate A_n for the x^2 and x^3 terms, we get

$$x_0 = -1.0$$
$$x_1 = -0.375$$
$$x_2 = -0.234375$$
$$x_3 = -0.1640625$$
$$x_4 = -0.1179199$$
$$x_5 = -0.0835876.$$

If we sum these terms we get approximately $x = -1.98$, which makes us guess $x = -2.0$ and try it in the equation. (It is interesting to note, however, that we actually have an oscillating convergence. If we sum 10 terms, we get $x = -2.0876342$, which is a peak departure from $x = -2$. At 20 terms we have a peak departure in the opposite direction with $x = -1.9656587$. At 100 terms we have -1.997313.)

EXAMPLE.

$$x^3 - 6x^2 + 11x - 6 = 0$$

$$x = \frac{6}{11} + \frac{6}{11}x^2 - \frac{1}{11}x^3$$

$$\sum_{n=0}^{\infty} x_n = x_0 + \frac{6}{11}\sum_{n=0}^{\infty} A_n - \frac{1}{11}\sum_{n=0}^{\infty} B_n$$

expanding the x^2 and x^3 terms in our usual polynomials but using A_n and B_n to distinguish the two.

$$x_0 = 6/11 = 0.5455$$

$$x_1 = \left(\frac{6}{11}\right)A_0 - \left(\frac{1}{11}\right)B_0 = \left(\frac{10}{11}\right)\left(\frac{6^3}{11^4}\right) = 0.147531$$

$$x_2 = \left(\frac{6}{11}\right)A_1 - \left(\frac{1}{11}\right)B_1 = (19)(10)\left(\frac{6^5}{11^7}\right) = 0.0758160$$

$$x_3 = \left(\frac{6}{11}\right)A_2 - \left(\frac{1}{11}\right)B_2 = \left(\frac{6^7}{11^{10}}\right)(3610) = 0.038962$$

$$\phi_1 = 0.5455$$

$$\phi_2 = 0.693031$$

$$\phi_3 = 0.768847$$

$$\phi_4 = 0.80780$$

$$\vdots$$

where $\phi_n \to 1.0$ as $n \to \infty$.

We can write $x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = 0$, then

$$x = \frac{(r_1r_2r_3)}{(r_1r_2 + r_1r_3 + r_2r_3)} + \frac{(r_1 + r_2 + r_3)}{(r_1r_2 + r_1r_3 + r_2r_3)}x^2 - \frac{1}{(r_1r_2 + r_1r_3 + r_2r_3)}x^3$$

choose $r_1 < r_2 < r_3$ or $r_2 = \alpha r_1$ and $r_3 = \beta r_1$ where α, β are appropriate real fractions. Then the equation for x will become

$$x = (r_1)\left\{\frac{\alpha\beta + (1+\alpha+\beta)(x^2/r_1^2) - (x^3/r_1^3)}{\alpha+\beta+\alpha\beta}\right\}$$

where the bracketed quantity $\to 1$ and the first solution found is r_1. Thus, letting $\psi = (x/r_1)$

$$\psi = \left(\frac{\alpha\beta}{\gamma}\right) + \left(\frac{1+\alpha+\beta}{\gamma}\right)\psi^2 - \left(\frac{1}{\gamma}\right)\psi^3$$

where $\gamma = \alpha + \beta + \alpha\beta$. Then if $\psi = \sum_{n=0}^{\infty} \psi_n$ and $\psi^2 = \sum_{n=0}^{\infty} A_n$ and $\psi^3 = \sum_{n=0}^{\infty} B_n$,

$$\psi_0 = \left(\frac{\alpha\beta}{\gamma}\right)$$

$$\psi_1 = \left(\frac{1+\alpha+\beta}{\gamma}\right)\left(\frac{\alpha\beta}{\gamma}\right)^2 - \left(\frac{1}{\gamma}\right)\left(\frac{\alpha\beta}{\gamma}\right)^3$$

$$\vdots$$

If, for example, $r_1 < \frac{1}{10} r_2$ and $r_2 < \frac{1}{10} r_3$,

$$\psi_0 = \frac{(10)(100)}{(10+100+1000)} = \frac{1000}{1110} \approx 1$$

and

$$\psi_1 = \left(\frac{111}{1110}\right)\left(\frac{1000}{1110^2}\right) - \left(\frac{1}{1110}\right)\frac{1000^3}{1116^3}$$

so terms are indeed rapidly approaching zero and r_1 will be the root calculated.

III. Polynomial Operators

Higher-degree polynomial equations are similarly solved. In the cubic case it is, as we see above, not necessary, of course, to eliminate the quadratic term. We can solve the original equation by simply substituting the appropriate A_n summations for both x^2 and x^3 terms. Even higher-degree equations (or nonintegral powers or negative powers) can be equally well handled by substituting appropriate A_n for each nonlinearity (see Ref. [1]). Let's consider an equation in the form

$$\sum_{\mu=0}^{n} \gamma_\mu x^\mu = 0$$

where γ_μ are given constants and seek the roots $r_1, r_2, \ldots, r_n$ (assumed to be real) satisfying $\prod_{\nu=1}^{n} (x - r_\nu) = 0$. We found if Nx or $f(x) = x^2$, for example, $A_0 = x_0^2$ or $f(x_0)$.

Now we have $Nx = \sum_{\mu=0}^{n} \gamma_\mu x^\mu$ so $A_0 = \sum_{\mu=0}^{n} \gamma_\mu x_0^\mu$. Since $\sum_{\mu=0}^{n} \gamma_\mu x^\mu = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \cdots + \gamma_n x^n$, we can write the A_n for each term or for the entire polynomial. Let's consider a specific example of the form $f(x) = \sum_{\mu=0}^{n} \gamma_\mu x^\mu = \gamma_n x^n + \gamma_{n-1} x^{n-1} + \cdots + \gamma_1 x + \gamma_0$ with $\gamma_n \neq 0$ and γ_i constant for $0 \leqslant i \leqslant n$.

Example. It is interesting to consider a 5th-order polynomial operator since no formula exists for $n = 5$ or higher. The equation $x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 = 0$ has the roots 1, 2, 3, 4, 5. To calculate all the roots we rewrite the equation in Adomian's usual form as

$$\begin{aligned} x &= (120/274) + (225/274)\,x^2 - (85/274)\,x^3 \\ &\quad + (15/274)\,x^4 - (1/274)\,x^5 \\ x &= 0.43796 + 0.82117x^2 - 0.31022x^3 \\ &\quad + 0.054745x^4 - 0.0036496x^5 \end{aligned}$$

or

$$x = k + \sum_{n=2}^{5} \gamma_n x^n$$

where

$$\begin{aligned} k &= 0.43796 \\ \gamma_2 &= 0.82117 \\ \gamma_3 &= -0.31022 \\ \gamma_4 &= 0.054745 \\ \gamma_5 &= -0.0036496. \end{aligned}$$

We have the first approximation $\phi_1 = x_0 = k = 0.43796$.

Then

$$x_1 = \gamma_2 A_0(x^2) + \gamma_3 A_0(x^3) + \gamma_4 A_0(x^4) + \gamma_5 A_0(x^5).$$

The notation $A_0(x^2)$ means the A_0 for the x^2 term, etc. Thus

$$x_1 = 0.82117x_0^2 - 0.31022x_0^3 + 0.054745x_0^4 - 0.0036496x_0^5$$

$$x_1 = 0.15751 - 0.026060 + 0.0020141 - 0.00005881$$

$$x_1 = 0.13341.$$

Hence $\phi_2 = x_0 + x_1 = 0.57137$.

$$x_2 = \gamma_2 A_1(x^2) + \gamma_3 A_1(x^3) + \gamma_4 A_1(x^4) + \gamma_5 A_1(x^5)$$

where

$$A_1(x^2) = 2x_0 x_1$$

$$A_1(x^3) = 3x_0^2 x_1$$

$$A_1(x^4) = 4x_0^3 x_1$$

$$A_1(x^5) = 5x_0^4 x_1.$$

Consequently,

$$x_2 = \{(0.82117)(2)\,x_0 - (0.31022)(3)\,x_0^2 + (0.05745)(4)\,x_0^3 - (0.0036496)(5)\,x_0^4\}\,x_1$$

$$x_2 = 0.0746299.$$

Then $\phi_3 = 0.6459999 = 0.6460$. Continuing

$$x_3 = \gamma_2 A_2(x^2) + \gamma_3 A_2(x^3) + \gamma_4 A_2(x^4) + \gamma_5 A_2(x^5)$$

etc., as necessary. The A_n are generated by rather complex rules necessarily dealt with elsewhere since they are applicable to differential and partial differential equations as well and require much discussion. They are to be viewed here as a special set of polynomials proposed by Adomian for the expression of nonlinear terms in his decomposition method which are given or can be obtained. The first few of these are:

$$A_0 = h_0(x_0)$$

$$A_1 = h_1(x_0)\,x_1$$

$$A_2 = \frac{1}{2}\{h_2(x_0)\,x_1^2 + 2h_1(x_0)\,x_2\}$$

$$A_3 = \frac{1}{6}\{h_3(x_0)\,x_1^3 + 6h_2(x_0)\,x_1 x_2 + 6h_1(x_0)\,x_3\}$$

$$A_4 = \frac{1}{24}\{h_4(x_0)\,x_1^4 + 12h_3(x_0)\,x_1^2x_2 + h_2(x_0)[12x_2^2 + 24x_1x_3]$$
$$+ 24h_1(x_0)\,x_4\}$$

$$A_5 = \frac{1}{120}\{h_5(x_0)\,x_1^5 + 20h_4(x_0)\,x_1^3x_2 + 60h_3(x_0)[x_1x_2^2 + x_1^2x_3]$$
$$+ 120h_2(x_0)[x_2x_3 + x_1x_4] + 120h_1(x_0)\,x_5\}$$
$$\vdots$$

where $h_i = d^if/dx_i$ for the function $f(x)$.

Let's list final results for the 5th-degree equation above to 10 digit accuracy.

$$\begin{aligned}
x_0 &= 0.4379562044 & \phi_1 &= 0.4379562044\\
x_1 &= 0.1334006838 & \phi_2 &= 0.5713568882\\
x_2 &= 0.0745028484 & \phi_3 &= 0.6458597366\\
x_3 &= 0.0500356263 & \phi_4 &= 0.6958953629\\
x_4 &= 0.0449342233 & \phi_5 &= 0.7408295862\\
x_5 &= 0.0446966625 & \phi_6 &= 0.7855262487\\
x_6 &= 0.0331390668 & \phi_7 &= 0.8186653155\\
x_7 &= 0.0272374949 & \phi_8 &= 0.8459028104\\
x_8 &= 0.022258001 & \phi_9 &= 0.8681608114\\
x_9 &= 0.0196274208 & \phi_{10} &= 0.8877882322\\
x_{10} &= 0.0166467228 & \phi_{11} &= 0.904434955
\end{aligned}$$

$$\begin{aligned}
\psi_1 &= 56.2\%\\
\psi_2 &= 42.9\%\\
\psi_3 &= 35.4\%\\
\psi_4 &= 30.4\%\\
\psi_5 &= 25.9\%\\
\psi_6 &= 21.4\%\\
\psi_7 &= 18.1\%\\
\psi_8 &= 15.4\%\\
\psi_9 &= 13.9\%\\
\psi_{10} &= 11.2\%\\
\psi_{11} &= 9.56\%
\end{aligned}$$

The error ψ decreases gradually to less than 10% by ψ_{11} but it can easily be carried further by computer. The convergence for inversion in this case of a quintic operator is relatively poor because of the greater number of more closely spaced roots and the case of equal roots will be the poorest case.

For $f(x)=x^k$ where k is an integer $\geqslant 2$, let's write $h_n=d^nf/dx^n$ for $n\geqslant 0$. (We will write $h_n(x_0)$ for $(d^nf/dx^n)|_{\lambda=0}$ for the computation of the A_n.) Then for x^k,

$$h_0=x^k$$
$$h_1=kx^{k-1}$$
$$\vdots$$
$$h_n=k(k-1)\cdots(k-n+1)\,x^{k-n}=\binom{k}{n}x^{k-n}$$

where $\binom{k}{n}=k!/(k-n)!$. Consequently, the A_n for $f(x)=x^k$ are given by

$$A_0=x_0^k$$
$$A_1=\left\{\binom{k}{1}x_0^{k-1}\right\}x_1$$
$$A_2=\frac{1}{2}\left\{\binom{k}{2}x_0^{k-2}\right\}x_1^2+\left\{\binom{k}{1}x_0^{k-1}\right\}x_2$$
$$A_3=\frac{1}{6}\left\{\binom{k}{3}x_0^{k-3}\right\}x_1^3+\left\{\binom{k}{2}x_0^{k-2}\right\}x_1x_2+\left\{\binom{k}{1}x_0^{k-1}\right\}x_3$$
$$A_4=\frac{1}{24}\left\{\binom{k}{4}x_0^{k-4}\right\}+\frac{1}{2}\left\{\binom{k}{3}x_0^{k-3}\right\}x_1^2x_2 + \left\{\binom{k}{2}x_0^{k-2}\right\}\left[\frac{1}{2}x_2^2+x_1x_3\right]+\left\{\binom{k}{1}x_0^{k-1}\right\}x_4$$
$$A_5=\frac{1}{120}\left\{\binom{k}{5}x_0^{k-5}\right\}x_1^5+\frac{1}{6}\left\{\binom{k}{4}x_0^{k-4}\right\}x_1^3x_2+\frac{1}{2}\left\{\binom{k}{3}x_0^{k-3}\right\} \times[x_1x_2^2+x_1^2x_3]+\left\{\binom{k}{2}x_0^{k-2}\right\}[x_2x_3+x_1x_4]+\left\{\binom{k}{1}x_0^{k-1}\right\}x_5.$$

We observe that the subscripts for A_n always add to n and the superscripts of the x_i's always add to k.

The above work will yield the lowest root, or 1, reducing the equation to a 4th power then the root 2, etc. We can do the problem more rapidly as follows.

Let's write a general polynomial in x with constant nonzero coefficients.

$$f(x)=\sum_{i=k}^{0} \gamma_i x^i = \gamma_k x^k + \cdots + \gamma_0.$$

Now

$$h_0 = \sum_{i=k}^{0} \gamma_i x^i$$

$$h_1 = \sum_{i=k}^{1} i\gamma_i x^{i-1}$$

$$\vdots$$

$$h_n = \sum_{i=k}^{n} \binom{i}{n} \gamma_i x^{i-n} \qquad (k>n)$$

$$\vdots$$

$$h_k = \binom{k}{k} \gamma_k x^{k-k} = \gamma_k k! \qquad (k=n)$$

$$h_{k+1}=0 \qquad \text{or} \qquad h_n = 0 \qquad \text{for } n>k.$$

The A_n can now be given

$$A_0 = \sum_{i=k}^{0} \gamma_i x_0^i$$

$$A_1 = \left\{ \sum_{i=k}^{1} \binom{i}{1} \gamma_i x_0^{i-1} \right\} x_1$$

$$A_2 = \frac{1}{2} \left\{ \sum_{i=k}^{2} \binom{i}{2} \gamma_i x_0^{i-2} \right\} x_1^2 + \left\{ \sum_{i=k}^{1} \binom{i}{1} \gamma_i x_0^{i-1} \right\} x_2$$

$$A_3 = \frac{1}{6} \left\{ \sum_{i=k}^{3} \binom{i}{3} \gamma_i x_0^{i-3} \right\} x_1^3 + \left\{ \sum_{i=k}^{2} \binom{i}{2} \gamma_i x_0^{i-2} \right\} x_1 x_2 + \left\{ \sum_{i=k}^{1} \binom{i}{1} \gamma_i x_0^{i-1} \right\} x_3$$

$$A_4 = \frac{1}{24} \left\{ \sum_{i=k}^{4} \binom{i}{4} \gamma_i x_0^{i-4} \right\} + \frac{1}{2} \left\{ \sum_{i=k}^{3} \binom{i}{3} \gamma_i x_0^{i-3} \right\} x_1^2 x_2 + \left\{ \sum_{i=k}^{2} \binom{i}{2} \gamma_i x_0^{i-2} \right\} \left(\frac{1}{2} x_2^2 + x_1 x_3 \right) + \left\{ \sum_{i=k}^{1} \binom{i}{1} \gamma_i x_0^{i-1} \right\} x_4$$

$$A_5=\frac{1}{120}\left\{\sum_{i=k}^{5}\binom{i}{5}\gamma_i x_0^{i-5}\right\}x_1^5+\frac{1}{6}\left\{\sum_{i=k}^{4}\binom{i}{4}\gamma_i x_0^{i-4}\right\}x_1^3x_2$$
$$+\frac{1}{2}\left\{\sum_{i=k}^{3}\gamma_i x_0^{i-3}\right\}(x_1x_2^2+x_1^2x_3)$$
$$+\left\{\sum_{i=k}^{2}\binom{i}{2}\gamma_i x_0^{i-2}\right\}(x_2x_3+x_1x_4)+\left\{\sum_{i=k}^{1}\binom{i}{1}\gamma_i x_0^{i-1}\right\}x_5$$

etc.

from which polynomial equations can be solved more rapidly than with individual substitutions for the various powers as we did earlier in this paper.

Negative powers. Consider an example like $x=2+x^{-2}$ or the more general form

$$x=k+x^{-m}.$$

We write

$$\sum_{n=0}^{\infty} x_n=k+\sum_{n=0}^{\infty} A_n$$

with $x_0=k$ and $x_n=A_{n-1}$ for $n\geqslant 1$. Then

$$x_1=A_0=x_0^{-m}=k^{-m}$$
$$x_2=A_1=-mx_0^{-(m+1)}x_1$$
$$x_3=A_2=\tfrac{1}{2}m(m+1)\,x_0^{-(m+2)}x_1^2-mx_0^{-(m+1)}x_2$$
$$x_4=A_3=-\tfrac{1}{6}m(m+1)(m+2)\,x_0^{-(m+3)}\,x_1^3+m(m+1)\,x_0^{-(m+2)}\,x_1x_2$$
$$-\,mx_0^{-(m+1)}x_3$$
$$x_5=A_4=\tfrac{1}{24}m(m+1)(m+2)(m+3)\,x_0^{-(m+4)}x_1^4-\tfrac{1}{2}m(m+1)$$
$$\times\,(m+2)\,x_0^{-(m+3)}x_1^2x_2+m(m+1)\,x_0^{-(m+2)}[\tfrac{1}{2}x_2^2+x_1x_3]$$
$$-\,mx_0^{-(m+1)}x_4.$$

If $k=2$ and $m=2$, then $x_0=2$ and

$$x_1=2^{-2}=0.25$$
$$x_2=-2(2)^{-3}\,(0.25)=-\,0.0625$$
$$x_3=\tfrac{1}{2}(2)(3)(2)^{-4}(0.25)^2-(2)(2)^{-3}\,(-0.0625)=0.02734375$$
$$x_4=-0.0146484375$$
$$x_5=-0.0087280273.$$

By ϕ_6 we get an excellent approximation to the solution (2.205569431) and note again rapid convergence.

$$\begin{aligned}
\phi_1 &= 2 & \psi_1 &= 9.32\% \\
\phi_2 &= 2.25 & \psi_2 &= -2.02\% \\
\phi_3 &= 2.1875 & \psi_3 &= 0.82\% \\
\phi_4 &= 2.21484375 & \psi_4 &= -0.42\% \\
\phi_5 &= 2.200189063 & \psi_5 &= 0.24\% \\
\phi_6 &= 2.20891709 & \psi_6 &= -0.15\%
\end{aligned}$$

Non integer powers. Let's now consider inversion of algebraic operator equations involving fractional or noninteger powers, e.g., consider

$$x = k + x^{1/2}.$$

Write

$$\sum_{n=0}^{\infty} x_n = k + \sum_{n=0}^{\infty} A_n$$

where $x_0 = k$ and $x_n = A_{n-1}$ for $n \geqslant 1$. Then

$$\begin{aligned}
x_1 &= A_0 = x_0^{1/2} = k^{1/2} \\
x_2 &= A_1 = \tfrac{1}{2} x_0^{1/2} x_1 = \tfrac{1}{2}(k)^{-1/2}(k)^{1/2} = \tfrac{1}{2} \\
x_3 &= A_2 = \tfrac{1}{2} x_0^{-1/2} x_2 - \tfrac{1}{8} x_0^{-3/2} x_1^2 = \tfrac{1}{8} k^{-1/2} \\
x_4 &= A_3 = 0 \\
x_5 &= A_4 = -\tfrac{1}{128} k^{-3/2} \\
&\vdots \\
\phi_6 &= (k) + (k)^{1/2} + \tfrac{1}{2} + \tfrac{1}{8} k^{-1/2} + 0 - \tfrac{1}{128} k^{-3/2}.
\end{aligned}$$

If $k = 2$ we expect the solution to converge to $x = 4$. As verification, we get

$$\begin{aligned}
x_0 &= 2 \\
x_1 &= 1.414213562 \\
x_2 &= 0.50 \\
x_3 &= 0.0883883476 \\
x_4 &= 0 \\
x_5 &= -0.0027621359
\end{aligned}$$

and

$$\begin{aligned}
\phi_1 &= 2 & \psi_1 &= 50\% \\
\phi_2 &= 3.414213562 & \psi_2 &= 14.65\% \\
\phi_3 &= 3.914213562 & \psi_3 &= 2.15\% \\
\phi_4 &= 4.00260191 & \psi_4 &= -0.065\% \\
\phi_5 &= \phi_4 + 0 & \psi_5 &= \psi_4 \\
\phi_6 &= 3.999839774 & \psi_6 &= 0.0040\%.
\end{aligned}$$

Thus ϕ_6 is an excellent approximation (in fact ϕ_4 is!). With ϕ_6 we have 4/1000 of 1% error. Let's be more general now and consider nonlinear terms $Nx = x^{1/m}$ where m belongs to the set of positive integers.

EXAMPLE.

$$x = k + x^{1/m}$$

$$\sum_{n=0}^{\infty} x_n = k + \sum_{n=0}^{\infty} A_n$$

where the A_n are the usual Adomian polynomials generated for the specific nonlinearity under consideration. Then

$$x_0 = k$$

$$x_n = A_{n-1} \qquad \text{for} \quad n \geqslant 1.$$

The A_n are given by

$$\begin{aligned}
A_0 &= x_0^{1/m} \\
A_1 &= (1/m)\, x_0^{(1/m)-1} x_1 \\
A_2 &= 1/2(1/m)((1/m)-1)\, x_0^{(1/m)-2} x_1^2 + (1/m)\, x_0^{(1/m)-1} x_2 \\
A_3 &= 1/6(1/m)((1/m)-1)((1/m)-2)\, x_0^{(1/m)-3} x_1^3 \\
&\quad + (1/m)((1/m)-1)\, x_0^{(1/m)-2} x_1 x_2 \\
&\quad + (1/m)\, x_0^{(1/m)-1} x_3 \\
&\vdots
\end{aligned}$$

Since $x_n = A_{n-1}$ in this problem, we now have the x_n. Their general form of x_n is $\sigma(1/m)\, k^{(n/m)-n+1)}$ where $\sigma(1/m)$ has the form $\sum_i \prod_j \alpha_i(1/m - \beta_j)$. If $m = 2$ and $k = 2$ we get precisely the previous results in the preceding example.

We now also see that we can consider operators involving a cube root, a fourth root, etc.

Decimal powers. Now consider solution of algebraic equations involving (rational or irrational) decimal powers, first taking up the case of rational powers.

$$x = k + x^{a/b}$$

where k is real and a, b are positive integers. Write

$$\sum_{n=0}^{\infty} x_n = k + \sum_{n=0}^{\infty} A_n$$

where $x_0 = k$ and $x_n = A_{n-1}$ for $n \geqslant 1$. (Thus we have $Nx = x^{a/b} = \sum_{n=0}^{\infty} A_n$.) Now

$$\begin{aligned}
A_0 &= x_0^{a/b} \\
A_1 &= (a/b)\, x_0^{(a/b)-1} x_1 \\
A_2 &= (a/b)\, x_0^{(a/b)-1} x_2 + \tfrac{1}{2}(a/b)((a/b)-1)\, x_0^{(a/b)-2} x_1^2 \\
A_3 &= (a/b)\, x_0^{(a/b)-1} x_3 + (a/b)((a/b)-1)\, x_0^{(a/b)-2} x_1 x_2 \\
&\quad + \tfrac{1}{6}(a/b)((a/b)-1)((a/b)-2)\, x_0^{(a/b)-3} x_1^3.
\end{aligned}$$

Hence,

$$\begin{aligned}
x_0 &= k \\
x_1 &= A_0 = k^{a/b} \\
x_2 &= A_1 = (a/b)\, k^{2(a/b)-1} \\
x_3 &= A_2 = \{(a/b)^2 + \tfrac{1}{2}(a/b)((a/b)-1)\}\, k^{3(a/b)-2} \\
&\;\vdots
\end{aligned}$$

As a specific case, choose $a = 3$, $b = 11$, $k = 2$. Then $a/b = 0.272727...$ and $x = 2 + x^{3/11}$

$x_0 = 2$	$\phi_1 = 2$
$x_1 = 1.208089444$	$\phi_2 = 3.208089444$
$x_2 = 0.1990200144$	$\phi_3 = 3.407109458$
$x_3 = -0.0109288172$	$\phi_4 = 3.396180641.$

Notice ϕ_3 and ϕ_4 differ very little as we try substitution into the original equation of ϕ_4 as an approximate solution. Then

$$\tilde{\phi}_4 = 2 + \phi_4^{3/11} = 3.395775811.$$

We see $\tilde{\phi}_4$ is very close to ϕ_4, differing by about 0.01%. (We have defined $\tilde{\phi}_n \equiv k + \phi_n^{a/b}$ to see if the approximate solution satisfies the original equation.)

Now we consider the case of irrational powers. Write

$$x = k - x^{\gamma}$$

letting k be real and γ an irrational number such as e or π. Now

$$\sum_{n=0}^{\infty} x_n = k - \sum_{n=0}^{\infty} A_n$$

where $x_0 = k$ and $x_n = -A_{n-1}$ for $n \geqslant 1$. Since $Nx = x^{\gamma} = \sum_{n=0}^{\infty} A_n$,

$$\begin{aligned} A_0 &= x_0^{\gamma} \\ A_1 &= \gamma x_0^{\gamma-1} x_1 \\ A_2 &= \gamma x_0^{\gamma-1} x_2 + \tfrac{1}{2}\gamma(\gamma-1)\, x_0^{\gamma-2} x_1^2 \\ A_3 &= \gamma x_0^{\gamma-1} x_3 + \gamma(\gamma-1)\, x_0^{\gamma-2} x_1 x_2 \\ &\quad + \tfrac{1}{6}\gamma(\gamma-1)(\gamma-2)\, x_0^{\gamma-3} x_1^3. \end{aligned}$$

Now the components x_n of the solution $x = \sum_{n=0}^{\infty} x_n$ can be computed

$$\begin{aligned} x_0 &= k \\ x_1 &= -k^{\gamma} \\ x_2 &= \gamma k^{2\gamma-1} \\ x_3 &= -\{\gamma^2 + \tfrac{1}{2}\gamma(\gamma-1)\}\, k^{3\gamma-2} \\ x_4 &= \{\gamma^2(\gamma-1) + \tfrac{1}{6}\gamma(\gamma-1)(\gamma-2) + \gamma^3 + \tfrac{1}{2}\gamma^2(\gamma-1)\}\, k^{4\gamma-3}. \end{aligned}$$

For a specific example we now choose $k = 1/\pi$ and $\gamma = \pi$

$$x = (1/\pi) - x^{\pi}$$

(letting $\pi = 3.1415927$ and $1/\pi = 0.3183099$ for the computation). We get $x = 0.296736$ which is within a hundredth of 1%.

Random algebraic equations. The treatment of algebraic equations by the decomposition method suggests further generalization to random algebraic equations. Such equations, with random coefficients, arise in engineering, physics, and statistics whenever random errors are involved. Random matrices[2] too are found in finite-dimensional approximation models

[2] Application of the decomposition method also solves equations with random matrix operators. See [5–7].

for random Hamiltonian operators and various engineering applications concerned with systems of linear random equations. Thus suppose one has the equation

$$x^3 + \alpha x^2 + \beta x + \gamma = 0$$

where α is stochastic. Then we have

$$x = -(\gamma/\beta) - (1/\beta)\, x^3 - (\alpha/\beta)\, x^2$$

where α is a stochastic process and β, γ are constants. We now write

$$x = x_0 - (1/\beta) \sum_{n=0}^{\infty} A_n - (\alpha/\beta) \sum_{n=0}^{\infty} B_n$$

where the A_n, B_n are the appropriate Adomian polynomials computed for the nonlinear terms x^3 and x^2. For example,

$$x_1 = -(1/\beta)\, x_0^3 - (\alpha/\beta)\, x_0^2$$

hence x_1 involves a stochastic coefficient in the second term. Continuing one writes ϕ_n and appropriate statistics such as $\langle \phi_n \rangle$.

If we consider a quadratic operator and a forcing term in the form

$$y^2 + by + c = x(t)$$

where b and c can be functions of t, we can write immediately $y = (1/b)(t-c) - (1/b) \sum_{n=0}^{\infty} A_n(y^2)$, or, since $y_0 = (1/b)(t-c)$ and $y_1 = -(1/b) A_0$, etc.,

$$y_1 = -\frac{1}{b^3}(x-c)^2$$

$$y_2 = \frac{2}{b^5}(x-c)^3$$

$$y_3 = -\frac{5}{b^7}(x-c)^4$$

$$\vdots$$

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n k_n}{b^{2n+1}}(x-c)^{n-1}$$

with k_n as appropriately defined constants. Clearly $x(t)$ can be stochastic (or $x(t, \omega)$). The $b(t)$ can have a fluctuating or random component and be written as $b_0(t) + \beta(t, \omega)$, in which case

$$y^2 + (b_0 + \beta) y + c = x$$

$$y^2 + b_0 y + c = x - \beta y$$

$$b_0 y = (x - c) - \beta y - y^2$$

$$y = \frac{1}{b}(x - c) - \frac{\beta}{b} y - \frac{1}{b} \sum_{n=0}^{\infty} A_n(y^2)$$

$$y_0 = \frac{1}{b}(x - c)$$

$$y_1 = -\frac{1}{b} \beta y_0 - \frac{1}{b} y_0^2$$

$$\vdots$$

Conclusions. We have seen that algebraic equations can be handled by the decomposition method and it provides a useful method for computation of roots of polynomial equations often yielding a very rapid convergence. As discussed earlier by Adomian [1], we have a computational and highly convergent system to solve problems of the real world more realistically without assumptions changing the essential nonlinear nature. Whether we deal with differential or partial differential equations [1, 2, 4, 5] or algebraic systems as demonstrated in this paper (and in Ref. [5][3]), an accurate methodology is available for physical applications and more realistic modeling.

REFERENCES

1. G. ADOMIAN, "Stochastic Systems," Academic Press, New York, 1983.
2. R. E. BELLMAN AND G. ADOMIAN, "Partial Differential Equations," Reidel, Dordrecht, 1984.
3. R. RACH, A convenient computational form for the Adomian polynomials, *J. Math. Anal. Appl.* **102** (1984), 415–419.
4. G. ADOMIAN, A new approach to nonlinear partial differential equations, *J. Math. Anal. Appl.* **102** (1984), 420–434.
5. G. ADOMIAN, "Stochastic Systems II," in press.
6. G. ADOMIAN AND R. RACH, Application of the decomposition method to inversion of matrices, *J. Math. Anal. Appl.*, in press.
7. G. ADOMIAN, "Applications of Stochastic Systems Theory to Physics and Engineering," to appear.

[3] Reference [5] continues the theory of Ref. [1], applying it to algebraic equations of all kinds, differential and partial differential equations, delay equations, coupled systems, matrix equations, etc. Applications to physics and engineering appear in [7].