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Simple Constructions for Balanced Incomplete Block Designs with Block Size Three

C. ST. J. A. NASH-WILLIAMS

University of Waterloo, Waterloo, Ontario, Canada

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Let S be a finite set with v elements. It is known that there exists a sequence of three-element subsets of S such that each two-element subset of S is contained in exactly λ terms of the sequence if and only if $\lambda(v-1)/2$ and $\lambda v(v-1)/6$ are integers. The known proof is somewhat complicated when $v \equiv 2 \pmod 6$, and this paper provides a simpler proof for this case. Proofs are also given for all other values of v by reviewing known constructions or providing new ones.

We shall introduce the following notation and terminology. Throughout this paper, λ, k, v denote integers such that $\lambda \geqslant 1$, $k \geqslant 2$, $v \geqslant 3$. A *2-subset* of a set S is a subset of S with cardinal number 2. An *array* is a finite sequence $B_1, \ldots, B_b$ of non-empty finite sets such that $b > 0$ and $|B_1| = |B_2| = \cdots = |B_b|$. The sets B_i are the *blocks* of the array and the array is said to have *block number* b, *block size* $|B_1|$ and *order* $|B_1 \cup \cdots \cup B_b|$. The *replication number* of an element ξ of $B_1 \cup \cdots \cup B_b$ in the array is the number of values of i for which $\xi \in B_i$. If all elements of $B_1 \cup \cdots \cup B_b$ have the same replication number in the array $B_1, \ldots, B_b$ then this number is called the *replication number* of the array. A (v, k, λ)-*array* is an array $B_1, \ldots, B_b$ of order v and block size k such that, for each 2-subset A of $B_1 \cup \cdots \cup B_b$, there are exactly λ values of i for which $A \subseteq B_i$. If $B_1, \ldots, B_b$ is a (v, k, λ)-array and $S = B_1 \cup \cdots \cup B_b$, then S has $\frac{1}{2}v(v-1)$ 2-subsets each of which is a 2-subset of λ blocks, and each block has $\frac{1}{2}k(k-1)$ 2-subsets, so that $\frac{1}{2}\lambda v(v-1) = \frac{1}{2}bk(k-1)$, i.e., $b = \lambda v(v-1)/k(k-1)$. Furthermore, if $\xi \in S$ and $f(1), \ldots, f(r)$ are the distinct values of i for which $\xi \in B_i$ and $A_1, \ldots, A_{v-1}$ are the 2-subsets of S which include ξ, then each A_j is contained in exactly λ of the sets $B_{f(1)}, \ldots, B_{f(r)}$ and each $B_{f(l)}$ contains exactly $k-1$ of the sets $A_1, \ldots, A_{v-1}$, so that $\lambda(v-1) = r(k-1)$, i.e., $r = \lambda(v-1)/(k-1)$. This argument shows that a (v, k, λ)-array has block number $\lambda v(v-1)/k(k-1)$ and replication number $\lambda(v-1)/(k-1)$. Therefore necessary conditions for the existence of a (v, k, λ)-array are that

$$\lambda(v-1)/(k-1) \quad \text{and} \quad \lambda v(v-1)/k(k-1) \quad \text{are integers.} \tag{1}$$

Evidently a further necessary condition is that $v \geqslant k$. It is well known (see [5], for example) that these necessary conditions are not in general sufficient; but they tend to be sufficient for small values of k. Specifically, the existence of a $(v, 2, \lambda)$-array is trivially demonstrable for all v, λ, and Hanani [6, 7] has proved the existence of a (v, k, λ)-array whenever v, k, λ satisfy (1) and $v \geqslant k$ and $k \in \{3, 4, 5\}$, with the sole exception [4; 5, p. 291] that no (15, 5, 2)-array exists. Hanani's arguments are somewhat complicated, and the present paper concerns a simpler treatment of the easiest case $k = 3$.

A (v, k, λ)-array is in standard terminology called a *balanced incomplete block design* with *parameters* b, v, r, k, λ, where b, r, denote the integers $\lambda v(v-1)/k(k-1)$, $\lambda(v-1)/(k-1)$, which have been shown to be the block number and replication number of the array. The non-standard term "(v, k, λ)-array" is used here for brevity. Balanced incomplete block designs are one of the main themes of combinatorial analysis, and much of the relevant work has been concerned with trying to determine the set of ordered triples (v, k, λ) for which (v, k, λ)-arrays exist. The first value of k for which this problem becomes non-trivial is $k = 3$, and hence a natural introduction to the subject would be a connected account of how, *using reasonably simple and direct constructions only*, one can obtain $(v, 3, \lambda)$-arrays for all possible choices of v, λ. Unfortunately, such an account does not exist: the required constructions are so widely scattered through the literature as to be very difficult to locate, and no simple construction seems to have been known hitherto for the case $v \equiv 2 \pmod 6$. The literature contains evidence that other mathematicians besides myself have been caused difficulty by the lack of an organized account of such material: for example, some authors seem to have been unaware of the construction for $(v, 3, 1)$-arrays with $v \equiv 1 \pmod 6$ which was given by Skolem [10] and is described below.

The purpose of this paper is to assemble in one place a set of simple constructions which establish the existence of a $(v, 3, \lambda)$-array for all possible choices of v, λ. Some of these constructions are merely collected from other literature, and hence much of this paper should be viewed as expository. The construction given for the case $v \equiv 2 \pmod 6$ is, however, believed to be new and substantially simpler than any previously known. The constructions given for the cases $v \equiv 0, 4 \pmod 6$ are also new so far as I know, although reasonably simple constructions for these cases were given by Bhattacharya [1] with the help of an idea of Bose [2]:

it might be arguable that the constructions in the present paper are very slightly simpler than those of [1].

Our task, therefore, is to demonstrate the existence of a (v, k, λ)-array if $k = 3$ and (1) holds. In other words, we have to prove the existence of a $(v, 3, \lambda)$-array whenever $\lambda = n\lambda_v$ for some positive integer n, where λ_v assumes the values 2, 1, 6, 1, 2, 3 when v is congruent to 0, 1, 2, 3, 4, 5, respectively, modulo 6. To prove this, it suffices (as was observed by Hanani [6]) to prove the existence of a $(v, 3, \lambda_v)$-array, because if $B_1, ..., B_b$ is a $(v, 3, \lambda_v)$-array and B_{pb+i} is defined to be B_i for $i = 1, ..., b$ and $p = 1, ..., n - 1$ then $B_1, B_2, ..., B_{nb}$ is a $(v, 3, n\lambda_v)$-array. Accordingly, the remainder of this note discusses the construction of a $(v, 3, \lambda_v)$-array.

A $(v, 3, 1)$-array is known as a *Steiner triple system* of order v, and the result to be proved when $v \equiv 1$ or 3 (mod 6) is a classical theorem on Steiner triple systems which was conjectured by Steiner [11] and first proved by Reiss [9]: since that time, many methods of constructing Steiner triple systems have been devised. For $v \equiv 3$ (mod 6), we remind the reader of the following well known simple construction for a $(v, 3, 1)$-array, which seems to have been given independently in [2] and [10]. Let t be a positive integer and let $\alpha_0, \alpha_1, ..., \alpha_{2t}, \beta_0, \beta_1, ..., \beta_{2t}, \gamma_0, \gamma_1, ..., \gamma_{2t}$ be $6t + 3$ distinct elements. For $i \in \{0, 1, ..., 2t\}$ and any non-zero integer q define $\alpha_{i+(2t+1)q}, \beta_{i+(2t+1)q}, \gamma_{i+(2t+1)q}$ to be $\alpha_i, \beta_i, \gamma_i$, respectively, so that the meaning of the symbols $\alpha_u, \beta_u, \gamma_u$ now depends only on the congruence class of u modulo $2t + 1$. Then it is an easy exercise to show that, if the sets

$$\{\alpha_i, \beta_i, \gamma_i\}, \{\alpha_i, \beta_{i-r}, \beta_{i+r}\}, \{\beta_i, \gamma_{i-r}, \gamma_{i+r}\}, \{\gamma_i, \alpha_{i-r}, \alpha_{i+r}\} \qquad (i = 0, 1, ..., 2t;\ r = 1, 2, ..., t)$$

are arranged in a sequence, this sequence will be a $(6t + 3, 3, 1)$-array. Moreover, if we introduce three additional elements δ, ϵ, ζ we can establish the existence of a $(6t + 4, 3, 2)$-array and a $(6t + 6, 3, 2)$-array. First, it is an easy exercise to show that a sequence of sets, obtained by writing down each of the sets

$$\{\alpha_i, \beta_{i-r}, \beta_{i+r}\}, \{\beta_i, \gamma_{i-r}, \gamma_{i+r}\}, \{\gamma_i, \alpha_{i-r}, \alpha_{i+r}\} \qquad (i = 0, 1, ..., 2t;\ r = 1, 2, ..., t)$$

twice and each of the sets

$$\{\alpha_i, \beta_i, \gamma_i\}, \{\delta, \beta_i, \gamma_i\}, \{\delta, \gamma_i, \alpha_i\}, \{\delta, \alpha_i, \beta_i\} \qquad (i = 0, 1, ..., 2t)$$

once is a $(6t + 4, 3, 2)$-array. Second, it is easily shown that a sequence of sets, obtained by writing down each of the sets

$$\{\delta, \epsilon, \zeta\}, \{\alpha_i, \beta_i, \gamma_i\}, \{\alpha_i, \beta_{i-s}, \beta_{i+s}\}, \{\beta_i, \gamma_{i-s}, \gamma_{i+s}\},$$
$$\{\gamma_i, \alpha_{i-s}, \alpha_{i+s}\} \qquad (i = 0, 1,..., 2t;\ s = 2, 3,..., t)$$

twice and each of the sets

$$\{\alpha_i, \beta_{i-1}, \beta_{i+1}\}, \{\beta_i, \gamma_{i-1}, \gamma_{i+1}\}, \{\gamma_i, \alpha_{i-1}, \alpha_{i+1}\}, \{\delta, \alpha_i, \beta_{i+1}\},$$
$$\{\delta, \beta_i, \gamma_{i+1}\}, \{\delta, \gamma_i, \alpha_{i+1}\}, \{\epsilon, \alpha_i, \beta_{i-1}\}, \{\epsilon, \beta_i, \gamma_{i-1}\},$$
$$\{\epsilon, \gamma_i, \alpha_{i-1}\}, \{\zeta, \alpha_{i-1}, \alpha_{i+1}\}, \{\zeta, \beta_{i-1}, \beta_{i+1}\}, \{\zeta, \gamma_{i-1}, \gamma_{i+1}\}$$
$$(i = 0, 1,..., 2t)$$

once, is a $(6t + 6, 3, 2)$-array. Thus the existence of a $(v, 3, \lambda_v)$-array has been established for $v \equiv 0, 3, 4 \pmod 6$, except in the simple cases $v = 3, 4, 6$ whose consideration may be left to the reader.

When $v \equiv 1 \pmod 6$, the simplest direct construction for a $(v, 3, 1)$-array known to the author is one of Skolem [10]. Let $v = 6t + 1$. Let $\theta_1,..., \theta_{2t}, \phi_1,..., \phi_{2t}, \psi_1,..., \psi_{2t}, \chi$ be $6t + 1$ distinct elements. For any $i \in \{1,..., 2t\}$ and any non-zero integer q define $\theta_{i+2tq}, \phi_{i+2tq}, \psi_{i+2tq}$ to be θ_i, ϕ_i, ψ_i, respectively. Then it is fairly easy to see that the sets

$$\{\theta_i, \phi_i, \psi_i\}, \{\theta_i, \phi_{t+i}, \chi\}, \{\phi_i, \psi_{t+i}, \chi\}, \{\psi_i, \theta_{t+i}, \chi\},$$
$$\{\theta_i, \phi_{i-r}, \phi_{i+r}\}, \{\phi_i, \psi_{i-r}, \psi_{i+r}\}, \{\psi_i, \theta_{i-r}, \theta_{i+r}\},$$
$$\{\theta_{t+i}, \phi_{i-s+1}, \phi_{i+s}\}, \{\phi_{t+i}, \psi_{i-s+1}, \psi_{i+s}\}, \{\psi_{t+i}, \theta_{i-s+1}, \theta_{i+s}\}$$
$$(i = 1, 2,..., t;\ r = 1, 2,..., t - 1;\ s = 1, 2,..., t),$$

when arranged in a sequence, constitute a $(v, 3, 1)$-array.

The theme of this paper emphasizes proofs by direct construction; but the reader may be referred to [3, pp. 91–94] for an elegant proof by induction of the existence of a Steiner triple system of order v whenever v is congruent to 1 or 3 modulo 6. A simpler, but at present unpublished, inductive proof of this fact has been obtained by Professor H. Hanani and was described at the Calgary International Conference on Combinatorial Structures and Their Applications in June 1969.

We have now only to establish the existence of $(v, 3, \lambda_v)$-arrays when $v \equiv 2, 5 \pmod 6$. This will be done by giving constructions for a $(2t + 1, 3, 3)$-array and a $(2t + 2, 3, 6)$-array which work for all positive integers t. The construction for a $(2t + 1, 3, 3)$-array was, in essentials, given on page 323 in Skolem's appendix to the second edition of [8], and I am indebted to Dr. D. A. Sprott for bringing it to my notice. So far as I know, the construction for a $(2t + 2, 3, 6)$-array is new and substantially simpler than any other known method of obtaining $(v, 3, \lambda_v)$-arrays when $v \equiv 2 \pmod 6$.

Let $\pi_0, \pi_1, \ldots, \pi_{2t}, \rho$ be $2t + 2$ distinct elements. For any $i \in \{0, 1, \ldots, 2t\}$ and any non-zero integer q define $\pi_{i+(2t+1)q}$ to be π_i. We make the convention that if Σ_h is a sequence with $2t + 1$ terms for $h = 1, \ldots, m$ then $\Sigma_1 \Sigma_2 \cdots \Sigma_m$ denotes the sequence with $(2t + 1)m$ terms obtained by writing down the terms of Σ_1 and then the terms of Σ_2 and so on, ending with the terms of Σ_m. Now let Φ_r denote the sequence

$$\{\pi_{0-r}, \pi_0, \pi_{0+r}\}, \{\pi_{1-r}, \pi_1, \pi_{1+r}\}, \{\pi_{2-r}, \pi_2, \pi_{2+r}\}, \ldots, \{\pi_{2t-r}, \pi_{2t}, \pi_{2t+r}\}$$

and Γ denote the sequence

$$\{\rho, \pi_0, \pi_1\}, \{\rho, \pi_1, \pi_2\}, \{\rho, \pi_2, \pi_3\}, \ldots, \{\rho, \pi_{2t}, \pi_{2t+1}\}$$

and Δ denote the sequence

$$\{\rho, \pi_0, \pi_2\}, \{\rho, \pi_1, \pi_3\}, \{\rho, \pi_2, \pi_4\}, \ldots, \{\rho, \pi_{2t}, \pi_{2t+2}\}.$$

Then it is an easy exercise to show that $\Phi_1 \Phi_2 \cdots \Phi_t$ is a $(2t + 1, 3, 3)$-array and that

$$\Phi_1 \Phi_2 \Phi_2 \Phi_3 \Phi_3 \Phi_4 \Phi_4 \cdots \Phi_{t-1} \Phi_{t-1} \Phi_t \Phi_t \Gamma \Gamma \Delta$$

is a $(2t + 2, 3, 6)$-array.

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