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Polynomial and Poisson dependence in free Poisson algebras and free Poisson fields

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ABSTRACT

We prove that any two Poisson dependent elements in a free Poisson algebra and a free Poisson field of characteristic zero are algebraically dependent, thus answering positively a question from Makar-Limanov and Umirbaev (2007) [8]. We apply this result to give a new proof of the tameness of automorphisms for free Poisson algebras of rank two (see Makar-Limanov and Umirbaev (2011) [9], Makar-Limanov et al. (2009) [10]).

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1. Introduction

The free Poisson algebras were first considered in [12]. They are naturally and closely related to polynomial algebras, free associative algebras, and free Lie algebras. For example, the free Poisson algebra and Poisson brackets were used in [13,14] to prove that the Nagata automorphism of the polynomial algebra of rank three is wild.

A systematic study of free Poisson algebras was started in [8] where several open questions on their structure were formulated. It was proved in [8] that the centralizer of a nonconstant element of a free Poisson algebra in the case of characteristic zero is a polynomial algebra in a single variable; this is an analogue of the famous Bergman Centralizer Theorem [1]. Then in [10] it was proved that locally nilpotent derivations of free Poisson algebra of rank two in the case of characteristic zero are triangulable and that automorphisms of these algebras are tame; these are analogues of the well-known

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Rentschler Theorem [11] and Jung Theorem [5] respectively. Finally, in [9] the Freiheitssatz was proved for free Poisson algebras over a field of characteristic zero.

In this paper we continue the study of free Poisson algebras and solve positively a question formulated in [8] by proving that every two Poisson dependent elements in a free Poisson algebra over a field of characteristic zero are algebraically dependent. In fact we prove a bit more: any two Poisson dependent elements which are rational over a free Poisson algebra are algebraically dependent.

As a corollary, we give another proof of the result from [10] that the automorphisms of free Poisson algebra of rank two in the case of characteristic zero are tame (see also [9]).

2. The main part

Below $\mathcal{P} = \mathcal{P}(X)$ denotes a free Poisson algebra on a set of generators $X = \{x_1, \dots, x_n\}$ over a field F of characteristic zero. Recall (see, for example [12]) that \mathcal{P} is isomorphic to $S(\text{Lie}(X))$, where $\text{Lie}(X)$ is the free Lie algebra over X and $S(V)$ means the symmetric algebra over a vector space V . Denote by $y_1 = x_1, y_2 = x_2, \dots, y_n = x_n, y_{n+1} = \{x_1, x_2\}, \dots$ a basis of $\text{Lie}(X)$ consisting of the Lie monomials which are ordered by increasing Lie degree (and arbitrary for monomials of the same degree). Hence as a commutative algebra \mathcal{P} is a polynomial algebra $F[y_1, y_2, \dots]$ with infinitely many generators. For elements $f, g \in \mathcal{P}$ we denote by fg their product as elements of $F[y_1, y_2, \dots]$ and by $\{f, g\}$ their Poisson product (the Poisson bracket) which is defined on y_1, y_2, \dots as elements of $\text{Lie}(X)$, extended on monomials of \mathcal{P} by the Leibnitz law, and then on \mathcal{P} by linearity.

A family of polynomial weight degree functions can be defined on $F[y_1, y_2, \dots]$ by giving arbitrary real weights $w_i = w(y_i)$ to the generators and extending it on monomials $M = y_1^{j_1} y_2^{j_2} \dots$ by $w(M) = \sum_i j_i w(y_i)$. Then for $f \in F[y_1, y_2, \dots]$ degree can be defined as $D(f) = \max(w(M) \mid M \in f)$, i.e. maximum by all monomials contained in f with non-zero coefficients. Of course not all of these functions make sense for \mathcal{P} as a Poisson algebra. We say that a weight degree function D on \mathcal{P} is compatible with the Poisson structure if it satisfies the following natural condition:

for any two monomials $M_1, M_2 \in \mathcal{P}$ (as a polynomial algebra) the bracket $\{M_1, M_2\}$ is D -homogeneous.

For example the weight which is defined on a Lie monomial y as the number of appearances of a free generator x_k in y defines a compatible degree function d_{x_k} . It is easy to check that in order to define a compatible degree function the weight should be given on a Lie monomial y by

$$w(y) = \sum_i (w(x_i) - c)d_{x_i}(y) + c$$

where $w(x_i)$ and c are arbitrary real numbers. To see it define $\Delta(i, j) = w(y_i) + w(y_j) - D(\{y_i, y_j\})$ for two different Lie monomials y_i, y_j . Take $\{y_i y_j, y_k\} = \{y_i, y_k\}y_j + y_i\{y_j, y_k\}$ where y_i, y_j, y_k are pairwise different Lie monomials. Then $D(\{y_i, y_k\}y_j) = D(y_i\{y_j, y_k\})$, $\Delta(i, k) = \Delta(j, k)$, and $\Delta(i, j) = c$ is a constant. Therefore weight w is completely determined by $w_i = w(x_i)$ and $c = \Delta(i, j)$.

Examples of compatible degree functions are d_{x_k} defined above and the Poisson degree which corresponds to $w(x_1) = \dots = w(x_n) = 1, c = 0$ (i.e. to the Lie degree). Total polynomial degree deg on $F[y_1, y_2, \dots]$ is also compatible and corresponds to $w(x_1) = \dots = w(x_n) = c = 1$.

Recall that $\text{deg}(\{f, g\}) = \text{deg}(f) + \text{deg}(g) - 1$ for homogeneous f and g if $\{f, g\} \neq 0$. Similar relation is true for any compatible weight degree function: $D(\{f, g\}) = D(f) + D(g) - c$ if $\{f, g\} \neq 0$ and f and g are D -homogeneous.

Below we will consider only the weights for which all parameters are integers.

Elements a_1, \dots, a_m of a Poisson algebra \mathcal{S} are called *Poisson dependent* if there exists a non-zero element $p(x_1, \dots, x_m)$ in the free Poisson algebra $\mathcal{P}(x_1, \dots, x_m)$ such that $p(a_1, \dots, a_m) = 0$ in \mathcal{S} ; the elements a_1, \dots, a_m are called *algebraically dependent* if there exists a non-zero polynomial $f(x_1, \dots, x_m) \in F[x_1, \dots, x_m]$ such that $f(a_1, \dots, a_m) = 0$.

Element u is called *algebraic* over a Poisson algebra \mathcal{S} if u belongs to a commutative algebra \mathcal{T} containing \mathcal{S} (as a commutative subalgebra) and $p(u) = 0$ in \mathcal{T} for a non-zero polynomial $p \in \mathcal{S}[t]$.

If S is a domain the bracket can be extended uniquely from S to the field $S(u)$. Indeed, take a non-zero polynomial $p(t) = \sum_i p_i t^i$ where $p_i \in S$ for which $p(u) = 0$ of the minimal degree possible. If an extension of the bracket exists and we use the same notation for it then $0 = \{f, p(u)\} = \{f, u\}p'(u) + \sum_i \{f, p_i\}u^i$ for any $f \in S$ which defines $\{f, u\}$ provided $p'(u) \neq 0$, i.e. in the zero characteristic case or for a separable extension. It is a straightforward computation to check that this bracket makes $S(u)$ a Poisson algebra.

Denote by \mathcal{Q} the field of fractions of \mathcal{P} considered as a commutative polynomial algebra. We can extend the bracket from \mathcal{P} to \mathcal{Q} as we saw above. A compatible weight degree function D can be extended from \mathcal{P} to \mathcal{Q} by $D(\frac{a}{b}) = D(a) - D(b)$. We will call \mathcal{Q} a free Poisson field.

Lemma 1. *Let f, g be elements algebraic over a free Poisson algebra \mathcal{P} . If $f, g \in \mathcal{P}[f, g]$ are Poisson dependent and $r_1(x_1, x_2), r_2(x_1, x_2) \in F(x_1, x_2)$ are rational functions then $r_1(f, g), r_2(f, g) \in \mathcal{P}(f, g)$ are also Poisson dependent.*

Proof. Elements f, g are Poisson dependent if the basic Lie monomials of f, g are algebraically dependent. Denote by $y_1, \dots, y_{N(A)}$ the set of all basic Lie monomials with $d(y_j) \leq a$. Consider the smallest A for which $y_1(f, g), y_2(f, g), \dots, y_{N(A)}(f, g)$ are algebraically dependent. It is easy to check using induction on $a_i = d(y_i(x_1, x_2))$ that $y_i(r_1(f, g), r_2(f, g)) \in F(f, g)[y_3(f, g), \dots, y_{N(A_i)}(f, g)]$. Hence there is an algebraic dependence between $y_1(r_1(f, g), r_2(f, g)), \dots, y_{N(A)}(r_1(f, g), r_2(f, g))$. \square

For $f \in \mathcal{Q}$ denote by $\text{supp}(f)$ the minimal set of polynomial variables on which f depends.

Lemma 2. *Let $f, g \in \mathcal{Q}$ be elements which are algebraically independent. Then for a given polynomial weight degree function D there exists an element $h \in F[f, g]$ such that the leading forms f_D, h_D are algebraically independent.*

Proof. A standard proof of this fact would be based on the notion of Gelfand–Kirillov dimension (see [4]) and is well known for the polynomial case. We give a proof using Poisson brackets which is possible in the case of zero characteristic.

Consider $\text{supp}(f) \cup \text{supp}(g) = \{y_{i_1}, \dots, y_{i_k}\}$. Since f, g are algebraically independent we may assume without loss of generality that $f, g, y_{i_3}, \dots, y_{i_k}$ are algebraically independent and introduce on $F(y_{i_1}, \dots, y_{i_k})$ a deficiency function (somewhat similar to the one introduced in [7]) by

$$\text{def}(f, h) = D(J_{y_{i_1}, \dots, y_{i_k}}(f, h, y_{i_3}, \dots, y_{i_k})) - D(h)$$

where $J_{y_{i_1}, \dots, y_{i_k}}(f, h, y_{i_3}, \dots, y_{i_k})$ is the Jacobian of $f, h, y_{i_3}, \dots, y_{i_k}$, i.e. the determinant of the corresponding Jacobi matrix. This function is defined and has values in \mathbb{Z} when $J_{y_{i_1}, \dots, y_{i_k}}(f, h, y_{i_3}, \dots, y_{i_k}) \neq 0$.

Since $J_{y_{i_1}, \dots, y_{i_k}}(f, p(f, g), y_{i_3}, \dots, y_{i_k}) = J_{y_{i_1}, \dots, y_{i_k}}(f, g, y_{i_3}, \dots, y_{i_k}) \frac{\partial p}{\partial g}$ for any $p \in F[f, g]$ and $J_{y_{i_1}, \dots, y_{i_k}}(f, g, y_{i_3}, \dots, y_{i_k}) \neq 0$ function def is defined on any algebraically independent pair from $F[f, g]$.

Observe that

$$\begin{aligned} \text{def}(f, h^k) &= \text{def}(f, h), \\ \text{def}(f, hr(f)) &= \text{def}(f, h), \quad r(f) \in F(f) \setminus 0, \\ \text{def}(f, h) &\leq D(f) - c, \end{aligned}$$

where c is a constant which depends only on degrees of $f, y_{i_1}, \dots, y_{i_k}$.

If f_D and g_D are algebraically dependent then there exists a non-zero polynomial $q = \sum_{i=0}^k q_i(x)y^i \in F[x, y]$ for which all monomials with non-zero coefficients have the same D degree, $k = \deg_y(q)$ is minimal possible, and $q(f_D, g_D) = 0$. In our setting elements $f, g' = q(f, g)$ are algebraically independent. Denote in this lemma only $J_{y_1, \dots, y_{i_k}}(f, h, y_{i_3}, \dots, y_{i_k})$ by $\{f, h\}$.

We have

$$\begin{aligned} \text{def}(f, g') &= D(\{f, g'\}) - D(g') = D\left(\sum_i \{f, q_i(f)g^i\}\right) - D(g') \\ &> D(\{f, q_k(f)g^k\}) - D(q_k(f)g^k) = \text{def}(f, g^k) = \text{def}(f, g) \end{aligned}$$

since $D(g') < D(q_k(f)g^k)$ while $D(\{f, q_k(f)g^k\}) = D(kq_k(f)g^{k-1}) + D(\{f, g\}) = D(\sum_i iq_i(f)g^{i-1}) + D(\{f, g\}) = D(\sum_i \{f, q_i(f)g^i\})$ (recall that $\sum_i iq_i(f_D)g_D^{i-1} \neq 0$). If f_D, g'_D are algebraically dependent, we repeat the procedure and obtain a pair f, g'' with $\text{def}(f, g'') > \text{def}(f, g')$. Since $\text{def}(f, h) \leq D(f) - c$ for any h and $\text{def}(f, h) \in \mathbb{Z}$, the process will stop after a finite number of steps and we will get an element $h \in F[f, g]$ for which h_D is algebraically independent with f_D . \square

Lemma 3. *Let $f, g \in \mathcal{Q}$ be elements which are Poisson dependent but not algebraically dependent. Then there exists a pair of elements which are homogeneous relative to any compatible weight degree function D with the same property.*

Proof. Denote by h_D the leading form of $h \in \mathcal{Q}$ relative to D . From the definition of compatibility $y_i(f, g)_D = y_i(f_D, g_D)$ if $y_i(f_D, g_D) \neq 0$. Since $P(f, g) = 0$ for a Poisson polynomial, $P_D(f_D, g_D) = 0$ for a polynomial P_D consisting of monomials M of P for which $D(M(f_D, g_D))$ is maximal possible. Hence f_D, g_D are Poisson dependent.

If f_D and g_D are algebraically dependent then we can use Lemma 2 to find an element $h \in F[f, g]$ such that h_D and f_D are algebraically independent and to obtain a D -homogeneous pair of Poisson dependent elements which are algebraically independent. The space of compatible weights is finite dimensional lattice, hence we can obtain a pair of Poisson dependent elements which are algebraically independent and are homogeneous relative to all compatible degree functions. \square

We will call elements which are homogeneous relative to all compatible degree functions *completely homogeneous*.

Lemma 4. *Let $f, g \in \mathcal{Q}$ be a Poisson dependent pair and x be the smallest element in $\text{supp}(f)$. Write $f = x^n f_x + \dots, g = x^m g_x + \dots$, where f_x, g_x do not contain x and dots stand for terms with smaller (polynomial) degrees in x . Then the pair f_x, g_x is Poisson dependent as well.*

Proof. Consider the Poisson polynomial $P(x_1, x_2)$ for which $P(f, g) = 0$; it is a sum of monomials of the type

$$u = y_1^{k_1} y_2^{k_2} \dots y_s^{k_s},$$

where y_i are Lie monomials in x_1, x_2 . We have

$$\begin{aligned} \{f, g\} &= x^{n+m} \{f_x, g_x\} + \dots, \\ y_i(f, g) &= x^{N_i} y_i(f_x, g_x) + \dots, \quad N_i = nd_{x_1}(y_i) + md_{x_2}(y_i), \\ u(f, g) &= x^{N(u)} u(f_x, g_x) + \dots, \quad N(u) = \sum_i k_i N_i, \end{aligned}$$

where again dots mean terms of smaller degree in x . Observe that x cannot appear in $\{f_x, g_x\}$ or in $y_i(f_x, g_x)$ when $d(y_i) > 1$ since for any $y \in \text{supp}(f)$, $z \in \text{supp}(g)$ we have $\{y, z\} > y \geq x$. Therefore,

$$0 = P(f, g) = Q(f_x, g_x)x^N + \dots,$$

where $N = \max\{N(u) \mid u \text{ monomial in } P(x_1, x_2)\}$, $Q(x_1, x_2) = \sum_{N(u)=N} u(x_1, x_2)$. Since all monomials u in $P(x_1, x_2)$ are linearly independent, we have $Q(x_1, x_2) \neq 0$ and hence f_x, g_x are Poisson dependent. \square

Lemma 5. *In the conditions of Lemma 4, assume that $f_x = 1$, $f = x^n + \alpha x^{n-1} + \dots$, $\deg g_x > 0$. Then the pair $nx + \alpha, g_x$ is Poisson dependent.*

Proof. Let us check by induction on the Poisson degree that $y_i(f, g) = x^{N_i} y_i(nx + \alpha, g_x) + \dots$ for $i > 1$, where $N_i = (n - 1)d_{x_1}(y_i) + md_{x_2}(y_i)$ for any Lie monomial with $i > 1$ and dots stand for the terms of smaller degree in x (recall that $\deg_x(g) = m$). The base of induction for $y_2(f, g) = g$ is clear. A Lie monomial $y_k(f, g)$ with $k > 2$ can be presented as either $\{y_l(f, g), f\}$ or $\{y_l(f, g), g\}$ where y_l is a monomial with a smaller Poisson degree. If $l = 1$ then $k = 3$ is the only interesting case and $y_3(f, g) = \{f, g\} = \{x^n + \alpha x^{n-1} + \dots, x^m g_x + \dots\} = nx^{n-1+m}\{x, g_x\} + x^{n-1+m}\{\alpha, g_x\} + \dots = x^{n-1+m}\{nx + \alpha, g_x\} + \dots$. If $l > 1$ then by induction $y_l(f, g) = x^{N_l} y_l(nx + \alpha, g_x) + \dots$ and similar computations verify the claim. It is essential that x is the smallest element in $\text{supp}(f)$ because $\text{supp}(\{y, z\})$ does not contain x if $y \geq x$ and no additional powers of x may appear as results of Poisson brackets.

Therefore for $u = y_1^{k_1} y_2^{k_2} \dots y_s^{k_s}$ the leading form of $u(f, g)$ relative to x is $x^{nk_1 + N_u} y_2(nx + \alpha, g_x)^{k_2} \dots y_s(nx + \alpha, g_x)^{k_s}$ where $N_u = (n - 1)d_{x_1}(y_2^{k_2} \dots y_s^{k_s}) + md_{x_2}(y_2^{k_2} \dots y_s^{k_s})$. Hence different monomials of $P(f, g)$ cannot cancel in the x -leading form of $P(f, g)$ and the elements $nx + \alpha, g_x$ are Poisson dependent. \square

Consider now a pair of algebraically independent elements $f, g \in \mathcal{Q}$. By the Shirshov–Witt Theorem a subalgebra of a free Lie algebra is a free Lie algebra (see [15,17]) so the elements of $\text{supp}(f) \cup \text{supp}(g)$ generate a free Lie algebra \mathcal{L} with the free basis which contains two smallest elements x, y of $\text{supp}(f) \cup \text{supp}(g)$. Elements x and y are different since otherwise $\text{supp}(f) \cup \text{supp}(g) = x$ and f, g are algebraically dependent. If \mathcal{P} is the free Poisson algebra which correspond to \mathcal{L} and \mathcal{Q} is the field of fractions of \mathcal{P} then $f, g \in \mathcal{Q}$. Though f, g are possibly written through different generators, the size of $\text{supp}(f) \cup \text{supp}(g)$ did not change.

Assume that there exists a pair of algebraically independent Poisson dependent elements in a free Poisson field \mathcal{Q} . Then we can find a pair which is minimal in the following sense: the size $|f, g|$ of $\text{supp}(f) \cup \text{supp}(g)$ is minimal possible, \mathcal{Q} is generated by $\text{supp}(f) \cup \text{supp}(g)$, elements f and g are completely homogeneous.

As we observed $|f, g|$ does not change when we replace the original Poisson field with the “minimal” one. The elements may stop being completely homogeneous but by Lemma 3 we can produce a completely homogeneous pair which belongs to $F[f, g]$, hence the union of supports of these two elements belongs to the union of supports of the original elements. Since $|f, g|$ is minimal it implies that the size cannot become smaller, so the union of supports of a completely homogeneous pair is the same as for the original pair.

Recall that if two homogeneous polynomials $f, g \in F[X]$ are algebraically dependent then there exists a homogeneous polynomial $h \in F[X]$ such that $f = \alpha h^k, g = \beta h^l$ for some $\alpha, \beta \in F$ and natural numbers k, l (see, for example, [2]). Similar statement is true for two algebraically dependent homogeneous rational functions $f, g \in F(X)$ if one of them, say f , has a non-zero degree. Indeed, since we may assume that $D(f^i g^j)$ is the same for all monomials of q

$$q(f, g) = \sum_{iD(f)+jD(g)=d} q_{ij} f^i g^j = 0.$$

If $D(g) = 0$ then $q(f, g) = f^a q_a(g)$, $q_a(g) = 0$ and $g \in F$. If $D(g) \neq 0$ then $q(f, g) = f^a g^b \tilde{q}(f^\rho g^{-\sigma})$ where ρ, σ are relatively prime integers for which $\rho D(f) = \sigma D(g)$ and $\tilde{q}(x) \in F[x]$. Hence $q(f, g) = f^a g^b \prod (f^\rho g^{-\sigma} - c_i)$ where $c_i \in \bar{F}$, an algebraic closure of F . So $f^\rho - c g^\sigma = 0$ for some $c \in \bar{F}$. Furthermore, if $r\rho + s\sigma = 1$, $r, s \in \mathbb{Z}$, and $h = f^s g^r$ then h^σ is proportional to f and h^ρ is proportional to g . If we also assume that h is not a proper power (of a rational function) then all rational functions which are algebraically dependent with h belong to $F(h)$.

Lemma 6. *Let $f, g \in \mathcal{Q}$ be a minimal pair. If x is the smallest element in $\text{supp}(f) \cup \text{supp}(g)$, then there exists a minimal pair $\tilde{f} = x + f_1, \tilde{g}$ where $\text{supp}(f_1) \cup \text{supp}(\tilde{g}) \not\ni x$.*

Proof. Write $f = x^n f_x + \dots, g = x^m g_x + \dots$, where f_x, g_x do not contain x and dots stand for terms with smaller (polynomial) degrees in x . Then the pair f_x, g_x is Poisson dependent by Lemma 4 and is algebraically dependent since $|f_x, g_x| < |f, g|$. If $D(f_x) = 0$ for any compatible degree function consider the second smallest element $y \in \text{supp}(f) \cup \text{supp}(g)$ and present $f = y^{n_1} f_y + \dots$ where f_y does not contain y and dots stand for terms with smaller degrees in y . If $D(f_y) = 0$ for any compatible degree function then $D(x^{n_1}) = D(y^{n_1})$ for any compatible degree. But x, y are elements of a free basis, so the Poisson degrees d_x and d_y are compatible degree functions and either $x = y$ which is impossible or $n = n_1 = 0$. Similar considerations for g show that either $D(g_x)$ or $D(g_y)$ is not identically zero or $m = m_1 = 0$. If $n = m = 0$ consider polynomial dependence q between f_x, g_x and a minimal pair $f, g_1 = q(f, g)$. Then $g_1 = x^k g_{1x} + \dots$ where $k < 0$. So either $D(g_{1x})$ or $D(g_{1y})$ is not identically zero. Since x, y are elements of a free basis we can reorder them as well as f, g_1 and assume that $D(f_x) \neq 0$ for some compatible degree function. Then by remarks above there exists a completely homogeneous element $h \in \mathcal{Q}$ such that $f_x = c_1 h^a, g_x = c_2 h^b$ where $c_1, c_2 \in F \setminus 0, D(h) \neq 0$ for some compatible degree function, and $a \neq 0$. Without loss of generality we may assume that $c_1 = c_2 = 1$. Hence $f = x^n h^a + \dots, g = x^m h^b + \dots$. The pair $f^b g^{-a}, f \in \mathcal{Q}$ is Poisson dependent by Lemma 1. We can write $f^b g^{-a} = x^{ln-km} + \alpha x^{ln-km-1} + \dots$. Hence by Lemma 5 the pair $(ln - km)x + \alpha, h^a$ is Poisson dependent. Recall that $\text{supp}(\alpha) \cup \text{supp}(h) \not\ni x$. Therefore $(ln - km)x + \alpha$ and h are algebraically independent if $(ln - km) \neq 0$. So if $(ln - km) \neq 0$ we proved the lemma.

If $ln - km = 0$ then algebraically independent rational functions f, g have algebraically dependent leading forms relative to polynomial deg_x . According to Lemma 2 ring $F[f, g]$ contains an element g' such that deg_x -leading forms of f and g' are algebraically independent. Since $\text{supp}(g') \subset \text{supp}(f) \cap \text{supp}(g)$ the pair f, g' is minimal and we can use it to prove the lemma. \square

Theorem 1. *Every two Poisson dependent elements in the free Poisson field \mathcal{Q} are algebraically dependent.*

Proof. Assume that the theorem is not true. Then by the previous lemmas there exists a completely homogeneous Poisson dependent algebraically independent pair $f = x + f_1, g \in \mathcal{Q}$, where the size $|f, g|$ is minimal possible, x is the minimal element in $\text{supp}(f) \cup \text{supp}(g)$ and $\text{supp}(f_1) \cup \text{supp}(g) \not\ni x$, and x is an element of the free basis of \mathcal{P} .

Consider the smallest element $y \in \text{supp}(g)$ and write $f = y^n f_y + \dots, g = y^m g_y + \dots$ where $\text{supp}(f_y) \cup \text{supp}(g_y) \not\ni y$. Elements f_y and g_y should be Poisson dependent by Lemma 4 and algebraically dependent since $|f_y, g_y| < |f, g|$.

If $n = 0$ then $f_y = x + f_{1y}$ and g_y are algebraically dependent and $d_x(f_y) = 1$. Hence $g_y = c f_y^b$ and $b = 0$ since otherwise $\text{supp}(g_y) \ni x$. If furthermore $m = 0$ then $g = c + \dots$ where $c \in F$ and we will replace g by $\tilde{g} = g - c$. Then $\tilde{g} = y^{\tilde{m}} \tilde{g}_y + \dots$ where $\tilde{m} \neq 0$. Furthermore, $f_y = x + f_{1y}$ and \tilde{g}_y are Poisson and algebraically dependent, which as above is possible only if $\tilde{g}_y \in F$. Since y is a Lie monomial and $y \neq x$ there is an element $z \neq x$ in the free basis for which $d_z(y) \neq 0$ and $d_z(\tilde{g}) = \tilde{m} d_z(y) \neq 0$. But $D(\tilde{g}) = 0$ for any compatible weight degree function. Therefore $\tilde{g}_y \notin F, \tilde{g}_y = c f_y^b$ where b is a non-zero integer, and $\text{supp}(\tilde{g}_y) \ni x$, a contradiction. We can conclude that $m \neq 0$ and that $f_y, m y + m\beta$ are Poisson dependent by Lemma 5. Since $f_y = x + f_{1y}$ where all elements of $\text{supp}(f_{1y})$ are larger than x and all elements of $\text{supp}(\beta)$ are larger than y we can see that $y_i(x + f_{1y}, y + \beta) = y_i(x, y) + \dots$ for any Lie monomial y_i where \dots stand for Lie monomials larger than $y_i(x, y)$. Hence these elements are Poisson independent and $n = 0$ is impossible.

Since x is an element of the free basis it follows from the complete homogeneity that $0 = d_z(x) = d_z(y^n f_y)$. Therefore $d_z(f_y) = -nd_z(y) \neq 0$. Elements f_y, g_y are algebraically dependent, hence $f_y = c_1 h^a, g_y = c_2 h^b$ for some element h where $a \neq 0$ and we may assume that $c_1 = c_2 = 1$.

If $b = 0$ and $m \neq 0$ then $g = y^m + \dots$ and f_y and $my + \beta$ are Poisson dependent. They are algebraically independent since $y \notin \text{supp}(f_y), y \in \text{supp}(my + \beta)$. But $\text{supp}(f_y) \cap \text{supp}(g_y) \neq \emptyset$ and we have a contradiction with the minimality of the pair f, g . If $m = 0$ consider $\tilde{g} = g - 1$. Then $\tilde{g} = y^{\tilde{m}} \tilde{g}_y + \dots$ where $\tilde{m} \neq 0$ and $\tilde{g}_y = h^{\tilde{b}}$ because $f_y = h^a$ where $a \neq 0$. Now, $d_x(f) = d_x(x) = 1 = d_x(y^n f_y), d_z(f) = d_z(x) = 0 = d_z(y^n f_y)$ and $d_x(f_y) = 1 - nd_x(y), d_z(f_y) = -nd_z(y)$. Since $d_x(y^{\tilde{m}} h^{\tilde{b}}) = 0, d_z(y^{\tilde{m}} h^{\tilde{b}}) = 0$ and $d_x(f_y) = 1 - nd_x(y), d_z(f_y) = -nd_z(y)$ both $\tilde{m} = \tilde{b} = 0$, which is impossible. Therefore $b \neq 0$.

Replace now g by $\tilde{g} = g^{-a} f^b$. Then $\tilde{g} = y^k + y^{k-1} \tilde{g}_1 + \dots$. The case $k = 0$ could be brought to a contradiction just as the case $b = m = 0$ above. Therefore $k \neq 0$.

Elements $ky + \tilde{g}_1, f_y$ are algebraically independent since $\text{supp}(f_y) \not\ni y$ and $k \neq 0$. Since

$$\text{supp}(ky + \tilde{g}_1) \cup \text{supp}(f_y) \subseteq \text{supp}(f) \cup \text{supp}(g)$$

we should have $\text{supp}(ky + \tilde{g}_1) \cup \text{supp}(f_y) = \text{supp}(f) \cup \text{supp}(g)$ by the minimality condition and $x \in \text{supp}(\tilde{g}_1)$ ($x \notin \text{supp}(f_y)$ since $n > 0$). Recall that $\tilde{g} = g^{-a} f^b$ and therefore $x \in \text{supp}(\tilde{g}_1)$ only if $n = 1$, i.e. if $f = yh^a + (x + \delta) + \dots$ where \dots stand for the terms with negative powers in y . Hence $\tilde{g} = \frac{(yh^a + (x + \delta) + \dots)^b}{(y^m h^b + \varepsilon y^{m-1} + \dots)^a} = y^k + [b(x + \delta)h^{-a} - a\varepsilon h^{-b}]y^{k-1} + \dots$ and $\tilde{g}_1 = b(x + \delta)h^{-a} - a\varepsilon h^{-b}$.

The elements $(ky + \tilde{g}_1)f_y, f_y$ are Poisson dependent by Lemma 1. Hence $[ky + b(x + \delta)h^{-a} - a\varepsilon h^{-b}]h^a = b(x + \delta) - a\varepsilon h^{a-b} + kyh^a$ and h^a are Poisson dependent.

It is clear that $\text{supp}(h^a)$ is a proper subset of $\text{supp}(g)$. So we may apply induction on the size of $\text{supp}(g)$ to prove the theorem. The base of induction when $|g| = 1$ corresponds to $g = y^m, m \neq 0$. As we have seen above in order to avoid a contradiction we should have $f = y^n f_y + \dots$ where $n \neq 0, f_y \notin F$, and $g_y \notin F$. But $g_y = 1$ and we have a contradiction which proves the theorem. \square

It was shown in [8] that $f, g \in \mathcal{P}$ is algebraically dependent if and only if $\{f, g\} = 0$ i.e. f and g are Poisson commuting. Of course if two elements f, g of a Poisson algebra which is a domain are algebraically dependent they Poisson commute: $p(f, g) = 0$ implies that $p_g(f, g)\{f, g\} = 0$. The theorem shows that for $f, g \in \mathcal{Q}$ Poisson commuting implies an algebraic dependence. Hence for the pairs from \mathcal{Q} the notions of Poisson commuting, algebraic dependence, and Poisson dependence are equivalent.

Corollary 1. *Let $f, g \in \mathcal{Q}, \{f, g\} \neq 0$. Then f, g generate a free Lie algebra with respect to the bracket $\{, \}$, and they generate a free Poisson subalgebra in \mathcal{Q} in complete analogy to the case of free associative algebras.*

Observe that the theorem is evidently not true for more than two elements: the elements $x_1, x_2, \{x_1, x_2\}$ are Poisson dependent but are algebraically independent. It is not true as well if $\text{char } F = p > 0$; the elements x_1, x_2^p are algebraically independent but $\{x_1, x_2^p\} = px_2^{p-1}\{x_1, x_2\} = 0$.

3. Application to automorphisms

It is well known [3,5,16,6] that the automorphisms of polynomial algebras and free associative algebras in two variables are tame. The automorphisms of free Poisson algebras in two variables over a field of characteristic zero are also tame [10]. In [9] this result was obtained as a corollary of the Freiheitssatz for Poisson algebras. Here we show that the result follows from our theorem as well.

Theorem 2. *(See [10].) Automorphisms of the free Poisson algebra $\mathcal{P}(x, y)$ of rank two over a field F of characteristic 0 are tame.*

Proof. Let α be an automorphism of $\mathcal{P}_2 = \mathcal{P}(x, y)$. Since any (tame) automorphism of $F[x, y]$ can be lifted to a (tame) automorphism of \mathcal{P}_2 , we can assume without loss of generality that the abelianization of α (that is, its homomorphic image under the natural epimorphism $\text{Aut}(\mathcal{P}_2) \rightarrow \text{Aut}(F[x, y])$) is the identity automorphism of $F[x, y]$. It remains to show that then α is the identity automorphism of \mathcal{P}_2 .

Let $\alpha(x) = f$, $\alpha(y) = g$. Assume that either $f \neq x$ or $g \neq y$. If we take weights $w(x) = \rho$, $w(y) = 1$ where $\rho \geq 0$ then $f_{\sim} = x$ and $g_{\sim} = y$ where f_{\sim} and g_{\sim} are the lowest Poisson forms of f and g with respect to w . If we start now to decrease ρ then for some non-positive value of ρ either $f_{\sim} \neq x$ or $g_{\sim} \neq y$ for the corresponding f_{\sim} and g_{\sim} . Let us take the largest ρ with this property. Then f_{\sim} and g_{\sim} are Poisson w -homogeneous, $d_w(f_{\sim}) = \rho$, $d_w(g_{\sim}) = 1$, $f_{\sim} = x + f_1$, $g_{\sim} = y + g_1$, where at least one of f_1, g_1 is non-zero and their abelianizations in $F[x, y]$ are both zero. Clearly, f_{\sim} and g_{\sim} are Poisson independent.

Let $x = X(f, g)$ for some Poisson polynomial $X(x_1, x_2)$, then $x = (X(f, g))_{\sim} = X_{\sim}(f_{\sim}, g_{\sim})$ since f_{\sim} and g_{\sim} are Poisson independent. Similarly, y belongs to the Poisson subalgebra generated by f_{\sim} and g_{\sim} . Therefore, the w -homogeneous Poisson forms f_{\sim}, g_{\sim} generate \mathcal{P}_2 .

Consider now the Poisson leading forms $(\overline{f_{\sim}})$ and $(\overline{g_{\sim}})$ of f_{\sim} and g_{\sim} with respect to the Poisson degree, when $d(x) = d(y) = 1$. If they were Poisson independent, then as above they would generate \mathcal{P}_2 . But this is impossible since otherwise their abelianizations, the images under the epimorphism $\mathcal{P}_2 \rightarrow F[x, y]$, would generate $F[x, y]$, while at least one of them is 0.

Next we can use our theorem and conclude that $(\overline{f_{\sim}})$ and $(\overline{g_{\sim}})$ are algebraically dependent. Therefore up to scalars they are h^a, h^b for a certain Poisson-homogeneous element $h \in \mathcal{P}_2$ and non-negative integers a, b . Then we have $ad_w(h) = \rho$, $bd_w(h) = 1$ where $\rho < 0$, which is impossible. \square

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