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Coherent states of the deformed Heisenberg–Weyl algebra in non-commutative space

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Abstract

In two-dimensional space a subtle point that for the case of both space–space and momentum–momentum non-commuting, different from the case of only space–space non-commuting, the deformed Heisenberg–Weyl algebra in non-commutative space is not completely equivalent to the undeformed Heisenberg–Weyl algebra in commutative space is clarified. It follows that there is no well-defined procedure to construct the deformed position–position coherent state or the deformed momentum–momentum coherent state from the undeformed position–momentum coherent state. Identifications of the deformed position–position and deformed momentum–momentum coherent states with the lowest energy states of a cold Rydberg atom in special conditions and a free particle, respectively, are demonstrated.

In recent hinting at new physics, motivated by studies of the low energy effective theory of D-brane with a non-zero Neveu–Schwarz B field background, it shows that physics in non-commutative space [1–7] is a possible candidate. Based on the incomplete decoupling mechanism one expects that quantum mechanics in non-commutative space (NCQM) may clarify some low energy phenomenological consequences, and may lead to deeper understanding of effects of spatial non-commutativity. In literature NCQM have been studied in detail [8–29]. Many interesting topics of NC quantum theories have been extensively investigated, from the Aharonov–Bohm effect to the quantum Hall effect [30–36]. Recent investigation of the non-perturbation aspect of the deformed Heisenberg–Weyl algebra (the NCQM algebra) [27] in non-commutative space explored that when the state vector space of identical bosons is constructed by generalizing one-particle quantum mechanics, in order to maintain Bose–Einstein statistics at the non-perturbation level described by deformed annihilation–creation opera-

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tors, the consistent ansatz of commutation relations of the phase space variables should include both space–space non-commutativity and momentum–momentum non-commutativity. This explores some new features of spatial non-commutativity: The spectrum of the angular momentum of a two-dimensional system possesses fractional eigenvalues and fractional intervals [27]; for a cold Rydberg atom arranged in appropriate external electric and magnetic fields, in the limits of vanishing kinetic energy and diminishing magnetic field the unusual value of the lowest orbital angular momentum shows a clear signal of spatial non-commutativity [28]; variances of a two-photon squeezed state in different degrees of freedom correlates each other [29].

In this Letter we clarify a subtle point related to the equivalency between the NCQM algebra in non-commutative space and the undeformed Heisenberg–Weyl algebra in commutative space. For the case of only space–space non-commuting, the phase space variables of the NCQM algebra is related to the ones of the undeformed Heisenberg–Weyl algebra by a singular-free linear transformation, i.e., two algebras are equivalent. By a well-defined procedure, the deformed position–position coherent state in non-commutative space can be obtained from the undeformed position–momentum coherent state in commutative space [19]. But for the case of both space–space and momentum–momentum non-commuting the situation is different. The point is that there is no singular-free linear transformation to relate phase space variables between two algebras, i.e., two algebras are not equivalent. As is well known, in this case three minimal uncertainties, respectively, corresponding to the position–momentum, position–position and momentum–momentum commutation relations are saturated by corresponding coherent states. It only relates to the NCQM algebra and has nothing to do with dynamics. Because of the non-equivalency between two algebras, there is no well-defined procedure to construct the deformed position–position coherent state or the deformed momentum–momentum coherent state from the undeformed position–momentum coherent state. We show an example of the deformed position–position coherent state: A cold Rydberg atom arranged in appropriate electric and magnetic fields in the limit of vanishing kinetic energy possesses non-trivial dynamics; its lowest energy state saturates the deformed position–position uncertainty relation. An example of the deformed momentum–momentum coherent state realized by the lowest energy state of a free particle is briefly demonstrated.

In order to develop the NCQM formulation we need to specify the phase space and the Hilbert space on which operators act. The Hilbert space is consistently taken to be exactly the same as the Hilbert space of the corresponding commutative system [8]. As for the phase space we consider both space–space non-commutativity (space–time non-commutativity is not considered) and momentum–momentum non-commutativity [10,27,37]. In this case the consistent NCQM algebra is as follows:

$$[\hat{x}_i, \hat{x}_j] = i\xi^2\theta\epsilon_{ij}, \qquad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij},$$
$$[\hat{p}_i, \hat{p}_j] = i\xi^2\eta\epsilon_{ij} \quad (i, j = 1, 2), \tag{1}$$

where θ and η are the constant, frame-independent parameters; ϵ_{ij} is an antisymmetric unit tensor, $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$; $\xi = (1 + \theta\eta/4\hbar^2)^{-1/2}$ is the scaling factor. When $\eta = 0$, we have $\xi = 1$. The NCQM algebra (1) reduces to the one of only space–space non-commuting.

We consider a Rydberg atom with mass μ in the following electric and magnetic fields [28,38,39]: The electric field $\vec{E}$ acts radially in the x–y plane, $E_i = -\mathcal{E}\hat{x}_i$ $(i = 1, 2)$, where $\mathcal{E}$ is a constant, and the constant magnetic field $\vec{B}$ aligns the z axis. The motion is constrained in the x–y plane and has rotational symmetry. The Rydberg atom is treated as a structureless dipole moment. In reality it has the internal atomic structure. For the following discussions effects of the internal structure are extremely small and hence can be forgotten. The Hamiltonian of such a Rydberg atom is (henceforth, summation convention is used):

$$\hat{H}_{\text{Ryd}} = \frac{1}{2\mu}\left(\hat{p}_i + \frac{1}{2}g\epsilon_{ij}\hat{x}_j\right)^2 + \frac{1}{2}\kappa\hat{x}_i^2$$
$$= \frac{1}{2\mu}\hat{p}_i^2 + \frac{1}{2\mu}g\epsilon_{ij}\hat{p}_i\hat{x}_j + \frac{1}{2}\mu\omega^2\hat{x}_i^2, \tag{2}$$

where the co-ordinates $\hat{x}_i$ refer to the laboratory frame of the Rydberg atom. The constant parameters $g = 2qB/c$ and $\kappa = 2q\mathcal{E}$, $q(> 0)$ is dipole's electric charge. The term $g\epsilon_{ij}\hat{p}_i\hat{x}_j/2\mu$ takes the Chern–Simons interaction. The frequency $\omega = [g^2/4\mu^2 + \kappa/\mu]^{1/2}$, where the dispersive "mass" term $g/2\mu$ comes from the presence of the Chern–Simons term.

The NCQM algebra (1) changes the boson algebra of deformed annihilation–creation operators $(\hat{a}_i, \hat{a}_j^\dagger)$ which are related to deformed phase space variables $(\hat{x}_i, \hat{p}_j)$. For the Rydberg system (2) the deformed annihilation operator $\hat{a}_i$ is defined as:

$$\hat{a}_i = \sqrt{\frac{\mu\omega}{2\hbar}}\left(\hat{x}_i + \frac{i}{\mu\omega}\hat{p}_i\right). \tag{3}$$

When the state vector space of identical bosons is constructed by generalizing one-particle quantum mechanics, the maintenance of Bose–Einstein statistics at the deformed level of $\hat{a}_i$ $([\hat{a}_i, \hat{a}_j] \equiv 0)$ leads to a consistency condition [27]

$$\eta = \mu^2\omega^2\theta, \tag{4}$$

and the deformed Boson algebra of $\hat{a}_i$ and $\hat{a}_j^\dagger$ reads

$$[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1, \qquad [\hat{a}_1, \hat{a}_2] = 0;$$
$$[\hat{a}_1, \hat{a}_2^\dagger] = i\xi^2\mu\omega\theta/\hbar. \tag{5}$$

The first three equations in (5) are the same boson algebra as the one in commutative space. Thus Eq. (3) is a correct definition of the deformed annihilation operator.

The last equation in (5) is a new one which correlates $\hat{a}_i$ and $\hat{a}_j^\dagger$ in deferent degrees of freedom, codes effects of spatial non-commutativity and has some influence on dynamics [27–29]. It is worth noting that it is consistent with *all* principles of quantum mechanics and Bose–Einstein statistics.

If momentum–momentum is commuting, $\eta = 0$, we could not obtain $[\hat{a}_i, \hat{a}_j] = 0$. It is clear that in order to maintain Bose–Einstein statistics for identical bosons at the deformed level described by $\hat{a}_i$ and $\hat{a}_i^\dagger$ we should consider both space–space and momentum–momentum non-commutativities.

The NCQM algebra (1) has different possible perturbation realizations [10]. Here we consider the following consistent ansatz of the perturbation expansions of $\hat{x}_i$ and $\hat{p}_i$

$$\hat{x}_i = \xi\left(x_i - \frac{1}{2\hbar}\theta\epsilon_{ij}p_j\right),$$
$$\hat{p}_i = \xi\left(p_i + \frac{1}{2\hbar}\eta\epsilon_{ij}x_j\right), \tag{6}$$

where x_i and p_i satisfy the undeformed Heisenberg–Weyl algebra $[x_i, x_j] = [p_i, p_j] = 0$, $[x_i, p_j] = i\hbar\delta_{ij}$. It is worth noting that the determinant $\mathcal{R}_s$ of the transformation matrix R_s between $(\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2)$ and (x_1, x_2, p_1, p_2) is $\mathcal{R}_s = \xi^4(1 - \theta\eta/4\hbar^2)^2$. When $\theta\eta = 4\hbar^2$, the matrix R_s is singular. Thus the NCQM algebra (1) and the undeformed Heisenberg–Weyl algebra are not completely equivalent.[1] Eq. (6) should be correctly explained as perturbation expansions of $\hat{x}_i$ and $\hat{p}_i$.

The perturbation expansions of $\hat{a}_i$ and $\hat{a}_i^\dagger$ are as follows

$$\hat{a}_i = \xi\left(a_i + \frac{i}{2\hbar}\mu\omega\theta\epsilon_{ij}a_j\right),$$
$$\hat{a}_i^\dagger = \xi\left(a_i^\dagger - \frac{i}{2\hbar}\mu\omega\theta\epsilon_{ij}a_j^\dagger\right), \tag{7}$$

where a_i and $a_j^\dagger$ satisfy the undeformed boson algebra $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = 0$. Eq. (7) are consistent with the NCQM algebra (1) and (6). The determinant $\mathcal{R}'_s$ of the transformation matrix R'_s between $(\hat{a}_1, \hat{a}_2, \hat{a}_1^\dagger, \hat{a}_2^\dagger)$ and $(a_1, a_2, a_1^\dagger, a_2^\dagger)$ is also singular at $\theta\eta = 4\hbar^2$. Eq. (7) should be correctly explained as perturbation expansions of $\hat{a}_i$ and $\hat{a}_j^\dagger$.

In the following we study dynamics of a cold Rydberg atom described by Eq. (2). This system is exactly solvable. But here we are interested in the limiting case of vanishing kinetic energy. In this limit the Hamiltonian (2) shows non-trivial dynamics. First we identify the limit of vanishing kinetic energy in the Hamiltonian formulation with the limit of the mass $\mu \to 0$ in the Lagrangian formulation. In the limit of vanishing kinetic energy, $\frac{1}{2\mu}(\hat{p}_i + \frac{1}{2}g\epsilon_{ij}\hat{x}_j)^2 = \frac{1}{2}\mu\dot{\hat{x}}_i\dot{\hat{x}}_i \to 0$, the Hamiltonian (2) reduces to $H_0 = \frac{1}{2}\kappa\hat{x}_i\hat{x}_i$. The Lagrangian corresponding to the Hamiltonian (2) is $L_{\mathrm{Ryd}} = \frac{1}{2}\mu\dot{\hat{x}}_i\dot{\hat{x}}_i - \frac{1}{2}g\epsilon_{ij}\dot{\hat{x}}_i\hat{x}_j - \frac{1}{2}\kappa\hat{x}_i\hat{x}_i$. In the limit of $\mu \to 0$ this Lagrangian reduces to $L_0 = \frac{1}{2}g\epsilon_{ij}\hat{x}_i\dot{\hat{x}}_j - \frac{1}{2}\kappa\hat{x}_i\hat{x}_i$. From L_0 the corresponding canonical momentum is $\hat{p}_{0i} = \partial L_0/\partial\dot{\hat{x}}_i = \frac{1}{2}g\epsilon_{ji}\hat{x}_j$, and the corresponding Hamiltonian is $H'_0 = p_{0i}\dot{\hat{x}}_i - L_0 = \frac{1}{2}\kappa\hat{x}_i\hat{x}_i = H_0$. Thus we identify the two limiting

[1] For the case of only space–space non-commuting, $\eta = 0$, the situation is different. The determinant $\mathcal{R}_{\mathrm{uns}}$ of the transformation matrix R_{uns} between $(\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2)$ and (x_1, x_2, p_1, p_2) is $\mathcal{R}_{\mathrm{uns}} \equiv 1$ which is singular-free. Thus for the case of only space–space non-commuting the NCQM algebra (1) with $\eta = 0$ and the undeformed Heisenberg–Weyl algebra are equivalent.

processes. It is worth noting that when the potential is velocity dependent, the limit of vanishing kinetic energy in the Hamiltonian does not corresponds to the limit of vanishing velocity in the Lagrangian. If the velocity approached zero in the Lagrangian there would be no dynamics. The Hamiltonian (2) and its massless limit have been studied by Dunne, Jackiw and Trugenberger [40].

The first equation of (2) shows that in the limit $E_k \to 0$ there are constraints[2]

$$C_i = \hat{p}_i + \frac{1}{2} g \epsilon_{ij} \hat{x}_j = 0, \tag{8}$$

which should be carefully considered [42]. Poisson brackets of these constraints are $\{C_i, C_j\}_P = g\epsilon_{ij} \neq 0$, so that the corresponding Dirac brackets of canonical variables $\hat{x}_i$ and $\hat{p}_j$ can be determined, $\{\hat{x}_1, \hat{p}_1\}_D = \{\hat{x}_2, \hat{p}_2\}_D = 1/2$, $\{\hat{x}_1, \hat{x}_2\}_D = -1/g$, $\{\hat{p}_1, \hat{p}_2\}_D = -g/4$. The Dirac brackets of C_i with any variables $\hat{x}_i$ or $\hat{p}_j$ are zero, so that the constraints (8) are strong conditions and can be used to eliminate the dependent variables. For example, if we choose $\hat{x}_1$ and $\hat{p}_1$ as the independent variables, from (8) we obtain $\hat{x}_2 = -2\hat{p}_1/g$, $\hat{p}_2 = g\hat{x}_1/2$. But for our purpose in the following we choose $\hat{x}_1$ and $\hat{x}_2$ as the independent variables. From the perturbation expansion (6) it follows that

$$\begin{aligned} H_0 &= \frac{1}{2}\kappa \hat{x}_i \hat{x}_i \\ &= \frac{1}{2\mu^*} p_i p_i + \frac{1}{2}\mu^* \omega^{*2} x_i x_i + \omega^* \epsilon_{ij} p_i x_j, \end{aligned} \tag{9}$$

where the effective mass $\mu^* \equiv 4\hbar^2/\xi^2\kappa\theta^2$, and the effective frequency $\omega^* \equiv \xi^2\kappa|\theta|/2\hbar$. The term $\omega^*\epsilon_{ij} p_i x_j$ is the induced Chern–Simons interaction.

In order to solve Eq. (9) we define the "coordinate" and the "momentum" (X, P) and the annihilation–creation operators $(A, A^\dagger)$ as follows [38,39]

$$\begin{aligned} X &= \frac{1}{2}\sqrt{\mu^*} x_1 - \frac{1}{2\omega^*}\sqrt{\frac{1}{\mu^*}} p_2, \\ P &= \sqrt{\frac{1}{\mu^*}} p_1 + \omega^* \sqrt{\mu^*} x_2, \end{aligned} \tag{10}$$

$$\begin{aligned} A &= \frac{i}{2}\sqrt{\frac{1}{\omega^*}} P + \sqrt{\omega^*} X, \\ A^\dagger &= -\frac{i}{2}\sqrt{\frac{1}{\omega^*}} P + \sqrt{\omega^*} X. \end{aligned} \tag{11}$$

Where X and P satisfy $[X, P] = i\hbar$, and A and $A^\dagger$ satisfy $[A, A^\dagger] = 1$. The number operator $N = A^\dagger A$ has eigenvalues $n = 0, 1, 2, \ldots$. The Hamiltonian (9) is rewritten in the form of a harmonic oscillator of the unit mass and the frequency $2\omega^*$, $H_0 = 2\omega^*\hbar(A^\dagger A + \frac{1}{2})$. The zero-point energy

$$E_0 = \omega^*\hbar = \frac{1}{2}\xi^2\kappa|\theta|. \tag{12}$$

This zero-point energy can be understood on the basis of the position–position non-commutativity (1) and the corresponding deformed $\hat{x}$–$\hat{x}$ minimum uncertainty relation. From Eq. (1) it follows that the deformed $\hat{x}$–$\hat{x}$ uncertainty relation reads $\Delta\hat{x}_1\Delta\hat{x}_2 \geqslant \frac{1}{2}\xi^2|\theta|$. Here for any normalized state ψ, $\Delta\hat{F} \equiv [(\psi, (\hat{F} - \bar{\hat{F}})^2\psi)]^{1/2}$, $\bar{\hat{F}} \equiv (\psi, \hat{F}\psi)$. Taking $(\Delta\hat{x}_1)_{\min} = (\Delta\hat{x}_2)_{\min} = (\frac{1}{2}\xi^2|\theta|)^{1/2}$ it follows that the minimal energy $(\Delta E)_{\min}$ corresponding to $(\Delta\hat{x}_i)_{\min}$ is $(\Delta E)_{\min} = \frac{1}{2}\kappa[(\Delta\hat{x}_1)^2_{\min} + (\Delta\hat{x}_2)^2_{\min}] = \frac{1}{2}\xi^2\kappa|\theta|$. This shows $(\Delta E)_{\min} = E_0$. From this result we conclude that the deformed $\hat{x}$–$\hat{x}$ coherent state is realized by the lowest energy state of the cold Rydberg atom described by Eq. (2) in the limiting case of vanishing kinetic energy.

According to Eq. (6) the perturbation expansion of the kinetic energy term $\frac{1}{2\mu}\hat{p}_i^2$ leads to a perturbation induced Chern–Simons interaction, i.e., a term like $\epsilon_{ij} p_i x_j$. The existence of this term is a general characteristics of the NCQM algebra (1). This term plays essential role in dynamics.[3] From this observation we show that the deformed $\hat{p}$–$\hat{p}$ coherent state is realized, as an example, by the lowest energy state of a free particle. From Eq. (6) it follows that the perturbation expansion of the Hamiltonian of a free particle $\hat{H}_{\text{free}}(\hat{x}, \hat{p}) = \frac{1}{2\mu}\hat{p}_i\hat{p}_i$ reads $\hat{H}_{\text{free}}(\hat{x}, \hat{p}) = \frac{1}{2\tilde{\mu}} p_i p_i +$

[2] In this example the symplectic method [41] leads to the same results as the Dirac method for constrained quantization, and the representation of the symplectic method is much streamlined.

[3] Physical systems confined to a space–time of less than four dimensions show a variety of interesting properties. There are well-known examples, such as the quantum Hall effect, high T_c superconductivity, cosmic string in planar gravity, etc. In many of these cases the Chern–Simons interaction, which exists in $2+1$ dimensions and is associated with the topologically massive gauge fields, plays a crucial role.

$\frac{1}{2}\tilde{\mu}\tilde{\omega}^2 x_i x_i + \tilde{\omega}\epsilon_{ij} p_i x_j$, where the effective mass $\tilde{\mu} \equiv \xi^{-2}\mu$ and the effective frequency $\tilde{\omega} \equiv \xi^2|\eta|/2\mu\hbar$. In the above equation there are an effective harmonic potential $\frac{1}{2}\tilde{\mu}\tilde{\omega}^2 x_i x_i$ and an effective Chern–Simons interaction $\tilde{\omega}\epsilon_{ij} p_i x_j$. This means that a "free" particle in non-commutative space is not free; it moves in the above effective potentials. Based on this result we may guess that the non-commutativity of space originates from some *intrinsic* background fields. By a similar procedure of solving Eq. (9) we obtain $\hat{H}_{\mathrm{free}} = 2\tilde{\omega}\hbar(\tilde{A}^\dagger\tilde{A} + \frac{1}{2})$, where $\tilde{A}$ and $\tilde{A}^\dagger$ are defined by a similar equation (11), in which (X, P) and (μ^*, ω^*) are replaced, respectively, by $(\tilde{X}, \tilde{P})$ and $(\tilde{\mu}, \tilde{\omega})$. Here $\tilde{X}$ and $\tilde{P}$ are defined by a similar equation (10), in which μ^* and ω^* are replaced, respectively, by $\tilde{\mu}$ and $\tilde{\omega}$. It is interesting to notice that the spectrum of $\hat{H}_{\mathrm{free}}$ is *not* continuous, the interval of the spectrum is $2\tilde{\omega}$. For the case $\theta \to 0$ we have $\tilde{\omega} \to 0$, $2\tilde{\omega}\tilde{A}^\dagger\tilde{A} \to \frac{1}{2\mu} p_i p_i$. The Hamiltonian of a free particle in commutative space is recovered. The zero-point energy $\tilde{E}_0 = \tilde{\omega}\hbar = \frac{1}{2\mu}\xi^2|\eta|$, which can also be understood on the basis of the deformed momentum–momentum non-commutativity. From Eq. (1) it follows that the deformed $\hat{p}$–$\hat{p}$ uncertainty relation reads $\Delta\hat{p}_1\Delta\hat{p}_2 \geqslant \frac{1}{2}\xi^2|\eta|$. Taking $(\Delta\hat{p}_1)_{\min} = (\Delta\hat{p}_2)_{\min} = (\frac{1}{2}\xi^2|\eta|)^{1/2}$, it follows that the minimal energy $(\Delta\tilde{E})_{\min}$ corresponding to $(\Delta\hat{p}_i)_{\min}$ is $(\Delta\tilde{E})_{\min} = \frac{1}{2\mu}[(\Delta\hat{p}_1)^2_{\min} + (\Delta\hat{p}_2)^2_{\min}] = \frac{1}{2\mu}\xi^2|\eta|$. This shows that $(\Delta\tilde{E})_{\min} = \tilde{E}_0$. We conclude that the deformed $\hat{p} - \hat{p}$ coherent state is realized by the lowest energy state of a free particle.

In summary, in this Letter first we clarify a subtle point related to the equivalency between the deformed Heisenberg–Weyl algebra in non-commutative space and the undeformed Heisenberg–Weyl algebra in commutative space. For the case of both space–space and momentum–momentum non-commuting, different from the case of only space–space non-commuting, there is no singular-free linear transformation to relate phase space variables between two algebras, i.e., two algebras are not completely equivalent. It follows that there is no well-defined procedure to construct the deformed position–position coherent state or the deformed momentum–momentum coherent state from the undeformed position–momentum coherent state. Then we demonstrate the identification of the deformed position–position coherent state with the lowest energy state of a cold Rydberg atom arranged in appropriate electric and magnetic fields in the limit of vanishing kinetic energy, and briefly show that the deformed momentum–momentum coherent state is realized by the lowest energy state of a free particle.

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