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Zeros of Quaternion Polynomials

R. SERÔDIO*
Departmento de Matemática, Universidade da Beira Interior
6200 Covilhã, Portugal
rserodio@noe.ubi.pt

LOK-SHUN SIU
Mathematics Department, University of Hong Kong
Pofulam Road, Hong Kong
h8401689@submaths.hku.hk

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Abstract—It is well known that, over a division ring, every zero of a polynomial $f(x) = (x - x_1) \cdots (x - x_n)$ is congruent to x_r for some r. In this note, we show further that, over the quaternion field, there exists at least one quaternion q_r congruent to each x_r, and that, through this result, a constructive method for determining the zeros of quaternion polynomials can be established.

Keywords—Division algebra, Polynomials, Zeros of polynomials

1. PRELIMINARY KNOWLEDGE

Let $\mathbb{R}$ be the real field, $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ be the complex field, and $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ be the quaternion field, where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k, jk = -kj = i$, and $ki = -ik = j$. Let $q = a + bi + cj + dk \in \mathbb{H}$. The conjugate of q is defined as $\bar{q} = a - bi - cj - dk$. Thus, a, the real part of q denoted by $\operatorname{Re} q$, is given by $a = (q + \bar{q})/2$ and $q\bar{q} = \bar{q}q = a^2 + b^2 + c^2 + d^2 \in \mathbb{R}$. The norm $|q|$ of q is defined to be $\sqrt{\bar{q}q}$. By routine computation, we can show the following basic property.

PROPOSITION 1. *For any* $q_1, q_2 \in \mathbb{H}$, $\overline{q_1 + q_2} = \overline{q_1} + \overline{q_2}$, *and* $\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$.

An equivalence relation $\sim$ over $\mathbb{H}$ is defined as follows. For any two quaternions q and $q', q \sim q'$ (read q is congruent to q'), if there exists $x \in \mathbb{H} \setminus \{0\}$ such that $q' = xqx^{-1}$. The congruence class of q, denoted by $[q]$, is the set $\{x \in \mathbb{H} : x \sim q\}$.

PROPOSITION 2. *(See [1].) For any* $q \in \mathbb{H}$, $[q] = \{x \in \mathbb{H} : \operatorname{Re} x = \operatorname{Re} q$ *and* $|x| = |q|\}$.

Let $\mathbb{H}[x]$ denote the polynomial ring in the variable x over $\mathbb{H}$. Then, every polynomial $f(x) \in \mathbb{H}[x]$ can be written as $a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ for some nonnegative integer n, where $a_r \in \mathbb{H}$

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*Member of the Centro de Matemática da UBI, Covilhã, Portugal.

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and $a_0 \neq 0$. The addition and the multiplication of polynomials are defined in the same way as the commutative case, where the variable x is assumed to commute with quaternion coefficients. For every $q \in \mathbb{H}$, we define $f(q)$ (the evaluation of f at q) to be $a_0q^n + a_1q^{n-1} + \cdots + a_{n-1}q + a_n$. A quaternion q is said to be a zero of $f(x)$ if $f(q) = 0$. Since $a_0 \neq 0$, $f(q) = 0$ if and only if $a_0^{-1}f(q) = 0$. So, for the sake of simplicity, we can always assume that the polynomial is monic, i.e., $a_0 = 1$. We denote the set of all zeros of $f(x)$ by $\mathrm{Zero}(f)$.

For any $q \in \mathbb{H} \setminus \mathbb{R}$, let $f_q(x) = (x - \bar{q})(x - q) = x^2 - 2\,\mathrm{Re}\,q x + |q|^2$ be the characteristic polynomial of the congruence class $[q]$. Note that $f_q(x)$ is a real quadratic polynomial with negative discriminant.

PROPOSITION 3. *(See [1].) For any $q \in \mathbb{H} \setminus \mathbb{R}$, $[q] = \mathrm{Zero}(f_q)$.*

We will have to apply some general result on the polynomial ring over a division ring. For more details of the general theory, we refer the reader to [2].

PROPOSITION 4. *(See [2].)*

(i) *Factor theorem. Let $f(x) \in \mathbb{H}[x]$ and $q \in \mathbb{H}$. Then $q \in \mathrm{Zero}(f)$ if and only if there exists $g(x) \in \mathbb{H}[x]$ such that $f(x) = g(x)(x - q)$.*

(ii) *Let $f(x) = g(x)h(x) \in \mathbb{H}[x]$. Then*

$$\mathrm{Zero}(f) = \mathrm{Zero}(h) \cup \left\{x \in \mathbb{H} : h(x) \neq 0 \text{ and } h(x)xh(x)^{-1} \in \mathrm{Zero}(g)\right\}.$$

2. MAIN RESULTS

Suppose that $f(x)$ can be decomposed into a product of linear factors:

$$f(x) = (x - x_1)\cdots(x - x_n). \qquad (*)$$

Then, by Proposition 4, we can inductively show that $\mathrm{Zero}(f) \subset [x_1] \cup \cdots \cup [x_n]$. Hence, $\mathrm{Zero}(f) = (\mathrm{Zero}(f) \cap [x_1]) \cup \cdots \cup (\mathrm{Zero}(f) \cap [x_n])$. In the following, we will show that each $\mathrm{Zero}(f) \cap [x_r]$ is at least nonempty.

LEMMA 1. *Given $p, q \in \mathbb{H}$. Then*

$$(x - p)(x - q) = \begin{cases} (x - (p - \bar{q})q(p - \bar{q})^{-1})(x - (q - \bar{p})^{-1}p(q - \bar{p})), & \text{if } q \neq \bar{p}, \\ (x - q)(x - p), & \text{if } q = \bar{p}. \end{cases}$$

PROOF. By direct verification. ∎

We can now prove the following theorem.

THEOREM 1. *For each r, the subset $\mathrm{Zero}(f) \cap [x_r]$ is nonempty.*

PROOF. By Lemma 1, each linear factor $(x - x_r)$ can freely move to the right of the factorization without changing the set of congruence classes. Therefore, the result follows from the factor theorem. ∎

In [3], Niven has shown the existence of the zeros of quaternion polynomials. Hence, by the factor theorem, every $f(x)$ can be decomposed into a linear product form as $(*)$.

PROPOSITION 5. *(See [3].) Suppose that q is a zero of $f(x)$. Then dividing $f(x)$ on the right by $f_q(x)$, we have*

$$f(x) = g(x)f_q(x) + ux + v.$$

Hence, $[q] \subset \mathrm{Zero}(f)$, when $u = v = 0$; otherwise, $q = -u^{-1}v$ is the only zero in $[q]$.

We can determine the subset $\mathrm{Zero}(f) \cap [x_r]$, provided all the characteristic polynomials are known. Thus, in order to solve the equation $f(x) = 0$, we first need to determine $\mathrm{Re}\, x_r$ and

$|x_r|^2$, for each r. According to [3], in order to do so, we need to solve a nonlinear system of two equations of degree $2n+1$ in two real unknowns, and this is not practical. For this reason, we consider instead the polynomial $F(x) = \bar{f}(x)f(x)$, where $\bar{f}(x)$ is formed by replacing all coefficients a_r of $f(x)$ by their conjugates.

THEOREM 2. *The set of zeros of* $F(x)$ *is the congruence closure of* $\mathrm{Zero}(f)$, *i.e.,*

$$\mathrm{Zero}(F) = \bigcup_{x\in \mathrm{Zero}(f)} [x].$$

PROOF. By Proposition 1, we have $\bar{f}(x) = (x - \overline{x_n})\cdots(x - \overline{x_1})$, and by simple computation we can show that $F(x) = f_{x_1}(x)\cdots f_{x_n}(x)$. Thus the result follows immediately from Proposition 4 (ii). ∎

By Proposition 2, for each congruence class $[x_r]$, there exists a complex number z_r such that $\mathrm{Re}\, q = \mathrm{Re}\, z$ and $|q| = |z|$. Theorem 2 guarantees that we can determine each characteristic polynomial $f_{x_r}(x)$ by solving directly the real equation $F(x) = 0$ for its complex zeros.

In [4], a similar procedure of the above result was applied in a computational algorithm.

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