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Standard and Nonstandard Polynomial Approximation

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As foundation of polynomial approximation, uniform convergence is replaced with basic nonstandard notions like S-continuity and standard part. In the real case, Weierstrass' approximation theorem is generalized to G_δ-sets. In the complex case the standard compactness requirements also disappear. Standard applications include a direct proof of a generalized Bernstein theorem on analyticity of C^∞ and of continuous functions. © 1992 Academic Press, Inc.

0. (ABSOLUTE) MICROCONTINUITY OF NONSTANDARD POLYNOMIALS

Nonstandard Analysis is used here in the superstructure approach, described, e.g., in [5]. The sets of natural, real, and complex numbers are denoted by **N**, **R**, and **C**, respectively, and their nonstandard extensions of *hypernatural*, *hyperreal* resp. *hypercomplex* numbers by ***N**, ***R**, and ***C**. The relation of being infinitely close is denoted by $\approx$, and the standard part of a finite number z by $^\circ z$.

A fundamental interpretation of continuity of a function (standard or not) is that variations of the value, caused by infinitesimal fluctuations of the argument, are themselves infinitesimal. Hence the following definitions:

an internal everywhere defined hyperreal function f is *S-continuous* at $x_0 \in {}^*\mathbf{R}$ if and only if

$$\varepsilon \in {}^*\mathbf{R} \ \& \ \varepsilon \approx 0 \Rightarrow f(x_0+\varepsilon) \approx f(x_0) \tag{1}$$

an internal everywhere defined hypercomplex function f is *S-continuous* at $x_0 \in {}^*\mathbf{C}$ if and only if

$$\eta \in {}^*\mathbf{C} \ \& |\eta| \approx 0 \Rightarrow |f(x_0+\eta) - f(x_0)| \approx 0. \tag{2}$$

For a hyperreal polynomial

$$P(x) = \sum_{j=0}^{N} a_j x^j \qquad (N \in {}^*\mathbf{N};\, a_0, a_1, ..., a_n \in {}^*\mathbf{R};\, x \in {}^*\mathbf{R})$$

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the complex continuity requirement (2) also makes sense if complex arguments are temporarily allowed. In that case the real form (1) is called *microcontinuity* at x_0, and the stronger complex form (2) *absolute microcontinuity* at $x_0 \in {}^*\mathbf{R}$. For a hypercomplex polynomial "absolute microcontinuity," "microcontinuity," and "S-continuity" are to be considered as synonyms. Properties concerning absolute microcontinuity of hyperreal polynomials also hold for S-continuity of hypercomplex ones. In general we refrain from mentioning both cases separately. (Absolute) microcontinuity at $x_0 \neq 0$ can be expressed as (absolute) microcontinuity at 0. It suffices to write

$$P(x) = \sum_{j=0}^{N} a_j x^j = \sum_{k=0}^{N} a_k(x_0)(x - x_0)^k, \tag{3}$$

where $a_k(x_0) = \sum_{j=k}^{N} j(j-1) \cdots (j-k+1)/k! a_j x_0^{j-k}$. Then $P(x)$ is (absolutely) microcontinuous at x_0 if and only if $\sum_{k=0}^{N} a_k(x_0)\, x^k$ is so at 0.

Many equivalent definitions can be given of absolute microcontinuity (see [6]). The name "absolute" microcontinuity was suggested by the fact that $P(x) = \sum_{j=0}^{N} a_j x^j$ is absolutely microcontinuous at 0 if and only if $\sum_{j=0}^{N} |a_j|\, x^j$ is microcontinuous at 0.

"Microcontinuity" and its stronger form "Absolute Microcontinuity" are very different. Nonabsolute microcontinuity is highly unstable, and may be confined to isolated monads. It is intrinsically impossible to characterize it by the sole magnitudes $|a_j|$ of the coefficients. As shown in [7], it cannot be realized with standard power series, because a power series either has all its infinite partial sums absolutely microcontinuous at 0 (if the radius of convergence is nonzero) or has no partial sum at all that is microcontinuous at 0 (if the radius of convergence is 0).

Absolute microcontinuity, on the other hand, is stable, and self-propagating over noninfinitesimal distances [7, Th. 3.2]. Easy examples are obtained by truncating standard power series with nonzero radius of convergence. Finally, it can be characterized by the size of the coefficients, since [6, Th. 2] $P(x) = \sum_{j=0}^{N} a_j x^j$ is absolutely microcontinuous at $x \in {}^*\mathbf{R}$ if and only if $|a_j(x)|^{1/j}$ is finite for $j = 1, 2, ..., N$, i.e., if and only if

$$C(x) = \max_{1 \leqslant j \leqslant N} |a_j(x)|^{1/j} \text{ is finite.} \tag{4}$$

An interesting example of a polynomial that is microcontinuous at 0, but not absolutely, is $T_{2\omega}/\omega$, where $T_{2\omega}$ is the Chebyshev polynomial of infinite order 2ω. Since any Chebyshev polynomial is, in absolute value, bounded by 1 on the unit interval, the nonstandard polynomial $T_{2\omega}/\omega$ is infinitesimal on the unit interval, hence in particular microcontinuous at 0. But the very first nonzero coefficients of $T_{2\omega}/\omega$ are infinite, so that there cannot be absolute microcontinuity at 0. A generic way to construct nonabsolutely microcontinuous polynomials will be deduced in Section 2.

Remark. Outside the origin, strange things may happen even with infinite partial sums of standard power series. If the radius of convergence is zero there may be infinite partial sums that are (though not microcontinuous at 0) microcontinuous on the S-interior of the whole interval $]0, 1[$. If the radius of convergence is nonzero there may be infinite partial sums that are absolutely microcontinuous outside the circle of convergence. (See [7].)

Nonabsolute microcontinuity of polynomials must be considered as a genuine nonstandard notion without standard analog. Absolute microcontinuity, on the contrary, has some analogy with convergence of power series. This similarity is easily accounted for. As (4) shows, absolute microcontinuity at 0 depends on the finiteness of the hyperreal number $\max_{1 \leqslant j \leqslant N} |a_j|^{1/j}$. For power series the three properties of convergence, absolute convergence, and continuity (all three more or less equivalent) amount to a nonzero convergence radius, i.e., to the finiteness of $\limsup |a_j|^{1/j}$. The similar role of similar numbers (one standard, one nonstandard) explains the occasional similarity between convergence and absolute microcontinuity.

This analogy is far from consistent. Als already remarked, an infinite partial sum may be absolutely microcontinuous outside the circle of convergence and many other properties, either for absolute microcontinuity or for convergence, have no counterpart. (See [6, 7].)

In Section 3 we return to absolute microcontinuity for hyperreal polynomials. First we deal with mere S-continuity, successively for the real and for the complex case. Given a nonstandard polynomial P, we define the *standard S-continuity set* of P, to be denoted $\mathscr{S}_P$, as the set of standard (real resp. complex) points at which P is both finite and S-continuous. The standard part $f = {}^\circ P$ is a standard function defined on $\mathscr{S}_P$.

1. Real Polynomial Approximation

According to Weierstrass' classical theorem, a continuous real function with a compact domain is a uniform limit of polynomials. In this uniform-limit formulation, it is impossible to go beyond compactness, the theorem cannot even be extended to bounded intervals of type $]a, b]$. It is readily verified that, if every P_n is continuous on $[a, b]$, and $\lim_n P_n = f$ uniformly on $]a, b]$, then P_n converges also in the endpoint a. In these circumstances, f has a right-hand limit at a, viz. $\lim_n P_n(a)$. Therefore, representing f as a uniform limit of polynomials on $]a, b]$ is impossible if f does not have a finite right-hand limit at a. This is the case with such elementary functions as $\tan x$ (unbounded) on $]-\pi/2, 0]$, or $\sin 1/x$ (bounded) on $]0, 1]$.

From a purely practical point of view it is, however, easy enough to obtain a satisfactory polynomial approximation for a continuous function f on an interval $]a, b]$. It suffices to consider a compact subinterval $[a+\varepsilon, b]$ (ε a positive real number), and to approximate f on this compact interval. If ε is small enough, the polynomial obtained will not be distinguishable from f on the interval $]a, b]$.

In the same way one might construct a polynomial approximation for a continuous function on an unbounded interval, say $\mathbf{R}$, by applying Weierstrass' Theorem to a compact interval $[-M, M]$ (M positive and big enough).

In standard analysis, formalizing these simple ideas would eventually result in sequences of uniformly convergent sequences. A nonstandard description is much simpler. First, a single infinitely small or infinitely big number does away with a whole sequence of shrinking ε's or expanding M's. Secondly, being uniformly close can be directly expressed by means of the $\approx$ relation, and does not require sequences at all. In the nonstandard framework the polynomial approximation problem is thereby reduced to finding a nonstandard polynomial P that is infinitely close to a given standard function f on a standard set A. Due to the continuity of f, the polynomial P will be S-continuous and finite on A. In the notation of Section 0, a polynomial P is wanted such that $A \subset \mathcal{S}_P$ and $f = {}^\circ P$ on A. In fact, more than that is achieved. From [8] we quote

THEOREM 1. *Let $P(x)$ be any hyperreal polynomial, and let $\mathcal{S}_P$ denote its standard S-continuity set. Then* (i) *$\mathcal{S}_P$ is a G_δ subset of the real line*; (ii) *the real function $f(x) = {}^\circ P(x)$ is defined and continuous on $\mathcal{S}_P$.*

Conversely, let A be a G_δ subset of the real line and let $f(x)$ be a real function, defined and continuous in A. Then there is, in an enlargement, a hyperreal polynomial $P(x)$ such that (i) *$\mathcal{S}_P = A$*; (ii) *$f = {}^\circ P$ on A.*

This nonstandard approximation theorem represents a considerable upgrading of Weierstrass' Theorem, from a sufficient condition on compact sets to a necessary and sufficient condition on G_δ-sets. For compact A it is equivalent to the standard requirement that $f = \lim_n P_n$ uniformly on A, and the examples concerning $]a, b]$ show that this equivalence does not hold for noncompact G_δ-sets. In the next few considerations, we want to make clear at what exact point compactness enters and allows to reduce the S-continuity version to the uniform-limit form.

A sequence $(P_n)_{n \in \mathbf{N}}$ of real polynomials, converging uniformly to f on $A \subset \mathbf{R}$, essentially leads to a nonstandard polynomial P for which

$$^*f(x) \approx P(x), \qquad \text{for all} \quad x \in {}^*A. \tag{5}$$

(Take P_ν, for any infinite hypernatural ν.) On the other hand, if f is the standard part of the nonstandard polynomial P, we have

$$P \text{ is finite and microcontinuous at } x \;\&\; f(x) = {}^\circ P(x), \qquad \text{for all} \quad x \in A. \tag{6}$$

Both descriptions of the nonstandard polynomial P agree on A, where P is to be microcontinuous and to have finite values. But (5) requires that P remain infinitely close to $^*f$ even on $^*A \setminus A$. Therefore, a Weierstrass approximation reflects a stronger requirement, and can be achieved for a smaller class of functions, than the nonstandard approximation, described in (6). To see why (5) and (6) are nonetheless equivalent if A is compact, recall Robinson's characterization of compactness: $A \subset \mathbf{R}$ *is compact if and only if every point of* $^*A$ *is infinitely close to a point of* A. If, then, we suppose A compact and take any $a \in {}^*A$, there is a standard point a' in A with $a' \approx a$. The continuity of f at a' leads to $^*f(a) \approx f(a')$, and the assumptions of (6) to $P(a) \approx P(a') \approx f(a')$; hence it follows that $^*f(a) \approx P(a)$.

For noncompact A there may be a big difference between A and $^*A$. In particular, there is always at least one point $a \in {}^*A$ that has no points of A infinitely close by. At such points the difference between (5) and (6) is essential. Let us reconsider a noncompact interval $I =]a, b]$ and on I a continuous function f without right-hand limit at a.

That no Weierstrass approximation exists for such function, is entirely due to the fact that no nonstandard polynomial can have the behaviour of f at the points of $^*I$ infinitely close to a. Any nonstandard polynomial is $*$-continuous, and so $P(x)$ must be close to $P(a)$ if x is sufficiently close to a. This is in contradiction with the diverging or oscillating behaviour of f near a.

A nonstandard approximation in the sense of (6) does exist, however. Of course, this nonstandard polynomial P does not follow the behaviour of f at a either. But P is not S-continuous at a, and so the behaviour of P on the monad of a (fatal for uniform convergence) has no other effect than to remove a from the domain of P's standard part f.

Remark. The coefficients of a polynomial P with a given standard part f are by no means unique, nor unique up to infinitesimals. The contrary is true. The same f is the standard part of other S-continuous polynomials whose coefficients differ infinitely from the coefficients of P. On the unit interval, e.g., it suffices to add to P the infinitesimal polynomial $T_{2\omega}/\omega$ considered in Section 0. In that sense, absolute microcontinuity is most unstable: the infinitesimal bundle of polynomials around a given standard function invariably contains both nonabsolutely and absolutely S-continuous ones!

2. Complex Polynomial Approximation

In this section we deal with the interaction between holomorphic functions and complex S-continuity. A fundamental fact was already proved by Robinson, viz. that the standard part of an S-continuous $*$-holomorphic function is analytic. In a more precise form [9, Th. 6.2.3]: *Let B be an S-open subset of the hypercomplex plane which contains only finite points. Let $f(z)$ be an internal function which is $*$-analytic and finite (hence S-continuous) in B. Then the standard part ${}^\circ f(z)$ of $f(z)$ is analytic in ${}^\circ B$, the set of complex points z_0 whose monad intersects B.* We formulate Robinson's Theorem pointwise and give it the following quantitative form for hypercomplex polynomials:

Proposition 1. *Let the hypercomplex polynomial P be finite and S-continuous at the standard complex point z_0. Then the standard part ${}^\circ P$ of P is defined and analytic at least in the standard open disc D_0 with center z_0 and radius $1/{}^\circ C(z_0)$ (interpreted as an arbitrary standard radius if $C(z_0) \approx 0$).*

Proof. As shown in [7, Th. 3.2], the absolute microcontinuity of P at any point z implies that P is also absolutely microcontinuous at all hypercomplex z' such that $|z' - z|\, C(z) \lessapprox 1$. For any (standard) z in the disc D_0 we have ${}^\circ(|z - z_0|\, C(z_0)) = |z - z_0|\, {}^\circ C(z_0) < 1$ and so P is absolutely microcontinuous at z. Due to the propagation of absolute microcontinuity, P is then absolutely microcontinuous on the whole hypercomplex disc D with center z_0 and radius $1/{}^\circ C(z_0)$. But the absolute microcontinuity of P implies in particular that P' is finite. From the mean value inequality and the finiteness of $P(z_0)$ we infer that $P(z)$ is finite everywhere in the hypercomplex disc D. Applying Robinson's Theorem with $B = D_0$ we find the result stated. ∎

Proposition 1 provides, among other things, an easy way to generate hyperreal polynomials that are microcontinuous, but not absolutely microcontinuous, at 0:

Corollary. *Let I be a compact interval with 0 in its interior. Let the standard function f be continuous on I, but not derivable at 0. The nonstandard version of Weierstrass' Approximation Theorem generates nonstandard polynomials P that are infinitely close to f on I. All of these polynomials are microcontinuous on I (with f as standard part), but none is absolutely microcontinuous at 0.*

(If P were absolutely microcontinuous at 0, its standard part f would be analytic on a neighbourhood of 0.)

The main theorem in this section gives a two-way relationship between analytic functions and nonstandard continuity. An essential role is played by the condition that a set has a connected complement. Intuitively, this means that the set has no holes, and we shall use this informal way of speaking instead of its formal content. One should be aware, however, that complements are considered here with respect to the extended (compactified) complex plane $\mathbf{C}_\infty$, and this relevant fact is not evident from the mere expression *without holes*. The complement of a set A with respect to the extended (hyper)plane will be denoted as $\complement A$.

THEOREM 2. *Let $P(z)$ be any hypercomplex polynomial, and let $\mathscr{S}_P$ denote its standard S-continuity set. Then* (i) *$\mathscr{S}_P$ is an open subset, without holes, of the complex plane*; (ii) *the complex function $f(z) = {}^\circ P(z)$ is defined and analytic on $\mathscr{S}_P$.*

Conversely, let Ω be an open subset, without holes, of the complex plane, and let $f(z)$ be a complex function, defined and analytic in Ω. Then there is, in an enlargement, a hypercomplex polynomial $P(z)$ such that (i) $\mathscr{S}_P = \Omega$; (ii) $f = {}^\circ P$ *on Ω. Moreover, the finite-order coefficients of P are uniquely determined up to infinitesimals.*

Proof. The openness of $\mathscr{S}_P$ and the analyticity of ${}^\circ P(z)$ are contained in Proposition 1. For the direct part of the theorem there only remains to be shown that $\mathscr{S}_P$ has no holes.

If $\mathscr{S}_P$ had holes, the disconnected closed set $\complement\, \mathscr{S}_P$ would be the union of two nonempty disjoint closed sets H and K. Let H be the one that contains ∞, and K the other one, which is a compact, though not necessarily connected, subset of $\mathbf{C}$. Now take a bounded open subset U of $\mathbf{C}$ with $K \subset U$ and $\bar{U} \cap H = \varnothing$. The boundary ∂U has no points in common with H or K, and is therefore a subset of $\mathscr{S}_P$. This means that $P(z)$ is S-continuous and finite on the standard set ∂U. By Robinson's characterization of compactness, every point of $^*\partial U$ is infinitely close to a point of ∂U, and so $P(z)$ is S-continuous and finite on $^*\partial U$ also. Equivalently, $|P^{(j)}(z)/j!|^{1/j}$ is finite for $j = 0, 1, ..., N$ and $z \in {}^*\partial U$. Due to the Maximum Modulus Principle a complex-valued function cannot have a maximum at a point in whose neighbourhood it can be represented by a nonconstant power series. After $*$-transform we find that the maximum of $|P^{(j)}(z)|$ $(j = 0, 1, ..., N)$ on the $*$-compact set $^*\bar{U}$ must be located on the boundary, i.e., for every $j = 0, 1, ..., N$ there is a $z_0 \in {}^*\partial U$ such that $|P^{(j)}(z)| \leqslant |P^{(j)}(z_0)|$ for all hypercomplex $z \in {}^*U$. (Nothing prevents U from being disconnected, but the Maximum Modulus Principle as used here does not require connectedness.) It follows that $|P^{(j)}(z)/j!|^{1/j}$ $(j = 0, 1, ..., N)$ is finite on the whole of $^*U$, which implies that $P(z)$ is S-continuous and finite on U. This shows that U (hence also K) is a subset of $\mathscr{S}_P$, and contradicts the assumption that K is a subset of $\complement\, \mathscr{S}_P$.

For the converse, we use the enlargement to fix a hyperfinite set $\mathbf{C}' = \{z_1, z_2, ..., z_\omega\}$ ($\omega \in {}^*\mathbf{N} \setminus \mathbf{N}$) with $\mathbf{C} \subset \mathbf{C}' \subset {}^*\mathbf{C}$. If V_ω denotes the open square $]-\omega, +\omega[\times]-\omega, +\omega[$, the $*$-open set $V_\omega \cap {}^*\Omega$ is the union of a hypersequence of mutually disjoint half-open intervals I_k, each of which has the form $I_k = \{x + iy \in {}^*\mathbf{C} : a_k \leqslant x < b_k \ \& \ c_k \leqslant y < d_k\}$ for some hyperreal a_k, b_k, c_k, d_k. Define A as the union of the I_k's that contain at least one element of $\mathbf{C}'$. Then A is an elementary figure (i.e., a hyperfinite union of intervals) for which $A \cap \mathbf{C} = \Omega$. (Due to the openness of Ω, all points of Ω are even S-interior to A.) Decompose A in connected components, say $A = A_1 \cup A_2 \cdots \cup A_M$. Each A_k is a connected elementary figure, and two different A_k's are separated, i.e., the closure of the one does not intersect the other. For the elementary figures A_k, composed of halfopen intervals, separation reduces to disjoint closures.

By assumption, ${}^*\Omega$ has no holes, and we verify that the same holds for $A \subset {}^*\Omega$. We have

$$\complement A = \complement\, {}^*\Omega \cup \complement\, V_\omega \cup A_{M+1} \cup A_{M+2} \cdots$$
$$= \complement\, {}^*\Omega \cup \complement\, V_\omega \cup \overline{A_{M+1}} \cup \overline{A_{M+2}} \cdots.$$

The last equality relies on the mutual separation of components, which implies that the line segments forming $\overline{A_k} \setminus A_k$ lie outside A, i.e., in $\complement\, {}^*\Omega$. The union of the first two sets is readily seen to be $*$-connected, since each of the two is, and they intersect (∞ is in the intersection). This $*$-connectivity is preserved by adding $\overline{A_k}$ ($k > M$), because $\overline{A_k}$ is $*$-connected and $\overline{A_k} \cap \complement\, {}^*\Omega \neq \varnothing$ ($\overline{A_k} \setminus A_k$ is a subset of the intersection).

In the same way it follows from

$$\complement A_k = \complement A \cup \left(\bigcup_{j \neq k} A_j \right) = \complement A \cup \left(\bigcup_{j \neq k} \overline{A_j} \right)$$

and $\overline{A_j} \cap \complement A \neq \varnothing$ that A_k ($k \in {}^*\mathbf{N}$) has no holes.

So we have at our disposal M connected elementary figures A_k, without holes, with mutually disjoint closures. Connect the boundary of A_1 with that of A_2 by means of a polygonal line Γ_1 whose sides are parallel to the axes, and which does not intersect the closure of any other A_k ($1 \leqslant k \leqslant M$). Repeat this procedure for the boundaries of A_2 and A_3, whose polygonal connection Γ_2 may intersect the first one. Due to the hyperfiniteness of the number of sets involved, it is a trivial matter to interconnect in this way the boundaries of all M sets A_k to form one single connected closed set $\partial A_1 \cup \Gamma_1 \cup \cdots \cup \partial A_{M-1} \cup \Gamma_{M-1} \cup \partial A_M$. We connect this closed set with the boundary of V_ω by means of one more polygonal line Γ_M. This step too is trivial, because the number of figures considered is hyperfinite. It suffices, e.g., to join vertically the lower left vertex of the lower left figure (whose lower and under sides cannot coincide with boundaries of V_ω) to the lower side of V_ω.

Define $\Gamma = \partial A_1 \cup \Gamma_1 \cup \cdots \cup \partial A_M \cup \Gamma_M$ and $C = V_\omega \backslash \Gamma$. By construction, C is an open elementary figure, composed of the interiors of A_k $(1 \leqslant k \leqslant M)$ and of an extra hyperfinite number, say L, of disjoint connected open figures B_k, whose union we call B. Moreover, $\complement C = \Gamma \cup \complement V_\omega$ is connected by construction.

A compact subset K of C is defined in the following way. Inside every A_k $(1 \leqslant k \leqslant M)$ we construct a closed figure A'_k whose boundary is an infinitesimal distance away from the boundary of A_k. Choosing this infinitesimal small enough, A'_k and A_k are similar figures, in particular A'_k is connected. The same is done for the components B_k $(1 \leqslant k \leqslant L)$, and K is the union of the $M+L$ compact figures A'_k and B'_k thus constructed. In the L figures B'_k, a hyperfinite number of intervals is involved. Let δ be an infinitesimal, smaller than the smallest horizontal dimension in these intervals, and set $\Lambda = 2\pi/\delta$.

On C we define a $*$-analytic hypercomplex function $g(z)$, which coincides with $^*f(z)$ on A and with $\sin(\Lambda z)$ on B. To C and g we can apply the $*$-transform of Runge's theorem: *If C is an open set in the plane, with connected complement $\mathbf{C}_\infty \backslash C$ in the compactified complex plane, and if g is analytic in C, then there is a sequence $(Q_k)_{k \in \mathbf{N}}$ of polynomials such that $Q_k \to g$ uniformly on compact subsets of C* [10, p. 290].

Starting with a positive infinitesimal, Runge's transferred theorem yields a hypercomplex polynomial $P(z)$ that is infinitely close to $g(z)$ on K. We verify that $P(z)$ satisfies the requirements of the converse part of our theorem.

To see that $P(z)$ is S-continuous on Ω, take any $z_0 \in \Omega$. This z_0 belongs to an A_k $(1 \leqslant k \leqslant M)$ and, being S-interior to $^*\Omega$, it is an S-internal point of the compact subfigure A'_k. Hence the monad of z_0 lies entirely inside A'_k, and the S-continuity of P at z_0 follows from the continuity of $f(z)$. On the other hand, $P(z)$ cannot be S-continuous at a standard point z_0 outside Ω. Since standard points of A are in Ω, such a z_0 does not belong to A, and it must lie in some B_k $(1 \leqslant k \leqslant L)$ or on the polygonal line Γ. In each case, the monad of z_0 contains at least a horizontal line segment, inside some B'_k, of length δ. By our choice of Λ this segment covers at least one period of $\sin \Lambda x$. Hence $|\sin \Lambda z|$ oscillates over noninfinitesimal amplitudes on the monad of z_0. The same holds for $|P(z)|$, since $P(z)$ is infinitely close to $\sin \Lambda z$ on B'_k. This proves that Ω is the set of standard points at which P is S-continuous. That $\Omega = \mathscr{S}_P$, and $f = {}^\circ P$ on $\mathscr{S}_P$, results from $P(z)$ being infinitely close to the standard function $f(z)$.

Finally, the almost-uniqueness of the coefficients of finite order follows from a theorem of Robinson's [9, Th. 6.2.7], stating that differentiation and standard part may be interchanged for S-continuous $*$-analytic functions. Here we infer that $°(P^{(k)}(z)) = f^{(k)}(z)$ for any finite index k and any $z \in \Omega$. ∎

Remarks. (1) The above proof relies on the fact that a standard open set in the plane consists of the standard points of some nonstandard elementary figure. (Note that we intersect an internal set with $\mathbf{R}^2$, which is not the same as taking its standard part. The latter operation invariably yields a closed set, which is of no use here.)

One might consider to use directly the open components of the set $V_\omega \cap {}^*\Omega$, and to work not with a hyperfinite number of halfopen intervals, but with a hyperfinite number of general open sets. Besides being intrinsically interesting (after all, a possibly intricate open set is reduced to an elementary figure), the first choice is also technically rewarding. Reconsider squeezing, inside a connected set, a slightly smaller subset that is still connected. In an elementary figure there is no difficulty in doing that, because there is a smallest nonzero side of the composing intervals. For some open sets of a general structure, however, there may not be a uniform way to construct such connected subsets. Consider, e.g., the plane open subset of the first quadrant, bounded by the coordinate axes and by the sawtooth line passing through the points (0, 1), (1, 1/2), (2, 1), (3, 1/3), (4, 1/2), ..., $(2k-1, 1/(2k-1))$, $(2k, 1/k)$, ... No matter how little is peeled off, the smaller set is disconnected. To obtain a *bounded* open set with this property, it suffices to have it not diverge along the x-axis, but spiral towards the origin.

(2) It is not possible to extend the almost-uniqueness of the coefficients to infinite orders. Consider, e.g., z^ω and the trivial polynomial 0. Both have zero as standard part on the unit disc, their finite-order coefficients are identical, but the difference between the ωth order coefficients 1 and 0 is not infinitesimal.

Concerning the very notion of analyticity some interesting conclusions may be drawn from the above theorem. Suppose that the function $f(z)$, analytic on the open set without holes $\Omega \ni 0$, has the local development

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots \qquad (|z| < R).$$

Theorem 2 shows that $f(z)$ is the standard part of a nonstandard polynomial $P(z)$ whose finite-order coefficients are the same as for $f(z)$,

$$P(z) = c_0 + c_1 z + c_2 z^2 + \cdots + a_N z^N.$$

Loosely speaking, one could say that the nonstandard polynomial describing f is obtained by adding an infinite tail to the standard development of f near some starting point, say 0. This tail of infinite order terms, whose coefficients have no simple relation with the given function f, makes the whole difference. The one polynomial obtained describes the given function f on the whole open set Ω. In the standard approach both the notions of infinite polynomial and of S-continuity are lacking, and standard developments are very "short" compared to the length of a non-standard polynomial. To compensate for these defects, standard analysis must resort to the use of an infinite patchwork of such short-ranged developments. The intrinsically "uniform" nature of Nonstandard Analysis is confirmed here: a single, finished object is at hand to replace a multitude of local limits.

The strange thing is (see Proposition 2 below) that this tail does not contribute much in the neighbourhood of the starting point 0. Its effect only shows up in the neighbourhood of points $z_0 \neq 0$, where the infinite-order coefficients $a_j(0)$ contribute considerably to the new coefficients $a_k(z_0)$.

The characterizations in Proposition 2 involve "tails" and "partial differences" that resemble the tails and partial sums of series. Note, however, that in any of our partial differences the constant term a_0 of the polynomial must be left out, while all other terms are significant. (Compare with convergence, where any finite number of first terms in a partial sum can be left out.)

PROPOSITION 2. *Given a hypercomplex polynomial* $P(z) = \sum_{j=0}^{N} a_j z^j$ *with* a_0 *finite, the following properties are equivalent*:

(i) *P is absolutely microcontinuous at* 0;

(ii) *P is absolutely microcontinuous on some noninfinitesimal* 0-*centered ball B*;

(iii) *on some noninfinitesimal* 0-*centered ball all the partial differences* $s_k(z) = \sum_{j=1}^{k} a_j z^j$ $(k \geqslant 1)$ *have finite modulus*;

(iv) *on some noninfinitesimal* 0-*centered ball all the tails* $t_k(z) = \sum_{j=k+1}^{N} a_j z^j$ $(k \geqslant 0)$ *have finite modulus*;

(v) *on some noninfinitesimal* 0-*centered ball all the tails* $t_k(z) = \sum_{j=k+1}^{N} a_j z^j$ $(k \geqslant 0)$ *have a finite and, for infinite k, infinitesimal modulus*;

(vi) *on some noninfinitesimal* 0-*centered ball all the partial differences* $s_k(z) = \sum_{j=1}^{k} a_j z^j$ $(k \geqslant 1)$ *have finite modulus and are, for infinite k, infinitely close together.*

Proof. (i) $\Rightarrow$ (ii) by Proposition 1.

(ii) $\Rightarrow$ (iii). As shown in Proposition 1, we can take for B the ball described by $|Cz| < \sigma < 1$ (σ standard). Then $|\sum_{j=1}^{k} a_j z^j| < \sum_{j=1}^{k} |Cz|^j < 1/(1-\sigma)$.

(iii) $\Rightarrow$ (iv) since $t_k = s_N - s_k$.

(iv) $\Rightarrow$ (v). For infinite k and $|Cz| < \sigma < 1$ it follows from Cauchy's Criterion that $|t_k| \leqslant \sum_{j=k+1}^{N} |Cz|^j \leqslant \sum_{j=k+1}^{N} \sigma^j \approx 0$.

(v) $\Rightarrow$ (vi) since $s_k = s_N - t_k$.

(vi) $\Rightarrow$ (i). $|a_j z^j| = |s_j - s_{j-1}|$ is finite on the interval considered, hence certainly for all $|z| \approx 0$. This implies that $|a_j|^{1/j}$ $(j \geqslant 1)$ is finite, which is equivalent to (i). ▮

3. Nonstandard Characterization of Real Analyticity, with Standard Applications

The foregoing sections contain nonstandard characterizations of real continuity on G_δ sets, and of holomorphy on open sets. We now turn to real functions analytic on open subsets of the real line.

THEOREM 3. *Let $P(x)$ be any hyperreal polynomial, and let $\mathscr{A}_P$ denote the set of standard points at which P is both finite and absolutely microcontinuous. Then* (i) *$\mathscr{A}_P$ is an open subset of the real line*; (ii) *the real function $f(x) = {}^\circ P(x)$ is defined and analytic on $\mathscr{A}_P$.*

Conversely, let A be an open subset of the real line and let $f(x)$ be a real function, defined and analytic in A. Then there is, in an enlargement, a hyperreal polynomial $P(x)$ such that (i) *$\mathscr{A}_P = A$*; (ii) *$f = {}^\circ P$ on A. Moreover, the finite-order coefficients of P are uniquely determined up to infinitesimals.*

Proof. The openness of $\mathscr{A}_P$ and the analyticity of ${}^\circ P(x)$ are contained in the real version of Proposition 1. For the converse we rely on Theorem 2. The real function $f(x)$ being analytic on A, we have, for every $x \in A$, a development $f(z) = \sum_{j=0}^{\infty} a_j(x)(z-x)^j$ $(z \in \,]x - \delta_x, x + \delta_x[)$. The same family of series, now for complex $z \in B(x, \delta_x)$, extends $f(x)$ to a complex analytic function $f(z)$ on an open complex superset, without holes, of A. Call this subset of the complex plane Ω. Let $P(z)$ be a hypercomplex polynomial for which $\mathscr{S}_P = \Omega$ and $f(z) = {}^\circ P(z)$ on Ω. The hyperreal polynomial $P(x) = \mathscr{R}(P(x))$ satisfies the requirements (i) and (ii) of the converse part. The almost-unicity of the finite-order coefficients is deduced as in Theorem 2. ▮

Theorem 3 allows one to characterize an analytic real function as the standard part of an absolutely microcontinuous hyperreal polynomial. Comparing with Theorem 1 we see that the standard part f is continuous resp. analytic at a point if the underlying polynomial is microcontinuous resp. absolutely microcontinuous. By formula (4) absolute microcontinuity is reduced to an easy property of the coefficients. Moreover, as shown in

[6], regularity in the coefficient sequence tends to lift microcontinuity to absolute microcontinuity. This happens when all the coefficients are of the same sign, or if they alternate signs. In Proposition 3 below we give another of these regularity patterns. First, separating positive and negative coefficients, we rearrange $P(x)$ in an internal way as $P(x) = P_+(x) + P_-(x)$, where P_+ has positive coefficients, and P_- negative ones. If P and P_+ are microcontinuous at x_0, so is P_-. But for P_+ and P_-, having coefficients of constant sign, microcontinuity and absolute microcontinuity are equivalent. Hence P is the difference of two absolutely microcontinuous polynomials, and is itself absolutely microcontinuous. Conversely, if P is absolutely microcontinuous at x_0, characterization (4) implies the absolute microcontinuity of P_+ and P_- at that point. Therefore we have the following: *The polynomial $P(x)$, microcontinuous at x_0, is absolutely microcontinuous at that point if and only if P_+ or P_- is (absolutely) microcontinuous there.*

Translating this simple observation in terms of the coefficients, we find

PROPOSITION 3. *The hyperreal polynomial $P(x) = \sum_{j=0}^{N} a_j x^j$ is absolutely microcontinuous at 0 if and only if one of the following conditions holds:* (i) *P is microcontinuous at 0 and $a_j \leqslant r^j$ $(j = 1, 2, ..., N)$ for some real $r \geqslant 0$;* (ii) *P is microcontinuous at 0 and $a_j \geqslant -r^j$ $(j = 1, 2, ..., N)$ for some real $r \geqslant 0$.*

This proposition has the intrinsic value of clarifying to some extent the elusive concept of nonabsolute microcontinuity. In principle, it must be possible to characterize microcontinuity in terms of the coefficients a_j and this characterization must contain more than the absolute values $|a_j|$; otherwise, microcontinuity and absolute microcontinuity could not be distinguished. From Proposition 3 we see that the characterization of microcontinuity must be such that, if one-sided geometric growth $a_j \leqslant r^j$ or $a_j \geqslant -r^j$ is added, two-sided geometric growth $|a_j| \leqslant r^j$ ensues.

We end this section with two standard applications. Both are related to the question: *When is a real function, initially only known to be C^∞ or continuous, in fact an analytic function*? The first characterization of this kind, due to Pringsheim, reduces analyticity to the uniform boundedness of $|f^{(n)}(x)/n!|^{1/n}$. Rather unexpectedly, Bernstein [1] established results in which the *sign* of the derivatives (even: of the differences) is important, rather than the modulus. A whole family of related results have been found since, by Boas, Widder, and others. (See the survey paper [2] for an account of these.) It is not difficult to prove the theorems below by standard means, if one takes Bernstein's sufficient condition for granted. The point is that the nonstandard approach gives a more general result in a straightforward way.

THEOREM 4 (Generalized Bernstein Theorem). *Let f be continuous on $[a, b]$ and C^∞ on $[a, b[$. If there exists a constant $r \geqslant 0$ such that*

$$f^{(j)}(x) \leqslant r^j j!, \qquad a \leqslant x < b \qquad (j = 1, 2, ...) \tag{7}$$

or

$$f^{(j)}(x) \geqslant -r^j j!, \qquad a \leqslant x < b \qquad (j = 1, 2, ...) \tag{8}$$

then f is analytic on $[a, b[$.

Proof. We rely on some properties of Bernstein polynomials, and therefore limit ourselves to the unit interval. (For a general interval modified Bernstein polynomials with identical properties could be used.) With f we associate the nonstandard Bernstein polynomial

$$B_N(f; x) = \sum_{j=0}^{N} {}^*f(j/N) \binom{N}{j} x^j (1-x)^{N-j}$$

of infinite order $N \in {}^*\mathbf{N} \backslash \mathbf{N}$.

Bernstein polynomials inherit, only slightly modified, the bounds of the function they approximate, and the same holds for derivatives of all orders. More precisely [4, p. 114]:

For every integer j with $0 \leqslant j \leqslant n$ we have that, if $f^{(j)}(x) \geqslant m$ (resp. $f^{(j)}(x) \leqslant M$) on $[0, 1]$, then $B_n^{(j)}(f; x) \geqslant F_n^{(j)} m$ (resp. $B_n^{(j)}(f; x) \leqslant F_n^{(j)} M$), with $F_n^{(j)} = n(n-1) \cdots (n-j+1)/n^j$ (interpreted as 1 for $j = 0$.)

The assumption (7) implies that ${}^*f^{(j)}(x) \leqslant r^j j!$ for all nonstandard x in ${}^*[0, 1[$ and every hypernatural $j \geqslant 1$. We infer that, for hypernatural $j \geqslant 1$, $B_N^{(j)}(f; x) \leqslant r^j j! F_N^{(j)} \leqslant r^j j!$ on ${}^*[0, 1[$. This means that, if $B_N^{(j)}(f; x)$ is rearranged around any centerpoint $x_0 \in {}^*[0, 1[$, the coefficients $a_j(x_0) = B_N^{(j)}(f; x_0)/j!$ satisfy the condition $a_j \leqslant r^j$ $(j = 1, 2, ..., N)$ of Proposition 3. Since $B_N(f; x)$ is microcontinuous and finite at every standard $x_0 \in {}^*]0, 1[$ (its standard part is $f(x_0)$ by Bernstein's form of Weierstrass' Theorem), $B_N(f; x)$ is absolutely microcontinuous and finite on $]0, 1[$. By Proposition 3, the standard part f is analytic on $]0, 1[$.

Microcontinuity at x_0 involves all values for $x \approx x_0$, both $x \leqslant x_0$ and $x \geqslant x_0$. Since we do not, a priori, know how $B_N(f; x)$ behaves for $x < 0$ or $x > 1$, this polynomial need not be microcontinuous at $x_0 = 0$ or $x_0 = 1$.

A short direct reasoning shows that $B_N(f; x) = a_0 + a_1 x + \cdots + a_N x^N$ is absolutely microcontinuous at 0. For hypercomplex infinitesimal z we have, assuming (7), that

$$\begin{aligned}
&|a_1 z + \cdots + a_N z^N| \\
&\quad \leqslant |(a_1 - r) z + \cdots + (a_N - r^N) z^N| + |rz + \cdots + r^N z^N| \\
&\quad \leqslant (r - a_1)|z| + \cdots + (r^N - a_N)|z|^N + r|z| + \cdots + r^N |z|^N \\
&\quad = 2(r|z| + \cdots + r^N |z|^N) - (a_1 |z| + \cdots + a_N |z|^N).
\end{aligned}$$

But

$$a_1|z| + \cdots + a_N|z|^N = B_N(f; |z|) - B_N(f; 0) \approx {}^*f(|z|) - {}^*f(0) \approx 0$$

since $0 \leqslant |z| < 1$. Also $r|z| + \cdots + r^N|z|^N \approx 0$. Hence $|a_1 z + \cdots + a_N z^N| \approx 0$.

This reasoning cannot be repeated for $x_0 = 1$ and, indeed, the theorem does not extend to compact $[a, b]$.[1] ▮

Bernstein proved [3, p. 160] that the existence and positivity of all derivatives in the sufficient condition $f^{(j)}(x) \geqslant 0$ can be replaced by the mere positivity of all differences

$$\Delta_h^j f(x) = \sum_{k=0}^{j} \binom{j}{k} (-1)^k f(x + jh).$$

In our context this variant also can easily be generalized. We shall rely on the well-known fact [4, pp. 108, 114] that the Bernstein polynomial $B_n(f; x)$ can be rearranged as

$$B_n(f; x) = \sum_{j=0}^{n} \Delta_{1/n}^j f(0) \binom{n}{j} x^j$$

and that, generally,

$$B_n^{(p)}(f; x) = n(n-1) \cdots (n-p+1) \sum_{j=0}^{n-p} \Delta_{1/n}^p f\left(\frac{j}{n}\right) \times \binom{n-p}{j} x^j (1-x)^{n-p-j}.$$

Theorem 5. *Let f be continuous on $[a, b]$. If there exists a real $r \geqslant 0$ such that*

$$\Delta_{1/n}^p f\left(\frac{j}{n}\right) \leqslant \frac{r^p p!}{n^p} \qquad (n = 1, 2, ...; \ p = 1, ..., n; \ 0 \leqslant j \leqslant n-p) \tag{9}$$

or

$$\Delta_{1/n}^p f\left(\frac{j}{n}\right) \geqslant \frac{-r^p p!}{n^p} \qquad (n = 1, 2, ...; \ p = 1, ..., n; \ 0 \leqslant j \leqslant n-p) \tag{10}$$

then f is analytic on $[a, b[$.

[1] Let $f(x) = \sum_{n=0}^{+\infty} 2^{-n^2} x^{2^n}$ for $0 \leqslant x \leqslant 1$. Then $f \in C^\infty([0, 1])$ and $f^{(j)} \geqslant 0$ on $[0, 1]$ for $j = 1, 2, ...$ yet f is not analytic at 1. The author thanks Professor J. Siciak of Krakov (Poland) for delivering this counterexample to the (originally ill-stated) theorem in proof.

Proof. Again, we limit ourselves to the unit interval. Consider the nonstandard Bernstein polynomial $B_n(f; x)$, with $N \in {}^*\mathbf{N}$ infinite. The assumed continuity of f implies that $B_N(f; x) \approx {}^*f(x)$ on ${}^*[0, 1]$, so that in particular $B_N(f; x)$ is finite on this interval. After transfer, condition (9) holds for $p = 1, ..., N$ and we infer that

$$\begin{aligned} B_N^{(p)}(f; x) &= N(N-1) \cdots (N-p+1) \sum_{j=0}^{N-p} \Delta_{1/N}^{p} f\left(\frac{j}{N}\right) \\ &\quad \times \binom{N-p}{j} x^j (1-x)^{N-p-j} \\ &\leqslant N^p \frac{r^p p!}{N^p} \sum_{j=0}^{N-p} \binom{N-p}{j} x^j (1-x)^{N-p-j} \\ &= r^p p! \qquad (x \in {}^*[0, 1];\ p = 1, ..., N). \end{aligned}$$

The rest of the reasoning is identical with that for Theorem 4. ∎

In Theorems 4 and 5, Bernstein's condition corresponds to $r = 0$ in the second form.

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