JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Journal of Computational and Applied Mathematics 125 (2000) 297–307

Introduction to the numerical analysis of stochastic delay differential equations

Evelyn Buckwar [1]

Department of Mathematics, The Victoria University of Manchester, Manchester M13 9PL, UK

Received 29 July 1999; received in revised form 17 February 2000

Abstract

We consider the problem of the numerical solution of stochastic delay differential equations of Itô form

$$dX(t) = f(X(t), X(t-\tau))dt + g(X(t), X(t-\tau))dW(t), \quad t \in [0, T]$$

and $X(t) = \Psi(t)$ for $t \in [-\tau, 0]$, with given f, g, Wiener noise W and given $\tau > 0$, with a prescribed initial function Ψ. We indicate the nature of the equations of interest and give a convergence proof for explicit single-step methods. Some illustrative numerical examples using a strong Euler–Maruyama scheme are provided. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

We are concerned here with the evolutionary problem for Itô stochastic delay differential equations or SDDEs. SDDEs generalise both deterministic delay differential equations (DDEs) and stochastic ordinary differential equations (SODEs). One might therefore expect the numerical analysis of DDEs and the numerical analysis of SODEs to have some bearing upon the problems that concern us here. We refer to [14] for an overview of the issues in the numerical treatment of DDEs. For a reprise of the basic issues in the numerical treatment of SODEs, see [4]; for more extensive treatments see [8,11]. In this article we will be interested in obtaining approximations to strong solutions of an SDDE. One reason to be interested in this kind of approximation is to examine the dependence of the solution on the initial function or on parameters that are contained in the definition of the SDDE. The article is based on [2].

[1] Supported by TMR Grant No. ERBFMBICT983282.

PII: S0377-0427(00)00475-1

We shall use a brief discussion of some model problems to introduce SDDEs to the reader.

- (*Cell population growth*) Consider a large (in order to justify continuous as opposed to discrete growth models) population $N(t)$ of cells at time t evolving with a proportionate rate $\rho_0 > 0$ of 'instantaneous' and a proportionate rate ρ_1 of 'delayed' cell growth. By 'instantaneous' cell growth, we mean that the rate of growth is dependent on the *current* cell population, and by 'delayed' cell growth, we mean that the rate of growth is dependent on some *previous* cell population. If the number $\tau > 0$ denotes the average cell-division time, the following equation provides a model
$$N'(t) = \rho_0 N(t) + \rho_1 N(t-\tau), \quad t \geqslant 0, \qquad N(t) = \Psi(t), \quad t \in [-\tau, 0].$$
Now assume that these biological systems operate in a noisy environment whose overall noise rate is distributed like white noise $\beta\, \mathrm{d}W(t)$. Then we will have a population $X(t)$, now a random process, whose growth is described by the SDDE
$$\mathrm{d}X(t) = (\rho_0 X(t) + \rho_1 X(t-\tau))\mathrm{d}t + \beta\, \mathrm{d}W(t), \qquad t > 0,$$
with $X(t) = \Psi(t)$ for $-\tau \leqslant t < 0$. This is a constant delay equation with additive noise (the delay is only in the drift term).
- (*Population growth again*) Assume now that in the above equation we want to model noisy behaviour in the system itself, e.g. the intrinsic variability of the cell proliferation or other individual differences and the interaction between individuals. This leads to the multiplicative noise term, as in
$$\mathrm{d}X(t) = (\rho_0 X(t) + \rho_1 X(t-\tau))\, \mathrm{d}t + \beta X(t)\, \mathrm{d}W(t), \qquad t > 0,$$
with $X(t) = \Psi(t)$ for $-\tau \leqslant t < 0$.
- (*More examples*) For some additional examples we can refer to examples in neural control mechanisms: neurological diseases [3], pupil light reflex [9] and human postural sway [6].

2. General formulation

Let $(\Omega, \mathscr{A}, P)$ be a complete probability space with a filtration $(\mathscr{A}_t)$ satisfying the usual conditions, i.e. the filtration $(\mathscr{A}_t)_{t \geqslant 0}$ is right-continuous, and each $\mathscr{A}_t$, $t \geqslant 0$, contains all P-null sets in $\mathscr{A}$. In this article we will prove convergence of a numerical method in the mean-square-sense, i.e. we say that $X \in \mathscr{L}^2 = \mathscr{L}^2(\Omega, \mathscr{A}, P)$ if $\mathscr{E}(|X|^2) < \infty$ and we define the norm $||X||_2 = (\mathscr{E}(|X|^2))^{1/2}$. We refer to [13] for the background on probability theory and to [1,7] for properties of a Wiener process and stochastic differential equations.

Let $0 = t_0 < T < \infty$. Let $W(t)$ be a one-dimensional Brownian motion given on the filtered probability space $(\Omega, \mathscr{A}, P)$. We consider the scalar autonomous stochastic delay differential equation (SDDE)

$$\begin{aligned} \mathrm{d}X(t) &= \overbrace{f(X(t), X(t-\tau))}^{\text{drift coefficient}}\, \mathrm{d}t + \overbrace{g(X(t), X(t-\tau))}^{\text{diffusion coefficient}}\, \mathrm{d}W(t), \quad t \in [0, T] \\ X(t) &= \Psi(t), \quad t \in [-\tau, 0] \end{aligned} \tag{1}$$

with one fixed delay, where $\Psi(t)$ is an $\mathscr{A}_{t_0}$-measurable $C([-\tau, 0], \mathbb{R})$-valued random variable such that $\mathscr{E}||\Psi||^2 < \infty$ ($C([-\tau, 0], \mathbb{R})$ is the Banach space of all continuous paths from $[-\tau, 0] \to \mathbb{R}$

equipped with the supremum norm). The functions $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are assumed to be continuous. Eq. (1) can be formulated rigorously as

$$X(t)=X(0)+\int_0^t f(X(s),X(s-\tau))\,\mathrm{d}s+\int_0^t g(X(s),X(s-\tau))\,\mathrm{d}W(s) \tag{2}$$

for $t\in[0,T]$ and with $X(t)=\Psi(t)$, for $t\in[-\tau,0]$. The second integral in (2) is a stochastic integral, which is to be interpreted in the Itô sense. If g does not depend on X the equation has *additive noise*, otherwise the equation has *multiplicative noise*. We refer to [10,12] for the following definition and a proof of Theorem 2.

Definition 1. An $\mathbb{R}$-valued stochastic process $X(t):[-\tau,T]\times\Omega\to\mathbb{R}$ is called a strong solution of (1), if it is a measurable, sample-continuous process such that $X|[0,T]$ is $(\mathscr{A}_t)_{0\leqslant t\leqslant T}$-adapted and X satisfies (1) or (2), almost surely, and satisfies the initial condition $X(t)=\Psi(t)$ $(t\in[-\tau,0])$. A solution $X(t)$ is said to be path-wise unique if any other solution $\hat{X}(t)$ is stochastically indistinguishable from it, that is $P(X(t)=\hat{X}(t)$ for all $-\tau\leqslant t\leqslant T)=1$.

Theorem 2. *Assume that there exist positive constants $L_{f,i}$, $i=1,2$ and K_f, such that both the functions f and g satisfy a uniform Lipschitz condition and a linear growth bound of the following form: For all $\xi_1,\xi_2,\eta_1,\eta_2,\xi,\eta\in\mathbb{R}$ and $t\in[0,T]$*

$$|f(\xi_1,\eta_1)-f(\xi_2,\eta_2)|\leqslant L_{f,1}|\xi_1-\xi_2|+L_{f,2}|\eta_1-\eta_2|,$$
$$|f(\xi,\eta)|^2\leqslant K_f(1+|\xi|^2+|\eta|^2)$$

and likewise for g with constants $L_{g,i}$, $i=1,2$, and K_g. Then there exists a path-wise unique strong solution to Eq. (1).

3. Numerical analysis for an autonomous SDDE

Define a mesh with a uniform step h on the interval $[0,T]$ and $h=T/N, t_n=n\cdot h, n=0,\dots,N$. We assume that there is an integer number N_τ such that the delay can be expressed in terms of the stepsize as $\tau=N_\tau\cdot h$. We consider strong approximations $\tilde{X}_n$ of the solution to (1), using a stochastic explicit single-step method with an increment function ϕ incorporating increments $\Delta W_{n+1}:=W_{(n+1)h}-W_{nh}$ of the driving Wiener process. For all indices $n-N_\tau\leqslant 0$ define $\tilde{X}_{n-N_\tau}:=\Psi(t_n-\tau)$, otherwise

$$\tilde{X}_{n+1}=\tilde{X}_n+\phi(h,\tilde{X}_n,\tilde{X}_{n-N_\tau},\Delta W_{n+1}),\quad n=0,\dots,N-1. \tag{3}$$

Notation 1. We denote by $X(t_{n+1})$ the value of the exact solution of Eq. (1) at the meshpoint t_{n+1} and by $\tilde{X}_{n+1}$ the value of the approximate solution using (3) and by $\tilde{X}(t_{n+1})$ the value obtained after just one step of (3), i.e., $\tilde{X}(t_{n+1})=X(t_n)+\phi(h,X(t_n),X(t_n-\tau),\Delta W_{n+1})$.

With this notation we can give the following definitions.

Definition 3. The *local error* of the above approximation $\{\tilde{X}(t_n)\}$ is the sequence of random variables $\delta_{n+1}=X(t_{n+1})-\tilde{X}(t_{n+1})$, $n=0,\dots,N-1$. The global error of the above approximation $\{\tilde{X}_n\}$ is the sequence of random variables $\varepsilon_n:=X(t_n)-\tilde{X}_n$, $n=1,\dots,N$.

Note that ε_n is $\mathscr{A}_{t_n}$-measurable since both $X(t_n)$ and $\tilde{X}_n$ are $\mathscr{A}_{t_n}$-measurable random variables.

Definition 4. Method (3) is *consistent with order* p_1 *in the mean and with order* p_2 *in the mean-square sense* if the following estimates hold with $p_2 \geqslant \frac{1}{2}$ and $p_1 \geqslant p_2 + \frac{1}{2}$:

$$\max_{0 \leqslant n \leqslant N-1} |\mathscr{E}(\delta_{n+1})| \leqslant Ch^{p_1} \quad \text{as } h \to 0, \tag{4}$$

$$\max_{0 \leqslant n \leqslant N-1} (\mathscr{E}|\delta_{n+1}|^2)^{1/2} \leqslant Ch^{p_2} \quad \text{as } h \to 0, \tag{5}$$

where the (generic) constant C does not depend on h, but may depend on T and the initial data.

We also will assume the following properties of the increment function ϕ: assume there exist positive constants C_1, C_2 such that for all $\xi, \xi', \eta, \eta' \in \mathbb{R}$

$$|\mathscr{E}(\phi(h, \xi, \eta, \Delta W_{n+1}) - \phi(h, \xi', \eta', \Delta W_{n+1}))| \leqslant C_1 h(|\xi - \xi'| + |\eta - \eta'|), \tag{6}$$

$$\mathscr{E}(|\phi(h, \xi, \eta, \Delta W_{n+1}) - \phi(h, \xi', \eta', \Delta W_{n+1})|^2) \leqslant C_2 h(|\xi - \xi'|^2 + |\eta - \eta'|^2). \tag{7}$$

We now state the main theorem of this article.

Theorem 5. *We assume that the conditions of Theorem* 2 *are fulfilled and that the increment function* ϕ *in* (3) *satisfies estimates* (6) *and* (7). *Suppose the method defined by* (3) *is consistent with order* p_1 *in the mean and order* p_2 *in the mean-square sense, so that* (4) *and* (5) *hold* (*where the constant* C *does not depend on* h). *Then, approximation* (3) *for Eq.* (1) *is convergent in* $\mathscr{L}^2$ (*as* $h \to 0$ *with* $\tau/h \in \mathbb{N}$) *with order* $p = p_2 - \frac{1}{2}$. *That is, convergence is in the mean-square sense and*

$$\max_{1 \leqslant n \leqslant N} (\mathscr{E}|\varepsilon_n|^2)^{1/2} \leqslant Ch^p \quad \text{as } h \to 0, \ \text{where } p = p_2 - \tfrac{1}{2}. \tag{8}$$

Proof. Since we have exact initial values we set $\varepsilon_n = 0$ for $n = -N_\tau, \ldots, 0$. Now beginning with $\varepsilon_{n+1} = X(t_{n+1}) - \tilde{X}_{n+1}$, using Notation 1, adding and subtracting $X(t_n)$ and $\phi(h, X(t_n), X(t_n - \tau), \Delta W_{n+1})$ and rearranging we obtain $\varepsilon_{n+1} \leqslant \varepsilon_n + \delta_{n+1} + u_n$, where

$$u_n := \phi(h, X(t_n), X(t_n - \tau), \Delta W_{n+1}) - \phi(h, \tilde{X}_n, \tilde{X}_{n-N_\tau}, \Delta W_{n+1}).$$

Squaring both sides, employing the conditional mean with respect to the σ-algebra $\mathscr{A}_{t_0}$, and taking absolute values, we obtain

$$\begin{aligned} \mathscr{E}(\varepsilon_{n+1}^2 | \mathscr{A}_{t_0}) \leqslant{}& \mathscr{E}(\varepsilon_n^2 | \mathscr{A}_{t_0}) + \underbrace{\mathscr{E}(|\delta_{n+1}|^2 | \mathscr{A}_{t_0})}_{1)} + \underbrace{\mathscr{E}(|u_n|^2 | \mathscr{A}_{t_0})}_{2)} \\ & + \underbrace{2|\mathscr{E}(\delta_{n+1} \cdot u_n | \mathscr{A}_{t_0})|}_{3)} + \underbrace{2|\mathscr{E}(\delta_{n+1} \cdot \varepsilon_n | \mathscr{A}_{t_0})|}_{4)} + \underbrace{2|\mathscr{E}(\varepsilon_n \cdot u_n | \mathscr{A}_{t_0})|}_{5)} \end{aligned} \tag{9}$$

which holds almost surely. We will now estimate the separate terms in (9) individually and in sequence; all the estimates hold almost surely. We will frequently use the Hölder inequality, the inequality $2ab \leqslant a^2 + b^2$ and properties of conditional expectation.

- For the term labelled 1) in (9) we have, due to the assumed consistency in the mean-square sense of the method,

$$\mathscr{E}(|\delta_{n+1}|^2|\mathscr{A}_{t_0}) = \mathscr{E}(\mathscr{E}(|\delta_{n+1}|^2|\mathscr{A}_{t_n})|\mathscr{A}_{t_0}) \leqslant c_1 h^{2p_2}.$$

- For the term labelled 2) in (9) we have, due to property (7) of the increment function,

$$\mathscr{E}(|u_n|^2|\mathscr{A}_{t_0}) \leqslant c_2 h\mathscr{E}(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + c_2 h\mathscr{E}(|\varepsilon_{n-N_\tau}|^2|\mathscr{A}_{t_0}).$$

- For the term labelled 3) in (9) we obtain, by employing the consistency condition and property (7) of the increment function ϕ,

$$\begin{aligned}2|\mathscr{E}(\delta_{n+1}\cdot u_n|\mathscr{A}_{t_0})| &\leqslant 2(\mathscr{E}(|\delta_{n+1}|^2|\mathscr{A}_{t_0}))^{1/2}(\mathscr{E}(|u_n|^2|\mathscr{A}_{t_0}))^{1/2}\\ &\leqslant \mathscr{E}(\mathscr{E}(|\delta_{n+1}|^2|\mathscr{A}_{t_n})|\mathscr{A}_{t_0}) + \mathscr{E}(|u_n|^2|\mathscr{A}_{t_0})\\ &\leqslant c_3 h^{2p_2} + hc_6\mathscr{E}(\varepsilon_n^2|\mathscr{A}_{t_0}) + hc_6\mathscr{E}(\varepsilon_{n-N_\tau}^2|\mathscr{A}_{t_0}).\end{aligned}$$

- For the term labelled 4) we have, due to the consistency condition,

$$\begin{aligned}2|\mathscr{E}(\delta_{n+1}\cdot\varepsilon_n|\mathscr{A}_{t_0})| &\leqslant 2\mathscr{E}(|\mathscr{E}(\delta_{n+1}|\mathscr{A}_{t_n})|\cdot|\varepsilon_n||\mathscr{A}_{t_0})\\ &\leqslant 2(\mathscr{E}|\mathscr{E}(\delta_{n+1}|\mathscr{A}_{t_n})|^2)^{1/2}\cdot(\mathscr{E}(|\varepsilon_n|^2|\mathscr{A}_{t_0}))^{1/2}\\ &\leqslant 2(\mathscr{E}(c_5 h^{p_1})^2)^{1/2}\cdot(\mathscr{E}(|\varepsilon_n|^2|\mathscr{A}_{t_0}))^{1/2}\\ &= 2(\mathscr{E}(c_5^2 h^{2p_1-1}))^{1/2}\cdot(h\mathscr{E}(|\varepsilon_n|^2|\mathscr{A}_{t_0}))^{1/2}\\ &\leqslant c_5^2 h^{2p_1-1} + h\mathscr{E}(|\varepsilon_n|^2|\mathscr{A}_{t_0}).\end{aligned}$$

- For the term labelled 5) in (9) we have, using property (6) of the increment function ϕ,

$$\begin{aligned}2|\mathscr{E}(\varepsilon_n\cdot u_n|\mathscr{A}_{t_0})| &\leqslant 2\mathscr{E}(|\mathscr{E}(u_n|\mathscr{A}_{t_n})||\varepsilon_n||\mathscr{A}_{t_0})\\ &\leqslant c_6 h\mathscr{E}(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + 2c_6 h\mathscr{E}(|\varepsilon_n||\varepsilon_{n-N_\tau}||\mathscr{A}_{t_0})\\ &\leqslant c_6 h\mathscr{E}(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + c_6 h2(\mathscr{E}(|\varepsilon_n|^2|\mathscr{A}_{t_0}))^{1/2}\cdot(\mathscr{E}(|\varepsilon_{n-N_\tau}|^2|\mathscr{A}_{t_0}))^{1/2}\\ &\leqslant c_6 h\mathscr{E}(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + c_6 h\mathscr{E}(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + c_6 h\mathscr{E}(|\varepsilon_{n-N_\tau}|^2|\mathscr{A}_{t_0})\\ &\leqslant c_6 h\mathscr{E}(|\varepsilon_n|^2|\mathscr{A}_{t_0}) + c_6 h\mathscr{E}(|\varepsilon_{n-N_\tau}|^2|\mathscr{A}_{t_0}).\end{aligned}$$

Combining these results, we obtain, with $2p_2 \leqslant 2p_1 - 1$,

$$\mathscr{E}(\varepsilon_{n+1}^2|\mathscr{A}_{t_0}) \leqslant (1+c_7 h)\mathscr{E}(\varepsilon_n^2|\mathscr{A}_{t_0}) + c_7 h^{2p_2} + c_8 h\mathscr{E}(|\varepsilon_{n-N_\tau}|^2|\mathscr{A}_{t_0}).$$

Now we will proceed by using an induction argument over consecutive intervals of length τ up to the end of the interval $[0,T]$.

Step 1: $t_n \in [0,\tau]$, i.e., $n = 1,\dots,N_\tau$ and $\varepsilon_{n-N_\tau} = 0$.

$$\begin{aligned}\mathscr{E}(\varepsilon_{n+1}^2|\mathscr{A}_{t_0}) &\leqslant (1+c_7 h)\mathscr{E}(\varepsilon_n^2|\mathscr{A}_{t_0}) + c_7 h^{2p_2}\\ &\leqslant c_7 h^{2p_2}\sum_{k=0}^{n}(1+c_6 h)^k = c_7 h^{2p_2}\frac{(1+c_6 h)^{n+1}-1}{(1+c_6 h)-1}\\ &\leqslant c_9 h^{2p_2-1}((e^{c_6 h})^{n+1}-1) \leqslant c_9 h^{2p_2-1}(e^{c_6 T}-1).\end{aligned}$$

Step 2: $t_n \in [k\tau,(k+1)\tau]$ and we make the assumption $\mathscr{E}(|\varepsilon_{n-N_\tau}|^2|\mathscr{A}_{t_0}) \leqslant c_{10} h^{2p_2-1}$.

$$\begin{aligned}\mathscr{E}(\varepsilon_{n+1}^2|\mathscr{A}_{t_0}) &\leqslant (1+c_7 h)\mathscr{E}(\varepsilon_n^2|\mathscr{A}_{t_0}) + c_7 h^{2p_2} + c_8 h\mathscr{E}(|\varepsilon_{n-N_\tau}|^2|\mathscr{A}_{t_0})\\ &\leqslant (1+c_7 h)\mathscr{E}(\varepsilon_n^2|\mathscr{A}_{t_0}) + c_7 h^{2p_2} + hc_{10} h^{2p_2-1}\end{aligned}$$

$$= (1 + c_7 h)\mathscr{E}(\varepsilon_n^2|\mathscr{A}_{t_0}) + c_{11} h^{2p_2}$$
$$\leqslant c_{12} h^{2p_2-1}(e^{c_6 T} - 1),$$

by the same arguments as above. This implies, almost surely,

$$(\mathscr{E}(\varepsilon_{n+1}^2|\mathscr{A}_{t_0}))^{1/2} \leqslant c_9 h^{p_2-1/2},$$

which proves the theorem. □

Remark 6. Assumption (6) reduces to the condition of Lipschitz-continuity for the increment function ϕ in the deterministic setting, i.e., without noise. This is a standard assumption for convergence in the theory of numerical analysis for deterministic ordinary differential equations, as it implies the zero-stability of the numerical method.

4. The Euler–Maruyama scheme

As a start we have considered strong Euler–Maruyama approximations with a fixed stepsize on the interval $[0, T]$, i.e., $h = T/N$, $t_n = n \cdot h$, $n = 0, \ldots, N$. In addition we have assumed that there is an integer number N_τ such that the delay can be expressed in terms of the stepsize as $\tau = N_\tau h$.

For Eq. (1) the increment function ϕ_{EM} of the Euler–Maruyama scheme has the following form in the method (3):

$$\phi_{\mathrm{EM}}(h, \tilde{X}_n, \tilde{X}_{n-N_\tau}, \Delta W_{n+1}) = hf(\tilde{X}_n, \tilde{X}_{n-N_\tau}) + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta W_{n+1} \tag{10}$$

for $0 \leqslant n \leqslant N-1$ and with $\Delta W_{n+1} := W_{(n+1)h} - W_{nh}$, denoting independent $N(0, h)$-distributed Gaussian random variables.

Theorem 7. *If the functions f and g in Eq.* (1) *satisfy the conditions of Theorem* 2, *then the Euler–Maruyama approximation is consistent with order $p_1 = 2$ in the mean and order $p_2 = 1$ in the mean square.*

We gave a complete proof in [2], based on the consistency analysis given in [11] for SODEs and using a theorem from Mao [10, Lemma 5.5.2], which provides the necessary moment inequalities for the solution of (1).

Lemma 8. *If the functions f and g in Eq.* (1) *satisfy the conditions of Theorem* 2, *then the increment function ϕ_{EM} of the Euler–Maruyama scheme* (*given by* (10)) *satisfies estimates* (6) *and* (7) *for all $\xi, \xi', \eta, \eta' \in \mathbb{R}$.*

$$|\mathscr{E}(\phi_{\mathrm{EM}}(h, \xi, \eta, \Delta W_{n+1}) - \phi_{\mathrm{EM}}(h, \xi', \eta', \Delta W_{n+1}))|$$
$$= |\mathscr{E}(hf(\xi, \eta) + g(\xi, \eta)\Delta W_{n+1} - hf(\xi', \eta') - g(\xi', \eta')\Delta W_{n+1})|$$
$$\leqslant h|f(\xi, \eta) - f(\xi', \eta')| + |g(\xi, \eta) - g(\xi', \eta')||\mathscr{E}(\Delta W_{n+1})|$$
$$\leqslant h(L_1|\xi - \xi'| + L_2|\eta - \eta'|)$$

Table 1

Time step	0.25	0.125	0.0625	0.03125
I ε	0.0184	0.00404	0.000973	0.000244
II ε	0.1088654	0.04912833	0.02437045	0.01213507

$$\begin{aligned}
&\mathscr{E}(|\phi_{\mathrm{EM}}(h,\xi,\eta,\Delta W_{n+1}) - \phi_{\mathrm{EM}}(h,\xi',\eta',\Delta W_{n+1})|^2) \\
&\quad = \mathscr{E}(|hf(\xi,\eta) + g(\xi,\eta)\Delta W_{n+1} - hf(\xi',\eta') - g(\xi',\eta')\Delta W_{n+1}|^2) \\
&\quad \leqslant h^2|f(\xi,\eta) - f(\xi',\eta')|^2 + |g(\xi,\eta) - g(\xi',\eta')|^2\mathscr{E}|\Delta W_{n+1}|^2 \\
&\quad \leqslant h^2(L_1^2|\xi-\xi'|^2 + L_2^2|\eta-\eta'|^2) + h(L_3^2|\xi-\xi'|^2 + L_4^2|\eta-\eta'|^2),
\end{aligned}$$

from which the estimates follow.

Remark 9. Theorem 7 and the last lemma imply that for the Euler–Maruyama method Theorem 5 is valid, with order of convergence $p=\frac{1}{2}$ in the mean-square-sense. If Eq. (1) has additive noise, then the Euler–Maruyama approximation is consistent with order $p_1 = 2$ in the mean and order $p_2 = \frac{3}{2}$ in the mean square, which implies an order of convergence $p = 1$ in the mean-square-sense.

5. Numerical experiments

We have used the equation

$$\mathrm{d}X(t) = \{aX(t) + bX(t-1)\}\,\mathrm{d}t + \{\beta_1 + \beta_2 X(t) + \beta_3 X(t-1)\}\,\mathrm{d}W(t)$$

as a test equation for our method. In the case of additive noise ($\beta_2 = \beta_3 = 0$) we have calculated an explicit solution on the first interval $[0,\tau]$ by the method of steps (see, e.g., [5]), using $\Psi(t) = 1+t$ for $t \in [-1, 0]$ as an initial function. The solution on $t \in [0,1]$ is given by

$$X(t) = \mathrm{e}^{at}\left(1 + \frac{b}{a^2}\right) - \frac{b}{a}t - \frac{b}{a^2} + \beta\mathrm{e}^{at}\int_0^t \mathrm{e}^{-as}\,\mathrm{d}W(s).$$

We have then used this solution as a starting function to compute an ‘explicit solution’ on the second interval $[\tau, 2\tau]$ with a standard SODE-method and a small stepsize. In the case of multiplicative noise we have computed an ‘explicit solution’ on a very fine grid (2048 steps) with the Euler–Maruyama scheme.

One of our tests concerned the illustration of the theoretical order of convergence. In this case the mean-square error $\mathscr{E}|X(T) - \tilde{X}_N|^2$ at the final time $T = 2\tau$ was estimated in the following way. A set of 20 blocks each containing 100 outcomes ($\omega_{i,j}; 1 \leqslant i \leqslant 20, 1 \leqslant j \leqslant 100$), were simulated and for each block the estimator $\varepsilon_i = \frac{1}{100}\sum_{j=1}^{100} |X(T,\omega_{i,j}) - \tilde{X}_N(\omega_{i,j})|^2$ was formed. In Table 1 ε denotes the mean of this estimator, which was itself estimated in the usual way: $\varepsilon = \frac{1}{20}\sum_{i=1}^{20} \varepsilon_i$.

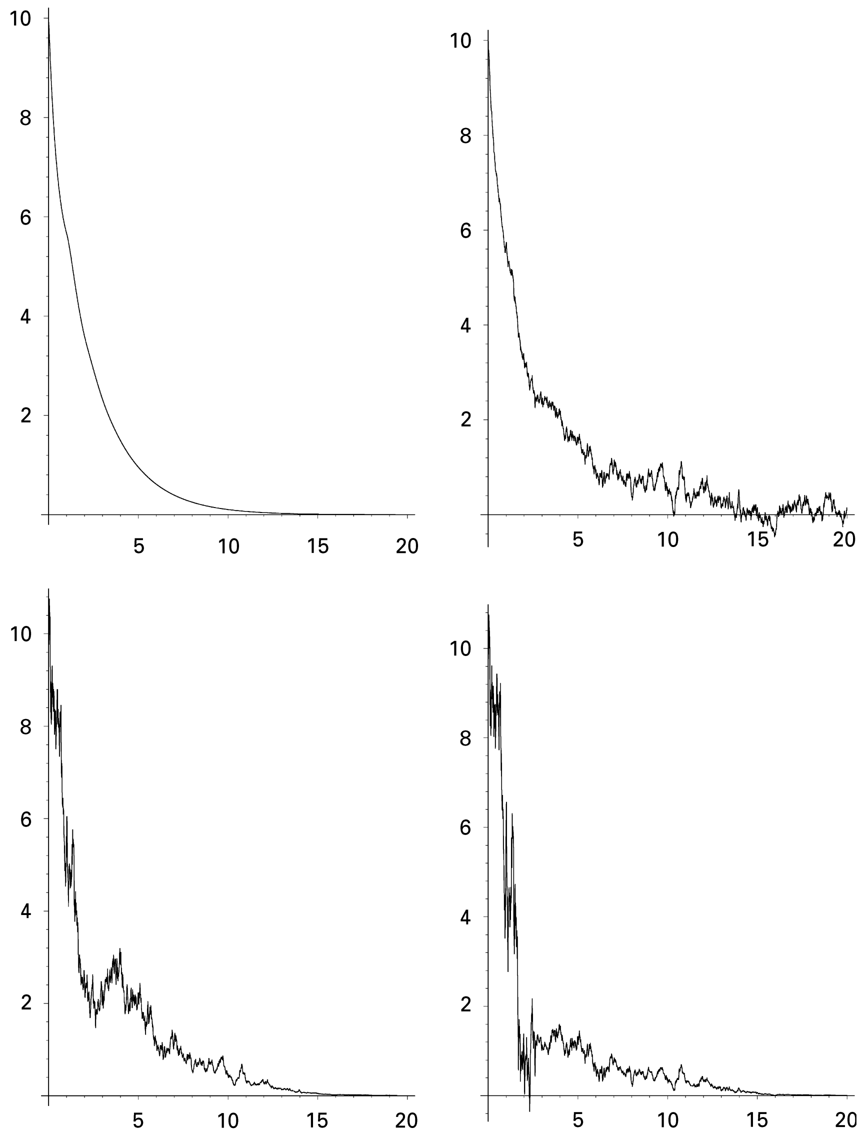


Fig. 1. Upper left: $\beta_i = 0,\ i = 1, \ldots, 3$, upper right: $\beta_1 = 0.5,\ \beta_i = 0,\ i \neq 1$, lower left: $\beta_2 = 0.5,\ \beta_i = 0,\ i \neq 2$, lower right: $\beta_3 = 0.5,\ \beta_i = 0,\ i \neq 3$.

We have used the set of coefficients I $a = -2, b = 0.1, \beta_1 = 1$ and II $a = -2, b = 0.1, \beta_2 = 1$ (the other coefficients in the diffusion term are set to 0). The figures display $\max_{1 \leqslant n \leqslant N} \mathscr{E}|X(T) - \tilde{X}_N|^2$, which according to (8) in Theorem 5 is bounded by $c^2 h^{2p}$, and they are compatible with the results given in Remark 9, i.e. $p = 1$ in (I), the example with additive noise, and $2p = \frac{1}{2}$ in (II), an example with multiplicative noise.

One may consider Eq. (1) as a deterministic delay equation perturbed by white noise. In this context Figs. 1 and 2 show the influence of the parameters β_i on the solution of the deterministic

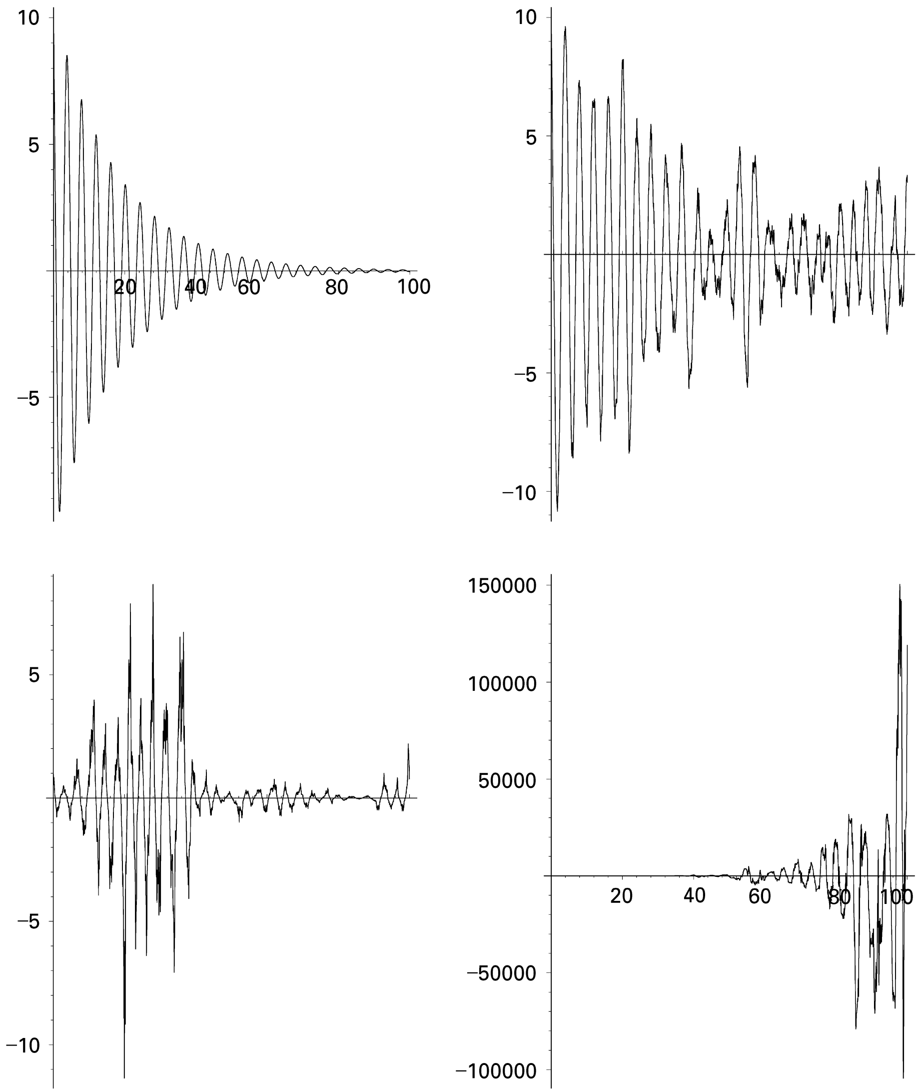
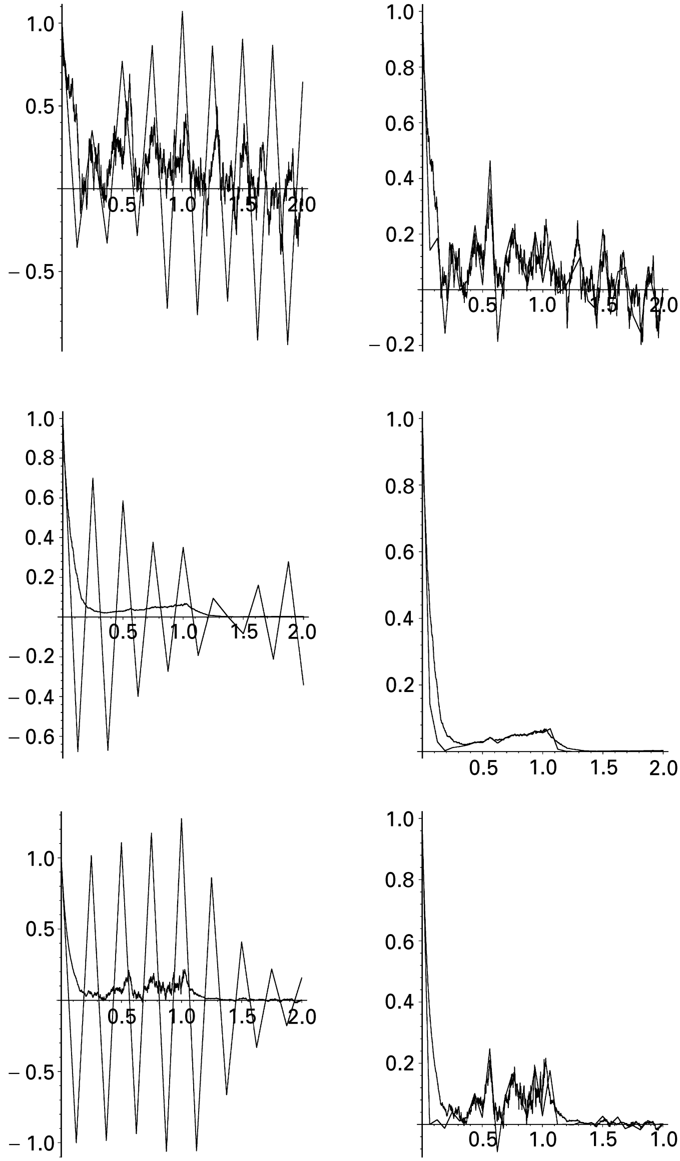


Fig. 2. Upper left: $\beta_i = 0,\ i = 1,\dots,3$, upper right: $\beta_1 = 1,\ \beta_i = 0,\ i \neq 1$, lower left: $\beta_2 = 1,\ \beta_i = 0,\ i \neq 2$, lower right: $\beta_3 = 1,\ \beta_i = 0,\ i \neq 3$.

test equation $x'(t) = ax(t) + bx(t - \tau)$. In the first four pictures $a = -2, b = 1$, in the second four pictures $a = 0, b = 1.45$.

As a last experiment we varied the stepsize in order to observe some stability behaviour of the Euler–Maruyama method. Using the coefficients $a = -16, b = 1$ and two stepsizes: $h = \frac{1}{16}$ (left figure) and $h = \frac{1}{32}$ (right figure), we observe the same stability behaviour as for the deterministic equation, i.e., a change from unstable to stable, when varying the coefficients of the diffusion term. In the pictures we have $\beta_1 = 0.5$ $(\beta_i = 0, i \neq 1)$, $\beta_2 = 0.5$ $(\beta_i = 0, i \neq 2)$, $\beta_3 = 0.5$ $(\beta_i = 0, i \neq 3)$, respectively.



6. Conclusions

This article provides an introduction to the numerical analysis of stochastic delay differential equations. When one seeks to advance the study further, one sees open a number of unanswered questions, involving (for example)

- the design of numerical methods for more general kinds of memory (e.g., time or state dependent time lags);
- the stability and dynamical properties of the numerical methods;
- the design of numerical methods for more general problems (e.g., stochastic integrodifferential equations).

We hope that such issues will be addressed in sequels to this report.

References

[1] L. Arnold, Stochastic Differential Equations: Theory and Applications, Wiley-Interscience, New York, 1974.
[2] C.T.H. Baker, E. Buckwar, Introduction to the numerical analysis of stochastic delay differential equations, MCCM Numerical Analysis Technical Report, Manchester University, ISSSN 1360–1725, 1999.
[3] A. Beuter, J. Bélair, Feedback and delays in neurological diseases: a modelling study using dynamical systems, Bull. Math. Biol. 55 (3) (1993) 525–541.
[4] J.M.C. Clark, The discretization of stochastic differential equations: a primer, in: H. Neunzert (Ed.), Road Vehicle Systems and Related Mathematics; Proceedings of the second DMV-GAMM Workshop, Torino, 1987, Teubner, Stuttgart, and Kluwer Academic Publishers, Amsterdam, pp. 163–179.
[5] R.D. Driver, Ordinary and Delay Differential Equations, Applied Mathematical Sciences, Vol. 20, Springer, New York, 1977.
[6] C.W. Eurich, J.G. Milton, Noise-induced transitions in human postural sway, Phys. Rev. E 54 (6) (1996) 6681–6684.
[7] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, Springer, New York, 1991.
[8] P.E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer, Berlin, 1992.
[9] M.C. Mackey, A. Longtin, J.G. Milton, J.E. Bos, Noise and critical behaviour of the pupil light reflex at oscillation onset, Phys. Rev. A 41 (12) (1990) 6992–7005.
[10] X. Mao, Stochastic Differential Equations and their Applications, Horwood Publishing Limited, Chichester, 1997.
[11] G.N. Milstein, Numerical Integration of Stochastic Differential Equations, Kluwer Academic Publishers, Dordrecht, 1995 (Translated and revised from the 1988 Russian original.)
[12] S.E.A. Mohammed, Stochastic Functional Differential Equations, Pitman (Advanced Publishing Program), Boston, MA, 1984.
[13] D. Williams, Probability with Martingales, Cambridge University Press, Cambridge, 1991.
[14] M. Zennaro, Delay differential equations: theory and numerics, in: M. Ainsworth, J. Levesley, W.A. Light, M. Marletta (Eds.), Theory and Numerics of Ordinary and Partial Differential Equations (Leicester, 1994), Oxford University Press, New York, 1995, pp. 291–333.