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DECIDABILITY OF A TEMPORAL LOGIC PROBLEM FOR PETRI NETS

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Abstract. The paper solves an open problem from [4] by showing a decision algorithm for a temporal logic language $L(Q', \mathrm{GF})$. It implies the decidability of the problem of the existence of an infinite weakly fair occurrence sequence for a given Petri net; thereby an open problem from [2] is solved.

1. Introduction

Real parallel systems should always satisfy some conditions of fairness; it means that the behaviour of a (global) system should be fair with respect to (local) components (processes). Therefore it is desirable to search for an exact expression of fairness in theoretical models and so also in Petri nets. Throughout the paper, we consider the "classical" place/transition Petri nets.

If we try to define a fair behaviour (execution) of a Petri net independently on a modelled problem, the following notions seem to be natural: an infinite occurrence sequence σ is (strongly) fair with respect to a transition t if t occurs infinitely many times or is enabled only finitely many times in σ; σ is weakly fair (or σ has the finite delay property) with respect to t if t occurs infinitely many times or is disabled infinitely many times in σ.

Carstensen uses more general notions in [2]; he considers fairness with respect to sets of transitions (where the above-mentioned fairness with respect to single transitions is a special case).

He proves that it is undecidable for a given Petri net and some sets of its transitions whether an infinite occurrence sequence being (strongly) fair with respect to the given sets of transitions exists; the same even holds in case of one given (single) transition (and in case of all (single) transitions as well).

On the other hand, [2] shows that it is decidable for a given Petri net and *one* set of its transitions whether an infinite occurrence sequence being weakly fair with respect to the given set exists. The idea of the proof cannot be generalized and the general problem (weak fairness with respect to *several* sets of transitions) is left open in [2]. The problem is also open in the later published paper [4] which provides a nice survey of complexities of similar problems for various definitions of fairness; [4] also defines a simple temporal logic language $L(Q', \mathrm{GF})$, in which the problem can be expressed.

This paper shows that there is a decision algorithm for formulas of $L(Q', \mathrm{GF})$, by which Carstensen's open problem is also solved.

Section 2 contains preliminaries, Section 3 shows the structure of the proof in an informal way, and Section 4 reduces our problem to the "key" problem which is to be solved. Sections 5 and 6 contain some auxiliary results—a decidable generalizing of the reachability problem and an application of a result from [3] on occurrence sequences. The main proof (of the decidability of the "key" problem) is given in Section 7.

2. Basic definitions and results

We use usual logical and set theoretical symbols $\neg$, &, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\exists$, $\forall$, $\in$, $\subseteq$, $\cap$, $\cup$; the quantifier $\exists_\infty$ means "there are infinitely many".

$A \backslash B$ denotes the set difference of sets A, B, $A \times B$ their cartesian product. $|A|$ denotes the cardinality of a set A, $f \upharpoonright A$ the restriction of a function f to a domain A.

$\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{Z}$ denote the sets of nonnegative, positive and all integers, respectively. $\mathbb{Q}_+$ denotes the set of nonnegative rational numbers.

Sometimes we implicitly regard a function $f \in \mathbb{Z}^S$ ($f: S \to \mathbb{Z}$), S being a finite set, as a vector from $\mathbb{Z}^r$ for $r = |S|$. The bold symbol $\mathbf{0}$ stands for the null vector $(0, 0, \ldots, 0)$; the dimension will be clear from the context.

A^* denotes the set of finite sequences of elements of A; ε denotes the empty sequence. For $u \in A^*$, $k \in \mathbb{N}$, $(u)^k$ stands for $uu \ldots u$, u being written k-times, $(u)^\omega$ stands for $uuu \ldots$.

The Petri net notation is taken mainly from [1].

A quadruple $\Sigma = (S, T, W, M_0)$ is called a *Petri net* if S and T are finite disjoint sets of *places* and *transitions*, respectively, $W: (S \times T) \cup (T \times S) \to \mathbb{N}$ is a *weight function* (for $W(s, t) > 0$, there is an *arc* from s to t with the *multiplicity* $W(s, t)$; similarly for $W(t, s) > 0$) and M_0 is an *initial marking*, where a *marking* M of Σ is a function $M: S \to \mathbb{N}$.

A transition t is *enabled by* M, $M[t\rangle$, if $M(s) \geq W(s, t)$ for every $s \in S$. If t is enabled by M, it can "occur" yielding a new marking M', $M[t\rangle M'$, where $M'(s) = M(s) - W(s, t) + W(t, s)$ for every $s \in S$. In a natural way, the definitions can be extended for the case $M[u\rangle$, $M[u\rangle M'$, where $u \in T^*$. By $M[>M'$ we mean that there is some $u \in T^*$ such that $M[u\rangle M'$.

Remark. Notice the trivial fact that $M \leq M'$ & $M[u\rangle$ implies $M'[u\rangle$; we often use it implicitly.

The *effect* of $u \in T^*$ (on markings), denoted by $\Delta(u)$, is given by the following: $\Delta: T^* \to \mathbb{Z}^S$, where $\Delta(\varepsilon) = \mathbf{0}$,

$$(\Delta(t))(s) = -W(s, t) + W(t, s), \qquad \Delta(tu) = \Delta(t) + \Delta(u).$$

For a Petri net, a sequence $\sigma = {}_{M_0} t_1 {}_{M_1} t_2 {}_{M_2} \ldots t_n {}_{M_n}$ is a (*finite*) *occurrence sequence* if $M_{i-1}[t_i\rangle M_i$ for all i; similarly for an *infinite occurrence sequence* $\sigma = {}_{M_0} t_1 {}_{M_1} t_2 {}_{M_2} \ldots$. The whole information in σ is given by the sequence of transitions; we write $\sigma_T = t_1 t_2 \ldots$.

Describing an occurrence sequence, we often write only some "passed through" markings explicitly (e.g. we write $\sigma = {}_{M_0} u_1 {}_{M_1} u_2 {}_{M_2} \ldots$ for $u_i \in T^*$).

We define weak fairness (finite delay property) as in [2].

Definition 2.1. Let a Petri net $\Sigma = (S, T, W, M_0)$ and some sets of transitions $T_1, T_2, \ldots, T_k \subseteq T$ be given. An infinite occurrence sequence $\sigma = {}_{M_0} t_1 {}_{M_1} t_2 {}_{M_2} \ldots$ is *weakly fair*, or has the *finite delay property* (has the *fdp*), *with respect to* $T_1, T_2, \ldots, T_k$ if $(\exists_\infty i)(t_i \in T_j)$ or $(\exists_\infty i)(M_i \not[T_j\rangle)$ for every T_j $(1 \leq j \leq k)$, where the expression $M_i \not[T_j\rangle$ means that all $t \in T_j$ are disabled (i.e. not enabled) by M_i.

We are interested in the following problem $\mathbb{P}_{\mathrm{fdp}}$.

Definition 2.2. *Problem* $\mathbb{P}_{\mathrm{fdp}}$ is specified in the following way:

Instance: A Petri net $\Sigma = (S, T, W, M_0)$ and some sets $T_1, T_2, \ldots, T_k \subseteq T$.

Question: Is there an infinite occurrence sequence σ of Σ which has the fdp with respect to $T_1, T_2, \ldots, T_k$?

We need the well-known result on the decidability of the reachability problem $\mathbb{RP}$:

Definition 2.3. *Problem* $\mathbb{RP}$ is specified in the following way:

Instance: A Petri net $\Sigma = (S, T, W, M_0)$, a set $S' \subseteq S$ and a function (submarking) $M' : S' \to \mathbb{N}$.

Question: Is there a marking M such that $M_0[\;\rangle M$ & $M \restriction S' = M'$?

Theorem 2.4. *Problem* $\mathbb{RP}$ *is decidable.*

Proof. In [6] or [8]. □

It is known from [7] that $\mathbb{RP}$ is exp-space-hard, but the known upper bound is not primitive recursive.

We know the following facts from [2].

Theorem 2.5. (1) $\mathbb{P}_{\mathrm{fdp}}$ *is exp-space-hard.*

(2) *In case of* one *given set* ($k = 1$ *in Definition* 2.2.), $\mathbb{P}_{\mathrm{fdp}}$ *is decidable.*

Proof. In [2]; (1) by the construction from [7], (2) by a reduction to $\mathbb{RP}$. □

The above results are shown more precisely in [4].

Theorem 2.6. (1) $\mathbb{RP} \leq_{\mathrm{PTIME}} \mathbb{P}_{\mathrm{fdp}}$ ($\mathbb{P}_{\mathrm{fdp}}$ *is at least as hard as* $\mathbb{RP}$).

(2) *In case of* one *given set*, $\mathbb{RP} \equiv_{\mathrm{PTIME}} \mathbb{P}_{\mathrm{fdp}}$.

Proof. In [4]. $\square$

In [4], a simple temporal language $L(Q', \mathrm{GF})$ is defined.

Definition 2.7. Let a Petri net $\Sigma = (S, T, W, M_0)$ be given. The *language* $L(Q', \mathrm{GF})$ is defined in the following way:

(a) *atomic formulas* (*predicates*) are $\mathrm{ge}(s, c)$, $\mathrm{fi}(t)$, where $s \in S$, $t \in T$, $c \in \mathbb{N}$, with the following interpretation: for any infinite occurrence sequence $\sigma = {}_{M_0} t_1 {}_{M_1} t_2 {}_{M_2} \ldots$ and for any $n \in \mathbb{N}$,

$$\langle \Sigma, \sigma, n\rangle \models \mathrm{ge}(s, c) \underset{\mathrm{df}}{\Leftrightarrow} M_n(s) \geq c,$$

$$\langle \Sigma, \sigma, n\rangle \models \mathrm{fi}(t) \underset{\mathrm{df}}{\Leftrightarrow} t_{n+1} = t,$$

(b) *formulas* are either *literals*, i.e. atomic formulas or their negations ($\mathrm{ge}(s, c)$, $\neg\mathrm{ge}(s, c)$, $\mathrm{fi}(t)$, $\neg\mathrm{fi}(t)$), or of the form $\mathrm{GF}f$, $f_1 \,\&\, f_2$, $f_1 \vee f_2$, where f, f_1, f_2 are formulas. $\mathrm{GF}f$ (it is always true that f will hold in future) can be defined as follows:

$$\langle \Sigma, \sigma, n\rangle \models \mathrm{GF}f \underset{\mathrm{df}}{\Leftrightarrow} \left(\exists_{\infty} i;\ i > n\right)(\langle \Sigma, \sigma, i\rangle \models f).$$

The rest of the interpretation is natural.

For technical reasons, we will also use literals $\mathrm{eq}(s, c)$, where

$$\langle \Sigma, \sigma, n\rangle \models \mathrm{eq}(s, c) \underset{\mathrm{df}}{\Leftrightarrow} M_n(s) = c.$$

The decidability of the next problem $\mathbb{P}_{\mathrm{temp}}$ was open in [4].

Definition 2.8. *Problem* $\mathbb{P}_{\mathrm{temp}}$ is specified in the following way:

Instance: A Petri net Σ, a formula $f \in L(Q', \mathrm{GF})$.

Question: Is there an infinite occurrence sequence σ of Σ such that $\langle \Sigma, \sigma, 0\rangle \models f$?

It is easy to verify that the finite delay property can be expressed by a formula from $L(Q', \mathrm{GF})$; it means that $\mathbb{P}_{\mathrm{fdp}}$ is reducible to $\mathbb{P}_{\mathrm{temp}}$. (It can be done in PTIME; thus $\mathbb{P}_{\mathrm{fdp}} \leq_{\mathrm{PTIME}} \mathbb{P}_{\mathrm{temp}}$.)

The main aim of this paper is to prove the decidability of $\mathbb{P}_{\mathrm{temp}}$. In fact, the proof shows an exponential reduction to $\mathbb{RP}$.

A crucial point of the proof is to show the decidability of the following problem $\mathbb{P}_{\mathrm{key}}$.

Definition 2.9. *Problem* $\mathbb{P}_{\mathrm{key}}$ is specified in the following way:

Instance: A Petri net $\Sigma = (S, T, W, M_0)$, n pairs ($n \in \mathbb{N}_+$),

$$(S_1, f_1), (S_2, f_2), \ldots, (S_n, f_n) \text{ where } S_i \subseteq S,\ f_i : S_i \to \mathbb{N}.$$

Question: Is there a *solution of the* $\mathbb{P}_{\text{key}}$-*instance*, i.e. an infinite occurrence sequence $\sigma = {}_{M_0}\, t_1\, {}_{M_1}\, t_2\, {}_{M_2} \ldots$ such that

$$(\forall j,\ 1 \leq j \leq n)(\overset{\infty}{\exists}\, i)(M_i \restriction S_j = f_j)?$$

3. Informal outline of the proof

Carstensen's proof of (2) in Theorem 2.5 depends on the fact that the existence of σ which has the fdp with respect to T_1 implies the existence of a "periodic" σ' (σ'_1 is in the form $u(w)^{\omega}$) which has the fdp with respect to T_1 as well.

Figure 1 (also given in [2]) shows that this idea cannot be extended. The occurrence sequence

$$\sigma = {}_{(0,1,1)}\, t_1\, {}_{(1,0,1)}\, t_2\, {}_{(0,2,1)} \cdots$$

$$\cdots {}_{(0,2^i,1)}\, (t_1)^{2^i}\, {}_{(2^i,0,1)}\, (t_2)^{2^i}\, {}_{(0,2^{i+1},1)} \cdots$$

has the fdp with respect to $\{t_3\}$, $\{t_4\}$ (also with respect to $\{t_1\}$, $\{t_2\}$) but there is not any periodic σ' with this property.

Notice that the $\mathbb{P}_{\text{fdp}}$-instance Σ, $\{t_3\}$, $\{t_4\}$ can be reduced to searching for an infinite occurrence sequence in which s_1, s_2 are empty infinitely many times, i.e. to a $\mathbb{P}_{\text{key}}$-instance.

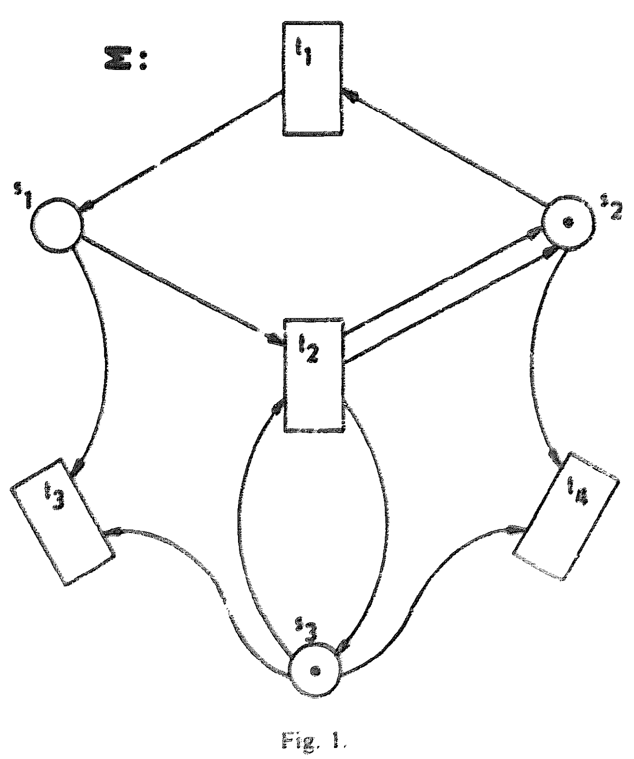

Fig. 1.

It is not difficult to show that $\mathbb{P}_{\text{temp}}$ (and so $\mathbb{P}_{\text{fdp}}$ also) is reducible to $\mathbb{P}_{\text{key}}$ (see Section 4).

We shall outline the proof of the decidability of $\mathbb{P}_{\text{key}}$. First notice that the described σ is regular in a certain way: if we denote $M_1^i = (0, 2^i, 1)$, $M_2^i = (2^i, 0, 1)$ then (1) holds for *all i*.

$$(\forall j, 1 \leq j \leq 2) \quad (\exists k_j \in \mathbb{N}_+) \quad (M_j^{i+2} - M_j^{i+1} = k_j(M_j^{i+1} - M_j^i)). \qquad (1)$$

It is important that a "regular beginning" (a finite sequence in which (1) holds for $i = 0$; cf. Definition 7.8.) can be lengthened to infinity in a "regular" way.

$\mathbb{P}_{\text{key}}$-solutions will be characterized as so-called ω-good sequences satisfying a certain condition INSERT; it will be done using the results from Section 6.

Within a (finite) $f(n|S|)$-good INSERT sequence, where f is a certain exponential function of the "size" of the relevant $\mathbb{P}_{\text{key}}$-instance, a series of modifications can be performed so that a regular sequence (i.e. a regular beginning) arises; it is the most technically difficult part of the proof.

Unfortunately, neither the existence of a regular sequence nor the existence of an $f(n|S|)$-good INSERT sequence can be reduced to the reachability problem in a straightforward way.

We shall use the fact that the infinite regular lengthening of a regular sequence yields an ω-good sequence which satisfies INSERT in a STRICT way.

The existence of an $f(n|S|)$-good STRICT INSERT sequence is reducible to the reachability problem using the results from Section 5.

Figure 2 shows the structure of the proof.

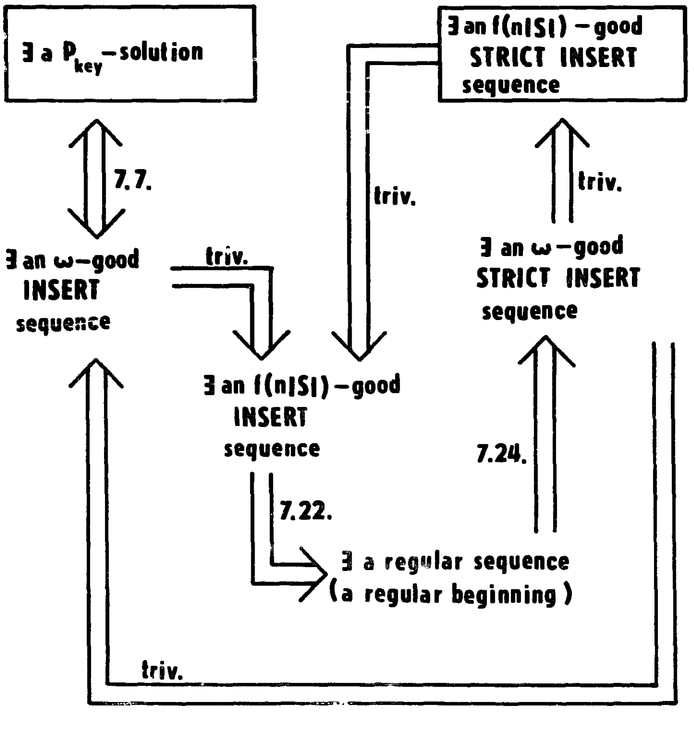

Fig. 2.

4. Reduction of $\mathbb{P}_{\text{temp}}$ to $\mathbb{P}_{\text{key}}$

Let us say that a formula $f \in L(Q', \text{GF})$ is "simple" if f can be written $f \equiv \&_{j \in J} \text{GF} f_j$, J being a finite index set and f_j being finite conjunctions of literals of the type $\text{eq}(s, c)$.

A $\mathbb{P}_{\text{temp}}$-instance Σ, f, where f is simple, can be rewritten into a $\mathbb{P}_{\text{key}}$-instance in an obvious way.

In what follows we show how to reduce any (general) $\mathbb{P}_{\text{temp}}$-instance to finitely many $\mathbb{P}_{\text{temp}}$-instances with simple formulas only.

Convention 4.1. Further we will implicitly assume that all literals $\neg\text{ge}(s, c)$ are replaced by $\text{eq}(s, 0) \vee \text{eq}(s, 1) \vee \cdots \vee \text{eq}(s, c-1)$ and so we will not consider them.

Let us define a natural equivalence on the set of formulas.

Definition 4.2 (of the *equivalence* $\cong$). Let a Petri net Σ be given. For any $f_1, f_2 \in L(Q', \text{GF})$, $f_1 \cong f_2$ iff $\langle \Sigma, \sigma, 0 \rangle \models f_1 \Leftrightarrow \langle \Sigma, \sigma, 0 \rangle \models f_2$ for all σ.

Lemma 4.3. *Any $\mathbb{P}_{\text{temp}}$-instance $\Sigma = (S, T, W, M_0)$, f can be reduced to finitely many $\mathbb{P}_{\text{temp}}$-instances where the formulas are in the form $\&_{j \in J} \text{GF} f_j$, f_j being conjunctions of literals.*

Proof. Using the distributive laws for $\vee$, $\&$ and obvious equivalences

$$\text{GF}(\text{GF}f) \cong \text{GF}f,$$

$$\text{GF}(f_1 \vee f_2) \cong \text{GF}f_1 \vee \text{GF}f_2,$$

$$\text{GF}(\text{GF}f_1 \,\&\, f_2) \cong \text{GF}f_1 \,\&\, \text{GF}f_2,$$

it is clear that f can be written as a finite disjunction of formulas of the type $f' \equiv \&_{j \in J} g_j$ where g_j are either literals or in the form $\text{GF}g$, g being a conjunction of literals.

Answering all instances Σ, f', where f' is a member of the disjunction, we obtain the answer for Σ, f easily.

Hence it suffices to get rid of literals in the conjunction $f' \equiv \&_{j \in J} g_j$.

(i) If none of g_j is a literal, we are done.

(ii) In the following four points we show how Σ', f'' can be constructed so that the answer for Σ', f'' is the same as for Σ, f' and f'' is "more simple" than f'.

It will be clear that Case (i) will be achieved by finitely many steps.

(1) $\quad f' \equiv \text{ge}(s, c) \,\&\, \&_{j \in J'} g_j$.

If $M_0(s) < c$ then the answer for Σ, f' is NO,
if $M_0(s) \geq c$ then put $\Sigma' = \Sigma$ and $f'' \equiv \&_{j \in J'} g_j$.

(2) $\quad f' \equiv \text{eq}(s, c) \,\&\, \&_{j \in J'} g_j$.

It is similar to (1).

(3) $\not{f}' \equiv \mathrm{fi}(t') \,\&\, \&_{j \in J'}\, g_j$.

To obtain Σ', do the following changes in Σ (cf. Fig. 3):

- add new places s_1, s_2 and put $W(s_1, t') = W(t', s_1) = 1$, $W(s_2, t) = W(t, s_2) = 1$ for all $t \in T\backslash\{t'\}$ (s_1 is a "run-place" for t', s_2 for $T\backslash\{t'\}$) and still $W(t', s_2) = 1$ (the other values of W are 0).
- put $M_0(s_1) = 1$, $M_0(s_2) = 0$. Finally put $\not{f}'' \equiv \&_{j \in J'}\, g_j$.

(4) $\not{f}' \equiv \neg\mathrm{fi}(t') \,\&\, \underset{j \in J'}{\&} g_j$.

We can proceed as in (3) ($W(t, s_1) = 1$ for all $t \in T\backslash\{t'\}$, $M_0(s_1) = 0$ and $M_0(s_2) = 1$ in this case). □

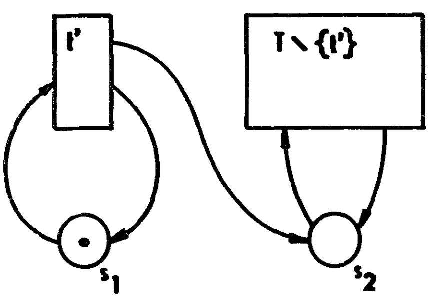

Fig. 3.

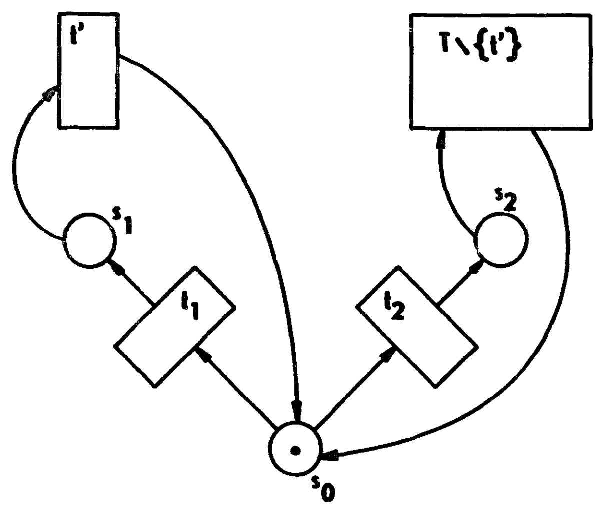

Fig. 4.

Lemma 4.4. *Any $\mathbb{P}_{\mathrm{temp}}$-instance $\Sigma = (S, T, W, M_0)$, $\not{f}$, where $\not{f} \equiv \&_{j \in J}\, \mathrm{GF}\not{f}_j$, $\not{f}_j$ being conjunctions of literals, can be reduced to another $\mathbb{P}_{\mathrm{temp}}$-instance Σ', $\not{f}'$, where $\not{f}' \equiv \&_{j \in J}\, \mathrm{GF}\not{f}'_j$, $\not{f}'_j$ being conjunctions of literals of the type* $\mathrm{eq}(s, c)$.

Proof. Again, it suffices to show how Σ', $\not{f}'$ with "more simple" $\not{f}'$ can be constructed.

(1) $\mathrm{fi}(t')$ or $\neg\mathrm{fi}(t')$ occurs in $\not{f}$. (We will slightly modify the construction from (3) in the proof of Lemma 4.3.)

To obtain Σ', do the following changes in Σ (cf. Fig. 4):

- add places s_0, s_1, s_2 and put $W(s_1, t') = 1$, $W(s_2, t) = 1$ for all $t \in T\backslash\{t'\}$ and $W(t, s_0) = 1$ for all $t \in T$.
- add transitions t_1, t_2 and put $W(s_0, t_1) = W(s_0, t_2) = 1$, $W(t_1, s_1) = 1$, $W(t_2, s_2) = 1$ (the other values of W are 0).
- put $M_0(s_0) = 1$, $M_0(s_1) = M_0(s_2) = 0$. (First t_1 or t_2 occurs by which it is decided whether t' or some $t \in T\backslash\{t'\}$ can occur next. Any $t \in T$ puts "a token" to s_0, then again t_1 or t_2 occurs, etc.)

To obtain f', replace the occurrences of $\text{fi}(t')$ by $\text{eq}(s_1, 1)$ and the occurrences of $\neg\text{fi}(t')$ by $\text{eq}(s_2, 1)$ in f.

In (2), we can suppose only literals $\text{ge}(s, c)$, $\text{eq}(s, c)$ occurring in f.

(2) $\text{ge}(s, c)$ occurs in f. To obtain Σ', do the following changes in Σ. Add places s_1, s_2, s_3, $M_0(s_1) = 1$, $M_0(s_2) = M_0(s_3) = 0$, and transitions t_1, t_2, t_3 as in Fig. 5 ($W(s_3, t) = W(t, s_1) = 1$ for all $t \in T$). To obtain f', replace the occurrences of $\text{ge}(s, c)$ by $\text{eq}(s, c)$ in f. □

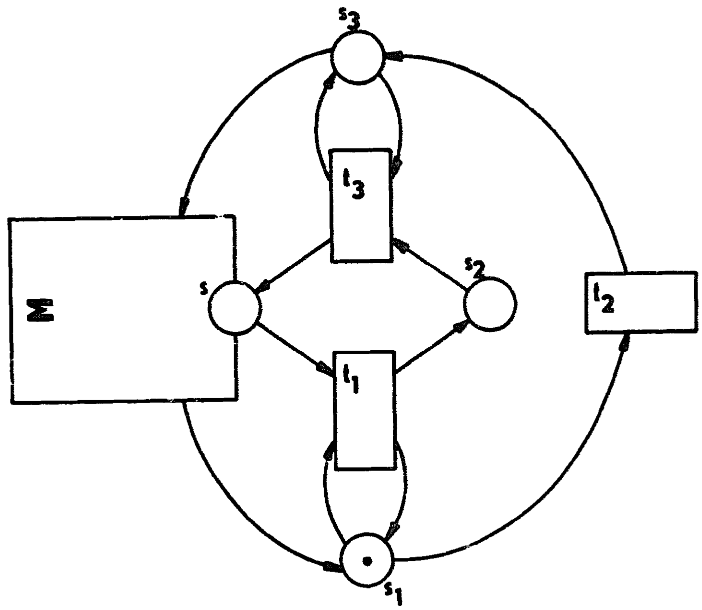

Fig. 5.

Proposition 4.5. *If* $\mathbb{P}_{\text{key}}$ *is decidable then* $\mathbb{P}_{\text{temp}}$ *is decidable.*

Proof. It follows from Lemmas 4.3. and 4.4. and from the considerations at the beginning of this section. □

5. A decidable extension of the reachability problem

Look at the proposition of Lemma 5.2. If $f(M)$ is a finite conjunction of conditions $M(s) = c$, $s \in S$, $c \in \mathbb{N}$, then the problem under discussion is, in fact, $\mathbb{RP}$ and the proposition follows from Theorem 2.4. But we need formulas from L_Σ, which are more general.

Definition 5.1. Let a Petri net $\Sigma = (S, T, W, M_0)$ be given. The *language* L_Σ is the set of the formulas which are defined as follows:

(1) there is one *variable* $\mathcal{M}$ for elements of $\mathbb{N}^S$;

(2) a *term* is either atomic, $\mathcal{M}(s)$ or c, where $s \in S$, $c \in \mathbb{N}$, or of the form $t_1 + t_2$, where t_1, t_2 are terms;

(3) a *formula* is either atomic, $t_1 < t_2$ or $t_1 \leq t_2$, where t_1, t_2 are terms, or of the form $f_1 \,\&\, f_2$, where f_1, f_2 are formulas.

The semantics is natural.

We also use $t_1 = t_2$ by which $t_1 \leq t_2 \,\&\, t_2 \leq t_1$ is abbreviated. For a concrete marking M, $f(M)$ denotes the instance of f in which M is substituted for $\mathcal{M}$.

Lemma 5.2. *There is an algorithm with the following specification:*

Input: A Petri net $\Sigma = (S, T, W, M_0)$ and a formula $f \in L_\Sigma$.

Output: YES if there is a marking M such that $M_0[>M$ and $f(M)$ is true, NO otherwise.

Proof. (i) The case with f as described before Definition 5.1, f is "simple", is obvious.

(ii) In the general case, we shall show how Σ' and $f' \in L_{\Sigma'}$ can be constructed so that the output for Σ', f' is the same as for Σ, f and f' is "more simple" than f.

The way of constructing Σ', f' is shown in the following five points; it will be clear that Case (i) will be achieved by finitely many steps.

(1) Some $c \in \mathbb{N}$ occurs in f: add a new isolated place s_N with $M_0(s_N) = c$ and replace an occurrence of c by $\mathcal{M}(s_N)$ in f.

(2) Some $s \in S$ occurs in f several times: add a new place $\hat{s}$ to Σ and put $M_0(\hat{s}) = M_0(s)$ and $W(\hat{s}, t) = W(s, t)$, $W(t, \hat{s}) = W(t, s)$ for every $t \in T$ ($\hat{s}$ is a duplicate of s); in f, replace an occurrence of $\mathcal{M}(s)$ by $\mathcal{M}(\hat{s})$.

Thus we can suppose in the following points that no $c \in \mathbb{N}$ occurs in f and that every $s \in S$ occurs in f at most once.

(3) A term $t \equiv \mathcal{M}(s_1) + \mathcal{M}(s_2)$ occurs in f: Σ' is shown in Fig. 6; in f, replace the occurrence of t by $\mathcal{M}(s_N)$ and add the conjunction $\mathcal{M}(s_1) = 0 \,\&\, \mathcal{M}(s_2) = 0$.

(4) a formula $g \equiv \mathcal{M}(s_1) < \mathcal{M}(s_2)$, or $g' \equiv \mathcal{M}(s_1) \leq \mathcal{M}(s_2)$, occurs in f: Σ' is shown in Fig. 7; in f, replace g by $\mathcal{M}(s_1) = 0 \,\&\, \mathcal{M}(s_2) = 1$ or g' by $\mathcal{M}(s_1) = 0 \,\&\, \mathcal{M}(s_2) = 0$. $\square$

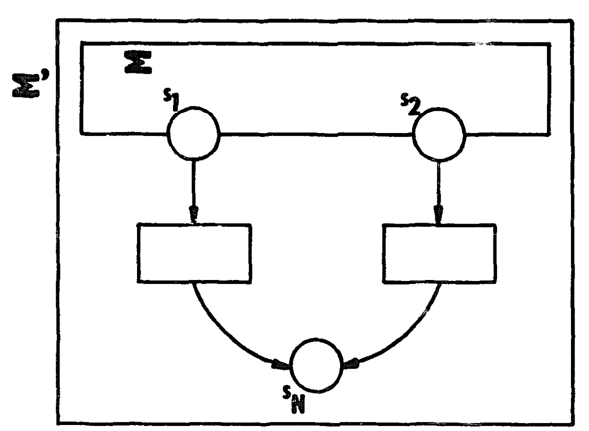

Fig. 6.

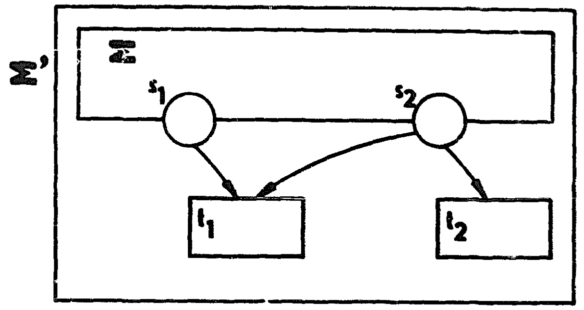

Fig. 7.

We shall still generalize the result by putting some conditions on "passed through" markings.

Definition 5.3. Let a Petri net $\Sigma=(S, T, W, M_0)$ be given. For $k\in\mathbb{N}_+$, the *language* L_Σ^k is defined in the same way as L_Σ, but it contains k variables $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ ($\mathcal{M}_i(s)$ is a term for $i=1,2,\ldots,k$).

Theorem 5.4. *There is an algorithm specified as follows*:

Input: *a Petri net* $\Sigma=(S, T, W, M_0)$ *and a formula* $f\in L_\Sigma^k$ (*for any* $k\in\mathbb{N}_+$).

Output: *YES if there are markings* $M_1, M_2, \ldots, M_k$ *such that* $M_0[>M_1[>M_2\ldots[>M_k$ *and* $f(M_1, M_2, \ldots, M_k)$ *is true, NO otherwise.*

Proof. The case $f\in L_\Sigma^1$ is clear from Lemma 5.2.

The case Σ, $f\in L_\Sigma^{k+1}$ will be reduced to the case Σ', $f'\in L_{\Sigma'}^k$ with the same output in the following way:

- Add a duplicate $\hat{s}$ for every $s\in S$ (as in (2) in the proof of Lemma 5.2.) and denote $\bar{S}=\{\hat{s}\mid s\in S\}$.
- For every $t\in T$, add a new transition $\bar{t}$ and put $W(s,\bar{t})=W(\bar{t},s)=0$, $W(\hat{s},\bar{t})=W(s,t)$, $W(\bar{t},\hat{s})=W(t,s)$ for every $s\in S$. Denote $\bar{T}=\{\bar{t}\mid t\in T\}$. (The transitions from (original) T work both on original and on duplicate places, the transitions from $\bar{T}$ work on duplicate places only.)
- Now add a place s_R putting $M_0(s_R)=1$, $W(s_R,t)=W(t,s_R)=\alpha$, where α is 1 for $t\in T$ and 0 for $t\in\bar{T}$, and a place s'_R putting $M_0(s'_R)=0$, $W(s'_R,t)=W(t,s'_R)=\beta$, where β is 0 for $t\in T$ and 1 for $t\in\bar{T}$. (s_R is a "run-place" for T, s'_R a "run-place" for $\bar{T}$).
- Finally add t_R as shown in Fig. 8; thereby the construction of Σ' is completed.

To obtain $f'\in L_{\Sigma'}^k$, do the following changes in f:

- add the conjunction

$$\mathcal{M}_1(s_R)=1\ \&\ \mathcal{M}_2(s_R)=1\ \&\ \cdots\ \&\ \mathcal{M}_{k-1}(s_R)=1\ \&\ \mathcal{M}_k(s_R)=0,$$

- replace every occurrence of $\mathcal{M}_{k+1}(s)$ by $\mathcal{M}_k(\hat{s})$. □

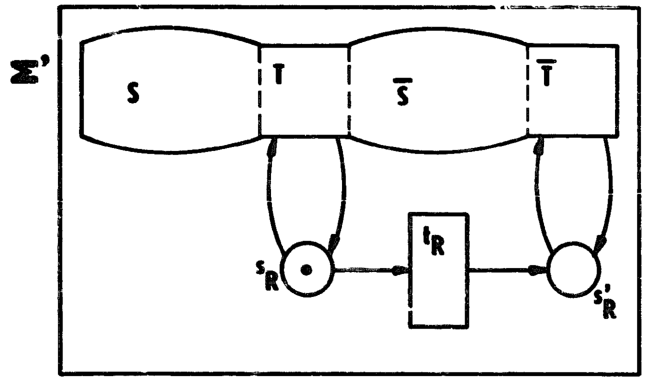

Fig. 8.

6. A special ordering of occurrence sequences

Recall the following well-known proposition.

Proposition 6.1. *Every infinite sequence of elements of* $\mathbb{N}^r$ $(r \in \mathbb{N}_+)$ *has an infinite ascending subsequence* ($\leqslant$ *being considered componentwise*).

Proof. By induction on r. $\square$

We shall use a nontrivial generalization for quasi-ordered sets (in such a set, $a \leqslant a$ and $a \leqslant b \;\&\; b \leqslant c \Rightarrow a \leqslant c$ hold for all a, b, c).

Definition 6.2. A quasi-ordered set $(A, \leqslant)$ has the *finite basis property*, the *fbp*, if every infinite sequence of elements of A has an infinite ascending subsequence.

For a quasi-ordered set $(A, \leqslant)$, let A^* be quasi-ordered in the following way: for $u, v \in A^*$, $u = a_1 a_2 \ldots a_m$ $(a_i \in A)$, $u \leqslant v$ holds iff v can be written $v = v_1 b_1 v_2 b_2 v_3 \ldots b_m v_{m+1}$ so that $b_i \in A$ and $a_i \leqslant b_i$ for $i = 1, 2, \ldots, m$.

Proposition 6.3. *If* $(A, \leqslant)$ *has the fbp then* $(A^*, \leqslant)$ *also has the fbp.*

Proof. In [3] as a corollary of a more general theorem. $\square$

For a Petri net, we shall show a quasi-order with the fbp on the set of couples (M, u), where M enables u.

Definition 6.4. Let a Petri net be given. Let us define the following *relation* $\leqslant$: for any markings M_1, M_2 and any finite sequences of transitions u_1, u_2 such that $M_1[u_1\rangle$, $M_2[u_2\rangle$, $(M_1, u_1) \leqslant (M_2, u_2)$ iff, for $u_1 = t_1 t_2 \ldots t_m$ (t_i being transitions), u_2 can be written $u_2 = w_1 t_1 w_2 t_2 w_3 \ldots t_m w_{m+1}$, where $M_2 - M_1 + \Delta(w_1 w_2 \ldots w_c) \geqslant \mathbf{0}$ for $c = 0, 1, 2, \ldots, m+1$.

Thus $(M_1, u_1) \leqslant (M_2, u_2)$ means that $M_1 \leqslant M_2$, $M_1 + \Delta(u_1) \leqslant M_2 + \Delta(u_2)$ and $M_1 + \Delta(t_1 t_2 \ldots t_c) \leqslant M_2 + \Delta(w_1 t_1 w_2 t_2 \ldots w_c t_c)$ for $c = 1, 2, \ldots, m$. It is easy to verify that $\leqslant$ is a quasi-order.

The next theorem gives the main result of this section.

Theorem 6.5. *For any Petri net* $\Sigma = (S, T, W, M_0)$, *the set* $\mathbb{M} = \{(M, u) \mid M[u\rangle\}$ *has the fbp in* $\leqslant$.

Proof. First notice that $(\mathbb{N}^r)^*$, $r \in \mathbb{N}_+$, has the fbp (cf. Propositions 6.1 and 6.3).

We can define a one-to-one map EMS of $\mathbb{M}$ into $(\mathbb{N}^r)^*$, where $r = |S| + |T| + 2$, in the following way:

- take the marking sequence corresponding to (M, u);
- extend each element of this sequence by $|T|$ components; the component corresponding to the transition just occurred will be equal to 1, the others will be equal to 0;
- add a special end-element which contains the last marking but all "the transition components" are equal to 0;
- extend each element by another two components, one indicating the begin-element and the other indicating the end-element.

An example makes it clear:

$$\begin{aligned}&\mathrm{EMS}[(M, t_1 t_2)]\\ &\quad = [M, 0, \ldots, 0, \underset{\substack{\uparrow\\ \text{Begin}}}{1}, 0][M + \Delta(t_1), 0, \ldots, 0, \underset{\substack{\uparrow\\ t_1}}{1}, 0, \ldots, 0, 0, 0]\\ &\quad \times [M + \Delta(t_1 t_2), 0, \ldots, 0, \overset{\substack{t_2\\ \downarrow}}{1}, 0, \ldots, 0, 0, 0][M + \Delta(t_1 t_2), 0, \ldots, 0, 0, \overset{\substack{\text{End}\\ \downarrow}}{1}].\end{aligned}$$

The following equivalence is easy to verify:

$$(M_1, u_1) \leqslant (M_2, u_2) \text{ iff } \mathrm{EMS}[(M_1, u_1)] \leqslant \mathrm{EMS}[(M_2, u_2)].$$

Hence the theorem is obvious. □

Later we shall use Lemma 6.7 for which Lemma 6.6 is needed.

Lemma 6.6. *Suppose* $(M, u) \leqslant (M', u')$ *for a given Petri net. For any "partition"* $u = u_1 u_2 \ldots u_m$ *of* u, u' *can be written* $u' = v_1 v_2 \ldots v_m$ *where* $M + \Delta(u_1 u_2 \ldots u_c) \leqslant M' + \Delta(v_1 v_2 \ldots v_c)$ *for* $c = 0, 1, 2, \ldots, m$.

Proof. It follows from the definition of $\leqslant$ in an obvious way. □

Lemma 6.7. *Let a Petri net* $\Sigma = (S, T, W, M_0)$ *be given. Suppose* $(M_1, u_1) \leqslant (M_2, u_2) \leqslant (M_3, u_3)$ *and* $M_i[u_i\rangle M_i'$ *for* $i = 1, 2, 3$. *Then there is* $v \in T^*$ *such that* $M_3 + (M_2 - M_1)[v\rangle M_3' + (M_2' - M_1')$ *and* $(M_3, u_3) \leqslant (M_3 + (M_2 - M_1), v)$.

Proof. Suppose $u_1 = t_1 t_2 \ldots t_m$, $u_2 = w_1 t_1 w_2 t_2 w_3 \ldots t_m w_{m+1}$ where $M_2 - M_1 + \Delta(w_1 w_2 \ldots w_c) \geq 0$ for $c = 0, 1, \ldots, m+1$. Due to the previous lemma, we can write $u_3 = v_1 v_2 \ldots v_{m+1}$ where $M_2 + \Delta(w_1 t_1 w_2 t_2 \ldots w_c t_c) \leq M_3 + \Delta(v_1 v_2 \ldots v_c)$ for $c = 0, 1, \ldots, m$. It can be easily verified that $v = w_1 v_1 w_2 v_2 \ldots w_{m+1} v_{m+1}$ proves the lemma. □

7. Decidability of $\mathbb{P}_{\text{key}}$

We use the following technical convention in this section.

Convention 7.1. We will assume implicitly that every Petri net $\Sigma = (S, T, W, M_0)$ has a special *step counter place*, denoted by s_{cnt}, where $M_0(s_{\text{cnt}}) = 0$ and $W(s_{\text{cnt}}, t) = 0$, $W(t, s_{\text{cnt}}) = 1$ for every $t \in T$.

We can immediately establish a result corresponding to Carstensen's result 2.5. (2).

Proposition 7.2. $\mathbb{P}_{\text{key}}$ *is decidable in case* $n = 1$ (*there is* one (S_1, f_1) *only*).

Proof. In such a case, a solution σ exists iff there is a finite sequence $\sigma' = {}_{M_0}\, u_1\, {}_{M_1}\, u_2\, {}_{M_2}$ such that

$$[M_1 \restriction S_1 = f_1 \;\&\; M_2 \restriction S_1 = f_1] \;\&\; [M_1 \leq M_2 \;\&\; M_1(s_{\text{cnt}}) < M_2(s_{\text{cnt}})].$$

(σ implies σ' due to Proposition 6.1, σ' implies a solution $\sigma_T = u_1(u_2)^\omega$.)

The existence of such σ' is decidable due to Theorem 5.4. □

In what follows we implicitly assume $n \geq 2$ in $\mathbb{P}_{\text{key}}$-instances. Now we show a more detailed structure of $\mathbb{P}_{\text{key}}$-solutions.

Definition 7.3. Let $\Sigma = (S, T, W, M_0)$, $(S_1, f_1), \ldots, (S_n, f_n)$ be a $\mathbb{P}_{\text{key}}$-instance. A (finite) occurrence sequence σ in the form

$$\sigma = {}_{M_0}\, u_0\, {}_{M_0^1}\, u^1\, {}_{M_n^1}\, w^1\, {}_{M_0^2}\, u^2\, {}_{M_n^2}\, w^2\, {}_{M_0^3} \cdots {}_{M_0^k}\, u^k\, {}_{M_n^k}\, w^k\, {}_{M_0^{k+1}},$$

where every subsequence ${}_{M_0^i}\, u^i\, {}_{M_n^i}$ can be written in more detail ${}_{M_0^i}\, u_1^i\, {}_{M_1^i}\, u_2^i\, {}_{M_2^i} \ldots u_n^i\, {}_{M_n^i}$, is *k-good* if the following conditions hold for all (relevant) i, j:

(1) $M_j^i \restriction S_j = f_j$ (i.e. $M_j^1 \restriction S_j = M_j^2 \restriction S_j = \cdots = M_j^k \restriction S_j = f_j$ for $1 \leq j \leq n$) and $M_0^i \restriction S_n = f_n$;

(2) $\Delta(u^i) \geq 0$, $\Delta(w^i) \geq 0$ (i.e. $M_0^1 \leq M_n^1 \leq M_0^2 \leq M_n^2 \leq \cdots \leq M_0^k \leq M_n^k \leq M_0^{k+1}$);

(3) $M_j^{i-1} \leq M_j^i$ (i.e. $M_j^1 \leq M_j^2 \leq M_j^3 \leq \cdots \leq M_j^k$ also for $1 \leq j \leq n-1$) and $M_j^{i-1}(s_{\text{cnt}}) < M_j^i(s_{\text{cnt}})$ (the "sector from M_j^{i-1} to M_j^i" is not empty).

We will say that a k-good sequence σ *satisfies INSERT*, σ is a *k-good INSERT sequence*, if in addition $(M_{j-1}^i, u_j^i) \lessapprox (M_{j-1}^{i+1}, u_j^{i+1})$ for $i = 1, 2, \ldots, k-1$, $j = 1, 2, \ldots, n$.

In a natural way, the definition can be extended for the case $k=\omega$ yielding the notions of an *ω-good sequence* and an *ω-good INSERT sequence.*

Notation 7.4. For technical convenience, we write $\text{sect}(i,j)$ for $u^{i-1}_{j+1}u^{i-1}_{j+2}\ldots u^{i-1}_{n}w^{i-1}u^{i}_{1}u^{i}_{2}\ldots u^{i}_{j}$ (i.e. for "sector from M^{i-1}_{j} to M^{i}_{j}"). Condition (3) in Definition 7.3. means that $\Delta[\text{sect}(i,j)]\geq\mathbf{0}$ and $\Delta[\text{sect}(i,j)](s_{\text{cnt}})>0$.

Remark 7.5. We will often suppose $w^{i}=\varepsilon$ for all i ($M^{i}_{n}=M^{i+1}_{0}$ in such a case). Considering a k-good (ω-good) sequence, we can always obtain the case $w^{i}=\varepsilon$ by the notation change $(u^{i+1}_{1})_{\text{NEW}}=(w^{i}u^{i+1}_{1})_{\text{OLD}}$; of course, the condition INSERT may be affected thereby. Doing a notation change $(w^{i})_{\text{NEW}}=(w^{i}u^{i+1}w^{i+1})_{\text{OLD}}$ for some i, we obtain a $(k-1)$-good sequence (another form of an ω-good sequence, respectively).

Lemma 7.6. *Any ω-good sequence can be rewritten into a form of an ω-good INSERT sequence.*

Proof. Because of Theorem 6.5, any infinite sequence of elements of the set $\{(M,u)\mid M[u\rangle\}$ has an infinite subsequence ascending in $\preccurlyeq$. The lemma can be proved by applying this fact n-times (and using the last notation change in the previous remark). □

Proposition 7.7. *Let a $\mathbb{P}_{\text{key}}$-instance $\Sigma=(S,T,W,M_0),(S_1,f_1),\ldots,(S_n,f_n)$ be given. The solutions of the $\mathbb{P}_{\text{key}}$-instance are exactly the ω-*good *INSERT sequences.*

Proof. It is obvious that an ω-good sequence is a solution.

(*) Now let $\sigma={}_{M_0}t_1\,{}_{M_1}t_2\,{}_{M_2}\ldots$ be a solution of the $\mathbb{P}_{\text{key}}$-instance. Due to the previous lemma it suffices to show that σ can be written in a form of an ω-good sequence.

Take any j, $1\leq j\leq n$. From Proposition 6.1. it is clear that there is an infinite subsequence φ_j of the sequence $M_0,M_1,M_2,\ldots$, where $M\restriction S_j=f_j$ for every member M of φ_j and φ_j is ascending (in the component s_{cnt} strictly ascending).

It is easy to establish (*) using sequences $\varphi_1,\varphi_2,\ldots,\varphi_n$. □

A special 3-good sequence will be called regular.

Definition 7.8. Let a $\mathbb{P}_{\text{key}}$-instance $\Sigma=(S,T,W,M_0),(S_1,f_1),\ldots,(S_n,f_n)$ be given. A 3-good sequence σ in the form

$$\sigma={}_{M_0}u_0\,{}_{M^1_0}u^1_1\,{}_{M^1_1}\ldots u^1_n\,{}_{M^1_n}u^2_1\,{}_{M^2_1}\ldots u^2_n\,{}_{M^2_n}u^3_1\,{}_{M^3_1}\ldots u^3_n\,{}_{M^3_n}$$

($w^{i}=\varepsilon$ for $i=1,2,3$) will be called a *regular sequence* if

$$(\forall j,1\leq j\leq n)\;(\exists k_j\in\mathbb{N}_+)\quad[M^3_j-M^2_j=k_j(M^2_j-M^1_j)].$$

Our next aim will be to prove that the existence of a k-good INSERT sequence, for sufficiently large k, implies the existence of a regular sequence. It is the most technical part of the whole proof.

First we define technical notions R_j^i and $\mathrm{MUL}(i,j)$.

Definition 7.9. Consider a k-good sequence σ as in Definition 7.3 (we can suppose $w^i=\varepsilon$ for all i).

(1) The *functions* (*vectors*) $R_j^i \in (\mathbb{Q}_+)^S$ ($2\leqslant i\leqslant k$, $1\leqslant j\leqslant n$) will be defined in the following way: for any $s\in S$,

$$R_j^i(s)=\frac{[M_j^i-M_j^{i-1}](s)}{[M_j^i-M_j^{i-1}](s_{cnt})}=\frac{[\Delta(\mathrm{sect}(i,j))](s)}{[\Delta(\mathrm{sect}(i,j))](s_{cnt})}.$$

(2) The *predicate* $\mathrm{MUL}(i,j)$ ($3\leqslant i\leqslant k$, $1\leqslant j\leqslant n$) will be defined in the following way:

$$\mathrm{MUL}(i,j)\underset{\mathrm{df}}{\Leftrightarrow}(\exists p\in\mathbb{N}_+)\{(M_j^i-M_j^{i-1})(s_{\mathrm{cnt}})=p[(M_j^{i-1}-M_j^{i-2})(s_{\mathrm{cnt}})]\}.$$

Remark 7.10. Notice that a 3-good sequence with $w^i=\varepsilon$ is regular iff $\mathrm{MUL}(3,j)$ and $R_j^2=R_j^3$ for $j=1,2,\ldots,n$.

We will use certain special k-good sequences (called "relatives"); for them, we will show certain modifications which keep the special k-goodness conditions and change R_j^i (and are able to establish $\mathrm{MUL}(i,j)$). We need the next two lemmas.

Notation 7.11. Let σ be a k-good sequence as in Definition 7.3. For any (relevant) i, j and any $r_1,r_2,\ldots,r_{i-1}\in\mathbb{N}$, $M_j^i(r_1,r_2,\ldots,r_{i-1})$ stands for $M_j^i+r_1(M_j^2-M_j^1)+r_2(M_j^3-M_j^2)+\cdots+r_{i-1}(M_j^i-M_j^{i-1})$.

Lemma 7.12. *Let σ be a k-good INSERT sequence as in Definition 7.3. For any i,j, $1\leqslant i\leqslant k$, $0\leqslant j\leqslant n-1$ and any $r_1,r_2,\ldots,r_{i-1}\in\mathbb{N}$, there is $u\in T^*$ such that $M_j^i(r_1,r_2,\ldots,r_{i-1})[u\rangle M_{j+1}^i(r_1,r_2,\ldots,r_{i-1})$.*

Proof. We show in addition that u can be chosen so that $(M_j^i,u_{j+1}^i)\leqslant(M_j^i(r_1,r_2,\ldots,r_{i-1}),u)$. The proof is made by induction on the sum $c=\sum_{p=1}^{i-1}r_p$ (assuming fixed i,j).

(i) In case of $c=0$, put $u=u_{j+1}^i$.

(ii) In the induction step, denote

$$M=M_j^i(r_1,r_2,\ldots,r_{i-1}),\qquad M'=M_{j+1}^i(r_1,r_2,\ldots,r_{i-1})$$

and suppose

$$M[u\rangle M',(M_j^i,u_{j+1}^i)\leqslant(M,u) \tag{2}$$

The proposition will be shown for

$$M_j^i(r_1,\ldots,r_{d-1},1+r_d,r_{d+1},\ldots,r_{i-1})=M+(M_j^{d+1}-M_j^d),$$

where $1\leqslant d\leqslant i-1$.

We have $(M_j^d, u_{j+1}^d) \leqslant (M_j^{d+1}, u_{j+1}^{d+1}) \leqslant (M, u)$ due to INSERT and (2). Because of Lemma 6.7, there is $v \in T^*$ such that

$$M + (M_j^{d+1} - M_j^d)[v\rangle M' + (M_{j+1}^{d+1} - M_{j+1}^d)$$

and

$$(M, u) \leqslant (M + (M_j^{d+1} - M_j^d), v).$$

Hence

$$M_j^i(r_1, \ldots, r_{d-1}, 1 + r_d, r_{d+1}, \ldots, r_{i-1})[v\rangle$$
$$\rangle M_{j+1}^i(r_1, \ldots, r_{d-1}, 1 + r_d, r_{d+1}, \ldots, r_{i-1})$$

and

$$(M_j^i, u_{j+1}^i) \leqslant [M_j^i(r_1, \ldots, r_{d-1}, 1 + r_d, r_{d+1}, \ldots, r_{i-1}), v]$$

by which the proof is finished. □

The next lemma adds the case $j = n$ to the previous one (we do not need INSERT in this case).

Lemma 7.13. *Let σ be a k-good sequence as in Definition 7.3. For any i, $1 \leqslant i \leqslant k$, and any $r_1, r_2, \ldots, r_{i-1} \in \mathbb{N}$, there is $w \in T^*$ such that*

$$M_n^i(r_1, r_2, \ldots, r_{i-1})[w\rangle M_0^{i+1}(r_1', r_2', \ldots, r_{i-1}', r_i'),$$

where $r_p \leqslant r_p'$ for $p = 1, 2, \ldots, i-1$.

Proof. (i) The case $\sum_{p=1}^{i-1} r_p = 0$ is clear (put $w = w^i$).

(ii) Suppose $M_n^i(r_1, r_2, \ldots, r_{i-1})[w\rangle M_0^{i+1}(r_1', r_2', \ldots, r_{i-1}', r_i')$. It is easy to verify that

$$M_n^i(r_1, r_2, \ldots, r_{i-1}) + (M_n^{d+1} - M_n^d)[wu^d w^{d+1}\rangle M,$$

where

$$\begin{aligned} M &= M_0^{i+1}(r_1', r_2', \ldots, r_{i-1}', r_i') + M_n^{d+1} - M_n^d + M_n^d - M_0^d + M_0^{d+2} - M_n^{d+1} \\ &= M_0^{i+1}(r_1', \ldots, r_{d-1}', 1 + r_d', 1 + r_{d+1}', r_{d+2}', \ldots, r_{i-1}'). \quad \square \end{aligned}$$

Now we define the notion of a "relative"; it is a k-good sequence which does not need to satisfy INSERT but which is closely related to some k-good INSERT sequence.

Definition 7.14. Let σ be a k-good INSERT sequence exactly as in Definition 7.3. Let σ' be an occurrence sequence in the form

$$\sigma' = {}_{M_0} u_0 {}_{M_0^1} u^1 {}_{M_n^1} \underline{u}^2 {}_{M_n^2} \underline{u}^3 {}_{M_n^3} \ldots \underline{u}^k {}_{M_n^k},$$

where every subsequence ${}_{M_0^i} \underline{u}^i {}_{M_n^i}$ can be written in more detail

$${}_{M_0^i} \underline{u}_1^i {}_{M_1^i} \underline{u}_2^i {}_{M_2^i} \ldots \underline{u}_n^i {}_{M_n^i}.$$

(Hence σ' has the same prefix as σ; of course, $M_n^1 = \underline{M}_n^1$).

σ' is a *relative of* σ if the following conditions hold:

(1) for all i, j, $1 \leq i \leq k$, $1 \leq j \leq n$, there are parameters $r_1, r_2, \ldots, r_{i-1}$ such that $\underline{M}_j^i = M_j^i(r_1, r_2, \ldots, r_{i-1})$;

(2) the parameters in (1) can be chosen so that if $(i < i') \vee (i = i' \,\&\, j < j')$ then $r_p \leq r_p'$ for $p = 1, 2, \ldots, i-1$ (r_p being the parameters belonging to i, j, r_p' to i', j').

Remark 7.15. It is easy to verify that a relative of σ is also a k-good sequence (where INSERT is not necessary and $w^i = \varepsilon$ for all i). Notice that any k-good INSERT sequence can be rewritten into a form of its relative by the notation change $(u_1^i)_{\mathrm{NEW}} = (w^{i-1} u_1^i)_{\mathrm{OLD}}$ for all $i \geq 2$.

Now we are ready to define the modifications mentioned before Notation 7.11. They use linear combinations of the "previous relevant sectors" in order to change R_j^i (and establish MUL(i, j)).

Definition 7.16. Let σ and its relative σ' be exactly as in Definition 7.14. An *i, j-modification* of σ' ($2 \leq i \leq k$, $1 \leq j \leq n$) consists of two steps:
(1) replacing $\underline{u}_j^i$ by

$$u_j^i = \underline{u}_j^i [\mathrm{sect}(2, j)]^{k_2} [\mathrm{sect}(3, j)]^{k_3} \ldots [\mathrm{sect}(i, j)]^{k_i}$$

for some $k_2, k_3, \ldots, k_i \in \mathbb{N}$ (the sectors from σ' are meant; $\underline{M}_j^i$ changes appropriately, but conditions from Definition 7.14. remain holding for it).
(2) a successive replacing of

$$\underline{u}_{j+1}^i, \underline{u}_{j+2}^i, \ldots, \underline{u}_n^i, \underline{u}_1^{i+1}, \underline{u}_2^{i+1}, \ldots, \underline{u}_n^{i+1}, \ldots, \underline{u}_1^k, \underline{u}_2^k, \ldots, \underline{u}_n^k$$

(and the appropriate changing of $\underline{M}_{j+1}^i, \ldots, \underline{M}_n^k$) in such a way that the new sequence is also a relative of σ.

Lemma 7.17. *For any i, j, $2 \leq i \leq k$, $1 \leq j \leq n$, and any $k_2, k_3, \ldots, k_i \in \mathbb{N}$, there is the i, j-modification described above.*

Proof. The correctness of Step (1) follows from the k-good sequence properties (a relative is k-good); the "new" $\underline{M}_j^i$ (i.e. M_j^i) is obviously equal to $M_j^i(r_1, r_2, \ldots, r_{i-1})$ for some parameters which are equal or greater than the "old" ones.

For Step (2), we can use Lemmas 7.12 and 7.13. If, during the successive replacing, we are to replace some $\underline{u}_j^{i'}$, where $2 \leq j \leq n$, we use u ensured by Lemma 7.12. If we are to replace some $\underline{u}_1^{i'}$, we use wu, where w is ensured by Lemma 7.13. and u by Lemma 7.12.

By such a successive replacing "from left to right", we really get another relative σ'' of (the original) σ. □

Remark 7.18. In fact, only i, j-modifications with at most two k_p not equal to zero will be used. Notice that any i, j-modification does not affect the prefix before $\underline{u}_j$ ($\underline{u}_j^i$ is changed in the defined way; as regards the suffix, we need only the correctness).

Now we show how $\mathrm{MUL}(i, j)$ can be achieved without affecting R_j^i.

Lemma 7.19. *Let a relative σ (of a k-good INSERT sequence) be given. For any fixed i, j, there is an i, j-modification after which $MUL(i, j)$ holds and R_j^i remains unchanged.*

Proof. It suffices to perform an i, j-modification which starts by replacing u_j^i by $u_j^i[\mathrm{sect}(i, j)]^r$ where $r = [M_j^{i-1} - M_j^{i-2}](s_{\mathrm{cnt}}) - 1$. □

A crucial point in changing R_j^i is Lemma 7.21; we need the following simple fact.

Lemma 7.20. *For any $m_1, m_2 \in \mathbb{N}$, $n_1, n_2 \in \mathbb{N}_+$, $q \in \mathbb{Q}_+$, where $m_1/n_1 < q < m_2/n_2$, there are $x_1, x_2 \in \mathbb{N}_+$ such that $(m_1x_1 + m_2x_2)/(n_1x_1 + n_2x_2) = q$.*

Proof. Let $q = c_1/c_2$ for $c_1, c_2 \in \mathbb{N}_+$. We want to solve the equation $c_2m_1x_1 + c_2m_2x_2 = c_1n_1x_1 + c_1n_2x_2$, i.e. $(c_2m_2 - c_1n_2)x_2 = (c_1n_1 - c_2m_1)x_1$. Due to $c_2m_1 < c_1n_1$, $c_1n_2 < c_2m_2$, a positive solution is clear. □

Lemma 7.21. *Let a relative σ (of a k-good INSERT sequence) be given. Let $S' \subseteq S$, $s' \in S \setminus S'$, $2 \leq i' < i'' < i \leq k$, $1 \leq j \leq n$ and*

$$R_j^{i'} \upharpoonright S' = R_j^{i''} \upharpoonright S' = R_j^i \upharpoonright S'. \tag{3}$$

In addition, let $q \in \mathbb{Q}_+$ be strictly between $R_j^{i'}(s')$ and $R_j^{i''}(s')$. Then it can be achieved by an i, j-modification that $R_j^i(s') = q$ and $R_j^i \upharpoonright S'$ remains the same.

Proof. If $R_j^i(s') \neq q$ then q is strictly between $R_j^{i'}(s')$ and $R_j^i(s')$ or strictly between $R_j^{i''}(s')$ and $R_j^i(s')$; suppose the first alternative (the second being similar).

Due to the previous lemma there are x_1, x_2 such that the following holds: performing an i, j-modification which starts by replacing u_j^i by $u_j^i[\mathrm{sect}(i', j)]^{x_1}[\mathrm{sect}(i, j)]^{x_2 - 1}$, we obtain $R_j^i(s') = q$ and $R_j^i \upharpoonright S'$ remains the same due to (3). □

Now we can reach the aim defined after Definition 7.8.

Proposition 7.22. *Let a $\mathbb{P}_{\text{key}}$-instance $\Sigma = (S, T, W, M_0), (S_1, f_1), \ldots, (S_n, f_n)$ be given. If there is a k-good INSERT sequence for $k = 1 + 2^c$, where $c = n|S| + 1$, then there is also a regular sequence.*

Proof. Let σ be a k-good INSERT sequence. Change the notation as in Remark 7.15 to obtain σ in a form of its relative. It suffices to show a series of modifications which will ensure that

$$(\exists m, 1 \leq m \leq k-2) \quad (\forall j, 1 \leq j \leq n) \quad [R_j^{m+1} = R_j^{m+2} \;\&\; \mathrm{MUL}(m+2, j)] \quad (4)$$

(cf. Remark 7.10; the exact form of a regular sequence will then be obtained by notation changes).

We shall show the following proposition P(r) for $r = 0, 1, \ldots, n|S|$ by induction. P($n|S|$) will imply (4) by which the proof will be finished.

Proposition 7.23 (P(r)). *There are*
- *a relative σ_r of σ,*
- *$m_r \in \mathbb{N}_+$, $1 \leq m_r \leq k - 2^{c-r}$ ($c = n|S| + 1$),*
- *two disjoint sets $M_r, N_r \subseteq S \times \{1, 2, \ldots, n\}$, where $|M_r \cup N_r| = r$ and some $q_j(s) \in \mathbb{Q}_+$ is given for every $(s, j) \in M_r$*

meeting the following two conditions (we write range$_r$ instead of $\{m_r + 1, m_r + 2, \ldots, m_r + 2^{c-r}\}$):

(1) *suppose* any *series of* any *i, j-modifications of σ_r,* i moving in range$_r$ only, *has been performed. Now for every pair i', j', $i' \in$ range$_r$, we can achieve by several i', j'-modifications that*

$$(\forall s \in S) \quad [(s, j') \in M_r \Rightarrow R_{j'}^{i'}(s) = q_{j'}(s)].$$

(2) *Again, suppose* any *series of* any *i, j-modifications of σ_r,* i moving in range$_r$ only, *has been performed. For any i', i'', j', i', $i'' \in$ range$_r$, if*

$$(\forall s \in S) \quad [(s, j') \in M_r \Rightarrow R_{j'}^{i'}(s) = R_{j'}^{i''}(s) = q_{j'}(s)]$$

holds, then

$$(\forall s \in S) \quad [(s, j') \in N_r \Rightarrow R_{j'}^{i'}(s) = R_{j'}^{i''}(s)]$$

is also ensured.

P(0) is obvious: take σ in the form of its relative as σ_0 and put $m_0 = 1$, $M_0 = N_0 = \emptyset$.

We shall show P(r)$\Rightarrow$P($r+1$) for $r < n|S|$. FH(range$_r$) will denote the first half of range$_r$, i.e. $\{m_r + 1, m_r + 2, \ldots, m_r + 2^{c-r-1}\}$, SH(range$_r$) the second half similarly. Take an arbitrary $(s', j') \notin M_r \cup N_r$. There are two possibilities, (a) and (b), only.

(a) After performing any series of any i, j-modifications of σ_r, i moving in FH(range$_r$) only, the following holds: for any i', $i'' \in$ FH(range$_r$), if

$$(\forall s \in S) \quad [(s, j') \in M_r \Rightarrow R_{j'}^{i'}(s) = R_{j'}^{i''}(s) = q_{j'}(s)]$$

holds then $R_{j'}^{i'}(s') = R_{j'}^{i''}(s')$ is also ensured. In such a case, we can put $\sigma_{r+1} = \sigma_r$, $m_{r+1} = m_r$, $M_{r+1} = M_r$, $N_{r+1} = N_r \cup \{(s', j')\}$ and $P(r+1)$ is clear ($\text{range}_{r+1} = \text{FH}(\text{range}_r)$, of course).

(b) There is a series φ of i, j-modifications of σ_r, where i moves in $\text{FH}(\text{range}_r)$ only, after the performing of which we have for some $i', i'' \in \text{FH}(\text{range}_r)$:

$$(\forall s \in S) \quad [(s, j') \in M_r \Rightarrow R_{j'}^{i'}(s) = R_{j'}^{i''}(s) = q_{j'}(s)]$$

and $R_{j'}^{i'}(s') \neq R_{j'}^{i''}(s')$.

In such a case, take σ_r modified by φ as σ_{r+1} and put $m_{r+1} = m_r + 2^{c-r-1}$, $N_{r+1} = N_r$, $M_{r+1} = M_r \cup \{(s', j')\}$ choosing $q_{j'}(s')$ strictly between $R_{j'}^{i'}(s')$ and $R_{j'}^{i''}(s')$.

$P(r+1)$ can then be verified by help of $P(r)$ and Lemma 7.21 (in this case, $\text{range}_{r+1} = \text{SH}(\text{range}_r)$).

In the end, we show $P(n|S|) \Rightarrow (4)$. Take σ_r and put $m = m_r$ for $r = n|S|$ ($\text{range}_r = \{m+1, m+2\}$). The i', j'-modifications ensured by (1) of $P(n|S|)$ can be performed successively for

$$(i', j') = (m+1, 1), (m+1, 2), \ldots, (m+1, n), (m+2, 1),$$
$$(m+2, 2), \ldots, (m+2, n);$$

in case $i' = m+2$, we always add the (i', j')-modification ensured by Lemma 7.19. Thereby (4) is established. □

Now we will show that a regular sequence can be lengthened to infinity in the "regular" way whereby an ω-good STRICT INSERT sequence arises.

Definition 7.24. Let σ be a k-good sequence as in Definition 7.3. σ will be called a *k-good STRICT INSERT sequence* if

(1) $w^i = \varepsilon$ for all i, and

(2) for any i, j, $2 \leq i \leq k$, $1 \leq j \leq n$, $u_j^i = u_j^{i-1} v$ for some v.

An *ω-good STRICT INSERT sequence* is defined in the same way.

It is obvious that a k-good (ω-good) STRICT INSERT sequence satisfies INSERT (i.e. the above notion is correct).

Proposition 7.25. *Let a $\mathbb{P}_{\text{key}}$-instance be given. If there is a regular sequence then there is also an ω-good STRICT INSERT sequence.*

Proof. Suppose σ as in Definition 7.8 and take some $x \in \mathbb{N}$ such that $k_n + x = ck_1$ for some $c \in \mathbb{N}$. Notice that

$$\Delta(u_1^3 u_2^1 u_3^1 \ldots u_n^1) = (M_1^3 - M_n^2) + (M_n^1 - M_1^1) = -(M_n^2 - M_n^1) + (M_1^3 - M_1^1)$$

and recall $M_n^3 = M_n^2 + k_n(M_n^2 - M_n^1)$.

Now we can easily verify what follows.

$$M_n^3[(u_1^3 u_2^1 u_3^1 \dots u_n^1)^{k_n}\rangle M_n^2 + k_n(M_1^3 - M_1^1)[u_1^3\rangle M_1^3 + k_n(M_1^3 - M_1^1)$$
$$\times [(u_2^1 u_3^1 \dots u_n^1 u_1^2)^x\rangle M',$$

where

$$\begin{aligned} M' &= M_1^3 + k_n(M_1^3 - M_1^2) + k_n(M_1^2 - M_1^1) + x(M_1^2 - M_1^1) \\ &= M_1^3 + k_n(M_1^3 - M_1^2) + ck_1(M_1^2 - M_1^1) \\ &= M_1^3 + (k_n + c)(M_1^3 - M_1^2) \underset{\mathrm{df}}{=} M_1^4. \end{aligned}$$

Thus we have lengthened σ by a certain $u_1^4 = u_1^3 v$ in the "regular way". This way can be continued to infinity. Then the notation change $(u^i)_{\mathrm{NEW}} = (u^{i+2})_{\mathrm{OLD}}$ finishes the proof. □

Proposition 7.26. *There is an algorithm specified as follows:*

Input: a $\mathbb{P}_{\mathrm{key}}$*-instance* $\Sigma = (S, T, W, M_0), (S_1, f_1), (S_2, f_2), \dots, (S_n, f_n)$.

Output: YES if there is a k-good STRICT INSERT sequence for $k = 1 + 2^c$ *where* $c = n|S| + 1$, *NO otherwise.*

Proof. In a straightforward way, the conditions satisfied by a k-good STRICT INSERT sequence can be described with a formula from L_Σ^m (defined in Definition 5.3.) for some m depending on k. The only difficulty is to describe the condition $u_j^i = u_j^{i-1} v$ (for some v); but it suffices to ensure $u_j^i = uv$ where $\Delta(u) = \Delta(u_j^{i-1})$; as such u could be immediately replaced by u_j^{i-1} (because of $M_{j-1}^{i-1} \leqslant M_{j-1}^i$).

Thus the existence of the above-specified algorithm follows from Theorem 5.4. □

Theorem 7.27. *Problem* $\mathbb{P}_{\mathrm{key}}$ *is decidable.*

Proof. It follows from Propositions 7.7, 7.22, 7.25 (cf. Fig. 2) that the algorithm from Proposition 7.26, decides $\mathbb{P}_{\mathrm{key}}$. □

Theorem 7.28. *Problem* $\mathbb{P}_{\mathrm{temp}}$ *(including* $\mathbb{P}_{\mathrm{fdp}}$*) is decidable.*

Proof. It follows from Proposition 4.5 and Theorem 7.27. □

Conclusion

Establishing the decidability of $\mathbb{P}_{\mathrm{fdp}}$ and of (more general) $\mathbb{P}_{\mathrm{temp}}$ we have solved an open problem from [2] and [4]. Unfortunately, the presented proof does not imply $\mathbb{P}_{\mathrm{fdp}} \equiv_{\mathrm{PTIME}} \mathbb{RP}$ (our reduction was exponential); that would answer the relevant question from [4] completely.

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